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EXISTENCE OF Ψ-BOUNDED SOLUTIONS FOR A SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. In this article, we present a necessary and sufficient condition for the existence of solutions to the linear nonhomogeneous system $x' = A(t)x + f(t)$. Under the condition stated, for every Lebesgue Ψ-integrable function f there is at least one Ψ-bounded solution on the interval $(0, +\infty)$.

1. Introduction

We give a necessary and sufficient condition for the nonhomogeneous system

$$x' = A(t)x + f(t) \tag{1.1}$$

to have at least one Ψ-bounded solution for every Lebesgue Ψ-integrable function f, on the interval $\mathbb{R}_+ = [0, +\infty)$. Here Ψ is a continuous matrix function, instead of a scalar function, which allows a mixed asymptotic behavior of the components of the solution.

The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations has been studied by many authors; see for example Akinyele [1], Constantin [3], Avramescu [2], Hallam [5], and Morchalo [6]. In these papers, the function Ψ is a scalar continuous function: Increasing, differentiable, and bounded in [1]; nondecreasing with $\Psi(t) \geq 1$ on $\mathbb{R}_+$ in [3]).

Let $\mathbb{R}^d$ be the Euclidean d-space. Elements in this space are denoted by $x = (x_1, x_2, \dots x_d)^T$ and their norm by $\|x\| = \max\{|x_1|, |x_2|, \dots |x_d|\}$. For $d \times d$ real matrices, we define the norm $|A| = \sup_{\|x\| \leq 1} \|Ax\|$.

Let $\Psi_i : \mathbb{R}_+ \to (0, \infty)$, $i = 1, 2, \dots d$, be continuous functions, and let

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots \Psi_d].$$

Then the matrix $\Psi(t)$ is invertible for each $t \geq 0$.

Definition. A function $\varphi : \mathbb{R}_+ \to R^d$ is said to be Ψ-bounded on $\mathbb{R}_+$ if $\Psi(t)\varphi(t)$ is bounded on $\mathbb{R}_+$.

Definition. A function $\varphi : \mathbb{R}_+ \to R^d$ is said to be Lebesgue Ψ-integrable on $\mathbb{R}_+$ if $\varphi(t)$ is measurable and $\Psi(t)\varphi(t)$ is Lebesgue integrable on $\mathbb{R}_+$.

By a solution of (1.1), we mean an absolutely continuous function satisfying the system for almost all $t \geq 0$.

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Let A be a continuous $d \times d$ real matrix and the associated linear differential system be
$$y' = A(t)y. \tag{1.2}$$
Also let Y be the fundamental matrix of (1.2) with $Y(0) = I_d$, the identity $d \times d$ matrix.

Let X_1 denote the subspace of $\mathbb{R}^d$ consisting of all vectors which are values of Ψ-bounded solutions of (1.2) at $t = 0$. Let X_2 be an arbitrary closed subspace of $\mathbb{R}^d$, supplementary to X_1. Let P_1, P_2 denote the corresponding projections of $\mathbb{R}^d$ onto X_1, X_2.

2. The Main Results

In this section, we give the main results of this Note.

Theorem 2.1. *If A is a continuous $d \times d$ real matrix, then* (1.1) *has at least one Ψ-bounded solution on $\mathbb{R}_+$ for every Lebesgue Ψ-integrable function f on $\mathbb{R}_+$ if and only if there is a positive constant K such that*
$$\begin{aligned} &|\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \le K, \quad \text{for } 0 \le s \le t,\\ &|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \le K, \quad \text{for } 0 \le t \le s. \end{aligned} \tag{2.1}$$

Proof. First, we prove the "only if" part. We define the sets:
$C_\Psi = \{x : \mathbb{R}_+ \to \mathbb{R}^d : x \text{ is } \Psi\text{-bounded and continuous on } \mathbb{R}_+\}$,
$B = \{x : \mathbb{R}_+ \to \mathbb{R}^d : x \text{ is Lebesgue } \Psi\text{-integrable on } \mathbb{R}_+\}$,
$D = \{x : \mathbb{R}_+ \to \mathbb{R}^d : x$ is absolutely continuous on all intervals $J \subset \mathbb{R}_+$, Ψ-bounded on $\mathbb{R}_+$, $x(0)$ in X_2, $x'(t) - A(t)x(t)$ in $B\}$.

It is well-known that C_Ψ is a real Banach space with the norm
$$\|x\|_{C_\Psi} = \sup_{t\ge0} \|\Psi(t)x(t)\|.$$
Also, it is well-known that B is a real Banach space with the norm
$$\|x\|_B = \int_0^\infty \|\Psi(t)x(t)\|dt.$$
The set D is obviously a real linear space and
$$\|x\|_D = \sup_{t\ge0}\|\Psi(t)x(t)\| + \|x' - A(t)x\|_B$$
is a norm on D.

Now, we show that $(D, \|\cdot\|_D)$ is a Banach space. Let $(x_n)_n$ be a fundamental sequence in D. Then, $(x_n)_n$ is a fundamental sequence in C_Ψ. Therefore, there exists a continuous and bounded function $x : \mathbb{R}_+ \to \mathbb{R}^d$ such that
$$\lim_{n\to\infty} \Psi(t)x_n(t) = x(t), \quad \text{uniformly on } \mathbb{R}_+.$$
Denote $\bar{x}(t) = \Psi^{-1}(t)x(t) \in C_\Psi$. From
$$\|x_n(t) - \bar{x}(t)\| \le |\Psi^{-1}(t)|\|\Psi(t)x_n(t) - x(t)\|,$$
it follows that $\lim_{n\to\infty} x_n(t) = \bar{x}(t)$, uniformly on every compact of $\mathbb{R}_+$. Thus, $\bar{x}(0) \in X_2$.

On the other hand, $(f_n(t))$, where $f_n(t) = \Psi(t)(x_n'(t) - A(t)x_n(t))$, is a fundamental sequence in L, the Banach space of all vector functions which are Lebesgue integrable on $\mathbb{R}_+$ with the norm

$$\|f\| = \int_0^\infty \|\Psi(t)f(t)\|dt.$$

Thus, there is a function f in L such that

$$\lim_{n\to\infty} \int_0^\infty \|f_n(t) - f(t)\|dt = 0.$$

Putting $\bar{f}(t) = \Psi^{-1}(t)f(t)$, it follows that $\bar{f}(t) \in B$

For a fixed, but arbitrary, $t \geq 0$, we have

$$\begin{aligned}
\bar{x}(t) - \bar{x}(0) &= \lim_{n\to\infty} (x_n(t) - x_n(0)) \\
&= \lim_{n\to\infty} \int_0^t x_n'(s)ds \\
&= \lim_{n\to\infty} \int_0^t [(x_n'(s) - A(s)x_n(s)) + A(s)x_n(s)]ds \\
&= \lim_{n\to\infty} \int_0^t \{\Psi^{-1}(s)[f_n(s) - f(s)] + \bar{f}(s) + A(s)x_n(s)\}ds \\
&= \int_0^t [\bar{f}(s) + A(s)\bar{x}(s)]ds.
\end{aligned}$$

It follows that $\bar{x}'(t) - A(t)\bar{x}(t) = \bar{f}(t) \in B$ and $\bar{x}(t)$ is absolutely continuous on all intervals $J \subset \mathbb{R}_+$. Thus, $\bar{x}(t) \in D$. From $\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)\bar{x}(t)$, uniformly on $\mathbb{R}_+$ and

$$\lim_{n\to\infty} \int_0^\infty \|\Psi(t)[(x_n'(t) - A(t)x_n(t)) - (\bar{x}'(t) - A(t)\bar{x}(t))]\|dt = 0,$$

it follows that $\lim_{n\to\infty} \|x_n - \bar{x}\|_D = 0$. Thus, $(D, \|\cdot\|_D)$ is a Banach space.

Now, we define

$$T : D \to B, \quad Tx = x' - A(t)x.$$

Clearly, T is linear and bounded, with $\|T\| \leq 1$. Let $Tx = 0$. Then, $x' = A(t)x, x \in D$. This shows that x is a Ψ-bounded solution of (1.2). Then, $x(0) \in X_1 \cap X_2 = \{0\}$. Thus, $x = 0$, such that the operator T is one-to-one.

Now, let $f \in B$ and let $x(t)$ be the Ψ-bounded solution of the system (1.1). Let $z(t)$ be the solution of the Cauchy problem

$$z' = A(t)z + f(t), \quad z(0) = P_2x(0).$$

Then, $x(t) - z(t)$ is a solution of (1.2) with $P_2(x(0) - z(0)) = 0$, i.e. $x(0) - z(0) \in X_1$. It follows that $x(t) - z(t)$ is Ψ-bounded on R_+. Thus, $z(t)$ is Ψ-bounded on $\mathbb{R}_+$. It follows that $z(t) \in D$ and $Tz = f$. Consequently, the operator T is onto.

From a fundamental result of Banach: "If T is a bounded one-to-one linear operator from Banach space onto another, then the inverse operator T^{-1} is also bounded, we have that there is a positive constant $K = \|T^{-1}\| - 1$ such that, for $f \in B$ and for the solution $x \in D$ of (1.1),

$$\sup_{t\geq 0} \|\Psi(t)x(t)\| \leq K \int_0^\infty \|\Psi(t)f(t)\|.$$

For $s \geq 0$, $\delta > 0$, $\xi \in \mathbb{R}^d$, we consider the function $f : \mathbb{R}_+ \to \mathbb{R}^d$,

$$f(t) = \begin{cases} \Psi^{-1}(t)\xi, & \text{for } s \leq t \leq s+\delta \\ 0, & \text{elsewhere.} \end{cases}$$

Then, $f \in B$ and $\|f\|_B = \delta\|\xi\|$. The corresponding solution $x \in D$ is

$$x(t) = \int_s^{s+\delta} G(t,u)du,$$

where

$$G(t,u) = \begin{cases} Y(t)P_1Y^{-1}(u), & \text{for } 0 \leq u \leq t \\ -Y(t)P_2Y^{-1}(u), & \text{for } 0 \leq t \leq u. \end{cases}$$

Clearly, G is continuous except on the line $t = u$, where it has a jump discontinuity. Therefore,

$$\|\Psi(t)x(t)\| = \| \int_s^{s+\delta} \Psi(t)G(t,u)\Psi^{-1}(u)\xi du\| \leq K\delta\|\xi\|.$$

It follows that

$$\|\Psi(t)G(t,s)\Psi^{-1}(s)\xi\| \leq K\|\xi\|.$$

Hence,

$$|\Psi(t)G(t,s)\Psi^{-1}(s)| \leq K,$$

which is equivalent with (2.1). By continuity, (2.1) remains true also in the case $t = s$.

Now, we prove the "if" part. We consider the function

$$x(t) = \int_0^t Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds, t \geq 0,$$

where f is a Lebesgue Ψ-integrable function on $\mathbb{R}_+$ It is easy to see that $x(t)$ is a Ψ-bounded solution on $\mathbb{R}_+$ of (1.1). The proof is now complete. ☐

Remark. By taking $\Psi(t) = I_d$ in Theorem 2.1, the conclusion in [4, Theorem 2, Chapter V] follows.

Theorem 2.2. *Suppose that:*

(1) *The fundamental matrix $Y(t)$ of (1.2) satisfies the conditions:*
   (a) $\lim_{t\to\infty} \Psi(t)Y(t)P_1 = 0$*;*
   (b) $|\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) \leq K$, *for* $0 \leq s \leq t$,
   $|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \leq K$, *for* $0 \leq t \leq s$,

   where K is a positive constant and P_1 and P_2 are as in the Introduction

(2) *The function $f : \mathbb{R}_+ \to \mathbb{R}^d$ is Lebesgue Ψ-integrable on $\mathbb{R}_+$.*

Then, every Ψ-bounded solution $x(t)$ of (1.1) is such that

$$\lim_{t\to\infty} \|\Psi(t)x(t)\| = 0.$$

Proof. Let $x(t)$ be a Ψ-bounded solution of (1.1). There is a positive constant M such that $\|\Psi(t)x(t)\| \leq M$, for all $t \geq 0$. We consider the function

$$y(t) = x(t) - Y(t)P_1x(0) - \int_0^t Y(t)P_1Y^{-1}(s)f(s)ds + \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds$$

for all t ≥ 0.

From the hypotheses, it follows that the function $y(t)$ is a Ψ-bounded solution of (1.2). Then, $y(0) \in X_1$. On the other hand, $P_1 y(0) = 0$. Therefore, $y(0) = P_2 y(0) \in X_2$. Thus, $y(0) = 0$ and then $y(t) = 0$ for $t \geq 0$.

Thus, for $t \geq 0$ we have

$$x(t) = Y(t)P_1 x(0) + \int_0^t Y(t)P_1 Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2 Y^{-1}(s)f(s)ds.$$

Now, for a given $\varepsilon > 0$, there exists $t_1 \geq 0$ such that

$$\int_t^\infty \|\Psi(s)f(s)\|ds < \frac{\varepsilon}{2K}, \quad \text{for } t \geq t_1.$$

Moreover, there exists $t_2 > t_1$ such that, for $t \geq t_2$,

$$|\Psi(t)Y(t)P_1| \leq \frac{\varepsilon}{2}\Big[\|x(0)\| + \int_0^{t_1} \|Y^{-1}(s)f(s)\|ds\Big]^{-1}.$$

Then, for $t \geq t_2$ we have

$$\begin{aligned}
\|\Psi(t)x(t)\| &\leq |\Psi(t)Y(t)P_1|\|x(0)\| + \int_0^{t_1} |\Psi(t)Y(t)P_1|\|Y^{-1}(s)f(s)\|ds \\
&\quad + \int_{t_1}^t |\Psi(t)Y(t)P_1 Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_2 Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|ds \\
&\leq |\Psi(t)Y(t)P_1|\Big[\|x(0)\| + \int_0^{t_1} \|Y^{-1}(s)f(s)\|ds\Big] \\
&\quad + K\int_{t_1}^\infty \|\Psi(s)f(s)\|ds < \varepsilon.
\end{aligned}$$

This shows that $\lim t \to \infty \|\Psi(t)x(t)\| = 0$. The proof is now complete. $\square$

Remark. Theorem 2.2 generalizes a result in Constantin [3].

Note that Theorem 2.2 is no longer true if we require that the function f be Ψ-bounded on $\mathbb{R}_+$, instead of condition (2) of the Theorem. Even if the function f is such that

$$\lim t \to \infty \|\Psi(t)f(t)\| = 0,$$

Theorem 2.2 does not apply. This is shown by the next example.

Example. Consider the linear system (1.2) with $A(t) = O_2$. Then $Y(t) = I_2$ is a fundamental matrix for (1.2). Consider

$$\Psi(t) = \begin{pmatrix} \frac{1}{t+1} & 0 \\ 0 & t+1 \end{pmatrix}$$

We have $\Psi(t)Y(t) = \Psi(t)$, such that

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that the first hypothesis of the Theorem is satisfied with $K = 1$. When we take $f(t) = (\sqrt{t+1}, (t+1)^{-2})^T$, then $\lim_{t\to\infty} \|\Psi(t)f(t)\| = 0$. On the other hand, the solutions of the system (1.1) are

$$x(t) = \begin{pmatrix} \frac{2}{3}(t+1)^{3/2} + c_1 \\ -\frac{1}{t+1} + c_2 \end{pmatrix}$$

It follows that the solutions of the system (1.1) are Ψ-unbounded on $\mathbb{R}_+$.

Remark. When in the above example we consider

$$f(t) = \big((t+1)^{-1}, (t+1)^{-3}\big)^T,$$

then we have

$$\int_0^\infty \|\Psi(t)f(t)\|dt = 1.$$

On the other hand, the solutions of the system (1.1) are

$$x(t) = \begin{pmatrix} \ln(t+1) + c_1 \\ -\frac{1}{2}(t+1)^{-2} + c_2 \end{pmatrix}$$

It is easy to see that these solutions are Ψ-bounded on $\mathbb{R}_+$ if and only if $c_2 = 0$. In this case, $\lim_{t\to\infty} \|\Psi(t)x(t)\| = 0$.

Note that the asymptotic properties of the components of the solutions are not the same. This is obtained by using a matrix Ψ rather than a scalar.

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