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INSTANTANEOUS BLOW-UP OF SEMILINEAR NON-AUTONOMOUS EQUATIONS WITH FRACTIONAL DIFFUSION

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Abstract. We consider the Cauchy initial value problem

$$\frac{\partial}{\partial t}u(t,x) = k(t)\Delta_\alpha u(t,x) + h(t)f(u(t,x)),$$
$$u(0,x) = u_0(x),$$

where Δ_α is the fractional Laplacian for $0 < \alpha \leq 2$. We prove that if the initial condition u_0 is non-negative, bounded and measurable then the problem has a global integral solution when the source term f is non-negative, locally Lipschitz and satisfies the generalized Osgood's condition

$$\int_{\|u_0\|_\infty}^{\infty} \frac{ds}{f(s)} \geq \int_0^\infty h(s)ds.$$

Also, we prove that if the initial data is unbounded then the generalized Osgood's condition does not guarantee the existence of a global solution. It is important to point out that the proof of the existence hinges on the role of the function h. Analogously, the function k plays a central role in the proof of the instantaneous blow-up.

1. Introduction

We consider integral solutions of the semilinear non-autonomous parabolic equation

$$\begin{gathered} \frac{\partial}{\partial t}u(t,x) = k(t)\Delta_\alpha u(t,x) + h(t)f(u(t,x)), \quad t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \end{gathered} \tag{1.1}$$

where the diffusion $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ is the fractional Laplacian (or α-Laplacian), $0 < \alpha \leq 2$, u_0 is the initial data and f is the source term. The diffusion and the source terms are multiplicatively perturbated by continuous functions k, h.

Basic references for the study of the fractional Laplacian are the books [7] and [12]. However, it is worth noticing that the systematic study of partial differential equations considering fractional diffusion is relatively new. This area of mathematics has been actively studied in the last decade by Caffarelli, Vázquez and many others (see for instance, [3, 14] and the references therein).

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We denote by p the real-valued function determined by

$$\int_{\mathbb{R}^d} p(t,z)e^{iz\cdot\xi}dz = e^{-t|\xi|^\alpha}, \quad \text{for all } t>0,\ \xi\in\mathbb{R}^d. \tag{1.2}$$

The space of all real-valued essentially bounded functions defined on $\mathbb{R}^d$ will be denoted by $L^\infty(\mathbb{R}^d)$. Let us consider the family $\{T_t, t\geq 0\}$ of bounded linear operators defined on $L^\infty(\mathbb{R}^d)$ as

$$T_t g(x) = \int_{\mathbb{R}^d} p(t,x-y)g(y)dy, \quad x\in\mathbb{R}^d.$$

It is well known that $\{T_t, t\geq 0\}$ is a strongly continuous semigroup with infinitesimal generator Δ_α (see [1]).

As in [6, 9], we introduce the following concept.

Definition 1.1. Let $u_0\geq 0$ be a measurable function. We say that (1.1) has a local integral solution on $[0,T)$ if there is a measurable function $u:[0,T)\times\mathbb{R}^d\to[0,\infty]$ that is finite almost everywhere and

$$u(t) = T_{K_0(t)}u_0 + \int_0^t h(s)T_{K(s,t)}f(u(s))ds, \quad t\in[0,T), \tag{1.3}$$

holds almost everywhere (a.e.) in $[0,T)\times\mathbb{R}^d$, where

$$K(s,t) = \int_s^t k(r)dr \quad \text{and} \quad K_0(t) = K(0,t).$$

We will say that (1.1) has a global integral solution if (1.1) has a local integral solution for all $T>0$.

A solution of the differential equation (1.1) is called a classical solution. It is clear that the non-existence of a local integral solution for (1.1) implies the non-existence of a classical solution (see [9]).

In this article we consider the following hypotheses:

- (H1) $u_0\geq 0$ and $0<\|u_0\|_q<\infty$, with $1\leq q\leq\infty$;
- (H2) $f:[0,\infty)\to[0,\infty)$ is non-decreasing, locally Lipschitz, $f(0)=0$, and $f>0$ on $(0,\infty)$;
- (H3) $h,k:(0,\infty)\to(0,\infty)$ are continuous functions:
  - (a) $\lim_{t\to 0}\int_0^t h(s)ds = 0$,
  - (b) $\lim_{t\to 0}\int_0^t k(s)ds = 0$.

In what follows we will use the notation

$$H(t) = \int_0^t h(s)ds, \quad F_{x_0}(x) = \int_{x_0}^x \frac{ds}{f(s)},$$

where $x_0\geq 0$.

It is not difficult to prove that under the hypotheses (H2) and (H3), the initial value problem

$$\begin{gathered} \frac{dy(t)}{dt} = h(t)f(y(t)), \quad t>0, \\ y(0) = x_0\geq 0, \end{gathered} \tag{1.4}$$

has a unique solution if and only if $\operatorname{im}(H)\subseteq\operatorname{im}(F_{x_0})$. Moreover, the solution is given by $y(t) = F_{x_0}^{-1}(H(t))$, $t\geq 0$. Such criterion, of existence and uniqueness for

(1.4), is called the generalized Osgood's test (for a proof see for example [4, Lemma 2.2]).

Assume the hypotheses (H2), (H3) and (H3)(a). Theorem 2.2 will establish that (1.1) has a global integral solution if $0 \leq u_0 \in L^\infty(\mathbb{R}^d)$ and $\operatorname{im}(H) \subseteq \operatorname{im}(F_{\|u_0\|_\infty})$. Thus the generalized Osgood's test is still valid in some sense.

The following question arouse naturally in [6] for the case $h \equiv k \equiv 1$ and $\alpha = 2$:

> Does $\operatorname{im}(H) \subseteq \operatorname{im}(F_{\|u_0\|_q})$ (with $1 \leq q < \infty$) guarantees the existence of a global integral solution for (1.1)? (1.5)

To answer the above question we define the closed ball with center at x and radius R by $B_R(x) = \{y \in \mathbb{R}^d : |x - y| \leq R\}$.

Definition 1.2. Let $u_0 \geq 0$ be a measurable function. We say that a measurable function $u \geq 0$ is a local p-integrable solution to (1.1) if there are $x \in \mathbb{R}^d$, $p \geq 1$, $R > 0$ and $T > 0$, such that $u(t, \cdot) \in L^p(B_R(x))$, for all $t < T$, with $u(0) = u_0$ and the equality (1.3) is satisfied a.e. on $[0, T) \times B_R(x)$.

Of course, the non-existence of p-integrable local solutions implies the non-existence of local integral solutions. Therefore we say that an equation *blows-up instantaneously* if it does not have a p-integrable local solution.

In [6], an initial condition u_0 satisfying (H1) with $1 \leq q < \infty$, and a source term f satisfying (H2) and $\operatorname{im}(H) \subseteq \operatorname{im}(F_{\|u_0\|_q})$ are assumed. Under these conditions it is readily proved that any local integral solution for (1.1) does not belong to $L^1(B_R(0))$ for all $t < \tau$ and $R > 1$.

The concept of local p-integrable solution is weaker than that of local integral solution introduced in [6]. In this new context we will have the following consequences:

(i) The answer given in [6] for the question (1.5) uses strongly the fact that $R > 1$ (this assumption is essential in the study of the set $\{|x| \leq (R-1)/2\sqrt{t-s}\}$ introduced in the proof of [6, Theorem 4.1]). We overcome this difficulty through Lemma 2.4: see the inequality (3.4) below.

(ii) Some new phenomenon are found.

(a) The existence of a global integral solution depends only of the combined source term h and f. More precisely, the existence of a global integral solution depends of (H1) ($q = \infty$), (H2), (H3), (H3)(a) and $\operatorname{im}(H) \subseteq \operatorname{im}(F_{\|u_0\|_q})$. This, at an intuitively level, implies that the diffusion term can be perturbed by a very large term k and we still have the existence of a global integral solution.

(b) Symmetrically, given (H1) ($1 \leq q < \infty$) and (H3), to construct a convenient source term $\tilde{f}$ satisfying $\int_{\|u_0\|_q}^\infty \frac{ds}{\tilde{f}(s)} = \infty$ one requires Hypotheses (H3)(b). In this case h is arbitrary, which means that $\int_0^\infty h(s)ds$ could be finite or infinite. In any case the generalized Osgood's condition is satisfied.

It is worth mention that Osgood-type conditions appears naturally in some applied problems (see the references in [6]). On the other hand, from [15] we have that if we require additionally that the source term f is convex then we have that the local integral solution of (1.1) blows-up in finite time when $\int_{\|u_0\|_\infty}^\infty \frac{ds}{f(s)} < \infty$ and $\int_0^\infty h(s)ds = \infty$. The case $\int_{\|u_0\|_\infty}^\infty \frac{ds}{f(s)} < \int_0^\infty h(s)ds < \infty$ is still open, but we believe that we also have blow-up in finite time. Concerning the global existence of (1.1), we need additional conditions to guarantee a solution in a strong sense. For example, when $h \equiv k \equiv 1$ and $\alpha \in (1, 2)$, the authors of [5] proved that (1.1) has a mild solution (or even such solution is classical). Moreover, (1.1) has a mild global solution if we assume (see [10])

$$\sup\Big\{\int_0^t\Big(\int_s^t k(r)dr\Big)^{-1/\alpha}ds : t\in[0,T]\Big\}<\infty,\quad \text{for all } T>0.$$

The importance of the study of equations like (2.1) with fractional diffusion is well known in applied mathematics. For example, they arise in fields like molecular biology, hydrodynamics and statistical physics [11]. Also, notice that generators of the form $k(t)\Delta_\alpha$ arise in models of anomalous growth of certain fractal interfaces [8]. The study of partial differential equations with fractional diffusion is becoming more popular. In fact the number of works both theoretical and practical are increasing: see for example [3, 5, 14] and the references therein.

This article is organized as follows. In Section 2 we prove the existence of global mild solutions of (2.1) and we present some basic properties of $p(t,x)$. Section 3 provides a negative answer to the question (1.5), that is, we give an initial condition $\phi_\theta\in L^q(\mathbb{R}^d)$, $1\le q<\infty$, $0<\theta<d/q$ and a source term $\tilde f$ that satisfy the generalized Osgood's condition for which there is instantaneous blow-up.

2. Global existence and preliminary results

The existence of local integral solutions for (1.3) follows form the Banach contraction principle, as we shall see. First, we state some well known properties of p.

Lemma 2.1. *Let $t>0$ and $x,y\in\mathbb{R}^d$ then:*

(a) $p(t,x)\ge 0$ *and* $\int_{\mathbb{R}^d}p(t,z)dz=1$ *(density property).*
(b) $p(t,x)=t^{-d/\alpha}p(1,t^{-1/\alpha}x)$ *(scaling property).*
(c) *If* $|x|\ge|y|$ *then* $p(t,x)\le p(t,y)$ *(radially decreasing).*
(d) *The function* $(t,x)\to p(t,x)$ *is in* $C^\infty((0,\infty)\times\mathbb{R}^d)$ *(regularity).*
(e) *There exists* $c_0=c_0(d,\alpha)>0$*, such that*

$$p(t,x)\ge c_0\min\Big\{\frac{t}{|x|^{d+\alpha}},\frac{1}{t^{d/\alpha}}\Big\}.$$

Proof. For the proofs of (a)–(c) see [13, Section 2]. The proof of property (d) can be found in [5], and that of property (e) can be found in [2]. □

In what follows, c will denote a positive constant whose specific value is unimportant and can change from place to place. On the other hand, if the constant $c.$ has a subindex then we refer to a specific constant.

Let us recall that, by $L^\infty(\mathbb{R}^d)$, we denote the space of all measurable functions $\varphi:\mathbb{R}^d\to\mathbb{R}$ such that

$$\|\varphi\|_\infty=\inf\{M\ge 0:|\varphi(x)|\le M \text{ holds for almost all } x\}<\infty.$$

Let $\tau>0$ be a real number that we will fix later. Define

$$E_\tau=\{u:[0,\tau]\to L^\infty(\mathbb{R}^d) \text{ and } |||u|||<\infty\},$$

where

$$|||u|||=\sup\{\|u(t)\|_\infty : 0\le t\le\tau\}.$$

Then E_τ is a Banach space and the sets $(r>0)$

$$P_\tau=\{u\in E_\tau : u\ge 0,\text{ a.e. }\},\quad B_{\tau,r}=\{u\in E_\tau : |||u|||\le r\},$$

are closed subsets of E_τ.

Theorem 2.2. *Let us assume* (H1)*, with* $q=\infty$*,* (H2), (H3), (H3)(a)*. Then* (1.1) *has a global integral solution if* $\operatorname{im}(H)\subseteq \operatorname{im}(F_{\|u_0\|_\infty})$*.*

Proof. Define the operator $\Psi : B_{\tau,r}\cap P_\tau \to E_\tau$ by

$$\Psi(u)(t) = T_{K_0(t)}u_0 + \int_0^t h(s)T_{K(s,t)}f(u(s))ds.$$

Since $u\geq 0$ and $u_0\geq 0$, it is clear that $\Psi(u)\geq 0$, so $\Psi(u)\in P_\tau$. Using $\int_{\mathbb{R}^d} p(t,x)dx = 1$, for almost all $x\in\mathbb{R}^d$ and $u\in B_{\tau,r}$, we have

$$\begin{aligned}
\Psi(u)(t,x) &\leq \|u_0\|_\infty \int_{\mathbb{R}^d} p(K_0(t), y-x)dy \\
&\quad + \int_0^t h(s)\int_{\mathbb{R}^d} p(K(s,t),y-x)f(\|u(s)\|_\infty)dy\,ds \\
&\leq \|u_0\|_\infty + \int_0^t h(s)\int_{\mathbb{R}^d} p(K(s,t),y-x)f(r)dy\,ds \\
&\leq \|u_0\|_\infty + f(r)\int_0^\tau h(s)ds\,.
\end{aligned}\tag{2.1}$$

Then

$$|||\Psi(u)||| \leq \|u_0\|_\infty + f(r)\int_0^\tau h(s)ds.$$

Let us take $r = 1+\|u_0\|_\infty$. By Hypothesis (H3)(a) we can choose $\tau>0$ small enough such that

$$f(r)\int_0^\tau h(s)ds < 1.$$

Then $\Psi(u)\in B_{\tau,r}\cap P_\tau$, therefore $\Psi(B_{\tau,r}\cap P_\tau)\subset B_{\tau,r}\cap P_\tau$.

Now let us see that Ψ is a contraction. Take $u,\tilde u\in B_{\tau,r}\cap P_\tau$,

$$\begin{aligned}
&|\Psi(u)(t,x)-\Psi(\tilde u)(t,x)| \\
&= \Big|\int_0^t h(s)\int_{\mathbb{R}^d} p(K(s,t),y-x)[f(u(s,y))-f(\tilde u(s,y))]dy\,ds\Big| \\
&\leq \sup_{t\in[0,\tau]}\Big\{\int_0^t h(s)\int_{\mathbb{R}^d} p(K(s,t),y-x)|f(u(s,y))-f(\tilde u(s,y))|dy\,ds\Big\}.
\end{aligned}$$

Since $\mathbb{R}$ is locally compact, then f is Lipschitz on each compact subset of $\mathbb{R}$. In particular, for $[0,r]$ there exists a constant $c>0$ such that

$$|f(s)-f(t)|\leq c|s-t|,\quad \text{for all } s,t\in[0,r].$$

From the above inequality we easily deduce

$$|f(u(s,y))-f(\tilde u(s,y))|\leq c|u(s,x)-\tilde u(s,x)|\leq c\|u(s)-\tilde u(s)\|_\infty.$$

Consequently,

$$\begin{aligned}
|||\Psi(u)-\Psi(\tilde u)||| &\leq \sup_{t\in[0,\tau]}\Big\{\int_0^t h(s)c\|u(s)-\tilde u(s)\|_\infty ds\Big\} \\
&\leq (c\int_0^\tau h(s)ds)|||u-\tilde u|||.
\end{aligned}$$

Using again Hypothesis (H3)(a), we can choose $\tau > 0$ small enough such that Ψ is a contraction. Therefore Ψ has a unique fixed point (on $B_{\tau,r} \cap P_\tau$): the local mild solution to the equation (1.3).

Let $[0, \tau_{\max})$ be the maximal interval for which the local integral solution u of (1.3) exists. Clearly, $\tau_{\max} \geq \tau$. Let us suppose that $\tau_{\max} < \infty$. Since f is non decreasing, we deduce

$$\begin{aligned} u(t,x) &\leq T_{K_0(t)} \|u_0\|_\infty (x) + \int_0^t h(s) T_{K(s,t)} f(\|u(s)\|_\infty) ds \\ &= \|u_0\|_\infty + \int_0^t h(s) f(\|u(s)\|_\infty) ds, \quad 0 \leq t < \tau_{\max},\ x \in \mathbb{R}^d. \end{aligned}$$

Then

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + \int_0^t h(s) f(\|u(s)\|_\infty) ds.$$

Now let us consider the integral equation

$$y(t) = \|u_0\|_\infty + \int_0^t h(s) f(y(s)) ds,$$

whose solution y is given by

$$y(t) = F^{-1}_{\|u_0\|_\infty}(H(t)), \quad 0 \leq t < \infty,$$

where we have used that $\operatorname{im}(H) \subseteq \operatorname{im}(F_{\|u_0\|_\infty})$. By the Comparison Theorem (see [4]) we have

$$\|u(t)\|_\infty \leq F^{-1}_{\|u_0\|_\infty}(H(t)), \quad 0 \leq t < \tau_{\max}.$$

The continuity of p (property (d) in Lemma 2.1) and the Bounded Convergence Theorem (notice that $p(s) \leq 2^{d/\alpha} p(\tau_{\max})$, $\tau_{\max}/2 \leq s \leq \tau_{\max}$, this is consequence of (b) and (c) in Lemma 2.1) allow us to take the limit $t \uparrow \tau_{\max}$ in (1.3),

$$\begin{aligned} u(\tau_{\max}, x) &:= \lim_{t \uparrow \tau_{\max}} u(t,x) \\ &= T_{K_0(\tau_{\max})} u_0(x) + \int_0^{\tau_{\max}} h(s) T_{K(s,\tau_{\max})} f(u(s,\cdot)) ds \\ &\leq F^{-1}_{\|u_0\|_\infty}(H(\tau_{\max})). \end{aligned}$$

From this, the measurability of $u(\tau_{\max}, \cdot)$ follows easily. Also that

$$0 < \|u_0\|_\infty \leq \|u(\tau_{\max})\|_\infty \leq F^{-1}_{\|u_0\|_\infty}(H(\tau_{\max})).$$

Accordingly we can consider the equation (1.3) with initial condition $u(\tau_{\max}, \cdot)(\in L^\infty(\mathbb{R}^d))$. By the first part of the proof, the solution u can be extended beyond $\tau_{\max}$, contradicting the definition of $\tau_{\max}$. $\square$

The following estimates will be essentials in the proof of Theorem 3.1.

Lemma 2.3. *Let $R > 0$. There exists a $\delta = \delta(\alpha, R) > 0$ such that*

$$\int_{|y| \leq R} p(t, y - x) dy \geq c_1, \quad \text{for all } 0 < t \leq \delta,\ |x| \leq R, \tag{2.2}$$

where $c_1 = c_1(d, \alpha)$ is a positive constant.

Proof. The change of variable theorem and the scaling property of p imply

$$\begin{aligned}\int_{|y|\le R} p(t,y-x)dy &= \int_{|y+x|\le R} p(t,y)dy \\ &= t^{-d/\alpha}\int_{|y+x|\le R} p(1,t^{-1/\alpha}y)dy \\ &= \int_{|t^{1/\alpha}y+x|\le R} p(1,y)dy \\ &= \int_{|y+t^{-1/\alpha}x|\le t^{-1/\alpha}R} p(1,y)dy.\end{aligned}$$

We are going to set $\delta = R^\alpha$ and

$$\tilde{x} = \begin{cases} 0, & x = 0, \\ -\frac{1}{2|x|}x, & x \neq 0. \end{cases}$$

If $0 < t \le \delta$ and $|x| \le R$, then

$$B_{1/2}(\tilde{x}) \subset B_1(0) \cap B_{t^{-1/\alpha}R}(-t^{-1/\alpha}x).$$

Therefore

$$\begin{aligned}\int_{|y|\le R} p(t,y-x)dy &\ge \int_{B_{1/2}(\tilde{x})} p(1,y)dy \\ &\ge \inf_{z\in B_1(0)} p(1,z)\int_{B_{1/2}(\tilde{x})} dy \\ &= \mathrm{Vol}(B_{1/2}(0)) \inf_{z\in B_1(0)} p(1,z),\end{aligned}$$

where $\mathrm{Vol}(B_{1/2}(0))$ is the volume of the ball $B_{1/2}(0)$. □

Let $1 \le q < \infty$. We will consider the real-valued function

$$\phi_\theta(x) = \frac{1}{|x|^\theta} 1_{B_1(0)\setminus\{0\}}(x), \quad x \in \mathbb{R}^d. \tag{2.3}$$

Observe that

$$\phi_\theta \in L^q(\mathbb{R}^d), \quad \text{if } 0 < \theta < \frac{d}{q}. \tag{2.4}$$

Lemma 2.4. *Let $0 < t \le 1$ and $|x| \le t^{1/\alpha}/2$, then*

$$T_t\phi_\theta(x) \ge c_2 t^{-\theta/\alpha}, \tag{2.5}$$

where $c_2 = c_2(d,\alpha)$ is a positive constant.

Proof. By property (e) in Lemma 2.1 we obtain

$$\int_{|y|\le 1} \frac{p(t,y-x)}{|y|^\theta} dy \ge c_0 \int_{|y|\le 1, |x-y|\le t^{1/\alpha}} \frac{\min\{t|y-x|^{-d-\alpha}, t^{-d/\alpha}\}}{|y|^\theta} dy.$$

Since $|x-y| \le t^{1/\alpha}$ is equivalent to

$$\frac{t}{|y-x|^{d+\alpha}} \ge \frac{1}{t^{d/\alpha}},$$

we have

$$T_t\phi_\theta(x) \ge c_0 t^{-d/\alpha} \int_{|y|\le 1, |x-y|\le t^{1/\alpha}} \frac{dy}{|y|^\theta}.$$

Moreover

$$\begin{aligned}\int_{|y|\le 1, |x-y|\le t^{1/\alpha}} \frac{dy}{|y|^\theta} &\ge \int_{|y|\le t^{1/\alpha}-|x|} \frac{dy}{|y|^\theta}\\ &= c\int_0^{t^{1/\alpha}-|x|} r^{d-1-\theta}dr \ge ct^{d/\alpha-\theta/\alpha}.\end{aligned}$$

In the last inequality we have used (2.4) and $|x| \le t^{1/\alpha}/2$. □

3. Instantaneous blow-up

Let us consider the function ϕ_θ defined in (2.3), with $d > q\theta > 0$. We choose

$$r_1 := \|\phi_\theta\|_q + 1 > \|\phi_\theta\|_q,$$

and define recursively the sequence (r_i) as

$$r_{i+1} := 2\Big\{r_i + \Big(\int_0^{K_0^{-1}((r_i/c_2)^{-\alpha/\theta})} h(s)K_0(s)^{d/\alpha}ds\Big)^{-1}\Big(K_0(\frac{1}{r_i})\Big)^{-2}\Big\},$$

for $i = 2, 3, \dots$. Inasmuch as $r_{i+1} > 2r_i$, then (r_i) is strictly increasing and $r_i \uparrow \infty$. We define

$$\tilde f(r_i) = (\frac{r_{i+1}}{2} - r_i)K_0(\frac{1}{r_i}),$$

for $i = 1, 2, \dots$. Notice that

$$\begin{aligned}\tilde f(r_i) &= \Big(\int_0^{K_0^{-1}((r_i/c_2)^{-\alpha/\theta})} h(s)K_0(s)^{d/\alpha}ds\Big)^{-1}\Big(K_0(\frac{1}{r_i})\Big)^{-1}\\ &\le \Big(\int_0^{K_0^{-1}((r_{i+1}/c_2)^{-\alpha/\theta})} h(s)K_0(s)^{d/\alpha}ds\Big)^{-1}\Big(K_0(\frac{1}{r_{i+1}})\Big)^{-1} = \tilde f(r_{i+1}),\end{aligned}$$

then we can define $\tilde f : [0,\infty) \to [0,\infty)$ as

$$\tilde f(x) = \begin{cases} \frac{\tilde f(r_1)}{r_1}x, & 0 \le x \le r_1,\\ \tilde f(r_i), & r_i \le x \le \frac12 r_{i+1},\\ \text{linear interpolation}, & \frac12 r_{i+1} \le x \le r_{i+1}.\end{cases} \tag{3.1}$$

In this way $\tilde f$ is non-decreasing. Also, from the definition of $\tilde f$ follows that it is locally Lipschitz. Moreover,

$$\begin{aligned}\int_{\|\phi_\theta\|_q}^\infty \frac{ds}{\tilde f(s)} &\ge \sum_{i=1}^\infty \int_{r_i}^{\frac12 r_{i+1}} \frac{1}{\tilde f(r_i)}ds\\ &= \sum_{i=1}^\infty \Big(K_0(\frac{1}{r_i})\Big)^{-1} = \infty,\end{aligned}$$

because of Hypothesis (H3)(b). In this way, the function $\tilde f$ satisfies Hypothesis (H2). Also, the generalized Osgood's condition is satisfied for all continuous functions $h : (0,\infty) \to (0,\infty)$ because $\int_{\|\phi_\theta\|_q}^\infty \frac{ds}{\tilde f(s)} = \infty$ $(1 \le q < \infty)$.

Our main result reads as follows.

Theorem 3.1. *Let us suppose $q \in [1,\infty)$, $\tilde f$ is as in* (3.1) *and Hypotheses* (H3), (H3)(b). *If $0 < \theta < d/q$, and $u_0 = \phi_\theta$, then equation* (1.1) *possesses no local p-integrable solution for all $p \ge 1$.*

Proof. We proceed by contradiction. Suppose that there are $p \geq 1$, $R > 0$ and $\tau > 0$, such that (1.3) has an p-integrable solution $u(t,\cdot)$ on $B_R(0)$, for all $t < \tau$, with $u_0 = \phi_\theta$. Let us take a $t < \min\{\tau, K_0^{-1}(1), {K_0}^{-1}(\delta), {K_0}^{-1}((2R)^\alpha)\}$, where δ is given in Lemma 2.3.

We observe that (1.3) implies

$$u(s,x) \geq T_{K_0(s)}\phi_\theta(x), \quad \text{for all } s \geq 0,\ x \in \mathbb{R}^d.$$

From (1.3), Jensen's inequality, and the above estimation we obtain

$$\begin{aligned}
&\int_{|x|\leq R} u(t,x)^p dx \\
&\geq \int_{|x|\leq R} (\int_0^t h(s) \int_{\mathbb{R}^d} p(K(s,t), y-x)\tilde{f}(T_{K_0(s)}\phi_\theta(y)) dy\, ds)^p dx \\
&\geq cR^{d(1-p)} (\int_{|x|\leq R} \int_0^t h(s) \int_{\mathbb{R}^d} p(K(s,t), y-x)\tilde{f}(T_{K_0(s)}\phi_\theta(y)) dy\, ds\, dx)^p.
\end{aligned}$$

Denoting the set $\{y : T_{K_0(s)}\phi_\theta(y) \geq r_i\}$ by A_i, and using that $\tilde{f}$ is non-decreasing we obtain

$$\begin{aligned}
&\int_{|x|\leq R} u(t,x)^p dx \\
&\geq c(\int_{|x|\leq R} \int_0^t h(s) \int_{A_i} p(K(s,t), y-x)\tilde{f}(T_{K_0(s)}\phi_\theta(y)) dy\, ds dx)^p \\
&\geq c(\int_0^t h(s) \int_{A_i} \int_{|x|\leq R} p(K(s,t), y-x) dx \tilde{f}(r_i) dy\, ds)^p.
\end{aligned}$$

Since $r_i \uparrow \infty$, there exists $i_0 \in \mathbb{N}$ such that

$$t_i := K_0^{-1}\Big((\frac{r_i}{c_2})^{-\alpha/\theta}\Big) \leq t, \quad \text{for all } i \geq i_0. \tag{3.2}$$

If $0 \leq s \leq t_i$, the inequality (3.2) yields

$$r_i < c_2 K_0(s)^{-\theta/\alpha}.$$

Moreover, if $|y| \leq K_0(s)^{1/\alpha}/2$, then (2.5) implies

$$r_i < T_{K_0(s)}\phi_\theta(y).$$

Hence,

$$A_i = \{y : T_{K_0(s)}\phi_\theta(y) \geq r_i\} \supset \{y : |y| \leq 2^{-1}K_0(s)^{1/\alpha}\}. \tag{3.3}$$

On the other hand, for $s < t$, we have $K(s,t) \leq K_0(t) \leq \delta$ and $2^{-1}K_0(s)^{1/\alpha} \leq 2^{-1}K_0(t)^{1/\alpha} < R$ which indicate that we are able to use (2.2). Then

$$\int_{|x|\leq R} p(K(s,t), y-x) dx \geq c_1, \quad \text{for all } |y| \leq 2^{-1}K_0(s)^{1/\alpha}. \tag{3.4}$$

Using (3.3) and (3.4) we obtain

$$\begin{aligned}
\Big(\int_{|x|\leq R} u(t,x)^p dx\Big)^{1/p} &\geq c\tilde{f}(r_i) \int_0^{t_i} h(s) \int_{\{|y|\leq 2^{-1}K_0(s)^{1/\alpha}\}} dy\, ds \\
&= c\tilde{f}(r_i) \int_0^{t_i} h(s) K_0(s)^{d/\alpha} ds
\end{aligned}$$

$$= c\Big(\int_0^{1/r_i} k(s)ds\Big)^{-1}.$$

Letting $i \to \infty$, $r_i \uparrow \infty$, therefore (H3)(b) implies $\int_{|x|\le R} u(t,x)^p dx = \infty$, $t < \tau$. The contradiction obtained proves the result. □

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