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SQUARE-MEAN ALMOST PERIODIC SOLUTIONS NONAUTONOMOUS STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. This paper concerns the square-mean almost periodic solutions to a class of nonautonomous stochastic differential equations on a separable real Hilbert space. Using the so-called 'Acquistapace-Terreni' conditions, we establish the existence and uniqueness of a square-mean almost periodic mild solution to those nonautonomous stochastic differential equations.

1. Introduction

Let $(\mathbb{H}, \|\cdot\|)$ be a real (separable) Hilbert space. The present paper is mainly concerned with the existence of mean-almost periodic solutions to the class of nonautonomous semilinear stochastic differential equations

$$dX(t) = A(t)X(t)\,dt + F(t, X(t))\,dt + G(t, X(t))\,dW(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of densely defined closed linear operators satisfying the so-called 'Acquistapace-Terreni' conditions [1], that is, there exist constants $\lambda_0 \geq 0, \theta \in (\frac{\pi}{2}, \pi), L, K \geq 0$, and $\alpha,\ \beta \in (0, 1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|} \tag{1.2}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t - s|^\alpha |\lambda|^\beta$$

for $t, s \in \mathbb{R}, \lambda \in \Sigma_\theta := \{\lambda \in \mathbf{C} - \{0\} : |\arg \lambda| \leq \theta\},\ F : \mathbb{R} \times L^2(\mathbf{P}, \mathbb{H}) \to L^2(\mathbf{P}, \mathbb{H})$ and $G : \mathbb{R} \times L^2(\mathbf{P}, \mathbb{H}) \to L^2(\mathbf{P}, L_2^0)$ are jointly continuous satisfying some additional conditions, and $W(t)$ is a Wiener process.

The existence of almost periodic (respectively, periodic) solutions to autonomous stochastic differential equations has been studied by many authors, see, e.g. [1, 3, 6, 12]. In Da Prato-Tudor [5], the existence of an almost periodic solution to (1.1) in the case when $A(t)$ is periodic, that is, $A(t+T) = A(t)$ for each $t \in \mathbb{R}$ for some $T > 0$ was established. In this paper, it goes back to study the existence and uniqueness of a square-mean almost periodic solution to (1.1) when the operators $A(t)$ satisfy

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'Acquistapace-Terreni' conditions (Theorem 3.3). Next, we make extensive use of our abstract result to establish the existence of mean-almost periodic solutions to a n-dimensional system of some stochastic (parabolic) partial differential equations.

The organization of this work is as follows: in Section 2, we recall some preliminary results that we will use in the sequel. In Section 3, we give some sufficient conditions for the existence and uniqueness of a square-mean almost periodic solution to (1.1). Finally, an example is given to illustrate our main results.

2. Preliminaries

Throughout the rest of this paper, we assume that $(\mathbb{K}, \|\cdot\|_K)$ and $(\mathbb{H}, \|\cdot\|)$ are real separable Hilbert spaces, and $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. The space $L_2(\mathbb{K}, \mathbb{H})$ stands for the space of all Hilbert-Schmidt operators acting from $\mathbb{K}$ into $\mathbb{H}$, equipped with the Hilbert-Schmidt norm $\|\cdot\|_2$.

For a symmetric nonnegative operator $Q \in L_2(\mathbb{K}, \mathbb{H})$ with finite trace we assume that $\{W(t), t \in \mathbb{R}\}$ is a Q-Wiener process defined on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $\mathbb{K}$. Recall that W can obtained as follows: let $\{W_i(t), t \in \mathbb{R}\}$, $i = 1, 2$, be independent K-valued Q-Wiener processes, then

$$W(t) = \begin{cases} W_1(t) & \text{if } t \geq 0, \\ W_2(-t) & \text{if } t \leq 0, \end{cases}$$

is Q-Wiener process with $\mathbb{R}$ as time parameter. We let $\mathcal{F}_t = \sigma\{W(s), s \leq t\}$.

The collection of all strongly measurable, square-integrable $\mathbb{H}$-valued random variables, denoted by $L^2(\mathbf{P}, \mathbb{H})$, is a Banach space when it is equipped with norm $\|X\|_{L^2(\mathbf{P},\mathbb{H})} = (\mathbf{E}\|X\|^2)^{1/2}$, where the expectation $\mathbf{E}$ is defined by

$$\mathbf{E}[g] = \int_\Omega g(\omega) d\mathbf{P}(\omega).$$

Let $\mathbb{K}_0 = Q^{1/2}K$ and let $L_2^0 = L_2(\mathbb{K}_0, \mathbb{H})$ with respect to the norm

$$\|\Phi\|^2_{\mathbb{L}_2^0} = \|\Phi\, Q^{1/2}\|_2^2 = \operatorname{Trace}(\Phi\, Q \Phi^*)\,.$$

Throughout, we assume that $A(t) : D(A(t)) \subset L^2(\mathbf{P}; \mathbb{H}) \to L^2(\mathbf{P}; \mathbb{H})$ is a family of densely defined closed linear operators on a common domain $D = D(A(t))$, which is independent of t and dense in $L^2(\mathbf{P}; \mathbb{H})$, and $F : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \mapsto L^2(\mathbf{P}; \mathbb{H})$ and $G : \mathbb{R} \times L^2(\mathbf{P}; \mathbb{H}) \mapsto L^2(\mathbf{P}; L_2^0)$ are jointly continuous functions.

We suppose that the system

$$\begin{aligned} u'(t) &= A(t)u(t) \quad t \geq s, \\ u(s) &= x \in L^2(\mathbf{P}; \mathbb{H}), \end{aligned} \tag{2.1}$$

has an associated evolution family of operators $\{U(t, s) : t \geq s \text{ with } t, s \in \mathbb{R}\}$, which is uniformly asymptotically stable.

If $\mathbb{B}_1, \mathbb{B}_2$ are Banach spaces, then the notation $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ stands for the Banach space of bounded linear operators from $\mathbb{B}_1$ into B_2. When $\mathbb{B}_1 = B_2$, this is simply denoted $\mathcal{L}(\mathbb{B}_1)$.

Definition 2.1. A family of bounded linear operators $\{U(t, s) : t \geq s \text{ with } t, s \in \mathbb{R}\}$ on $L^2(\mathbf{P}; \mathbb{H})$ is called an evolution family of operators for (2.1) whenever the following conditions hold:

(a) $U(t, s)U(s, r) = U(t, r)$ for every $r \leq s \leq t$;

(b) for each $x \in \mathbb{X}$ the function $(t,s) \to U(t,s)x$ is continuous and $U(t,s) \in \mathcal{L}(L^2(\mathbf{P};\mathbb{H}), D)$ for every $t > s$; and
(c) the function $(s,t] \to \mathcal{L}(L^2(\mathbf{P};\mathbb{H}))$, $t \to U(t,s)$ is differentiable with
$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s).$$

For additional details on evolution families, we refer the reader to the book by Lunardi [9].

For the reader's convenience, we review some basic definitions and results for the notion of square-mean almost periodicity.

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space.

Definition 2.2. A stochastic process $X : \mathbb{R} \to L^2(\mathbf{P};\mathbb{B})$ is said to be continuous whenever
$$\lim_{t\to s} \mathbf{E}\|X(t) - X(s)\|^2 = 0.$$

Definition 2.3. [3] A continuous stochastic process $X : \mathbb{R} \to L^2(\mathbf{P};\mathbb{B})$ is said to be square-mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which
$$\sup_{t\in\mathbf{R}} \mathbf{E}\|X(t+\tau) - X(t)\|^2 < \varepsilon.$$

The collection of all stochastic processes $X : \mathbb{R} \to L^2(\mathbf{P};\mathbb{B})$ which are square-mean almost periodic is then denoted by $AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{B}))$.

The next lemma provides with some properties of the square-mean almost periodic processes.

Lemma 2.4 ([3])**.** *If X belongs to $AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{B}))$, then*
(i) *the mapping $t \to \mathbf{E}\|X(t)\|^2$ is uniformly continuous;*
(ii) *there exists a constant $M > 0$ such that $\mathbf{E}\|X(t)\|^2 \le M$, for all $t \in \mathbb{R}$.*

Let $\mathbf{CUB}(\mathbb{R}; L^2(\mathbf{P};\mathbb{B}))$ denote the collection of all stochastic processes $X : \mathbb{R} \mapsto L^2(\mathbf{P};\mathbb{B})$, which are continuous and uniformly bounded. It is then easy to check that $\mathbf{CUB}(\mathbb{R}; L^2(\mathbf{P};\mathbb{B}))$ is a Banach space when it is equipped with the norm:
$$\|X\|_\infty = \sup_{t\in\mathbb{R}} \left(\mathbf{E}\|X(t)\|^2\right)^{1/2}.$$

Lemma 2.5 ([3])**.** *$AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{B})) \subset \mathbf{CUB}(\mathbb{R}; L^2(\mathbf{P};\mathbb{B}))$ is a closed subspace.*

In view of the above, the space $AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{B}))$ of square-mean almost periodic processes equipped with the norm $\|\cdot\|_\infty$ is a Banach space.

Let $(\mathbb{B}_1, \|\cdot\|_1)$ and $(\mathbb{B}_2, \|\cdot\|_2)$ be Banach spaces and let $L^2(\mathbf{P};\mathbb{B}_1)$ and $L^2(\mathbf{P};\mathbb{B}_2)$ be their corresponding L^2-spaces, respectively.

Definition 2.6. [3] *A function $F : \mathbb{R} \times L^2(\mathbf{P};\mathbb{B}_1) \to L^2(\mathbf{P};\mathbb{B}_2))$, $(t,Y) \mapsto F(t,Y)$, which is jointly continuous, is said to be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$ where $\mathbb{K} \subset L^2(\mathbf{P};\mathbb{B}_1)$ is a compact if for any $\varepsilon > 0$, there exists $l(\varepsilon,\mathbb{K}) > 0$ such that any interval of length $l(\varepsilon,\mathbb{K})$ contains at least a number τ for which*
$$\sup_{t\in\mathbf{R}} \mathbf{E}\|F(t+\tau,Y) - F(t,Y)\|_2^2 < \varepsilon$$
for each stochastic process $Y : \mathbb{R} \to \mathbb{K}$.

Theorem 2.7 ([3]). *Let $F : \mathbb{R} \times L^2(\mathbf{P};\mathbb{B}_1) \to L^2(\mathbf{P};\mathbb{B}_2)$, $(t,Y) \mapsto F(t,Y)$ be a square-mean almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$, where $\mathbb{K} \subset L^2(\mathbf{P};\mathbb{B}_1)$ is compact. Suppose that F is Lipschitz in the following sense:*

$$\mathbf{E}\|F(t,Y) - F(t,Z)\|_2^2 \le M\mathbf{E}\|Y - Z\|_1^2$$

for all $Y, Z \in L^2(\mathbf{P};\mathbb{B}_1)$ and for each $t \in \mathbb{R}$, where $M > 0$. Then for any square-mean almost periodic process $\Phi : \mathbb{R} \to L^2(\mathbf{P};\mathbb{B}_1)$, the stochastic process $t \mapsto F(t,\Phi(t))$ is square-mean almost periodic.

3. Main Result

Throughout this section, we require the following assumptions:

(H0) The operators $A(t)$, $U(r,s)$ commute and that the evolution family $U(t,s)$ is asymptotically stable. Namely, there exist some constants $M, \delta > 0$ such that
$$\|U(t,s)\| \le Me^{-\delta(t-s)} \quad \text{for every } t \ge s.$$
In addition, $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R};\mathcal{L}(L^2(\mathbf{P},\mathbb{H})))$ for λ_0 in (1.2);

(H1) The function $F : \mathbb{R} \times L^2(\mathbf{P};\mathbb{H}) \to L^2(\mathbf{P};\mathbb{H})$, $(t,X) \mapsto F(t,X)$ be a square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in \mathcal{O}$ ($\mathcal{O} \subset L^2(\mathbf{P};\mathbb{H})$ being a compact subspace). Moreover, F is Lipschitz in the following sense: there exists $K > 0$ for which
$$\mathbf{E}\|F(t,X) - F(t,Y)\|^2 \le K\mathbf{E}\|X - Y\|^2$$
for all stochastic processes $X, Y \in L^2(\mathbf{P};\mathbb{H})$ and $t \in \mathbb{R}$;

(H2) The function $G : \mathbb{R} \times L^2(\mathbf{P};\mathbb{H}) \to L^2(\mathbf{P};\mathbb{L}_2^0)$, $(t,X) \mapsto F(t,X)$ be a square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in \mathcal{O}'$ ($\mathcal{O}' \subset L^2(\mathbf{P};\mathbb{H})$ being a compact subspace). Moreover, G is Lipschitz in the following sense: there exists $K' > 0$ for which
$$\mathbf{E}\|G(t,X) - G(t,Y)\|_{\mathbb{L}_2^0}^2 \le K'\mathbf{E}\|X - Y\|^2$$
for all stochastic processes $X, Y \in L^2(\mathbf{P};\mathbb{H})$ and $t \in \mathbb{R}$.

In order to study (1.1) we need the following lemma which can be seen as an immediate consequence of [10, Proposition 4.4].

Lemma 3.1. *Suppose $A(t)$ satisfies the 'Acquistapace-Terreni' conditions, $U(t,s)$ is exponentially stable and $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R};\mathcal{L}(L^2(\mathbf{P},\mathbb{H})))$. Let $h > 0$. Then, for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval of length l contains at least a number τ with the property that*

$$\|U(t+\tau, s+\tau) - U(t,s)\| \le \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \ge h$.

Definition 3.2. A $\mathcal{F}_t$-progressively process $\{X(t)\}_{t\in\mathbb{R}}$ is called a mild solution of (1.1) on $\mathbb{R}$ if

$$\begin{aligned} X(t) = U(t,s)X(s) &+ \int_s^t U(t,\sigma)F(\sigma, X(\sigma))\,d\sigma \\ &+ \int_s^t U(t,\sigma)G(\sigma, X(\sigma))\,dW(\sigma) \end{aligned} \tag{3.1}$$

for all $t \ge s$ for each $s \in \mathbb{R}$.

Now, we are ready to present our main result.

Theorem 3.3. *Under assumptions* (H0)—(H2)*, then* (1.1) *has a unique square-mean almost period mild solution, which can be explicitly expressed as follows:*

$$X(t) = \int_{-\infty}^{t} U(t,\sigma)F(\sigma, X(\sigma))\,d\sigma + \int_{-\infty}^{t} U(t,\sigma)G(\sigma, X(\sigma))\,dW(\sigma) \quad \textit{for each } t \in \mathbb{R}$$

whenever

$$\Theta := M^2\Big(2\frac{K}{\delta^2} + \frac{K' \cdot \mathrm{Tr}(Q)}{\delta}\Big) < 1.$$

Proof. First of all, note that

$$X(t) = \int_{-\infty}^{t} U(t,\sigma)F(\sigma, X(\sigma))\,d\sigma + \int_{-\infty}^{t} U(t,\sigma)G(\sigma, X(\sigma))\,dW(\sigma) \tag{3.2}$$

is well-defined and satisfies

$$X(t) = U(t,s)X(s) + \int_{s}^{t} U(t,\sigma)F(\sigma, X(\sigma))\,d\sigma + \int_{s}^{t} U(t,\sigma)G(\sigma, X(\sigma))\,dW(\sigma)$$

for all $t \geq s$ for each $s \in \mathbb{R}$, and hence X given by (3.1) is a mild solution to (1.1).

Define

$$\Phi X(t) := \int_{-\infty}^{t} U(t,\sigma)F(\sigma, X(\sigma))\,d\sigma,$$
$$\Psi X(t) := \int_{-\infty}^{t} U(t,\sigma)G(\sigma, X(\sigma))\,dW(\sigma).$$

Let us show that $\Phi X(\cdot)$ is square-mean almost periodic whenever X is. Indeed, assuming that X is square-mean almost periodic and using (H1), Theorem 2.7, and Lemma 3.1, given $\varepsilon > 0$, one can find $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$\|U(t+\tau, s+\tau) - U(t,s)\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \geq \varepsilon$, and

$$\mathbf{E}\,\|F(\sigma+\tau, X(\sigma+\tau)) - F(\sigma, X(\sigma))\|^2 < \eta$$

for each $\sigma \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, it follows from Lemma 2.4 (ii) that there exists a positive constant K_1 such that

$$\sup_{\sigma \in \mathbb{R}} \mathbf{E}\|F(\sigma, X(\sigma))\|^2 \leq K_1\,.$$

Now

$$\begin{aligned}
&\big\|(\Phi X)(t+\tau) - (\Phi X)(t)\big\| \\
&= \Big\| \int_{-\infty}^{t+\tau} U(t+\tau, s)F(s, X(s))\,ds - \int_{-\infty}^{t} U(t,s)F(s,X(s))\,ds \Big\| \\
&= \Big\| \int_{0}^{\infty} U(t+\tau, t+\tau-s)\,F(t+\tau-s, X(t+\tau-s))\,ds \\
&\quad - \int_{0}^{\infty} U(t, t-s)\,F(t-s, X(t-s))\,ds \Big\| \\
&\leq \Big\| \int_{0}^{\infty} U(t+\tau, t+\tau-s)[F(t+\tau-s, X(t+\tau-s)) - F(t-s, X(t-s))]\,ds \Big\| \\
&\quad + \Big\| \Big(\int_{\varepsilon}^{\infty} + \int_{0}^{\varepsilon}\Big)[U(t+\tau, t+\tau-s)) - U(t,t-s)]F(t-s, X(t-s))\,ds \Big\|.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&\mathbf{E}\|\Phi X(t+\tau)-\Phi X(t)\|^2\\
&\le 3\mathbf{E}\Big[\int_0^\infty \|U(t+\tau,t+\tau-s)\|\|F(t+\tau-s,X(t+\tau-s))\\
&\quad -F(t-s,X(t-s))\|\,ds\Big]^2\\
&\quad +3\mathbf{E}\Big[\int_\varepsilon^\infty \|U(t+\tau,t+\tau-s)-U(t,t-s)\|\|F(t-s,X(t-s))\|\,ds\Big]^2\\
&\quad +3\mathbf{E}\Big[\int_0^\varepsilon \|U(t+\tau,t+\tau-s)-U(t,t-s)\|\|F(t-s,X(t-s))\|\,ds\Big]^2\\
&\le 3M^2\mathbf{E}\Big[\int_0^\infty e^{-\delta s}\|F(t+\tau-s,X(t+\tau-s))-F(t-s,X(t-s))\|\,ds\Big]^2\\
&\quad +3\varepsilon^2\mathbf{E}\Big[\int_\varepsilon^\infty e^{-\frac{\delta}{2}s}\|F(t-s,X(t-s))\|\,ds\Big]^2\\
&\quad +3M^2\mathbf{E}\Big[\int_0^\varepsilon 2e^{-\delta s}\|F(t-s,X(t-s))\|\,ds\Big]^2.
\end{aligned}
$$

Using Cauchy-Schwarz inequality it follows that

$$
\begin{aligned}
&\mathbf{E}\|\Phi X(t+\tau)-\Phi X(t)\|^2\\
&\le 3M^2\Big(\int_0^\infty e^{-\delta s}\,ds\Big)\\
&\quad \times\Big(\int_0^\infty e^{-\delta s}\mathbf{E}\|F(t+\tau-s,X(t+\tau-s))-F(t-s,X(t-s))\|^2\,ds\Big)\\
&\quad +3\varepsilon^2\Big(\int_\varepsilon^\infty e^{-\frac{\delta}{2}s}\,ds\Big)\Big(\int_\varepsilon^\infty e^{-\frac{\delta}{2}s}\mathbf{E}\|F(t-s,X(t-s))\|^2\,ds\Big)\\
&\quad +12M^2\Big(\int_0^\infty e^{-\delta s}\,ds\Big)\Big(\int_0^\varepsilon e^{-\delta s}\mathbf{E}\|F(t-s,X(t-s))\|^2\,ds\Big)^2\\
&\quad \le 3M^2\Big(\int_0^\infty e^{-\delta s}\,ds\Big)^2\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|F(\sigma+\tau,X(\sigma+\tau))-F(\sigma,X(\sigma))\|^2\\
&\quad +3\varepsilon^2\Big(\int_\varepsilon^\infty e^{-\frac{\delta}{2}s}\,ds\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|F(\sigma,X(\sigma))\|^2\\
&\quad +12M^2\Big(\int_0^\infty e^{-\delta s}\,ds\Big)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|F(\sigma,X(\sigma))\|^2\\
&\le 3\frac{M^2}{\delta^2}\eta+3\varepsilon^2\frac{4}{\delta^2}K_1+12M^2\varepsilon^2K_1\,,
\end{aligned}
$$

which implies that $\Phi X(\cdot)$ is square-mean almost periodic.

Similarly, assuming that X is square-mean almost periodic and using (H2), Theorem 2.7, and Lemma 3.1, given $\varepsilon>0$, one can find $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains at least τ with the property that

$$
\|U(t+\tau,s+\tau)-U(t,s)\|\le\varepsilon e^{-\frac{\delta}{2}(t-s)}
$$

for all $t-s\ge\varepsilon$, and

$$
\mathbf{E}\,\|G(\sigma+\tau,X(\sigma+\tau))-G(\sigma,X(\sigma))\|^2_{\mathbb{L}_2^0}<\eta
$$

for each $\sigma \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, it follows from Lemma 2.4 (ii) that there exists a positive constant K_2 such that

$$\sup_{\sigma\in\mathbb{R}} \mathbf{E}\|G(\sigma, X(\sigma))\|^2_{\mathbb{L}_2^0} \le K_2.$$

The next step consists of proving the square-mean almost periodicity of $\Psi X(\cdot)$. Of course, this is more complicated than the previous case because of the involvement of the Wiener process W. To overcome such a difficulty, we make extensive use of the properties of $\tilde{W}$ defined by $\tilde{W}(s) := W(s+\tau) - W(\tau)$ for each s. Note that $\tilde{W}$ is also a Wiener process and has the same distribution as W.

Now, let us make an appropriate change of variables to get

$$\begin{aligned}
&\mathbf{E}\|(\Psi X)(t+\tau) - (\Psi X)(t)\|^2 \\
&= \| \int_0^\infty U(t+\tau, t+\tau-s)\, G(t+\tau-s, X(t+\tau-s))\, d\tilde{W}(s) \\
&\quad - \int_0^\infty U(t, t-s)\, G(t-s, X(t-s))\, d\tilde{W}(s)\|^2 \\
&\le 3\mathbf{E}\| \int_0^\infty U(t+\tau, t+\tau-s)\, [G(t+\tau-s, X(t+\tau-s)) \\
&\quad - G(t-s, X(t-s))]\, d\tilde{W}(s)\|^2 \\
&\quad + 3\mathbf{E}\| \int_\varepsilon^\infty [U(t+\tau, t+\tau-s) - U(t,t-s)]\, G(t-s, X(t-s))\, d\tilde{W}(s)\|^2 \\
&\quad + 3\mathbf{E}\| \int_0^\varepsilon [U(t+\tau, t+\tau-s) - U(t,t-s)]\, G(t-s, X(t-s))\, d\tilde{W}(s)\|^2.
\end{aligned}$$

Then using an estimate on the Ito integral established in [7, Proposition 1.9], we obtain

$$\begin{aligned}
&\mathbf{E}\|(\Psi X)(t+\tau) - (\Psi X)(t)\|^2 \\
&\le 3\operatorname{Tr} Q \int_0^\infty \|U(t+\tau, t+\tau-s)\|^2 \mathbf{E}\|G(t+\tau-s, X(t+\tau-s)) \\
&\quad - G(t-s, X(t-s))\|^2_{\mathbb{L}_2^0}\, ds \\
&\quad + 3\operatorname{Tr} Q \int_\varepsilon^\infty \|U(t+\tau, t+\tau-s) - U(t,t-s)\|^2 \mathbf{E}\|G(t-s, X(t-s))\|^2_{\mathbb{L}_2^0}\, ds \\
&\quad + 3\operatorname{Tr} Q \int_0^\varepsilon \|U(t+\tau, t+\tau-s) - U(t,t-s)\|^2 \mathbf{E}\|G(t-s, X(t-s))\|^2_{\mathbb{L}_2^0}\, ds \\
&\le 3\operatorname{Tr} Q M^2 \Big(\int_0^\infty e^{-2\delta s}\, ds\Big) \sup_{\sigma\in\mathbb{R}} \|G(\sigma+\tau, X(\sigma+\tau)) - G(\sigma, X(\sigma))\|^2_{\mathbb{L}_2^0} \\
&\quad + 3\operatorname{Tr} Q \varepsilon^2 \Big(\int_\varepsilon^\infty e^{-\delta\, s}\, ds\Big) \sup_{\sigma\in\mathbb{R}} \mathbf{E}\|G(\sigma, X(\sigma))\|^2_{\mathbb{L}_2^0} \\
&\quad + 6\operatorname{Tr} Q M^2 \Big(\int_0^\varepsilon e^{-2\delta s}\, ds\Big) \sup_{\sigma\in\mathbb{R}} \mathbf{E}\|G(\sigma, X(\sigma))\|^2_{\mathbb{L}_2^0} \\
&\le 3\operatorname{Tr} Q \big[\eta \frac{M^2}{2\delta} + \varepsilon \frac{K_2}{\delta} + 2\varepsilon\, K_2\big],
\end{aligned}$$

which implies that $\Psi X(\cdot)$ is square-mean almost periodic. Define

$$(\Lambda X)(t) := \int_{-\infty}^{t} U(t,s)F(s,X(s))\,ds + \int_{-\infty}^{t} U(t,s)G(s,X(s))\,dW(s)\,.$$

In view of the above, it is clear that Λ maps $AP(\mathbb{R}; L^2(\mathbf{P};\mathbb{H}))$ into itself. To complete the proof, it suffices to prove that Λ has a unique fixed-point. Clearly,

$$\begin{aligned}
&\|(\Lambda X)(t) - (\Lambda Y)(t)\| \\
&= \|\int_{-\infty}^{t} U(t,s)[F(s,X(s)) - F(s,Y(s))]\,ds \\
&\quad + \int_{-\infty}^{t} U(t,s)[G(s,X(s)) - G(s,Y(s))]\,dW(s)\| \\
&\le M \int_{-\infty}^{t} e^{-\delta(t-s)} \|F(s,X(s)) - F(s,Y(s))\|\,ds \\
&\quad + \|\int_{-\infty}^{t} U(t,s)[G(s,X(s)) - G(s,Y(s))]\,dW(s)\|\,.
\end{aligned}$$

Since $(a+b)^2 \le 2a^2 + 2b^2$, we can write

$$\begin{aligned}
&\mathbf{E}\|(\Lambda X)(t) - (\Lambda Y)(t)\|^2 \\
&\le 2M^2 \mathbf{E}\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} \|F(s,X(s)) - F(s,Y(s))\|\,ds\Big)^2 \\
&\quad + 2\mathbf{E}\Big(\|\int_{-\infty}^{t} U(t,s)[G(s,X(s)) - G(s,Y(s))]\,dW(s)\|\Big)^2\,.
\end{aligned}$$

We evaluate the first term of the right-hand side as follows:

$$\begin{aligned}
&\mathbf{E}\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} \|F(s,X(s)) - F(s,Y(s))\|\,ds\Big)^2 \\
&\le \mathbf{E}\Big[\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} ds\Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} \|F(s,X(s)) - F(s,Y(s))\|^2\,ds\Big)\Big] \\
&\le \Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} \mathbf{E}\|F(s,X(s)) - F(s,Y(s))\|^2\,ds\Big) \\
&\le K \cdot \Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} \mathbf{E}\|X(s)) - Y(s))\|^2\,ds\Big) \\
&\le K \cdot \Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\Big)^2 \sup_{t\in\mathbb{R}} \mathbf{E}\|X(t) - Y(t)\|^2 \\
&= K \cdot \Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\Big)^2 \|X - Y\|_\infty \\
&\le \frac{K}{\delta^2} \cdot \|X - Y\|_\infty\,.
\end{aligned}$$

As to the second term, we use again an estimate on the Ito integral established in [7] to obtain:

$$
\begin{aligned}
&\mathbf{E}\Big(\| \int_{-\infty}^{t} U(t,s)\,[G(s,X(s)) - G(s,Y(s))]\,dW(s)\|\Big)^2 \\
&\quad \le \mathrm{Tr}\ \mathrm{Q} \cdot \mathbf{E}\Big[\int_{-\infty}^{t} \|U(t,s)\,[G(s,X(s)) - G(s,Y(s))]\|^2\,ds\Big] \\
&\le \mathrm{Tr}\, Q \cdot \mathbf{E}\Big[\int_{-\infty}^{t} \|U(t,s)\|^2 \|G(s,X(s)) - G(s,Y(s))\|^2_{\mathbb{L}_2^0}\,ds\Big] \\
&\le \mathrm{Tr}\, Q \cdot M^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} \mathbf{E}\|G(s,X(s)) - G(s,Y(s))\|^2_{\mathbb{L}_2^0}\,ds \\
&\le \mathrm{Tr}\, Q \cdot M^2 K' \cdot \Big(\int_{-\infty}^{t} e^{-2\delta(t-s)}\,ds\Big) \sup_{t\in R} \mathbf{E}\|X(s)) - Y(s))\|^2 \\
&\le \mathrm{Tr}\, Q \cdot \frac{M^2 K'}{2\delta} \cdot \|X - Y\|_\infty\,.
\end{aligned}
$$

Thus, by combining, it follows that

$$
\mathbf{E}\|(\Lambda X)(t) - (\Lambda Y)(t)\| \le M^2\Big(2\frac{K}{\delta^2} + \frac{K' \cdot \mathrm{Tr}\, Q}{\delta}\Big)\|X - Y\|_\infty,
$$

and therefore,

$$
\|\Lambda X - \Lambda Y\|_\infty \le M^2\Big(2\frac{K}{\delta^2} + \frac{K' \cdot \mathrm{Tr}\, Q}{\delta}\Big)\|X - Y\|_\infty = \Theta \cdot \|X - Y\|_\infty.
$$

Consequently, if $\Theta < 1$, then (1.1) has a unique fixed-point, which obviously is the unique square-mean almost periodic solution to (1.1). □

4. Example

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded subset whose boundary $\partial\mathcal{O}$ is of class C^2 and being locally on one side of $\mathcal{O}$.

Consider the parabolic stochastic partial differential equation

$$
d_t X(t,\xi) = \{A(t,\xi)X(t,\xi) + F(t,X(t,\xi))\}\,d_t + G(t,X(t,\xi))\,dW(t), \tag{4.1}
$$

$$
\sum_{i,j=1}^{n} n_i(\xi) a_{ij}(t,\xi) d_i X(t,\xi) = 0, \quad t \in \mathbb{R},\ \xi \in \partial\mathcal{O}, \tag{4.2}
$$

where $d_t = \frac{d}{dt}$, $d_i = \frac{d}{d\xi_i}$, $n(\xi) = (n_1(\xi), n_2(\xi), \dots, n_n(\xi))$ is the outer unit normal vector, the family of operators $A(t,\xi)$ are formally given by

$$
A(t,\xi) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(t,\xi)\frac{\partial}{\partial x_j}\Big) + c(t,\xi), \quad t \in \mathbb{R},\ \xi \in \mathcal{O},
$$

W is a real valued Brownian motion, and a_{ij}, c $(i,j = 1,2,\dots,n)$ satisfy the following conditions:

(H3)

(i) The coefficients $(a_{ij})_{i,j=1,\dots,n}$ are symmetric, that is, $a_{ij} = a_{ji}$ for all $i,j = 1,\dots,n$. Moreover, $a_{ij} \in C_b^\mu(\mathbb{R}, L^2(\mathbf{P}, C(\overline{\mathcal{O}}))) \cap C_b(\mathbb{R}, L^2(\mathbf{P}, C^1(\overline{\mathcal{O}}))) \cap AP(\mathbb{R}; L^2(\mathbf{P}, L^2(\mathcal{O})))$ for all $i,j = 1,\dots n$, and $c \in C_b^\mu(\mathbb{R}, L^2(\mathbf{P}, L^2(\mathcal{O}))) \cap C_b(\mathbb{R}, L^2(\mathbf{P}, C(\overline{\mathcal{O}}))) \cap AP(\mathbb{R}; L^2(\mathbf{P}, L^1(\mathcal{O})))$ for some $\mu \in (1/2, 1]$.

(ii) There exists $\varepsilon_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,\xi)\eta_i\eta_j \geq \varepsilon_0|\eta|^2,$$

for all $(t,\xi) \in \mathbb{R} \times \overline{\mathcal{O}}$ and $\eta \in \mathbb{R}^n$.

Under above assumptions, the existence of an evolution family $U(t,s)$ satisfying (H0) is obtained, see, eg., [10].

Set $\mathbb{H} = L^2(\mathcal{O})$. For each $t \in \mathbb{R}$ define an operator $A(t)$ on $L^2(\mathbf{P}; H)$ by

$$\mathcal{D}(A(t)) = \{X \in L^2(\mathbf{P}, H^2(\mathcal{O})) : \sum_{i,j=1}^{n} n_i(\cdot)a_{ij}(t,\cdot)d_iX(t,\cdot) = 0 \quad \text{on } \partial\mathcal{O}\}$$

and $A(t)X = A(t,\xi)X(\xi)$ for all $X \in \mathcal{D}(A(t))$.

Thus under assumptions (H1)–(H3), then the system (4.1)–(4.2) has a unique mild solution, which obviously is square-mean almost periodic, whenever M is small enough.

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