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SUBCRITICAL PERTURBATIONS OF RESONANT LINEAR PROBLEMS WITH SIGN-CHANGING POTENTIAL

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Abstract. We establish existence and multiplicity theorems for a Dirichlet boundary-value problem at resonance. This problem is a nonlinear subcritical perturbation of a linear eigenvalue problem studied by Cuesta, and includes a sign-changing potential. We obtain solutions using the Mountain Pass lemma and the Saddle Point theorem. Our paper extends some recent results of Gonçalves, Miyagaki, and Ma.

1. Introduction and main results

Let Ω be an arbitrary open set in $\mathbb{R}^N$, $N \geq 2$, and let $V : \Omega \to \mathbb{R}$ be a variable potential. Then we consider the eigenvalue problem

$$-\Delta u = \lambda V(x)u \quad \text{in } \Omega\,, \quad u \in H_0^1(\Omega). \tag{1.1}$$

Problems of this type have a long history. If Ω is bounded and $V \equiv 1$, problem (1.1) is related to the Riesz-Fredholm theory of self-adjoint and compact operators (see, e.g., Brezis [3, Theorem VI.11]). The case of a non-constant potential V was first considered in the pioneering papers of Bocher [2], Hess and Kato [7], Minakshisundaran and Pleijel [10] and Pleijel [11]. Minakshisundaran and Pleijel [10], [11] studied the case where Ω is bounded, $V \in L^\infty(\Omega)$, $V \geq 0$ in Ω and $V > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$. An important contribution in the study of Problem (1.1) if Ω and V are not necessarily bounded has been given recently by Cuesta [5] (see also Szulkin and Willem [14]) under the assumption that the sign-changing potential V satisfies

$$V^+ \not\equiv 0 \quad \text{and} \quad V \in L^s(\Omega)\,, \tag{1.2}$$

where $s > N/2$ if $N \geq 2$ and $s = 1$ if $N = 1$. As usual, we have denoted $V^+(x) = \max\{V(x), 0\}$. Obviously, $V = V^+ - V^-$, where $V^-(x) = \max\{-V(x), 0\}$.

To study the main properties (isolation, simplicity) of the principal eigenvalue of (1.1), Cuesta [5] proved that the minimization problem

$$\min\Big\{\int_\Omega |\nabla u|^2 dx;\ u \in H_0^1(\Omega),\ \int_\Omega V(x)u^2 dx = 1\Big\}$$

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has a positive solution $\varphi_1 = \varphi_1(\Omega)$ which is an eigenfunction of (1.1) corresponding to the eigenvalue $\lambda_1 := \lambda_1(\Omega) = \int_\Omega |\nabla\varphi_1|^2 dx$.

Our purpose in this paper is to study the existence of solutions of the perturbed nonlinear boundary-value problem

$$\begin{gathered} -\Delta u = \lambda_1 V(x)u + g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u \not\equiv 0 \quad \text{in } \Omega, \end{gathered} \tag{1.3}$$

where V satisfies (1.2) and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(x,0) = 0$ with subcritical growth, that is,

$$|g(x,s)| \le a_0 \cdot |s|^{r-1} + b_0, \quad \text{for all } s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{1.4}$$

for some constants a_0, $b_0 > 0$, where $2 \le r < 2^*$. We recall that 2^* denotes the critical Sobolev exponent; that is, $2^* := 2N/(N-2)$ if $N \ge 3$ and $2^* = +\infty$ if $N \in \{1,2\}$.

Problem (1.3) is resonant at infinity and has been first studied by Landesman and Lazer [8] in connection with concrete problems arising in Mechanics.

By multiplication with φ_1 in (1.3) and integration over Ω we deduce that this problem has no solution if $g \not\equiv 0$ does not change sign in Ω. The main purpose of this paper is to establish sufficient conditions on g in order to obtain the existence of one or several solutions of the nonlinear Dirichlet problem (1.3).

Set $G(x,s) = \int_0^s g(x,t)dt$. For the rest of this paper, we assume that there exist $k,\ m \in L^1(\Omega)$, with $m \ge 0$, such that

$$|G(x,s)| \le k(x), \quad \text{for all } s \in \mathbb{R}, \text{ a.e. } x \in \Omega \tag{1.5}$$

$$\liminf_{s\to 0} \frac{G(x,s)}{s^2} = m(x), \quad \text{a.e. } x \in \Omega\,. \tag{1.6}$$

The energy functional associated to Problem (1.3) is

$$F(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1 V(x)u^2)dx - \int_\Omega G(x,u)dx\,,$$

for all $u \in H_0^1(\Omega)$.

From the variational characterization of λ_1 and using (1.5) we obtain

$$F(u) \ge -\int_\Omega G(x,u(x))dx \ge -|k|_1 > -\infty\,,$$

for all $u \in H_0^1(\Omega)$ and, consequently, F is bounded from below. Let us consider $u_n = \alpha_n\varphi_1$, where $\alpha_n \to \infty$. Then the estimate $\int_\Omega |\nabla\varphi_1|^2 dx = \lambda_1 \int_\Omega V(x)\varphi_1^2 dx$ yields $F(u_n) = -\int_\Omega G(x,\alpha_n\varphi_1)dx \le |k|_1 < \infty$. Thus, $\lim_{n\to\infty} F(u_n) < \infty$. Hence the sequence $(u_n)_n \subset H_0^1(\Omega)$ defined by $u_n = \alpha_n\varphi_1$ satisfies $\|u_n\| \to \infty$ and $F(u_n)$ is bounded. In conclusion, if we suppose that (1.5) holds, then the energy functional F is bounded from below and is not coercive.

Our first result is the following.

Theorem 1.1. *Assume that for all $\omega \subset \Omega$ with $|\Omega \setminus \omega| > 0$ we have*

$$\int_\omega \limsup_{|s|\to\infty} G(x,s)dx \le 0 \quad \textit{and} \quad \int_{\Omega\setminus\omega} G(x,s)dx \le 0 \tag{1.7}$$

and

$$\int_\Omega \limsup_{|s|\to\infty} G(x,s)dx \le 0\,. \tag{1.8}$$

Then Problem (1.3) *has at lest one solution.*

Denote $V := \mathrm{Sp}(\varphi_1)$. Since $1 = \dim V < \infty$, there exists a closed complementary subspace W of V, that is, $W \cap V = \{0\}$ and $H_0^1(\Omega) = V \oplus W$. For such a closed complementary subspace $W \subset H_0^1(\Omega)$, denote

$$\lambda_W := \inf\left\{ \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega V(x) w^2 dx};\ w \in W,\ w \ne 0 \right\}.$$

The following result establishes a multiplicity result, provided G satisfies a certain subquadratic condition.

Theorem 1.2. *Assume that the conditions of Theorem 1.1 are fulfilled and that*

$$G(x,s) \le \frac{\lambda_W - \lambda_1}{2} V(x)\, s^2, \quad \text{for all } s \in \mathbb{R},\ \text{a.e. } x \in \Omega\,. \tag{1.9}$$

Then Problem (1.3) *has at least two solutions.*

In the next two theorems, we prove the existence of a solution if $V \in L^\infty(\Omega)$ and under the following assumptions on the potential G:

$$\limsup_{|s|\to\infty} \frac{G(x,s)}{|s|^q} \le b < \infty \quad \text{uniformly a.e. } x \in \Omega\,,\ q > 2; \tag{1.10}$$

$$\liminf_{|s|\to\infty} \frac{g(x,s)s - 2G(x,s)}{|s|^\mu} \ge a > 0 \quad \text{uniformly a.e. } x \in \Omega; \tag{1.11}$$

$$\limsup_{|s|\to\infty} \frac{g(x,s)s - 2G(x,s)}{|s|^\mu} \le -a < 0 \quad \text{uniformly a.e. } x \in \Omega\,. \tag{1.12}$$

Theorem 1.3. *Assume that conditions* (1.10), (1.11) *[or* (1.12)*] and*

$$\limsup_{s\to 0} \frac{2G(x,s)}{s^2} \le \alpha < \lambda_1 < \beta \le \liminf_{|s|\to\infty} \frac{2G(x,s)}{s^2} \quad \text{uniformly a.e. } x \in \Omega\,, \tag{1.13}$$

with $\mu > 2N/(q-2)$ *if* $N \ge 3$ *or* $\mu > q - 2$ *if* $1 \le N \le 2$*. Then Problem* (1.3) *has at least one solution.*

Theorem 1.4. *Assume that* (1.12) *[or* (1.11)*] is satisfied for some* $\mu > 0$*, and*

$$\lim_{|s|\to\infty} \frac{G(x,s)}{s^2} = 0 \quad \text{uniformly a.e. } x \in \Omega\,. \tag{1.14}$$

Then Problem (1.3) *has at least one solution.*

The above theorems extend to the anisotropic case $V \not\equiv$ const. some results of Gonçalves and Miyagaki [6] and Ma [9].

2. Compactness conditions and auxiliary results

Let E be a reflexive real Banach space with norm $\|\cdot\|$ and let $I : E \to \mathbb{R}$ be a C^1 functional. We assume that there exists a compact embedding $E \hookrightarrow X$, where X is a real Banach space, and that the following interpolation type inequality holds:

$$\|u\|_X \le \psi(u)^{1-t} \|u\|^t\,, \quad \text{for all } u \in E\,, \tag{2.1}$$

for some $t \in (0,1)$ and some homogeneous function $\psi : E \to \mathbb{R}_+$ of degree one. An example of such a framework is the following: $E = H_0^1(\Omega)$, $X = L^q(\Omega)$, $\psi(u) = |u|_\mu$, where $0 < \mu < q < 2^*$. Then, by the interpolation inequality (see Brezis [3, Remarque 2, p. 57]) we have

$$|u|_q \le |u|_\mu^{1-t} |u|_{2^*}^t , \quad \text{where } \frac{1}{q} = \frac{1-t}{\mu} + \frac{t}{2^*} .$$

The Sobolev inequality yields $|u|_{2^*} \le c\|u\|$, for all $u \in H_0^1(\Omega)$. Hence

$$|u|_q \le k|u|_\mu^{1-t} \|u\|^t , \quad \text{for all } u \in H_0^1(\Omega)$$

and this is a (H_1) type inequality.

We recall below the following Cerami compactness conditions.

Definition 2.1. (a) The functional $I : E \to \mathbb{R}$ is said to satisfy condition (C) at the level $c \in \mathbb{R}$ [denoted $(C)_c$] if any sequence $(u_n)_n \subset E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|) \cdot \|I'(u_n)\|_{E^*} \to 0$ possesses a convergent subsequence.
(b) The functional $I : E \to \mathbb{R}$ is said to satisfy condition $(\hat{C})$ at the level $c \in \mathbb{R}$ [denoted $(\hat{C})_c$] if any sequence $(u_n)_n \subset E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|) \cdot \|I'(u_n)\|_{E^*} \to 0$ possesses a bounded subsequence.

We observe that the above conditions are weaker than the usual Palais-Smale condition $(PS)_c$: any sequence $(u_n)_n \subset E$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_{E^*} \to 0$ possesses a convergent subsequence.

Suppose that $I(u) = J(u) - N(u)$, where J is 2-homogeneous and N is not 2-homogeneous at infinity. We recall that J is 2-homogeneous if $J(\tau u) = \tau^2 J(u)$, for all $\tau \in \mathbb{R}$ and for any $u \in E$. We also recall that the functional $N \in C^1(E, \mathbb{R})$ is said to be not 2-homogeneous at infinity if there exist a, $c > 0$ and $\mu > 0$ such that

$$|\langle N'(u), u\rangle - 2N(u)| \ge a\psi(u)^\mu - c , \quad \text{for all } u \in E . \tag{2.2}$$

We introduce the following additional hypotheses on the functionals J and N:

$$J(u) \ge k\|u\|^2 , \quad \text{for all } u \in E \tag{2.3}$$

$$|N(u)| \le b\|u\|_X^q + d , \quad \text{for all } u \in E , \tag{2.4}$$

for some constants k, b, $d > 0$ and $q > 2$.

Theorem 2.2. *Assume that* (2.1), (2.2), (2.3), (2.4) *are fulfilled, with* $qt < 2$. *Then the functional* I *satisfies condition* $(\hat{C})_c$, *for all* $c \in \mathbb{R}$.

Proof. Let $(u_n)_n \subset E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0$. We have

$$\begin{aligned} |\langle I'(u), u\rangle - 2I(u)| &= |\langle J'(u) - N'(u), u\rangle - 2J(u) + 2N(u)| \\ &= |\langle J'(u), u\rangle - 2J(u) - (\langle N'(u), u\rangle - 2N(u))| . \end{aligned}$$

However, J is 2-homogeneous and

$$\frac{J(u + tu) - J(u)}{t} = J(u)\, \frac{(1+t)^2 - 1}{t} .$$

This implies $\langle J'(u), u\rangle = 2J(u)$ and

$$|\langle I'(u), u\rangle - 2I(u)| = |\langle N'(u), u\rangle - 2N(u)| .$$

From (2.2) we obtain

$$|\langle I'(u),u\rangle - 2I(u)| = |\langle N'(u),u\rangle - 2N(u)| \geq a\psi(u)^{\mu} - c\,.$$

Letting $u = u_n$ in the inequality from above we have:

$$a\psi(u_n)^{\mu} \leq c + \|I'(u_n)\|_{E^*}\|u_n\| + 2|I(u_n)|\,.$$

Thus, by our hypotheses, for some $c_0 > 0$ and all positive integer n, $\psi(u_n) \leq c_0$ and hence, the sequence $\{\psi(u_n)\}$ is bounded. Now, from (H_1) and (H_4) we obtain

$$J(u_n) = I(u_n) + N(u_n) \leq b\|u_n\|_X^q + d_0 \leq b\psi(u_n)^{(1-t)q}\|u_n\|^{qt} + d_0\,.$$

Hence

$$J(u_n) \leq b_0\|u_n\|^{qt} + d_0\,, \quad \text{for all } n \in \mathbb{N}\,,$$

for some b_0, $d_0 > 0$. Finally, (H_3) implies

$$c\|u_n\|^2 \leq b_0\|u_n\|^{qt} + d_0\,, \quad \text{for all } n \in \mathbb{N}\,.$$

Since $qt < 2$, we conclude that $(u_n)_n$ is bounded in E. □

Proposition 2.3. *Assume that $I(u) = J(u) - N(u)$ is as above, where $N' : E \to E^*$ is a compact operator and $J' : E \to E^*$ is an isomorphism from E onto $J'(E)$. Then conditions $(C)_c$ and $(\hat{C})_c$ are equivalent.*

Proof. It is sufficient to show that $(\hat{C})_c$ implies $(C)_c$. Let $(u_n)_n \subset E$ be a sequence such that $I(u_n) \to c$ and $(1+\|u_n\|)\|I'(u_n)\|_{E^*} \to 0$. From $(\hat{C})_c$ we obtain a bounded subsequence $(u_{n_k})_k$ of $(u_n)_n$. But N' is a compact operator. Then $N'(u_{n_{k_l}}) \xrightarrow{l} f' \in E^*$, where $(u_{n_{k_l}})$ is a subsequence of (u_{n_k}). Since $(u_{n_{k_l}})$ is a bounded sequence and $(1 + \|u_{n_{k_l}}\|)\|I'(u_{n_{k_l}})\|_{E^*} \to 0$, it follows that $\|I'(u_{n_{k_l}})\| \to 0$. Next, using the relation

$$u_{n_{k_l}} = J'^{-1}(I'(u_{n_{k_l}}) + N'(u_{n_{k_l}}))\,,$$

we obtain that $(u_{n_{k_l}})$ is a convergent subsequence of $(u_n)_n$. □

3. Proof of Theorem 1.1

We first show that the energy functional F satisfies the Palais-Smale condition at level $c < 0$: any sequence $(u_n)_n \subset H_0^1(\Omega)$ such that $F(u_n) \to c$ and $\|F'(u_n)\|_{H^{-1}} \to 0$ possesses a convergent subsequence.

Indeed, it suffices to show that such a sequence $(u_n)_n$ has a bounded subsequence (see the Appendix). Arguing by contradiction, we suppose that $\|u_n\| \to \infty$. We distinguish the following two distinct situations.

Case 1: $|u_n(x)| \to \infty$ a.e. $x \in \Omega$. Thus, by our hypotheses,

$$\begin{aligned}
c &= \liminf_{n\to\infty} F(u_n)\\
&= \liminf_{n\to\infty} \left\{\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{\lambda_1}{2}\int_\Omega V(x)\,u_n^2 dx - \int_\Omega G(x,u_n(x))dx\right\}\\
&\geq \liminf_{n\to\infty}\left(-\int_\Omega G(x,u_n(x)))dx\right)\\
&= -\limsup_{n\to\infty}\int_\Omega G(x,u_n(x))dx\\
&= -\limsup_{|s|\to\infty}\int_\Omega G(x,s)dx.
\end{aligned}$$

Using Fatou's lemma we obtain

$$\limsup_{|s|\to\infty}\int_\Omega G(x,s)dx \le \int_\Omega \limsup_{|s|\to\infty} G(x,s)dx\,.$$

Our assumption (1.8) implies $c \ge 0$. This is a contradiction because $c<0$. Therefore, $(u_n)_n$ is bounded in $H_0^1(\Omega)$.

Case 2: There exists $\omega \subset\subset \Omega$ such that $|\Omega\setminus\omega|>0$ and $|u_n(x)| \not\to \infty$ for all $x\in \Omega\setminus\omega$. It follows that there exists a subsequence, still denoted by $(u_n)_n$, which is bounded in $\Omega\setminus\omega$. So, there exists $k>0$ such that $|u_n(x)|\le k$, for all $x\in\Omega\setminus\omega$. Since $I(u_n)\to c$ we obtain some M such that $I(u_n)\le M$, for all n. We have

$$\frac{1}{2}\|u_n\|^2 - \frac{\lambda_1}{2}\int_\Omega V(x)u_n^2dx - |k|_1 \le I(u_n)\le M \quad \text{as } \|u_n\|\to\infty\,.$$

It follows that $\int_\Omega V(x)u_n^2dx\to\infty$. We have

$$\int_\Omega V(x)u_n^2dx = \int_{\Omega\setminus\omega} V(x)u_n^2dx + \int_\omega V(x)u_n^2dx \le k^2|\Omega\setminus\omega|\,\|V\|_{L^1} + \int_\omega V(x)u_n^2dx\,.$$

This shows that $\int_\omega V(x)u_n^2dx\to\infty$. If $(u_n)_n$ is bounded in ω, this yields a contradiction. Therefore, $u_n\notin L^\infty(\omega)$. So, by Fatou's lemma and our assumptions (1.7) and (1.8),

$$\begin{aligned}
c &= \liminf_{n\to\infty} F(u_n)\\
&\ge -\limsup_{n\to\infty}\int_\Omega G(x,u_n(x))dx\\
&= -\limsup_{n\to\infty}\Big(\int_{\Omega\setminus\omega} G(x,u_n(x))dx + \int_\omega G(x,u_n(x))dx\Big)\\
&\ge -\limsup_{n\to\infty}\int_{\Omega\setminus\omega} G(x,u_n(x))dx - \limsup_{n\to\infty}\int_\omega G(x,u_n(x))dx\\
&\ge -\limsup_{n\to\infty}\int_{\Omega\setminus\omega} G(x,u_n(x))dx - \int_\omega \limsup_{|s|\to\infty} G(x,s)dx \ge 0\,.
\end{aligned}$$

This implies $c\ge 0$ which contradicts our hypothesis $c<0$. This contradiction shows that $(u_n)_n$ is bounded in $H_0^1(\Omega)$, and hence F satisfies the Palais-Smale condition at level $c<0$.

The assumption (1.6) is equivalent with: there exist $\delta_n \searrow 0$ and $\varepsilon_n \in L^1(\Omega)$ with $|\varepsilon_n|_1\to 0$ such that

$$\int_\Omega \frac{G(x,s)}{s^2}dx \ge \int_\Omega m(x)dx - \int_\Omega \varepsilon_n(x)dx\,, \quad \text{for all } 0<|s|\le\delta_n\,. \tag{3.1}$$

However, $|\varepsilon_n|_1\to 0$ implies that for all $\varepsilon>0$ there exists n_ε such that for all $n\ge n_\varepsilon$ we have $|\varepsilon_n|_1<\varepsilon$. Set $\varepsilon = \int_\Omega m(x)\varphi_1^2dx/\|\varphi_1\|^2_{L^\infty}$ and fix n large enough so that

$$L := \int_\Omega m(x)\varphi_1^2(x)dx - |\varepsilon_n|_1\|\varphi_1\|^2_{L^\infty} > 0\,.$$

Take $v\in V$ such that $\|v\|\le \delta_n/\|\varphi_1\|_{L^\infty}$. We have $F(v) = -\int_\Omega G(x,v(x))dx$. The inequality (3.1) is equivalent to

$$\int_\Omega G(x,s)dx \ge \int_\Omega m(x)s^2dx - \int_\Omega \varepsilon_n(x)s^2dx$$

and therefore,

$$F(v) = -\int_\Omega G(x, v(x))dx \le -\int_\Omega m(x)v^2(x)dx + \int_\Omega \varepsilon_n(x)v^2(x)dx\,. \tag{3.2}$$

By our choice of $v \in V = \mathrm{Sp}(\varphi_1)$ we have

$$|v(x)| = |\alpha|\,|\varphi_1(x)| \le |\alpha|\|\varphi_1\|_{L^\infty} \le |\alpha|\frac{\delta_n}{\|v\|}\,.$$

However, from (3.2),

$$\begin{aligned} F(v) &\le -\int_\Omega mv^2dx + \int_\Omega \varepsilon_n v^2 dx \le -\int_\Omega m|\alpha|^2\varphi_1^2dx + |\alpha|^2\int_\Omega \varepsilon_n||\varphi_1\|_{L^\infty}^2dx\\ &= |\alpha|^2\Big(-\int_\Omega m\varphi_1^2dx + |\varepsilon_n|_1\|\varphi_1\|_{L^\infty}^2\Big) = -L|\alpha|^2 = -L\|v\|^2\,. \end{aligned}$$

Therefore we obtain the existence of some $v_0 \in V$ such that $F(v_0) < 0$. This implies $l = \inf_{H_0^1(\Omega)} F < 0$. But the functional F satisfies the Palais-Smale condition (P-S)$_c$, for all $c < 0$. This implies that there exists $u_0 \in H_0^1(\Omega)$ such that $F(u_0) = l$. In conclusion, u_0 is a critical point of F and consequently it is a solution to (1.3). Our assumption $g(x, 0) = 0$ implies $F(0) = 0$ and we know that $F(u_0) = l < 0$, that is, $u_0 \not\equiv 0$. Therefore $u_0 \in H_0^1(\Omega)$ is a nontrivial solution of (1.3) and the proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

Let X be a real Banach space and $F : X \to \mathbb{R}$ be a C^1-functional. Denote

$$\begin{gathered} K_c := \{u \in X;\ F'(u) = 0 \text{ and } F(u) = c\},\\ F^c := \{u \in X;\ F(u) \le c\}\,. \end{gathered}$$

The proof of Theorem 1.2 uses the following deformation lemma (see Ramos and Rebelo [13]).

Lemma 4.1. *Suppose that F has no critical values in the interval (a, b) and that $F^{-1}(\{a\})$ contains at most a finite number of critical points of F. Assume that the Palais-Smale condition $(P - S)_c$ holds for every $c \in [a, b)$. Then there exists an F-decreasing homotopy of homeomorphism $h : [0, 1] \times F^b \setminus K_b \to X$ such that*

$$\begin{gathered} h(0, u) = u\,, \quad \textit{for all } u \in F^b \setminus K_b,\\ h(1, F^b \setminus K_b) \subset F^a,\\ h(t, u) = u\,, \quad \textit{for all } u \in F^a\,. \end{gathered}$$

We are now in position to give the proof of Theorem 1.2. Fix n large enough so that

$$F(v) \le -L\|v\|^2\,, \quad \text{for all } v \in V \text{ with } \|v\| \le \frac{\delta_n}{\|\varphi_1\|_{L^\infty}}\,.$$

Denote $d := \sup_{\partial B} F$, where $B = \{v \in V;\ \|v\| \le R\}$ and $R = \delta_n/\|\varphi_1\|_{L^\infty}$. We suppose that 0 and u_0 are the only critical points of F and we show that this yields a contradiction. For any $w \in W$ we have

$$F(w) = \frac{1}{2}\Big(\int_\Omega |\nabla w|^2dx - \lambda_1\int_\Omega V(x)w^2dx\Big) - \int_\Omega G(x, w(x))dx\,.$$

Integrating in (1.9), we find

$$-\int_\Omega G(x,w(x))dx \geq \frac{\lambda_1-\lambda_W}{2}\int_\Omega V(x)w^2dx\,. \tag{4.1}$$

Combining the definition of λ_W with relation (4.1) we obtain

$$\begin{aligned} F(w) &\geq \frac{1}{2}\int_\Omega |\nabla w|^2dx - \frac{\lambda_1}{2}\int_\Omega V(x)w^2dx + \frac{\lambda_1-\lambda_W}{2}\int_\Omega V(x)w^2dx \\ &= \frac{1}{2}\Big(\int_\Omega |\nabla w|^2dx - \lambda_W\int_\Omega V(x)w^2dx\Big) \geq 0\,. \end{aligned} \tag{4.2}$$

Using $0\in W$, $F(0)=0$ and relation (4.2) we find $\inf_W F=0$. If $v\in\partial B$ then $F(v)\leq -LR<0$ and, consequently,

$$d=\sup_{\partial B}F < \inf_W F = 0\,.$$

Obviously,

$$l=\inf_{H_0^1(\Omega)}F \leq \inf_{\partial B}F < d = \sup_{\partial B}F\,.$$

Denote

$$\alpha := \inf_{\gamma\in\Gamma}\sup_{u\in B}F(\gamma(u))\,,$$

where $\Gamma := \{\gamma\in C(B,H_0^1(\Omega));\gamma(v)=v \text{ for all } v\in\partial B\}$. It is known (see the Appendix) that $\gamma(B)\cap W\neq\emptyset$, for all $\gamma\in\Gamma$. Since $\inf_W F=0$, we have $F(w)\geq 0$ for all $w\in W$. Let $u\in B$ such that $\gamma(u)\in W$. It follows that $F(\gamma(u))\geq 0$ and hence $\alpha\geq 0$. The Palais-Smale condition holds true at level $c<0$ and the functional F has no critical value in the interval $(l,0)$, So, by Lemma 4.1, we obtain a F decreasing homotopy $h:[0,1]\times F^0\setminus K_0\to H_0^1(\Omega)$ such that

$$\begin{gathered} h(0,u)=u\,,\quad \text{for all } u\in F^0\setminus K_0=F^0\setminus\{0\}\,;\\ h(1,F^0)\setminus\{0\}\subset F^l=\{u_0\}\,;\\ h(t,u)=u\,,\quad \text{for all } u\in F^l\,. \end{gathered}$$

Consider the application $\gamma_0:B\to H_0^1(\Omega)$ defined by

$$\gamma_0=\begin{cases} u_0\,, & \text{if } \|v\|<R/2\\ h\Big(\frac{2(R-\|v\|)}{R},\frac{Rv}{2\|v\|}\Big)\,, & \text{if } \|v\|\geq R/2\,. \end{cases}$$

Since $\gamma_0(v)=h(1,v)=u_0$ if $\|v\|=R/2$, we deduce that γ_0 is continuous.

If $v\in\partial B$ then $v=R\varphi_1$ and $F(R\varphi_1)\leq 0$. Then $v\in F^0\setminus\{0\}$ and hence $\gamma_0(v)=v$. Therefore we obtain that $\gamma_0\in\Gamma$. The condition that h is F decreasing is equivalent with

$$s>t \quad\text{implies}\quad F(h(s,u))<F(h(t,u))\,.$$

Let us consider $v\in B$. We distinguish the following two situations.

Case 1: $\|v\|<\frac{R}{2}$. In this case, $\gamma_0(v)=u_0$ and $F(u_0)=l<d$.

Case 2: $\|v\|\geq\frac{R}{2}$. If $\|v\|=R/2$ then $\gamma_0(v)=h(1,v)$ and if $\|v\|=R$ then $\gamma_0(v)=h(0,v)$. But $0\leq t\leq 1$ and h is F decreasing. It follows that

$$F(h(0,v))\geq F(h(t,v))\geq F(h(1,v))\,,$$

that is, $F(\gamma_0(v))\leq F(h(0,v))=F(v)\leq d$.

¿From these two cases we obtain $F(\gamma_0(v)) \leq d$, for all $v \in B$ and from the definition of α we have $0 \leq \alpha \leq d < 0$. This is a contradiction. We conclude that F has a another critical point $u_1 \in H_0^1(\Omega)$ and, consequently, Problem (1.3) has a second nontrivial weak solution.

5. Proof of Theorems 1.3 and 1.4

We will use the following classical critical point theorems.

Theorem 5.1 (Mountain Pass, Ambrosetti and Rabinowitz [1]). *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies condition $(C)_c$, for all $c \in \mathbb{R}$ and, for some $\rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$,*

$$\max\{I(0), I(u_1)\} \leq \hat{\alpha} < \hat{\beta} \leq \inf_{\|u\|=\rho} I(u)\,.$$

Then I has a critical value $\hat{c} \geq \hat{\beta}$, characterized by

$$\hat{c} = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau))\,,$$

where $\Gamma := \{\gamma \in C([0,1], E);\ \gamma(0) = 0, \gamma(1) = u_1\}$.

Theorem 5.2 (Saddle Point, Rabinowitz [12]). *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies condition $(C)_c$, for all $c \in \mathbb{R}$ and, for some $R > 0$ and some $E = V \oplus W$ with $\dim V < \infty$,*

$$\max_{v \in V, \|v\|=R} I(v) \leq \hat{\alpha} < \hat{\beta} \leq \inf_{w \in W} I(w)\,.$$

Then I has a critical value $\hat{c} \geq \hat{\beta}$, characterized by

$$\hat{c} = \inf_{h \in \Gamma} \max_{v \in V, \|v\| \leq R} I(h(v))\,,$$

where $\Gamma = \{h \in C(V \bigcap \bar{B}_R, E);\ h(v) = v, \text{ for all } v \in \partial B_R\}$.

Lemma 5.3. *Assume that G satisfies conditions (1.10) and (1.11) [or (1.12)], with $\mu > 2N/(q-2)$ if $N \geq 3$ or $\mu > q-2$ if $1 \leq N \leq 2$. Then the functional F satisfies condition $(C)_c$ for all $c \in \mathbb{R}$.*

Proof. Let

$$N(u) = \frac{\lambda_1}{2} \int_\Omega V(x) u^2 dx + \int_\Omega G(x,u) dx \quad \text{and} \quad J(u) = \frac{1}{2}\|u\|^2\,.$$

Obviously, J is homogeneous of degree 2 and J' is an isomorphism of $E = H_0^1(\Omega)$ onto $J'(E) \subset H^{-1}(\Omega)$. It is known that $N' : E \to E^*$ is a compact operator. Proposition 2.3 ensures that conditions $(C)_c$ and $(\hat{C})_c$ are equivalent. So, it suffices to show that $(\hat{C})_c$ holds for all $c \in \mathbb{R}$. Hypothesis (2.3) is trivially satisfied, whereas (2.4) holds true from (1.10). Condition (1.10) implies that

$$\inf_{|s|>0} \sup_{|t|>|s|} \frac{G(x,t)}{|t|^q} \leq b\,.$$

Therefore, there exists $s_0 \neq 0$ such that

$$\sup_{|t|>|s_0|} \frac{G(x,t)}{|t|^q} \leq b \quad \text{and} \quad G(x,t) \leq b|t|^q, \quad \text{for all } t \text{ with } |t| > |s_0|\,.$$

The boundedness is provided by the continuity of the application $[-s_0, s_0] \ni t \longmapsto G(x,t)$. It follows that $\int_\Omega G(x,u)dx \leq b|u|_q^q + d$. By the definition of $N(u)$ and since $q > 2$, we deduce that (2.4) holds true, provided $|u|_q \leq 1$ then we obtain (2.4). Indeed, we have $|u|_2 \leq k|u|_q$ because Ω is bounded. Therefore, $|u|_2^2 \leq k|u|_q^2 \leq k|u|_q^q$ and finally (2.4) is fulfilled. Hypothesis (2.1) is a direct consequence of the Sobolev inequality. It remains to show that hypothesis (2.2) holds true, that is, the functional N is not 2-homogeneous at infinity. Indeed, using assumption (1.11) (a similar argument works if (1.12) is fulfilled) together with the subcritical condition on g yields

$$\sup_{|s|>0} \inf_{|t|>|s|} \frac{g(x,t)t - 2G(x,t)}{|t|^\mu} \geq a > 0\,.$$

It follows that there exists $s_0 \neq 0$ such that

$$\inf_{|t|>|s_0|} \frac{g(x,t)t - 2G(x,t)}{|t|^\mu} \geq a\,.$$

Hence

$$g(x,t)t - 2G(x,t) \geq a|t|^\mu\,, \quad \text{for all } |t| > |s_0|\,.$$

The application $t \mapsto g(x,t)t - 2G(x,t)$ is continuous in $[-s_0, s_0]$, therefore it is bounded. We obtain $g(x,t) - 2G(x,t) \geq a_1|t|^\mu - c_1$, for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. We deduce that

$$\begin{aligned} |\langle N'(u), u\rangle - 2N(u)| &= \left| \int_\Omega (g(x,u)u - 2G(x,u))dx \right| \\ &\geq a_1\|u\|_\mu^\mu - c_2\,, \quad \text{for all } u \in H_0^1(\Omega)\,. \end{aligned}$$

Consequently, the functional N is not 2-homogeneous at infinity.

Finally, when $N \geq 3$, we observe that condition $\mu > N(q-2)/2$ is equivalent with $\mu > 2^*(q-2)/2^* - 2$. From $1/q = (1-t)/\mu + t/2^*$ we obtain $(1-t)/\mu = (2^* - qt)/(2^*q)$. Hence $(2^* - qt)/q < (1-t)(2^* - 2)/(q-2)$ and, consequently, $(q-2^*)(2-tq) < 0$. But $q < 2^*$ and this implies $2 > tq$. Similarly, when $1 \leq N \leq 2$, we choose some $2^{**} > 2$ sufficiently large so that $\mu > 2^{**}(q-2)/(2^{**}-2)$ and $t \in (0,1)$ be as above. The proof of Lemma is complete in view of Theorem 2.2. $\square$

Our next step is to show that condition (1.13) implies the geometry of the Mountain Pass theorem for the functional F. The below assumptions have been introduced in Cuesta and Silva [4].

Lemma 5.4. *Assume that G satisfies the hypotheses*

$$\limsup_{|s|\to\infty} \frac{G(x,s)}{|s|^q} \leq b < \infty \quad \textit{uniformly a.e. } x \in \Omega\,, \tag{5.1}$$

$$\limsup_{s\to 0} \frac{2G(x,s)}{s^2} \leq \alpha < \lambda_1 < \beta \leq \liminf_{|s|\to\infty} \frac{2G(x,s)}{|s|^2} \quad \textit{uniformly a.e. } x \in \Omega\,. \tag{5.2}$$

Then there exists ρ, $\gamma > 0$ such that $F(u) \geq \gamma$ if $|u| = \rho$. Moreover, there exists $\varphi_1 \in H_0^1(\Omega)$ such that $F(t\varphi_1) \to -\infty$ as $t \to \infty$.

Proof. In view of our hypotheses and the subcritical growth condition, we obtain

$$\liminf_{|s|\to\infty} \frac{2G(x,s)}{s^2} \geq \beta \quad \text{is equivalent to} \quad \sup_{s\neq 0} \inf_{|t|>|s|} \frac{2G(x,t)}{t^2} \geq \beta\,.$$

There exists $s_0 \neq 0$ such that $\inf_{|t|>|s_0|} \frac{2G(x,t)}{t^2} \geq \beta$ and therefore $\frac{2G(x,t)}{t^2} \geq \beta$, for all $|t| > |s_0|$ or $G(x,t) \geq \frac{1}{2}\beta t^2$, provided $|t| > |s_0|$. We choose t_0 such that

$|t_0| \le |s_0|$ and $G(x,t_0) < \frac{1}{2}\beta|t_0|^2$. Fix $\varepsilon > 0$. There exists $B(\varepsilon,t_0)$ such that $G(x,t_0) \ge \frac{1}{2}(\beta-\varepsilon)|t_0|^2 - B(\varepsilon,t_0)$. Denote $B(\varepsilon) = \sup_{|t_0|\le|s_0|} B(\varepsilon,t_0)$. We obtain for any given $\varepsilon > 0$ there exists $B = B(\varepsilon)$ such that

$$G(x,s) \ge \frac{1}{2}\,(\beta-\varepsilon)|s|^2 - B\,, \quad \text{for all } s\in\mathbb{R}\,,\text{a.e. } x\in\Omega\,. \tag{5.3}$$

Fix arbitrarily $\varepsilon > 0$. In the same way, using the second inequality of (5.2) and (5.1) it follows that there exists $A = A(\varepsilon) > 0$ such that

$$2G(x,t) \le (\alpha+\varepsilon)t^2 + 2(b + A(\varepsilon))|t|^q\,, \quad \text{for all } t\in\mathbb{R}\,,\ \text{a.e. } x\in\Omega\,. \tag{5.4}$$

We now choose $\varepsilon > 0$ so that $\alpha+\varepsilon < \lambda_1$ and we use (5.4) together with the Poincaré inequality to obtain the first assertion of the lemma.

Set $H(x,s) = \lambda_1 V(x)s^2/2 + G(x,s)$. Then H satisfies

$$\limsup_{|s|\to\infty} \frac{H(x,s)}{|s|^q} \le b < \infty\,, \quad \text{uniformly a.e. } x\in\Omega\,, \tag{5.5}$$

$$\limsup_{s\to 0} \frac{2H(x,s)}{s^2} \le \alpha < \lambda_1 < \beta \le \liminf_{|s|\to\infty} \frac{2H(x,s)}{s^2}\,, \quad \text{uniformly a.e. } x\in\Omega\,. \tag{5.6}$$

In the same way, for any given $\varepsilon > 0$ there exists $A = A(\varepsilon) > 0$ and $B = B(\varepsilon)$ such that

$$\frac{1}{2}(\beta-\varepsilon)s^2 - B \le H(x,s) \le \frac{1}{2}(\alpha+\varepsilon)s^2 + A|s|^q\,, \tag{5.7}$$

for all $s\in\mathbb{R}$, a.e. $x\in\Omega$. Then we have

$$\begin{aligned} F(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega H(x,u)dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}(\alpha+\varepsilon)|u|_2^2 - A|u|_q^q \\ &\ge \frac{1}{2}\left(1 - \frac{\varepsilon+\alpha}{\lambda_1}\right)\|u\|^2 - Ak\|u\|^q\,. \end{aligned}$$

We can assume without loss of generality that $q > 2$. Thus, the above estimate yields $F(u) \ge \gamma$ for some $\gamma > 0$, as long as $\rho > 0$ is small, thus proving the first assertion of the lemma.

On the other hand, choosing now $\varepsilon > 0$ so that $\beta - \varepsilon > \lambda_1$ and using (5.7), we obtain

$$F(u) \le \frac{1}{2}\|u\|^2 - \frac{\beta-\varepsilon}{2}\,|u|_2^2 + B|\Omega|\,.$$

We consider φ_1 be the λ_1-eigenfunction with $\|\varphi_1\| = 1$. It follows that

$$F(t\varphi_1) \le \frac{1}{2}\left(1 - \frac{\beta-\varepsilon}{\lambda_1}\right)t^2 + B|\Omega| \to -\infty \quad \text{as } t\to\infty.$$

This proves the second assertion of our lemma. $\square$

Lemma 5.5. *Assume that $G(x,s)$ satisfies* (1.12) *(for some $\mu > 0$) and*

$$\lim_{|s|\to\infty} \frac{G(x,s)}{s^2} = 0\,, \quad \textit{uniformly a.e. } x\in\Omega\,. \tag{5.8}$$

Then there exists a subspace W of $H_0^1(\Omega)$ such that $H_0^1(\Omega) = V\oplus W$ and

(i) *$F(v)\to-\infty$, as $\|v\|\to\infty$, $v\in V$*
(ii) *$F(w)\to\infty$, as $\|w\|\to\infty$, $w\in W$.*

Proof. (i) The condition (1.12) is equivalent to: There exists $s_0 \neq 0$ such that

$$g(x,s)s - 2G(x,s) \leq -a|s|^\mu\,, \quad \text{for all } |s| \geq |s_0| = R_1\,,\ \text{a.e. } x \in \Omega\,.$$

Integrating the identity

$$\frac{d}{ds}\frac{G(x,s)}{|s|^2} = \frac{g(x,s)s^2 - 2|s|G(x,s)}{s^4} = \frac{g(x,s)|s| - 2G(x,s)}{|s|^3}$$

over an interval $[t,T] \subset [R,\infty)$ and using the above inequality we find

$$\frac{G(x,T)}{T^2} - \frac{G(x,t)}{t^2} \leq -a\int_t^T s^{\mu-3}ds = \frac{a}{2-\mu}\left(\frac{1}{T^{2-\mu}} - \frac{1}{t^{2-\mu}}\right).$$

Since we can assume that $\mu < 2$ and using the above relation, we obtain $G(x,t) \geq \hat{a}t^\mu$ for all $t \geq R_1$, where $\hat{a} = \frac{a}{2-\mu} > 0$. Similarly, we show that

$$G(x,t) \geq \hat{a}|t|^\mu\,, \quad \text{for } |t| \geq R_1\,.$$

Consequently, $\lim_{|t|\to\infty} G(x,t) = \infty$. Now, letting $v = t\varphi_1 \in V$ and using the variational characterization of λ_1, we have

$$F(v) \geq -\int_\Omega G(x,v)dx \to -\infty\,, \quad \text{as } \|v\| = |t|\|\varphi_1\| \to \infty\,.$$

This result is a consequence of the Lebesgue's dominated convergence theorem.
(ii) Let $V = \mathrm{Sp}(\varphi_1)$ and $W \subset H_0^1(\Omega)$ be a closed complementary subspace to V. Since λ_1 is an eigenvalue of Problem (1.1), it follows that there exists $d > 0$ such that

$$\inf_{0\neq w\in W}\frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega V(x)w^2 dx} \geq \lambda_1 + d\,.$$

Therefore,

$$\|w\|^2 \geq (\lambda_1 + d)|w|_2^2\,, \quad \text{for all } w \in W\,.$$

Let $0 < \varepsilon < d$. From (G_4) we deduce that there exists $\delta = \delta(\varepsilon) > 0$ such that for all s satisfying $|s| > \delta$ we have $2G(x,s)/s^2 \leq \varepsilon$, a.e. $x \in \Omega$. In conclusion

$$G(x,s) - \frac{1}{2}\varepsilon s^2 \leq M\,, \quad \text{for all } s \in \mathbb{R}\,,$$

where

$$M := \sup_{|s|\leq\delta}\left(G(x,s) - \frac{1}{2}\,\varepsilon\, s^2\right) < \infty\,.$$

Therefore,

$$\begin{aligned}
F(w) &= \frac{1}{2}\|w\|^2 - \frac{\lambda_1}{2}\int_\Omega V(x)w^2 - \int_\Omega G(x,w)dx \\
&\geq \frac{1}{2}\|w\|^2 - \frac{\lambda_1}{2}|w|_2^2 - \frac{1}{2}\varepsilon|w|_2^2 - M \\
&\geq \frac{1}{2}\left(1 - \frac{\lambda_1+\varepsilon}{\lambda_1+d}\right)\|w\|^2 - M = N\|w\|^2 - M\,, \quad \text{for all } w \in W\,.
\end{aligned}$$

It follows that $F(w) \to \infty$ as $\|w\| \to \infty$, for all $w \in W$, which completes the proof of the lemma. $\square$

Proof of Theorem 1.3. In view of Lemmas 5.3 and 5.4, we may apply the Mountain Pass theorem with $u_1 = t_1\varphi_1$, $t_1 > 0$ being such that $F(t_1\varphi_1) \le 0$ (this is possible from Lemma 5.4). Since $F(u) \ge \gamma$ if $\|u\| = \rho$, we have

$$\max\{F(0), F(u_1)\} = 0 = \hat{\alpha} < \inf_{\|u\|=\rho} F(u) = \hat{\beta}\,.$$

It follows that the energy functional F has a critical value $\hat{c} \ge \hat{\beta} > 0$ and, hence, (1.3) has a nontrivial solution $u \in H_0^1(\Omega)$. □

Proof of Theorem 1.4. In view of Lemmas 5.3 and 5.5, we may apply the Saddle Point theorem with $\hat{\beta} := \inf_{w\in W} F(w)$ and $R > 0$ being such that $\sup_{\|v\|=R} F(v) := \hat{\alpha} < \hat{\beta}$, for all $v \in V$ (this is possible because $F(v) \to -\infty$ as $\|v\| \to \infty$). It follows that F has a critical value $\hat{c} \ge \hat{\beta}$, which is a weak solution to (1.3). □

6. Appendix

Throughout this section we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. We start with the following auxiliary result.

Lemma 6.1. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and assume that there exist some constants a, $b \ge 0$ such that*

$$|g(x,t)| \le a + b|t|^{r/s}\,, \quad \textit{for all } t \in \mathbb{R}\,,\ \textit{a.e.} x \in \Omega\,.$$

Then the application $\varphi(x) \mapsto g(x, \varphi(x))$ is in $C(L^r(\Omega), L^s(\Omega))$.

Proof. For any $u \in L^r(\Omega)$ we have

$$\begin{aligned}\int_\Omega |g(x,u(x))|^s dx &\le \int_\Omega (a + b|u|^{r/s})^s dx\\ &\le 2^s \int_\Omega (a^s + b^s|u|^r) dx\\ &\le c\int_\Omega (1 + |u|^r) dx < \infty\,.\end{aligned}$$

This shows that if $\varphi \in L^r(\Omega)$ then $g(x,\varphi) \in L^s(\Omega)$. Let u_n, $u \in L^r$ be such that $|u_n - u|_r \to 0$. By Theorem IV.9 in Brezis [3], there exist a subsequence $(u_{n_k})_k$ and $h \in L^r$ such that $u_{n_k} \to u$ a.e. in Ω and $|u_{n_k}| \le h$ a.e. in Ω. By our hypotheses it follows that $g(u_{n_k}) \to g(u)$ a.e. in Ω. Next, we observe that

$$|g(u_{n_k})| \le a + b|u_{n_k}|^{r/s} \le a + b|h|^{r/s} \in L^s(\Omega)\,.$$

So, by Lebesgue's dominated convergence theorem,

$$|g(u_{n_k}) - g(u)|_s^s = \int_\Omega |g(u_{n_k}) - g(u)|^s dx \xrightarrow{k} 0\,.$$

This completes the proof of the lemma. □

The mapping $\varphi \mapsto g(x, \varphi(x))$ is the Nemitski operator of the function g.

Proposition 6.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $|g(x,s)| \le a + b|s|^{r-1}$ for all $(x,s) \in \Omega \times \mathbb{R}$, with $2 \le r < 2N/(N-2)$ if $N > 2$ or $2 \le r < \infty$ if $1 \le N \le 2$. Denote $G(x,t) = \int_0^t g(x,s)ds$. Let $I : H_0^1(\Omega) \to \mathbb{R}$ be the functional defined by*

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda_1}{2}\int_\Omega V(x)u^2 dx - \int_\Omega G(x,u(x))dx\,,$$

where $V \in L^s(\Omega)$ ($s > N/2$ if $N \geq 2$, $s = 1$ if $N = 1$).

Assume that $(u_n)_n \subset H_0^1(\Omega)$ has a bounded subsequence and $I'(u_n) \to 0$ as $n \to \infty$. Then $(u_n)_n$ has a convergent subsequence.

Proof. We have

$$\langle I'(u), v\rangle = \int_\Omega \nabla u \nabla v dx - \lambda_1 \int_\Omega V(x) uv dx - \int_\Omega g(x, u(x)) v(x) dx .$$

Denote by

$$\langle a(u), v\rangle = \int_\Omega \nabla u \nabla v dx ;$$

$$J(u) = \frac{\lambda_1}{2} \int_\Omega V(x) u^2 dx + \int_\Omega G(x, u(x)) dx .$$

It follows that

$$\langle J'(u), v\rangle = \lambda_1 \int_\Omega V(x) uv dx + \int_\Omega g(x, u(x)) v(x) dx$$

and $I'(u) = a(u) - J'(u)$. We prove that a is an isomorphism from $H_0^1(\Omega)$ onto $a(H_0^1(\Omega))$ and J' is a compact operator. This assumption yields

$$u_n = a^{-1}\langle(I'(u_n\rangle) + J'(u_n)) \to \lim_{n\to\infty} a^{-1}\langle(J'(u_n)\rangle) .$$

But J' is a compact operator and $(u_n)_n$ is a bounded sequence. This implies that $(J'(u_n))_n$ has a convergent subsequence and, consequently, $(u_n)_n$ has a convergent subsequence. Assume, up to a subsequence, that $(u_n)_n \subset H_0^1(\Omega)$ is bounded. From the compact embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$, we can assume, passing again at a subsequence, that $u_n \to u$ in $L^r(\Omega)$. We have

$$\begin{aligned}
&\|J'(u_n) - J'(u)\| \\
&\leq \sup_{\|v\|\leq 1} \left| \int_\Omega \left(g(x, u_n(x)) - g(x, u(x))\right) v(x) dx \right| + \sup_{\|v\|\leq 1} \lambda_1 \left| \int_\Omega V(x)(u_n - u) v dx \right| \\
&\leq \sup_{\|v\|\leq 1} \int_\Omega |g(x, u_n(x)) - g(x, u(x))||v(x)| dx + \lambda_1 \sup_{\|v\|\leq 1} \int_\Omega |V(x)(u_n - u)v| dx \\
&\leq \sup_{\|v\|\leq 1} \left(\int_\Omega |g(x, u_n) - g(x, u)|^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} |v|_r + \lambda_1 \sup_{\|v\|\leq 1} \int_\Omega |V(x)(u_n - u)v| dx \\
&\leq c \sup_{\|v\|\leq 1} \left(\int_\Omega |g(x, u_n) - g(x, u)|^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} \|v\| + \lambda_1 |V|_{L^s} \cdot |u_n - u|_\alpha \cdot |v|_\beta ,
\end{aligned}$$

where α, $\beta < 2N/(N-2)$ (if $N \geq 2$). Such a choice of α and β is possible due to our choice of s. By Lemma 6.1 we obtain $g \in C(L^r, L^{r/(r-1)})$. Next, since $u_n \to u$ in L^r and $u_n \to u$ in L^2, the above relation implies that $J'(u_n) \to J'(u)$ as $n \to \infty$, that is, J' is a compact operator. This completes our proof. □

Set

$$\Gamma := \{\gamma \in C(B, H_0^1(\Omega));\ \gamma(v) = v\, , \text{ for all } v \in \partial B\}$$

and $B = \{v \in \mathrm{Sp}\,(\varphi_1);\ \|v\| \leq R\}$. The following result has been used in the proof of Lemma 4.1.

Proposition 6.3. *We have $\gamma(B) \bigcap W \neq \emptyset$, for all $\gamma \in \Gamma$.*

Proof. Let $P : H_0^1(\Omega) \to \mathrm{Sp}\,(\varphi_1)$ be the projection of H_0^1 in $\mathrm{Sp}\,(\varphi_1)$. Then P is a linear and continuous operator. If $v \in \partial B$ then $(P \circ \gamma)(v) = P(\gamma(v)) = P(v) = v$ and, consequently, $P \circ \gamma = Id$ on ∂B. We have $P \circ \gamma\ , Id \in C(B, H_0^1)$ and $0 \notin Id(\partial B) = \partial B$. Using a property of the Brouwer topological degree we obtain $\deg\,(P \circ \gamma, \mathrm{Int} B, 0) = \deg\,(Id, \mathrm{Int} B, 0)$. But $0 \in \mathrm{Int}\, B$ and it follows that $\deg\,(Id, \mathrm{Int}\, B, 0) = 1 \neq 0$. So, by the existence property of the Brouwer degree, there exists $v \in \mathrm{Int}\, B$ such that $(P \circ \gamma)(v) = 0$, that is, $P(\gamma(v)) = 0$. Therefore $\gamma(v) \in W$ and this shows that $\gamma(B) \cap W \neq \emptyset$. $\square$

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