ZERO FORCING IN GRAPHS AND DIGRAPHS

by

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Her patience, guidance, and advice have been a big motivation throughout my mathematical career thus far.

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I. INTRODUCTION

Graphs

A *graph* is a pair $G = (V, E)$, where $V = V(G)$ is a non-empty set of objects called *vertices* (the singular is *vertex*). The set $E = E(G)$ is a collection of 2-element subsets of V called *edges*. The number of vertices in V is the *order* of G. The number of edges in E is the *size* of G. A graph of order 1 is called *trivial*. A graph of size 0 is called *empty*.

We often represent a graph G with a diagram, and refer to the diagram itself as G. The vertices are represented with circles or points, and the edges are represented with lines joining vertices.

Example 1. Here is the diagram of a graph G with
$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and
$E(G) = \{\{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_6\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_5, v_6\}\}$.
The order of G is 6, and the size of G is 9.

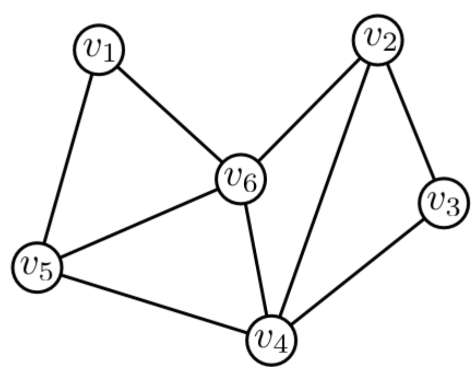


Figure 1: The diagram of a graph.

We say two vertices u and v of a graph G are *neighbors* if $\{u, v\}$ is an edge of G. The set of all neighbors of a vertex v is called the *neighborhood* of v and is denoted $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v.

If S is a set of vertices of G, the set of all of the neighbors of each vertex in S is called the *neighborhood of* S, and is denoted $N(S)$. The set $N[S] = N(S) \cup S$ is the *closed neighborhood of* S. The *degree* of a vertex v is the number of neighbors of v and is denoted $\deg(v)$. Hence, $\deg(v) = |N(v)|$. The minimum degree among all vertices of G is denoted $\delta(G)$. The maximum degree among all the vertices of G is $\Delta(G)$.

Example 2. In the graph of Example 1, $N(v_4) = \{v_2, v_3, v_5, v_6\}$, $\deg(v_4) = 4$, $\Delta(G) = 4$, and $\delta(G) = 2$.

Two graphs G and H are *isomorphic*, if there exists a bijective function $f : V(G) \to V(H)$, such that $\{u, v\}$ is an edge of G if and only if $\{f(u), f(v)\}$ is an edge of H.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a non-empty subset S of $V(G)$, the subgraph $G[S]$ of G *induced* by S has S as its vertex set. Two vertices u and v are neighbors in $G[S]$ if and only if u and v are neighbors in G. A subgraph H of a graph G is called an *induced subgraph* if there is a non-empty subset S of $V(G)$ such that $H = G[S]$.

(a) G (b) H

Figure 2: H is an induced subgraph of G.

For two vertices u and v of a graph G, a $u - v$ *path* is a sequence of distinct vertices of G such that consecutive vertices are neighbors. For a positive integer n, a $u-v$ path of length n can be expressed as $u = v_1, v_2, \ldots, v_n = v$, where $\{v_i, v_{i+1}\}$ is an edge of G for each $i = 1, 2, \ldots, n-1$. A *cycle* is a $u-u$ path. We say that a graph is *connected* if for any two distinct vertices u and v, there exists a $u-v$ path. If a graph is not connected, then it is *disconnected*. In that case, a maximal connected subgraph is a *component* of G. In Figure 3, the connected graph is a component of the disconnected graph.

Figure 3: (a) A connected graph and (b) a disconnected graph.

In a graph G, the *contraction* of an edge $\{u, v\}$ is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than $\{u, v\}$ that were incident with u or v. The resulting graph $G/\{u, v\}$ has one less edge than G. In Figure 4 we obtain different graphs when contracting the edge $\{u, v\}$, then $\{u, y\}$, followed by the edge $\{u, x\}$.

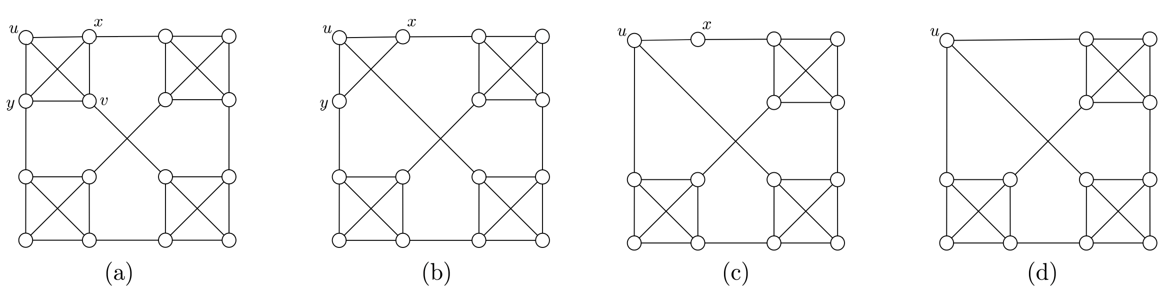
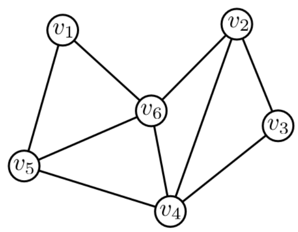


Figure 4: (a) G, (b) $G/\{u, v\}$, (c) $(G/\{u, v\})/\{u, y\}$, (d) $((G/\{u, v\})/\{u, y\})/\{u, x\}$.

Let $G = (V, E)$ be a graph of order n with $V = \{v_1, v_2, \ldots, v_n\}$. The *adjacency* *matrix* of G, denoted $\mathcal{A}(G)$, is the $n \times n$ matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$ where $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G and $a_{ij} = 0$ otherwise. Note that the entries of the diagonal of the adjacency matrix are 0. Figure 5 shows an example of a graph and its adjacency matrix.



(a)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(b)

Figure 5: (a) The graph G and (b) the adjacency matrix of G.

The *rank* of a matrix $M = [a_{ij}]$ is the number of linearly independent columns of M, denoted $\text{rank}(M)$. The *null space* of M is the set of all vectors $\mathbf{x}$ such that $M\mathbf{x} = \mathbf{0}$. The *nullity* of M is the dimension of its null space, denoted $\text{null}(M)$. For a positive integer n, if M has n columns, the rank-nullity theorem gives that $\text{rank}(M) + \text{null}(M) = n$. A matrix $M = [a_{ij}]$ is *symmetric* if and only if $a_{ij} = a_{ji}$. Note that the adjacency matrix of a graph is symmetric.

Families of Graphs

For an integer $n \geq 1$, the *complete graph* of order n, denoted K_n, is a graph in which every two distinct vertices are neighbors. Figure 6 shows the complete graphs of order 1, 2, 3, 4, and 5.

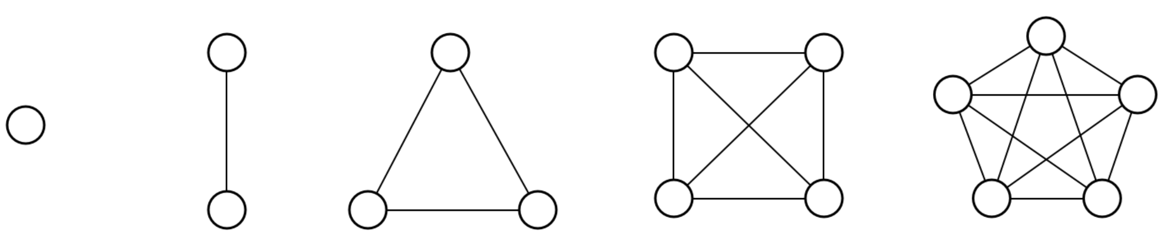


Figure 6: The complete graphs K_1, K_2, K_3, K_4, and K_5.

For an integer $n \geq 1$, the *path* of order n, denoted P_n is a graph whose vertices can be labeled $v_1, v_2, \ldots, v_n$ and whose edges are $\{v_i, v_{i+1}\}$ for each $i = 1, 2, \ldots, n-1$. Figure 7 shows the paths of order 1, 2, 3, and 4.

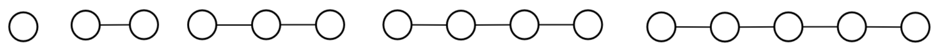


Figure 7: The graphs P_1, P_2, P_3, and P_4.

For an integer $n \geq 3$, the *cycle* of order n, denoted C_n is a graph whose vertices can be labeled $v_1, v_2, \ldots, v_n$ and whose edges are $\{v_1, v_n\}$ and $\{v_i, v_{i+1}\}$ for each $i = 1, 2, \ldots, n-1$. Figure 8 shows the cycles of order 3, 4, and 5.

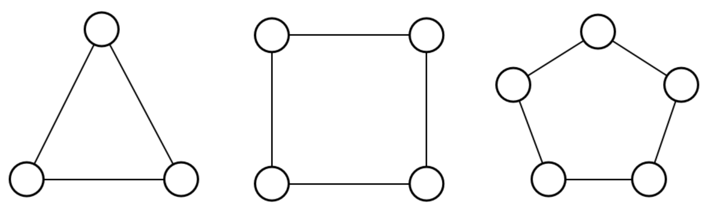


Figure 8: The graphs C_3, C_4, and C_5.

Digraphs

A *digraph* is a pair $D = (V, A)$, where $V = V(D)$ is a non-empty set of objects called *vertices*. The set $A = A(D)$ is a collection of ordered pairs of elements of V called *arcs*. An arc of the form (u, u) is called a *loop*. A *loopless* digraph is a digraph without loops. The number of vertices in V is the *order* of D. The number of arcs in A is the *size* of D. A digraph of order 1 is called *trivial*. A digraph of size 0 is called *empty*.

Example 3. A digraph D with $V(D) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $A(D) = \{(v_5, v_1), (v_6, v_1), (v_2, v_6), (v_4, v_2), (v_3, v_2), (v_3, v_4), (v_4, v_5), (v_4, v_6), (v_6, v_4), (v_5, v_6)\}$. The order of D is 6, and the size of D is 10.

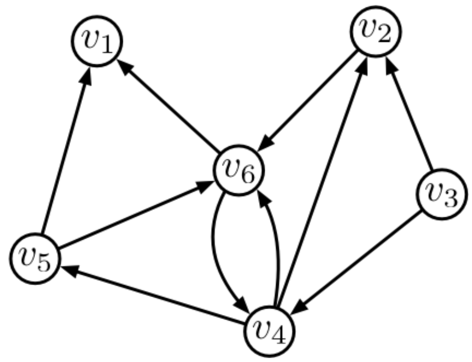


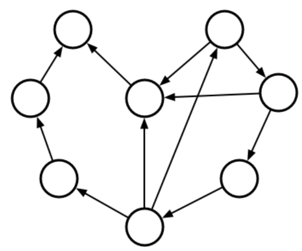
Figure 9: The diagram of a digraph.

Let $D = (V, A)$ be a digraph. A vertex u is an *out-neighbor* of a vertex v if (v, u) is an arc of D. The *out-neighborhood* of v, denoted $N^+(v)$, is the set of all out-neighbors of v. A vertex u is an *in-neighbor* of a vertex v if (u, v) is an arc of D. The *in-neighborhood* of v, denoted $N^-(v)$, is the set of all in-neighbors of v. The *out-degree* of a vertex v, denoted $\mathrm{d}^+(v)$, is the number of out-neighbors of v. The *in-degree* of a vertex v, denoted $\mathrm{d}^-(v)$, is the number of in-neighbors of v. The degree of v, denoted $\deg(v)$, is the sum of $\mathrm{d}^+(v)$ and $\mathrm{d}^-(v)$.

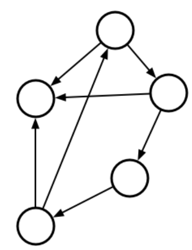
Example 4. In Figure 9, we see that $\mathrm{d}^+(v_5) = 2$ and $\mathrm{d}^-(v_5) = 1$.

Two digraphs D and H are *isomorphic*, if there exists a bijective function $f : V(D) \to V(H)$, such that (u, v) is an arc of D if and only if $(f(u), f(v))$ is an arc of H.

A digraph H is a *sub-digraph* of a digraph D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. For a non-empty subset S of $V(D)$, the *sub-digraph* $D[S]$ *of* D *induced by* S has S as its vertex set. For two vertices u and v in S, (u, v) is an arc of $D[S]$ if and only if (u, v) is an arc of D. A sub-digraph H of a digraph D is called an *induced sub-digraph* if there is a non-empty subset S of $V(D)$ such that $H = D[S]$.



(a) D

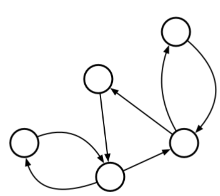


(b) H

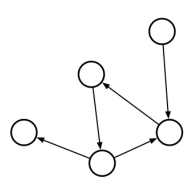
Figure 10: H is induced sub-digraph of D.

The graph G is the *underlying graph* of a digraph D if $V(G) = V(D)$ and $\{u, v\}$ is an edge of G if and only if (u, v) or (v, u), or (u, v) and (v, u) are arcs of D. A digraph is *weakly connected* if the underlying graph is connected. If a digraph is not weakly connected, then it is *disconnected*. In that case, a maximal weakly connected sub-digraph of D is a *weak component* of D.

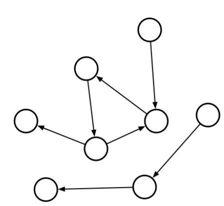
For any two vertices u and v of a digraph D, a $u - v$ *path* is a finite sequence $u = v_1, v_2, \ldots, v_n = v$ of distinct vertices of D such that (v_i, v_{i+1}) is an arc of D for each $i = 1, 2, \ldots, n-1$. A *cycle* is a $u - u$ path. We say that a digraph is *strongly connected* if for any two distinct vertices u and v, there exists a $u - v$ path, then a maximal strongly connected sub-digraph of D is a *strong component* of D. Figure 11 shows examples of connectivities in digraphs.



(a)



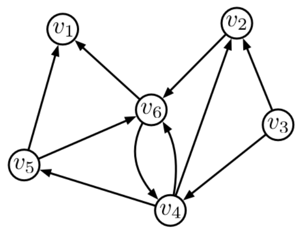
(b)



(c)

Figure 11: (a) A weakly connected digraph, (b) a strongly connected digraph, and (c) a disconnected digraph.

Let $D = (V, A)$ be a digraph of order n with $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of D, denoted $\mathcal{A}(D)$, is the $n \times n$ matrix $\mathcal{A}(D) = [a_{ij}]$ defined by $a_{ij} = 1$ if (v_i, v_j) is an arc of D and $a_{ij} = 0$ otherwise. Note that the entries of the diagonal of the adjacency matrix of a loopless digraph are 0. As opposed to graphs, the adjacency matrix of a digraph need not be symmetric. See Figure 12 for an example of a digraph and its adjacency matrix.



(a)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(b)

Figure 12: (a) A digraph D and (b) the matrix $\mathcal{A}(D)$.

Families of Digraphs

For an integer $n \geq 1$, the *directed path* of order n, denoted $\vec{P}_n$ is a graph whose vertices can be labeled $v_1, v_2, \ldots, v_n$ and whose arcs are (v_i, v_{i+1}) for each $i = 1, 2, \ldots, n-1$. Figure 13 shows the directed paths of order 1, 2, 3, and 4.

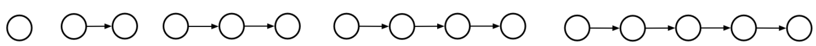


Figure 13: The digraphs $\vec{P}_1, \vec{P}_2, \vec{P}_3$, and $\vec{P}_4$.

For an integer $n \geq 3$, the *directed cycle* of order n, denoted $\vec{C}_n$ is a digraph whose vertices can be labeled $v_1, v_2, \ldots, v_n$ and whose arcs are (v_n, v_1) and (v_i, v_{i+1}) for each $i = 1, 2, \ldots, n-1$. Figure 14 shows the directed cycles of order 3, 4, and 5.

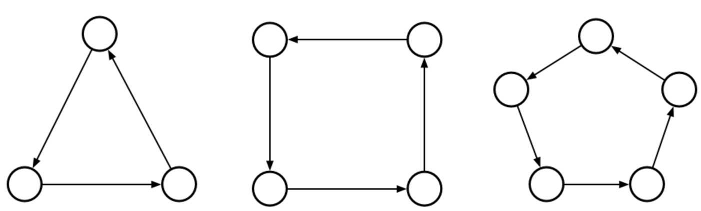


Figure 14: The digraphs $\vec{C}_3, \vec{C}_4$, and $\vec{C}_5$.

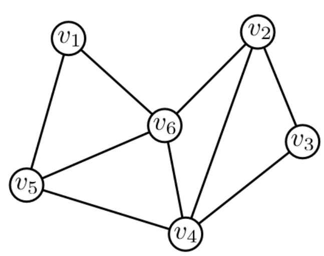
II. ZERO FORCING

Zero Forcing in Graphs

For a positive integer n, let $\mathcal{M}_n$ denote the set of all $n \times n$ real matrices. Let $G = (V, E)$ be a graph of order n, where $V = \{v_1, v_2, \ldots, v_n\}$. The set of *matrices described by* G is the set $\mathcal{S}(G)$ defined as

$$\mathcal{S}(G) = \{X = [a_{ij}] \in \mathcal{M}_n : a_{ij} = a_{ji} \text{ and } a_{ij} \neq 0 \Leftrightarrow \{i,j\} \in E(G)\}.$$

The difference between a matrix $M \in \mathcal{S}(G)$ and the adjacency matrix $\mathcal{A}(G)$ of G is that $\mathcal{A}(G)$ is a symmetric 0-1 matrix with zeros on the diagonal, while M is a real symmetric matrix that may have nonzero entries on the diagonal.



(a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 6 & 9 \\ 0 & 3 & 5 & 8 & 0 & 4 \\ 0 & 5 & 4 & 4 & 0 & 0 \\ 0 & 8 & 4 & 5 & 9 & 8 \\ 6 & 0 & 0 & 9 & 0 & 9 \\ 9 & 4 & 0 & 8 & 9 & 0 \end{pmatrix}$$

(b)

Figure 15: (a) A graph G and (b) a matrix in $\mathcal{S}(G)$.

The *maximum nullity of* G is $\mathrm{M}(G) = \max\{\mathrm{null}(X) : X \in \mathcal{S}(G)\}$. The *minimum rank of* G is the $\mathrm{mr}(G) = \min\{\mathrm{rank}(X) : X \in \mathcal{S}(G)\}$; clearly $\mathrm{M}(G) + \mathrm{mr}(G) = |V(G)|$. Zero forcing was introduced in linear algebra as a process to obtain an upper bound for $\mathrm{M}(G)$ [2].

In order to define zero forcing, we first define a sequence of sets of vertices in a given graph.

Let $G = (V, E)$ be a graph. Given an arbitrary non-empty set of vertices $S \subseteq V$, for every non-negative integer i, we define $B^i(S)$ by the following rules.

1. $B^0(S) = S$.

2. If $i \geq 0$, then

   $B^{i+1}(S) = B^i(S) \cup \{v \in V \setminus B^i(S) : \exists u \in B^i(S), N(u) \setminus B^i(S) = \{v\}\}$.

In terms of coloring, the set S can be referred to as the initial set of blue vertices. If a blue vertex has exactly one white neighbor, then we color that neighbor blue. This is equivalent to rule 2 and is often referred to as the *color changing rule*. For each non-negative integer i, the set $B^i(S)$ is the set of blue vertices at the i^{th} iteration of the color changing rule. Note that the color changing rule is applied to every vertex of G at each iteration.

We say that S is a *zero forcing set* of G if there exists a non-negative integer m such that $B^m(S) = V(G)$. A *minimum zero forcing set* of G is a zero forcing set of minimum cardinality. The cardinality of a zero forcing set is the *zero forcing number* of G and is denoted as $\mathrm{Z}(G)$. The *zero forcing problem* consists of finding $\mathrm{Z}(G)$ for a given graph G and a zero forcing set S such that $|S| = \mathrm{Z}(G)$.

Let S be a zero forcing set. If $u \in B^i(S)$ and $N(u) \setminus B^i(S) = \{v\}$ for some integer i, we say that u *forces* v. If there exists an integer m such that for all integers $i \geq m$, $B^m(S) = B^i(S)$, then $B^m(S)$ is called the *derived set*, which we denote as $B(S)$.

Example 5. In Figure 16 we show, from left to right, the effect on an initial blue/white coloring of each application of the color changing rule. In this example, the initial set of blue vertices is a zero forcing set of the graph depicted.

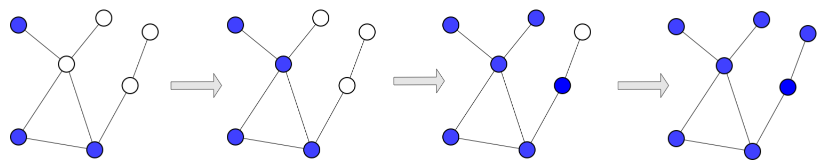


Figure 16: The zero forcing process.

Note that if a graph is not connected, then the zero forcing problem must be studied independently in each component.

Observe that zero forcing in a given graph G, models the process of forcing zeros in a null vector of any matrix X in $\mathcal{S}(G)$.

Example 6. In Figure 15, a zero forcing set of G is $\{v_1, v_5\}$. The color changing rule proceeds as follows:

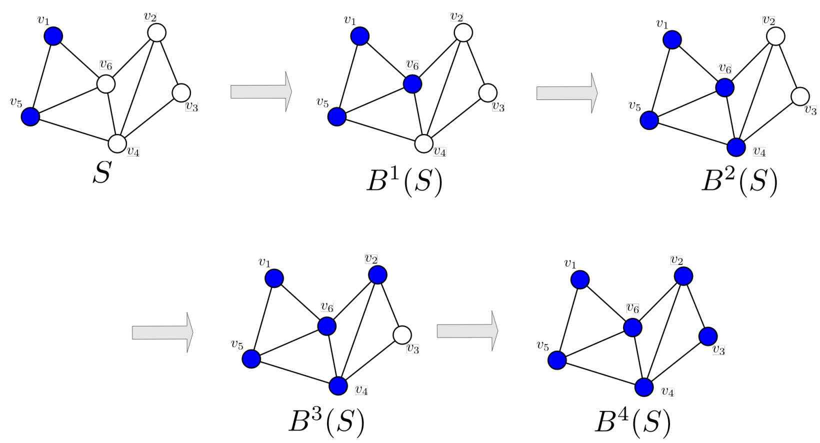


Figure 17: The color changing rule.

The following is the system of equations from the matrix described by G in Figure 15.

(1) $v_1 + 6v_5 + 9v_6 = 0$

(2) $3v_2 + 5v_3 + 8v_4 + 4v_6 = 0$

(3) $5v_2 + 4v_3 + 4v_4 = 0$

(4) $8v_2 + 4v_3 + 5v_4 + 9v_5 + 8v_6 = 0$

(5) $6v_1 + 9v_4 + 9v_6 = 0$

(6) $9v_1 + 4v_2 + 8v_4 + 9v_5 = 0$

Let i be a non-negative integer. Suppose

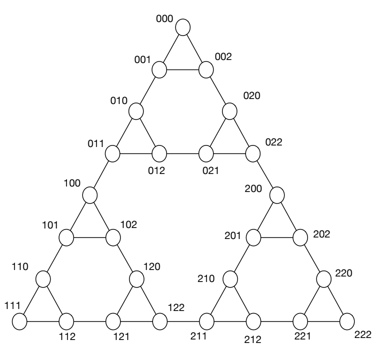
$B^i(S) = \{v_k : v_k = 0$ at the i^{th} step of solving the above system of equations $\}$.

$$
\begin{array}{lll}
 & & S = B^0(S) = \{v_1, v_5\} \\
(1) & v_1 + 6v_5 + 9v_6 = 0 \Rightarrow 9v_6 = 0 \Rightarrow v_6 = 0 & B^1(S) = \{v_1, v_5, v_6\} \\
(5) & 6v_1 + 9v_4 + 9v_6 = 0 \Rightarrow 9v_4 = 0 \Rightarrow v_4 = 0 & B^2(S) = \{v_1, v_5, v_6, v_4\} \\
(6) & 9v_1 + 4v_2 + 8v_4 + 9v_5 = 0 \Rightarrow 4v_2 = 0 \Rightarrow v_2 = 0 & B^3(S) = \{v_1, v_5, v_6, v_4, v_2\} \\
(2) & 3v_2 + 5v_3 + 8v_4 + 4v_6 = 0 \Rightarrow 5v_3 = 0 \Rightarrow v_3 = 0 & B^4(S) = \{v_1, v_5, v_6, v_4, v_2, v_3\}
\end{array}
$$

This observation leads to the result from [2]:

Theorem 7. *[2, Proposition 2.4] Let $G = (V, E)$ be a graph and let $S \subseteq V$ be a zero forcing set. Then $M(G) \leq |S|$, and thus $M(G) \leq \mathrm{Z}(G)$.*

Example 8. Figure 18 depicts a graph G and a matrix in $\mathcal{S}(G)$. Notice here that the nullity of the matrix is 0, whereas the zero forcing number of G is 6.



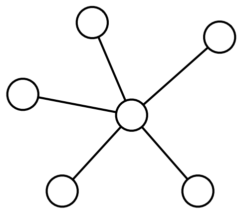
(a)

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
$$

(b)

Figure 18: (a) A graph G and (b) a matrix in $\mathcal{S}(G)$.

Example 9. Figure 19 depicts a graph G and a matrix in $\mathcal{S}(G)$. Notice here that the nullity of the matrix is 4, and the zero forcing number is 4 as well.



(a)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

(b)

Figure 19: (a) A graph G and (b) a matrix in $\mathcal{S}(G)$.

A computational approach to zero forcing is not feasible since the decision problem associated with the zero forcing problem is NP-complete [1]. As a consequence, one approach to the zero forcing problem consists of finding minimum zero forcing sets in particularly interesting families of graphs, or finding techniques to find or approximate the zero forcing number of graphs with specific properties. The American Institute of Mathematics maintains a catalog of graph families for which the zero forcing number has been determined [20]. The catalog also contains information on parameters related to zero forcing and references to the papers in which the results were obtained.

The concept of zero forcing was independently introduced by physicists studying control of quantum systems under the name of *graph infection* [6]. In addition, zero forcing is involved in the monitoring process of electrical power networks [4], and in fast-mixed graphs searching methods [19].

Zero Forcing in Digraphs

The definition of zero forcing is extended to loopless digraphs by replacing the neighborhood with the out-neighborhood.

Let $D = (V, A)$ be a loopless digraph. For any $S \subseteq V$ and any non-negative integer i, we define $B^i(S)$ by the following rules:

1. $B^0(S) = S$.

2. If $i \geq 0$, then
   $B^{i+1}(S) = B^i(S) \cup \{v \in V \setminus B^i(S) : \exists u \in B^i(S), N^+(u) \setminus B^i(S) = \{v\}\}$.

In terms of coloring, the set S can be referred to as the initial set of blue vertices. If a blue vertex has exactly one white neighbor, then we color that neighbor blue. This is equivalent to rule 2 and is referred to as the *color changing rule*. For each non-negative integer i, the set $B^i(S)$ is the set of blue vertices at the i^{th} iteration of the color changing rule. Note that the color changing rule is applied to every vertex of D at each iteration.

We say that S is a *zero forcing set* of D if there exists a non-negative integer m such that $B^m(S) = V(D)$. A *minimum zero forcing set* of D is a zero forcing set of minimum cardinality. The cardinality of a zero forcing set is the *zero forcing number* of D and is denoted as $\mathrm{Z}(D)$. The *zero forcing problem* consists of finding $\mathrm{Z}(D)$ for a given digraph D, and a zero forcing set S such that $|S| = \mathrm{Z}(D)$. If the vertices of a path P in D form a zero forcing set, then P is called a *zero forcing path* of D.

Let S be a zero forcing set. If $u \in B^i(S)$ and $N^+(u) \setminus B^i(S) = \{v\}$ for some integer i, we say that u *forces* v. If there exists an integer m, such that for all integers $i \geq m$, $B^m(S) = B^i(S)$, then $B^m(S)$ is called the *derived set* and we will denote it as $B(S)$.

Example 10. In Figure 20 we show, from left to right, the effect on an initial blue/white coloring of each application of the color changing rule. In this example, the initial set of blue vertices is a zero forcing set of the digraph depicted.

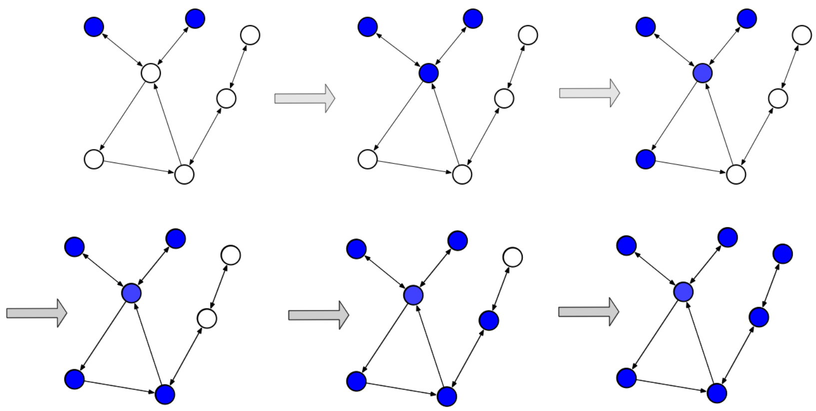


Figure 20: The zero forcing process in a digraph.

Example 11. In Figure 21 we show, from left to right, the effect on an initial blue/white coloring of each application of the color changing rule. In this example, the initial set of blue vertices is not a zero forcing set of the digraph depicted.

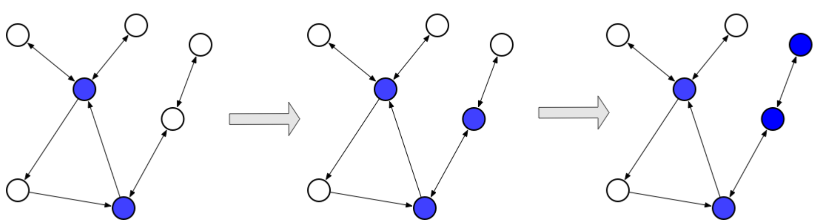


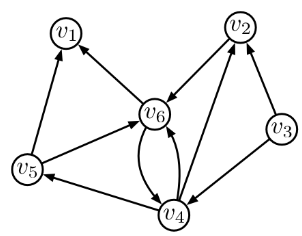
Figure 21: Failed zero forcing of Example 11.

Note that if a digraph is not weakly connected, then the zero forcing problem must be studied independently in each weak component.

For a positive integer n, let $\mathcal{M}_n$ denote the set of all $n \times n$ real matrices. Let $D = (V, A)$ be a loopless digraph of order n, where $V = \{v_1, v_2, \ldots, v_n\}$. The set of *matrices described by* D is the set $\mathcal{S}(D)$ defined as

$$\mathcal{S}(D) = \{X = [a_{ij}] \in \mathcal{M}_n : \text{ for } i \neq j, a_{ij} \neq 0 \Leftrightarrow (i,j) \in A(D)\}.$$

The difference between a matrix $M \in \mathcal{S}(D)$ and the adjacency matrix $\mathcal{A}(D)$ of D is that $\mathcal{A}(D)$ is a 0-1 matrix with zeros on the diagonal, while M is a real matrix that may have nonzero entries on the diagonal. In a digraph, the matrices described by a digraph need not be symmetric. See Figure 22 for an example of a matrix described by a digraph.



(a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 8 & 3 & 7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 10 \\ 4 & 0 & 0 & 0 & 4 & 1 \\ 2 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

(b)

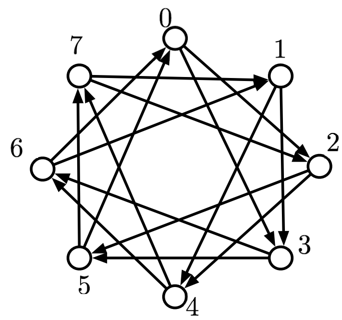
Figure 22: (a) A digraph D and (b) a matrix in $\mathcal{S}(D)$.

The *maximum nullity of* D is $\mathrm{M}(D) = \max\{\mathrm{null}(X) : X \in \mathcal{S}(D)\}$. The *minimum rank of* D is the $\mathrm{mr}(D) = \min\{\mathrm{rank}(X) : X \in \mathcal{S}(D)\}$, clearly

$$\mathrm{M}(D) + \mathrm{mr}(D) = |V(D)|.$$

As in the undirected case, zero forcing in a digraph D models the process to force zeros in a null vector of any matrix X in $\mathcal{S}(D)$, implying $M(D) \leq \mathrm{Z}(D)$ extending the result in Theorem 7 to digraphs [14].

Example 12. Figure 23 depicts a digraph D and a matrix in $\mathcal{S}(D)$. Notice here that the nullity of the matrix is 0, whereas the zero forcing number of D is 3.



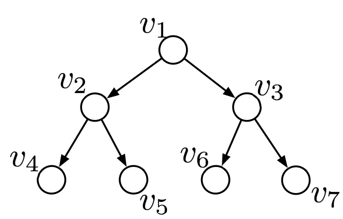
(a)

$$\begin{pmatrix} 0 & 0 & 5 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\ 10 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b)

Figure 23: (a) A digraph D and (b) a matrix in $\mathcal{S}(D)$.

Example 13. In Figure 24 the nullity of the adjacency matrix D is 4. Note that the zero forcing number of D is 4 as well.



(a)

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b)

Figure 24: (a) A digraph D and (b) the matrix $\mathcal{A}(D)$.

Let D be a digraph. The extension of the min-max relationship of Theorem 7 to digraphs [14] provides the nice property that if we find a matrix in $\mathcal{S}(D)$ and a zero forcing set S of D such that $|S| = \text{null}(D)$, then $M(D) = \text{Z}(D)$.

Unfortunately, such a matrix does not always exist. If one does exist, many times finding such a matrix is not a simple task. For example, Figure 23 shows a digraph and a matrix described by it, where the nullity of the matrix and the zero forcing number of the digraph are different.

III. SIERPIŃSKI GRAPHS

The Sierpiński graph family was introduced in [16]. An excellent survey of results in Sierpiński graphs can be found in [13, Chapter 4].

For integers $n \geq 3$, and $p \geq 1$, the *Sierpiński graph*, denoted $S(n,p)$, is defined by the following recursive process:

1. $S(n,1)$ is isomorphic to K_n.

   Label $V(S(n,1)) = \{0,1,\ldots,n-1\}$.

2. $S(n,p+1)$ is constructed by copying $S(n,p)$ n times and adding exactly one edge between each pair of copies following a rule we explain next.

   Label $V(S(n,p+1)) = \bigcup_{i=0}^{n-1} iV(S(n,p))$.

3. For all integers k, such that $1 \leq k \leq p-1$, add to $E(S(n,p))$ all the p-tuples $i = (i_p,\ldots,i_1)$ and $j = (j_p,\ldots,j_1)$ such that $i_p = j_k$, and $j_p = i_k$.

Note that at each iteration, we are adding a leftmost coordinate. Each vertex of $S(n,p)$ is a p-tuple of integers from $\{0,1,\ldots,n-1\}$. See Figure 25 for examples of Sierpiński graphs when $n = 3$.

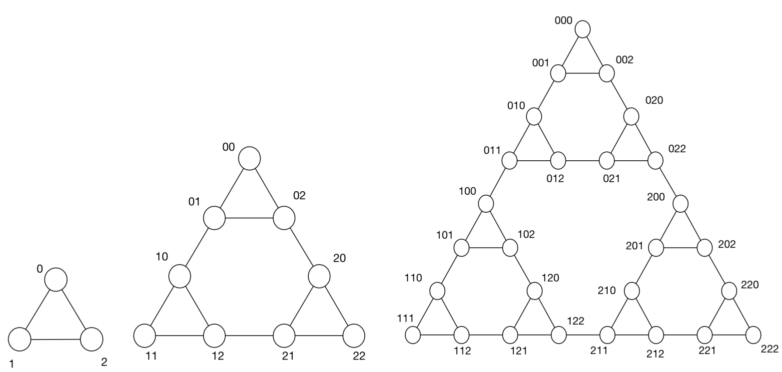


Figure 25: The graphs $S(3,1)$, $S(3,2)$, and $S(3,3)$.

Formally, we may define the Sierpiński graphs as follows:

Given two integers n and p with $n \geq 3$ and $p \geq 1$, the Sierpiński graph $S(n,p)$ [16] has as vertices all p-tuples of the integers $\{0, 1, \ldots, n-1\}$ denoted as $(s_p, \ldots, s_1)$. Two vertices $(s_p, \ldots, s_1)$ and $(t_p, \ldots, t_1)$ are neighbors in $S(n,p)$ if and only if there exists an integer r, $0 \leq r \leq p-1$, such that

i) $s_i = t_i$ for every i, $r+1 \leq i \leq p$,

ii) $s_r \neq t_r$,

iii) $s_i = t_r$ and $t_i = s_r$ for every i, $1 \leq i \leq r-1$.

For integers $n \geq 3$, and $p \geq 1$, the order of $S(n,p)$ is n^p. The graph $S(n,p)$ has n vertices of degree $n-1$ and $n^p - n$ vertices of degree n. The vertices of degree $n-1$ are called *extreme* vertices. If the vertex $(s_p, \ldots, s_1)$ is an extreme vertex of $S(n,p)$, then $s_i = s_{i+1}$ for every $i = 1, \ldots, p-1$.

For $p \geq 2$, $S(n,p)$ has n^i induced copies of $S(n, p-i)$ for every $i = 1, 2, \ldots, p-1$. In particular there are n^{p-1} induced copies of $S(n,1)$ and n induced copies of $S(n, p-1)$.

The p-tuples representing the vertices of each of those copies coincide in the i leftmost digits $s_p, \ldots, s_{p-i}$. If s is an i-tuple of integers in $\{0, 1, \ldots, n-1\}$, let $sS(n, p-i)$ denote the subgraph of $S(n,p)$ induced by the vertices whose leftmost i digits coincide with s. Thus, $sS(n,p-i)$ is isomorphic to $S(n,p-i)$. In particular, $tS(n,p-1)$ denotes the subgraph induced in $S(n,p)$ by all the vertices whose leftmost digit is t, for every integer $t = 0, 1, \ldots, n-1$. Figure 26 depicts the subgraphs of a Sierpiński graph.

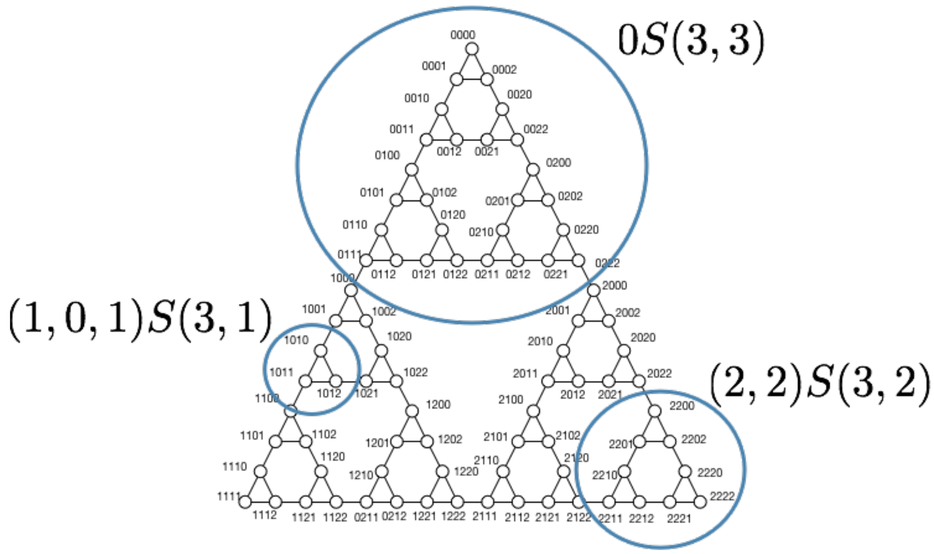


Figure 26: Subgraphs of $S(3,4)$ and their labelings.

Let n and p be integers such that $n \geq 3$, and $p \geq 2$. Let $G = S(n,p)$. We define the *border* of an induced subgraph $tS(n,p-1)$ in $S(n,p)$, denoted $b(tS(n,p-1))$, as the set containing the vertices of $tS(n,p-1)$ with neighbors outside $tS(n,p-1)$. Symbolically,

$$b(tS(n,p-1)) = \{v \in V(tS(n,p-1)) : N(v) \setminus V(tS(n,p-1)) \neq \varnothing\}.$$

By definition of a Sierpiński graph, for each vertex $v \in b(tS(n,p-1)$, we have $|N(v) \setminus V(tS(n,p-1))| = 1$. Moreover, if $N(v) \setminus V(tS(n,p-1)) = \{w\}$, then $w \in b(t'S(n,p-1))$, for some integer t', such that $0 \leq t' \leq n-1$ and $t \neq t'$.

For each subgraph $tS(n,p-1)$ let us also define the *interior* of an induced subgraph $tS(n,p-1)$ as the set

$$int(tS(n,p-1)) = V(tS(n,p-1)) \setminus b(tS(n,p-1)).$$

It is important to observe that $n-1$ of the n extreme vertices of $S(n,p-1)$

correspond to $b(tS(n,p-1))$, for every $t = 0,\ldots,n-1$. See Figure 27 as an example of the labeling of $S(n,1)$ subgraphs.

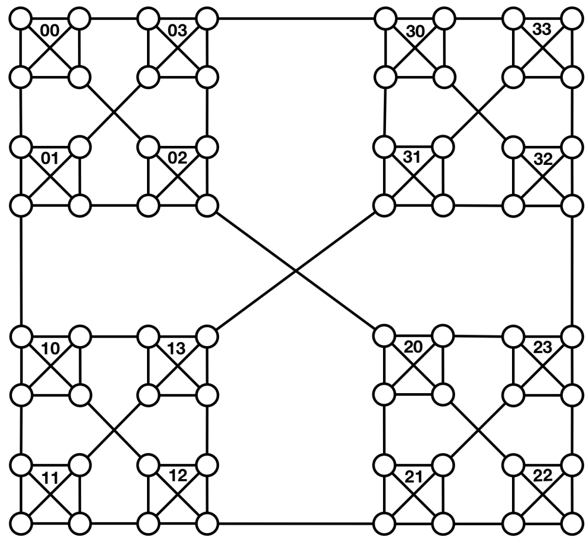


Figure 27: Labeling of the induced copies of K_4 in $S(4,3)$.

Observation 14. *Let $S(n,p)$ be a Sierpiński graph, note that there are n^{p-1} induced subgraphs in $S(n,p)$ that are isomorphic to K_n. We can define a total ordering $<$ to the set of K_n subgraphs in $S(n,p)$. For the $p-1$ tuples $i = (i_{p-1}, i_{p-2}, \ldots, i_1)$ and $j = (j_{p-1}, j_{p-2}, \ldots, j_1)$, we will say that $iK_n < jK_n$ if for some integer k, and all r such that $p-1 \leq r \leq k$, then $i_r = j_r$ and $i_{k-1} < j_{k-1}$.*

Zero Forcing in Sierpiński graphs

We now begin to study the zero forcing number of the Sierpiński graphs. When studying zero forcing of these graphs, it is natural to examine the rank of the matrices described by $S(n,p)$. Unfortunately, as seen in Figure 23, the nullity of the matrix is 0, whereas, the zero forcing number is 6. The off-diagonal entries that connect each level of $S(n,p)$ provide linearly independent vectors in the matrix. Thus, we take a different approach and study the zero forcing number of the Sierpiński graphs using the color changing process associated with zero forcing.

By definition, $S(n,1) = K_n$. Since $\mathrm{Z}(K_n) = n-1$, then $\mathrm{Z}(S(n,1)) = n-1$. From this point forward, we assume $p \geq 2$.

We begin by examining the Sierpiński graph when $p = 2$.

Lemma 15. *For an integer $n \geq 3$,* $\mathrm{Z}(S(n,2)) = \dfrac{n(n-1)}{2}$.

Proof. We shall first show that $\mathrm{Z}(S(n,2)) \geq \dfrac{n(n-1)}{2}$.

Let i and j be non-negative integers such that $i,j \leq n-1$ and $i \neq j$. Let S be a set of blue vertices such that $S \cap V(iS(n,1)) = \varnothing$ and $S \cap V(jS(n,1)) = \varnothing$. Note that for each k, such that $S \cap V(kS(n,1)) \neq \varnothing$, S can at most force $N[V(kS(n,1))]$.

Let X be a zero forcing set of $S(n,p)$. For some integer $j \leq n-3$, if the vertices in $\bigcup_{p=0}^{j} V(i_pS(n,1))$ are forced, and X has no vertices from any other copy of $S(n,1)$. Then each vertex of $i_pS(n,1)$ has no white neighbors. Also, for $k \neq i_p$, each vertex (k,i_p) has $n-j$ white neighbors. So, in order for X to be a zero forcing set, X must contain at least $n-j-1$ more vertices that are neighbors of some (k,i_p).

We have that X must contain vertices from $n-1$ copies of $S(n,1)$. Furthermore, for each $iS(n,1)$, where $i \leq n-2$, the zero forcing set X must contain $n-(i+1)$ vertices. Therefore,

$$|X| \geq \sum_{i=0}^{n-2} n-(i+1) = \sum_{i=1}^{n-1} n-i = \frac{n(n-1)}{2}.$$

To complete the proof, we construct a zero forcing set X, such that

$|X| = n - 2$. This construction is as follows:

$$\text{For } 0 \le i \le n-2, \text{ define } X_i = \{(i,i),(i,i+1),\ldots,(i,n-2)\}.$$
$$\text{Let } X = \bigcup_{i=0}^{n-2} X_i.$$

Note that for each i, j, such that $i \neq j$, we have $X_i \cap X_j = \varnothing$. Therefore, we have $|X|$ to be the following:

$$|X| = \sum_{i=0}^{n-2} |X_i| = \sum_{i=0}^{n-2} n - (i+1) = \frac{n(n-1)}{2}.$$

Next, we show that X is indeed a zero forcing set of $S(n,2)$. Notice that X_0 forces $0S(n,1)$, and the vertex $(0,1)$ forces $(1,0)$. Then $X_1 \cup \{(1,0)\}$ forces $1S(n,1)$. We continue this process until $(n-2)S(n,1)$ is forced. At that point, each vertex in $iS(n,1)$, for $0 \le i \le n-2$, forces a vertex in $(n-1)S(n,1)$. Finally, every vertex in $(n-1)S(n,1)$ has $(n-1,n-1)$ as its only white neighbor. Thus, $S(n,2)$ is forced.

□

Theorem 16. *For any positive integers $n \ge 3$ and $p \ge 2$,*

$$\mathrm{Z}(S(n,p)) = \mathrm{Z}(S(n,p-1)) + \frac{n^{p-2}(n-2)(n-1)}{2}.$$

Proof. We begin by showing that

$$\mathrm{Z}(S(n,p)) \ge \mathrm{Z}(S(n,p-1)) + \frac{n^{p-2}(n-2)(n-1)}{2}.$$

Let X be a zero forcing set of $S(n,p)$, and let ω be a $(p-2)$-tuple. We claim the following:

$$|X \cap int(wS(n,2))| \geq \frac{(n-2)(n-1)}{2}.$$

Let $W = X \cap int(wS(n,2))$, and let us suppose that every extreme vertex of $\omega S(n,2)$ is blue.

Let i, j, and k be different non-negative integers such that $i,j,k \leq n-1$. Let S be a set of blue vertices in the interior of $\omega S(n,2)$, such that the following are true:

$$|S \cap V(\omega i(Sn,1))| = 1,$$

$$|S \cap V(\omega j(Sn,1))| = 1, \text{ and}$$

$$|S \cap V(\omega k(Sn,1))| = 1.$$

Note that for each r such that $|S \cap V(wrS(n,1))| > 1$, S can at most force $N[V(wrS(n,1))]$.

For some integer $j \leq n-4$, if the vertices in $\bigcup_{p=0}^{j} V(\omega i_p S(n,1))$ are forced, and W has no other vertices from any other copy of $S(n,1)$ in $wS(n,2)$, then each vertex of $\omega i_p S(n,1)$ has no white neighbors. Furthermore, for $r \neq i_p$, each vertex $\omega(r,i_p)$ has $n-j-1$ white neighbors. Hence, in order for X to be a zero forcing set, W must contain at least $n-j-2$ more vertices that are neighbors of some $\omega(r,i_p)$.

We have that W must contain vertices from $n-2$ copies of $S(n,1)$ in $wS(n,2)$ and that for each $wiS(n,1)$, such that $i \leq n-3$, W must contain $n-(i+2)$ vertices. Therefore,

$$|W| \geq \sum_{i=0}^{n-3} n-(i+2) = \frac{(n-1)(n-2)}{2}.$$

Which proves our claim that

$$|X \cap int(wS(n,2))| \geq \frac{(n-2)(n-1)}{2}.$$

There are n^{p-2} copies of $S(n,2)$ in $S(n,p)$, this means that X contains $\frac{(n-1)(n-2)}{2}$ vertices in the interior of each copy of $S(n,2)$.

Now, we assumed that the extreme vertices of each copy of $S(n,2)$ were blue. Thus, by contracting the edges of each copy of $S(n,1)$ to the extreme vertices in $S(n,2)$ (see Figure 4), the contracted graph is now isomorphic to $S(n,p-1)$. Finding a zero forcing set for $S(n,p-1)$ would be equivalent to finding a zero forcing set that forces the extreme vertices of each copy of $S(n,2)$ in $S(n,p)$. Thus, we have that

$$\mathrm{Z}(S(n,p)) \geq \mathrm{Z}(S(n,p-1)) + \frac{n^{p-2}(n-2)(n-1)}{2}.$$

To complete the proof, we construct a zero forcing set X^p such that

$$|X^p| = \mathrm{Z}(S(n,p-1)) + \frac{n^{p-2}(n-2)(n-1)}{2}.$$

We will define X^p recursively for each $S(n,p)$:

If $p = 1$, $\quad X^1 = \{0, 1, \ldots, n-2\}$.

If $X^{p-1} = \{x_1, \ldots, x_s\}$, define $X_i^p = \{ix_{i+1}, \ldots, ix_s\}$.

Let $X^p = \bigcup_{i=0}^{n-1} X_i^p$.

Next, we prove by induction on p that X^p is a minimum zero forcing set of

$S(n,p)$. Now, X^1 is a minimum zero forcing set for $S(n,1)$, and by Lemma 15,

$$\begin{aligned}|X^2| &= \frac{n(n-1)}{2}\\ &= n-1+\frac{(n-2)(n-1)}{2}\\ &= \mathrm{Z}(S(n,1))+\frac{(n-2)(n-1)}{2}.\end{aligned}$$

Suppose then, that X^{p-1} is a zero forcing set of $S(n,p-1)$, such that

$$|X^{p-1}| = \mathrm{Z}(S(n,p-2))+\frac{n^{p-3}(n-2)(n-1)}{2}. \tag{1}$$

Then, $|X^{p-1}| = \mathrm{Z}(S(n,p-1))$.

By definition, $|X^p| = \sum_{i=0}^{n-1}|X_i^p|$, and since $|X_i^p| = |X^{p-1}|-i$, then

$$|X^p| = \sum_{i=0}^{n-1}|X^{p-1}|-i = n|X^{p-1}|-\frac{n(n-1)}{2}. \tag{2}$$

By substituting $|X^{p-1}|$ with (1), we have

$$\begin{aligned}|X^p| &= n\,\mathrm{Z}(S(n,p-2))+n\frac{n^{p-3}(n-2)(n-1)}{2}-\frac{n(n-1)}{2}\\ &= n|X^{p-2}|-\frac{n(n-1)}{2}+\frac{n^{p-2}(n-2)(n-1)}{2}.\end{aligned}$$

By substituting $n|X^{p-2}|-\dfrac{n(n-1)}{2}$ with $|X^{p-1}|$, we arrive at

$$\begin{aligned}|X^p| &= |X^{p-1}|+\frac{n^{p-2}(n-2)(n-1)}{2}\\ &= Z(S(n,p-1))+\frac{n^{p-2}(n-2)(n-1)}{2}.\end{aligned}$$

Note that there are exactly n^{p-1} induced subgraphs in $S(n,p)$ isomorphic to K_n. Since

$$\{(0,0,\ldots,0),(0,0,\ldots,0,1),\ldots,(0,0,\ldots,0,n-2)\} \subset X^p,$$

then, $(0,0,\ldots,0,0)$ has exactly one white neighbor, namely $(0,\ldots,0,n-1)$, and so $(0,\ldots,0)K_n$ and its neighbor K_n's have one blue vertex forced from $(0,\ldots,0)K_n$. That is to say that every vertex in

$$\{(0,0,\ldots,1,0),(0,0,\ldots,2,0),\ldots,(0,0,\ldots,n-1,0)\}$$

is forced. Since,

$$\{(0,0,\ldots,0,1,1),\ldots,(0,0,\ldots,0,1,n-2)\} \subseteq X^p,$$

then the vertices

$$(0,0,\ldots,1,0),(0,0,\ldots,0,1,1),\ldots,(0,0,\ldots,0,1,n-2)$$

are blue. At this point, $(0,0,\ldots,1,0)$ has only one white neighbor, namely $(0,0,\ldots,1,n-1)$. Thus, the vertices of $(0,\ldots,0,1)K_n$ are forced.

Suppose that if $j = (j_{p-1},j_{p-2},\ldots,j_1)$ is a $p-1$-tuple, and that for all $p-1$-tuples i, such that $iK_n < jK_n$, the vertices of iK_n are forced.

Now, $V(jK_n) = \{j0, j1, \ldots, j(n-1)\}$, and the neighboring K_n of jK_n are

$$(j_{p-1},j_{p-2},\ldots,0)K_n,\ (j_{p-1},j_{p-2},\ldots,1)K_n,\ \ldots,\ (j_{p-1},j_{p-2},\ldots,j_1-1)K_n,$$
$$(j_{p-1},j_{p-2},\ldots,j_1+1)K_n,\ \ldots,\ (j_{p-1},j_{p-2},\ldots,n-1)K_n.$$

However, the vertices, $jj_1, j(j_1+1), \ldots, j(n-2)$ are in X^p. Also, the vertices $j0, j2, \ldots, j(j_1-1)$ are forced by all the neighboring iK_n of jK_n such that $iK_n < jK_n$. Therefore, $j0$ forces $j(n-1)$ and every vertex of jK_n is forced. Hence we have shown by induction that all the vertices in each K_n subgraph have been forced. We conclude that X^p is a zero forcing set of $S(n,p)$ and that

$$\mathrm{Z}(S(n,p)) = Z(S(n,p-1)) + \frac{n^{p-2}(n-2)(n-1)}{2}.$$

□

Corollary 17. *Let $n \geq 3$ and $p \geq 2$ be integers, then*

$$\mathrm{Z}(S(n,p)) = n\,\mathrm{Z}(S(n,p-1)) - \frac{n(n-1)}{2} = \frac{n^p - 2n^{p-1} + n}{2}.$$

Proof. In the proof of Theorem 16 (see (2)), we showed that X^p is a minimum zero forcing set and that,

$$X^p = n\,\mathrm{Z}(S(n,p-1)) - \frac{n(n-1)}{2}.$$

So, we only need to prove the following:

$$Z(S(n,p)) = \frac{n^p - 2n^{p-1} + n}{2}.$$

We will prove the claim by induction on p. By Theorem 16,

$$\mathrm{Z}(S(n,p)) = \mathrm{Z}(S(n,p-1)) + \frac{n^{p-2}(n-2)(n-1)}{2}, \tag{3}$$

and Theorem 15 provides the base case,

$$Z(S(n,2)) = \frac{n(n-1)}{2} = \frac{n^2 - 2n^{2-1} + n}{2}.$$

Let us suppose that,

$$Z(S(n,p-1)) = \frac{n^{p-1} - 2n^{p-2} + n}{2}. \tag{4}$$

Then, by substituting (4) into (3), we arrive at

$$Z(S(n,p)) = \frac{n^p - 2n^{p-1} + n}{2}.$$

□

See Figure 28 for an example of a minimum zero forcing set for $S(4,4)$.

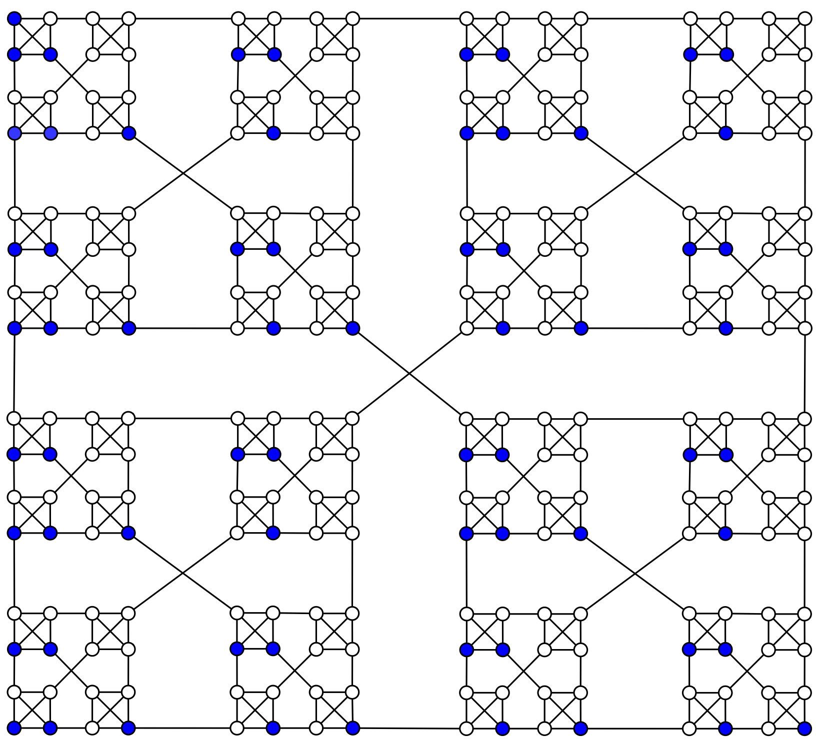


Figure 28: Minimum zero forcing set of $S(4,4)$.

IV. CIRCULANT DIGRAPHS

Definition 18. [11] Let n be an integer, such that $n \geq 3$, and let J denote a non-empty set such that $J \subseteq \{0, 1, \ldots, n-1\}$. The *circulant digraph* $D = DC_n\langle J\rangle$ has vertices $V(D) = \{0, 1, \ldots, n-1\}$ and arcs $A(D) = \{(u, v) : v = u + j \pmod{n}\ \forall j \in J\}$. We call J the *jump set*, and each element in J is a *jump*.

Observation 19. *For any vertex v in a circulant digraph with jump set J, $d^+(v) = d^-(v) = |J|$.*

Observation 20. *For two vertices u and v of a circulant digraph of order n with jump set $J = \{j_1, j_2, \ldots, j_k\}$, the existence of a $u - v$ path implies that there exists a sequence of non-negative integers $\{\alpha_i\}_{i=1}^k$ such that,*

$$v = u + \alpha_1 j_1, \alpha_2 j_2, \ldots, \alpha_k j_k \pmod{n}.$$

Figure 29 depicts a circulant digraph of order 8 with jump set $\{2, 3\}$

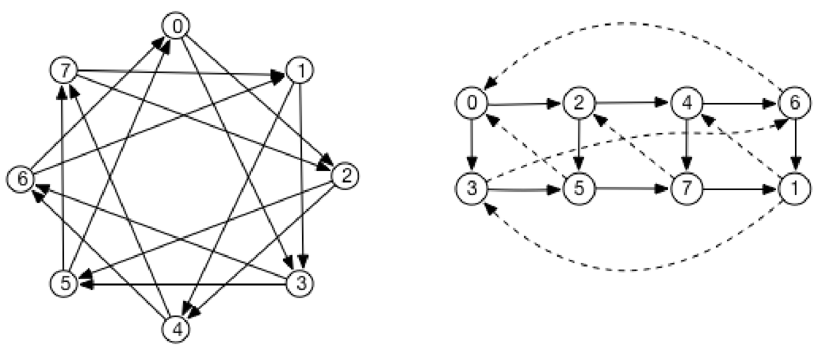


Figure 29: Labeled $DC_8\langle\{2, 3\}\rangle$.

We may draw a circulant digraph with 2 jumps as a lattice. When drawing a circulant digraph as a lattice, we will have repeated vertices. This representation allows one to observe the neighbors next to each other. See Figure 30 for an example.

$$\begin{tikzcd}
0 \arrow[r]\arrow[d] & s \arrow[r]\arrow[d] & 2s \arrow[r]\arrow[d] & \cdots \arrow[r]\arrow[d] & ks \arrow[r]\arrow[d] & \cdots \arrow[r]\arrow[d] & (n-1)s \arrow[r]\arrow[d] & ns \arrow[d] \\
t \arrow[r]\arrow[d] & t+s \arrow[r]\arrow[d] & t+2s \arrow[r]\arrow[d] & \cdots \arrow[r]\arrow[d] & t+ks \arrow[r]\arrow[d] & \cdots \arrow[r]\arrow[d] & t+(n-1)s \arrow[r]\arrow[d] & t+ns \arrow[d] \\
\vdots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \vdots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \vdots \arrow[d] \\
kt \arrow[r]\arrow[d] & kt+s \arrow[r]\arrow[d] & kt+2s \arrow[r]\arrow[d] & \cdots \arrow[r]\arrow[d] & kt+ks \arrow[r]\arrow[d] & \cdots \arrow[r]\arrow[d] & kt+(n-1)s \arrow[r]\arrow[d] & kt+ns \arrow[d] \\
\vdots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \ddots \arrow[r]\arrow[d] & \vdots \arrow[d] \\
nt \arrow[r] & nt+s \arrow[r] & nt+2s \arrow[r] & \cdots \arrow[r] & nt+ks \arrow[r] & \cdots \arrow[r] & nt+(n-1)s \arrow[r] & nt+ns
\end{tikzcd}$$

Figure 30: Lattice of $DC_n\langle\{s,t\}\rangle$.

These digraphs are called *circulant* because of their relation to circulant matrices. Indeed, the definition of the circulant digraph is motivated by the circulant matrix itself. Using the definition from [10], a *circulant matrix* of order n is a square matrix of the form:

$$X = \mathrm{circ}(c_0, c_1, \ldots, c_{n-1}) = \begin{pmatrix} c_0 & c_1 & \ldots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \ldots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1 & c_2 & \ldots & c_{n-1} & c_0 \end{pmatrix}.$$

The elements of each row of X are identical to those of the previous row, but are shifted one position to the right and wrapped around. Thus, we can characterize a circulant matrix by the first row.

Let $D = DC_n\langle J\rangle$. The adjacency matrix of D is a 0-1 circulant matrix $X = \mathrm{circ}(c_0, c_1, \ldots, c_{n-1})$, where $c_i = 1$ if i is in J. If $X = \mathrm{circ}(c_0, c_1, \ldots, c_{n-1})$, where $c_i \neq 0$ if and only if i is in J, then $X \in \mathcal{S}(D)$.

Zero Forcing in Circulant Digraphs

In order to study zero forcing in circulant digraphs, it is natural to investigate results obtained for the rank of a circulant matrix, since there is a large amount of research in the subject. As noted earlier, circulant matrices can be described with a finite sequence; thus we can also describe a circulant matrix with a polynomial.

Definition 21. For an integer $n \geq 3$, consider the circulant matrix $X = \text{circ}(c_0, c_1, \ldots, c_{n-1})$. The *associated polynomial of* $X = \text{circ}(c_0, c_1, \ldots, c_{n-1})$ is

$$f(x) = \sum_{i=0}^{n-1} c_i x^i.$$

The following result gives a relationship between the rank of a circulant matrix X and the associated polynomial of X.

Theorem 22. *[15, Proposition 1.1] For any integer $n \geq 3$, the rank of a circulant matrix X of order n is $n - d$, where d is the degree of the greatest common divisor of $1 - x^n$ and the associated polynomial of X.*

Example 23. Let $D = DC_{20}\langle\{7, 3\}\rangle$ be a circulant digraph. Then $p(x) = x^3 - x^7$ is the associated polyinomial of a circulant matrix from $\mathcal{S}(D)$ is. However, $\text{Z}(D) = 6$ and $\gcd(p(x), 1 - x^{20}) = x^4 - 1$, which means that the nullity is 4.

While testing examples by using Theorem 22, many times we were close to the zero forcing number of the circulant digraph like in Example 23. We arrived at the conjecture that the maximum nullity of a strongly connected circulant digraph D with 2 jumps is $M(D) = \text{Z}(D) - 1$. However, we have yet to provide a proof. So, we pursue the zero forcing problem via exploration of the coloring changing rule.

We begin our study of zero forcing in circulant digraphs by examining when these digraphs are weakly connected.

Example 24. Consider the circulant digraph $D = DC_{10}\langle\{2,6\}\rangle$. Note that there does not exist a path between an even vertex and an odd vertex. Indeed, D has two strongly connected components.

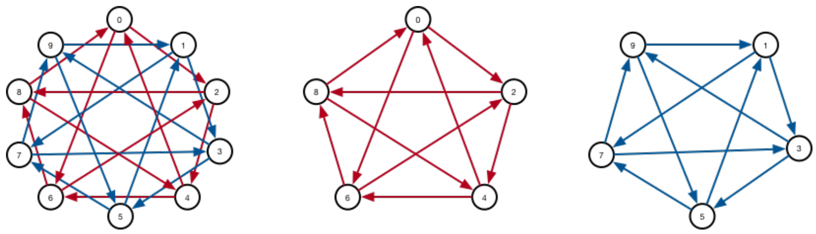


Figure 31: Labeled $DC_{10}\langle\{2,6\}\rangle$.

Example 25. Consider the circulant digraph $D' = DC_{10}\langle\{2,5\}\rangle$. Note that every vertex can be represented as a linear combination of 2 and 5 in $\mathbb{Z}_n$. Therefore, there exists a path between every two vertices, implying that the circulant digraph is strongly connected.

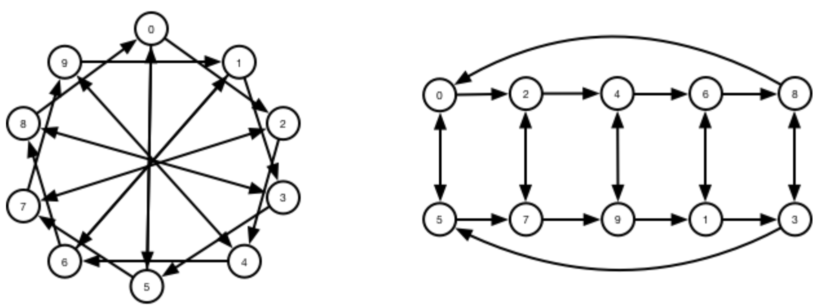


Figure 32: Labeled $DC_{10}\langle\{2,5\}\rangle$.

This is direct consequence of the following theorems:

Theorem 26. *[18, Corollary 1] Let $D = DC_n\langle\{a_1, a_2, \ldots, a_k\}\rangle$ be a circulant digraph. Then D is strongly connected if and only if* $\gcd(a_1, a_2, \ldots, a_k, n) = 1$.

Theorem 27. *[5, Proposition 1] Let $G = C_n\langle\{a_1, a_2, \ldots, a_k\}\rangle$ be a circulant graph. Then G is connected if and only if* $\gcd(a_1, a_2, \ldots, a_k, n) = 1$.

From Theorem 26 and Theorem 27 we immediately obtain the following result.

Corollary 28. *There does not exist a circulant digraph that is weakly connected but not strongly connected.*

Huang and Chang proved the following [7]:

Theorem 29. *[7, Corollary 4] Let* $n \geq 3$ *and* $D = DC_n\big\langle\{a_1, a_2, \ldots, a_k\}\big\rangle$ *be a circulant digraph. If* $\gcd(a_1, a_2, \ldots, a_k, n) = d > 1$*, then the components of* D *are isomorphic to* $DC_{\frac{n}{d}}\big\langle\{\frac{a_1}{d}, \frac{a_2}{d}, \ldots, \frac{a_k}{d}\}\big\rangle$*.*

Clearly, for a circulant digraph D as described in Theorem 29, D has d strongly connected components that are isomorphic to circulants themselves. Thus, we have the following observation:

Observation 30. *Let* D *be a circulant digraph as described in Theorem 29. Let* $D' = DC_{\frac{n}{d}}\big\langle\{\frac{a_1}{d}, \frac{a_2}{d}, \ldots, \frac{a_k}{d}\}\big\rangle$ *be a component of* D*. Then* $\mathrm{Z}(D) = d\,\mathrm{Z}(D')$*.*

Having established that a circulant digraph is disconnected or strongly connected, we shall simply use the term *connected*. At this point it is only necessary to consider connected digraphs. We begin by considering a connected circulant digraph with only one jump.

Proposition 31. *Let* $D = DC_n\big\langle\{s\}\big\rangle$ *be a connected circulant digraph, then* $\mathrm{Z}(D) = 1$*.*

Proof. Clearly D is isomorphic to the directed cycle C_n. Since $\mathrm{Z}(C_n) = 1$, then $\mathrm{Z}(D) = 1$. □

For the rest of the thesis, we consider the zero forcing problem on connected circulant digraphs with 2 jumps. We assume all arithmetic is done in $\mathbb{Z}_n$.

Lemma 32. *Let* $D = DC_n\langle\{s,t\}\rangle$ *be a connected circulant digraph. The set* $S = \{0, s, \ldots, ns\}$ *is a zero forcing set of* D.

Proof. Note that for each vertex v of D, we have $\mathrm{d}^+(v) = 2$. For each integer $i = 0, 1, \ldots, n$, let $S + it = \{it, it + s, \ldots, it + ns\}$. The induced subgraph $D[S + it]$ is a cycle, and for each vertex x in S, the out-neighbor of x, which is $x + s$ is in S. Therefore, the second out-neighbor of x, which is $x + t$ is in $B^1(S)$.

Inductively, for each $i = 0, 1, \ldots, n$, we have that $S + it \subseteq B^i(S)$. Therefore, for every non-negative integers $i, j \leq n$, $it + js$ is in $B(S)$. This is sufficient to conclude that $B(S) = V(D)$ and S is a zero forcing set. □

Next, we introduce some notation and definitions.

Definition 33. Let $D = DC_n\langle\{s,t\}\rangle$ be a connected circulant digraph. Let $k \leq n$ be an integer and $S = \{0, s, \ldots, ks\}$ be a set of vertices of D. For all non-negative integers, $i \leq k$, denote $T^i(S)$ and $T(S)$ as

$$T^i(S) = \{it, it + s, \ldots, it + (k - i)s\}.$$

$$T(S) = \bigcup_{i=0}^{k} T^i(S) = \{it + js : i \leq k,\ j \leq k - i\}.$$

We call $T(S)$ the *triangle generated by* S.

Figure 33 depicts a triangle of $DC_n\langle\{s,t\}\rangle$ when the circulant is drawn as a lattice.

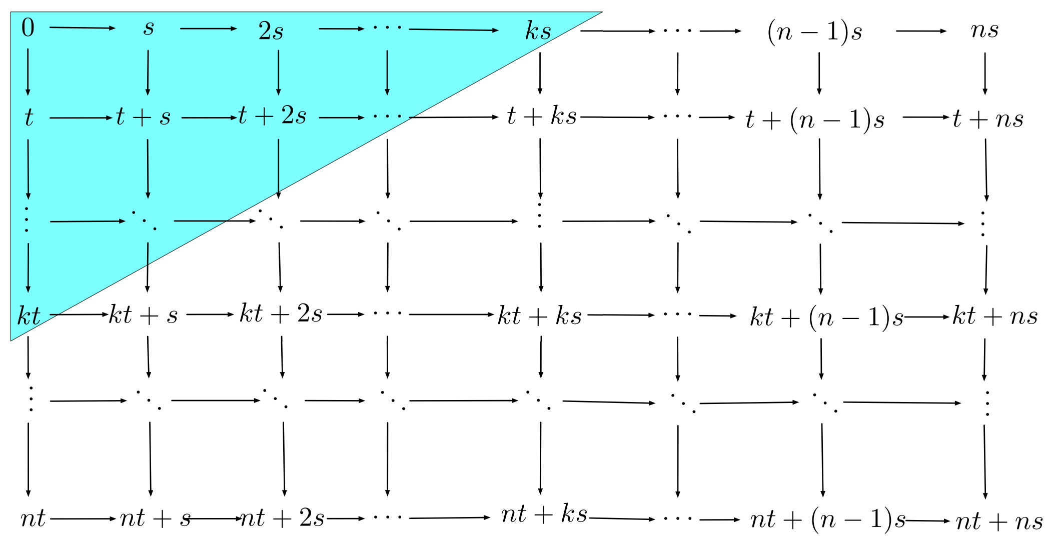
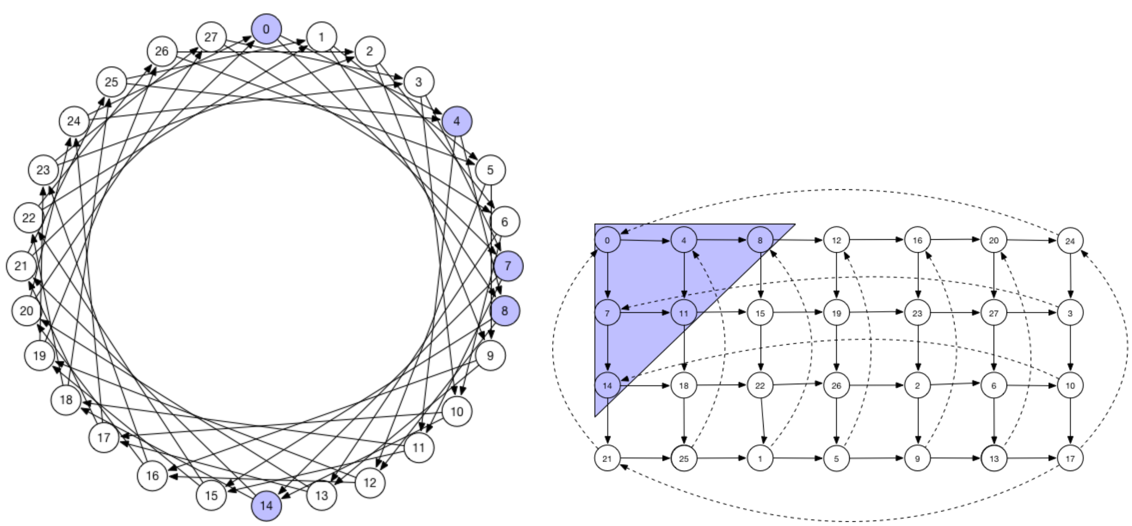


Figure 33: $T(\{0, s, \ldots, ks\})$ in $DC_n\langle\{s,t\}\rangle$.

The circulant digraph is commonly depicted as a circle as in Figure 34 (a). However, visualizing the graph as a grid or as a lattice as in (b) allows one to recognize why we use the term triangle.



(a) Standard drawing of $DC_{28}\langle\{4,7\}\rangle$. (b) Visualization as a triangle.

Figure 34: Two representations of $T(\{0,4,8\})$ in $DC_{28}\langle\{4,7\}\rangle$.

Let $D = DC_n\langle\{s,t\}\rangle$ be a connected circulant digraph. Let $k < n$ be an integer, and $S = \{0, s, \dots, ks\}$ be a subset of vertices of D. We make the following observations:

Observation 34. *If $it + js$ is in $T(S)$, then $j \le k - i$ and $i \le k - j$.*

Observation 35. *For each integer $i \geq 0$, the vertices in $T^i(S)$ form a $(it) - (it + (k - i)s)$ path.*

Observation 36. *If $S' \subseteq B(S)$, then $B(S') \subseteq B(S)$.*

If $S \subseteq S'$, then $B(S) \subseteq B(S')$.

If $S \subseteq S' \subseteq B(S)$, then $B(S) = B(S')$.

Observation 37. *If $T = \{0, t, \dots, kt\}$, then $T(T) = T(S)$.*

Observation 38. *Let v be a vertex in $B(S)$. Since $d^+(v) = 2$, we have that $v + s$ is in $B(S)$ if and only if $v + t$ is in $B(S)$.*

We will use the notion of the triangle to prove our main theorem:

Theorem 39. *Let $D = DC_n\big\langle\{s, t\}\big\rangle$ be a connected circulant digraph. Let k be the smallest positive integer that satisfies one of the following conditions:*

$$ks + t = it \qquad\qquad ks + s = it$$

$$ks + s = is \qquad\qquad kt + t = is.$$

Then $S = \{0, s, \dots, ks\}$, is a minimum zero forcing set of D.

Proof. To prove this theorem, we will prove a sequence of claims:

Claim: For all integers $i \ge 0$, $T^i(S) \subseteq B^i(S)$ for some integer $i \ge 0$.

We will prove this claim by induction on i.

As a base case, $T^0(S) = S = B^0(S)$. We assume that $T^i(S) \subseteq B^i(S)$.

For each vertex $x \neq it + (k-i)s$ in $T^i(S)$, $x + s$ is in $T^i(S)$ and thus in $B^i(S)$. Therefore $x + t$ is in $B^{i+1}(S)$. Thus, $T^{i+1}(S) \subseteq B^{i+1}(S)$. So, we have shown that $T(S) \subseteq B(S)$.

Claim: Suppose k is the smallest integer such that $N^+(ks) \cap T(S) \neq \varnothing$. Then $(k+1)s = it$ or $ks + t = it$.

Suppose that $(k+1)s$ is in $T(S)$, then $(k+1)s = it + js$ for $0 \leq i \leq k$ and $0 \leq j \leq k - i$. By way of contradiction, suppose $j \geq 1$. Let $S' = \{0, s, \ldots, (k-j)s\}$. We have

$$(k - j + 1)s = it + js - js = it.$$

Since $it \leq k - j$, then $(k - j + 1)s$ is in $T(S')$. Thus,

$$N^+((k-j)s) \cap T(S') \neq \varnothing.$$

This contradicts our assumption that k is the smallest integer where this happens. Thus, $(k+1)s = it$.

A similar contradiction occurs if $ks + t$ is in $T(S)$.

Claim: S is a zero forcing set of D.

Without loss of generality, let us suppose that $ks + t = it$. Let r be a non-negative integer such that $r \leq n - k$. We will prove by induction on r that

$$S^r = \{0, s, \ldots, (k+r)s\} \subseteq B(S).$$

As a base case, $S^0 = B^0(S)$. So, we suppose that $S^{r-1} \subseteq B(S)$ for some integer $r \geq 1$. We have that

$$(k + r - 1)s + t = it + (r-1)s.$$

Since $0 \leq k - i$, then

$$r - 1 \leq k + r - 1 - i.$$

Thus, $(k + r - 1)s + t$ is in $T(S^r)$. Therefore, $(k + r)s$ is in $B(S^{r-1})$. Since $S \subseteq S^{r-1} \subseteq B(S)$, then $B(S^{r-1}) = B(S)$. So, we have that $(k + r)s$ is in $B(S)$ for all integers r. By Lemma 32, S is a zero forcing set of D which proves the claim.

For any two vertices u and v, let P be the vertices of a u–v path in D. Then we may label the vertices of D accordingly, such that $u = 0$.

Claim: Let $P = 0, p_1, p_2, \ldots, p_k$ be a path in D, then $P \subseteq T(S)$.

Let $\{p_{j_z}\}$ be a subsequence of P such that if $j_{z-1} \leq i \leq j_z$, then

$$p_i = (z-1)t + (i-z+1)s \text{ and } p_{j_z} = zt + (j_z - z)s.$$

Note that each p_{j_z} represents each step down of the path. See Figure 35.

$$\begin{array}{lll}
0 \to s \to \cdots \to (j_1 - 1)s & & \\
\qquad\qquad\qquad\quad \downarrow & & \\
 & (j_1 - 1)s + t \to \cdots \to (j_2 - 1)s + t & \\
 & \qquad\qquad\qquad\qquad\qquad\quad \downarrow & \\
 & & (j_2 - 1)s + 2t \to \cdots
\end{array}$$

Figure 35: Partitioning a path P.

Since $i < j_z \leq k$ and $z \leq k$, then $i - z + 1 \leq k - (z-1)$ and $j_z \leq k - z$. Therefore, p_i is in $T(S)$ and p_{j_z} is in $T(S)$ proving the claim.

Up to this point, we have shown that for any path P of length $k + 1$, $|B(P)| \leq |B(S)| = |V(D)|$. Next, we show that S is a minimum zero forcing path.

Claim: A path of length k is not a zero forcing set.

Let $S' = \{0, s, \ldots, (k-1)s\}$ and $T' = \{0, t, \ldots, (k-1)t\}$. Note that for each i,

$$N^+(it + (k-i)s) \cap T(T^i(S')) = \varnothing \text{ and } N^+(is + (k-i)t) \cap T(T^i(T')) = \varnothing.$$

However, we will investigate if $N^+(it + (k-i)s) \cap T(S') \neq \varnothing$ elsewhere.

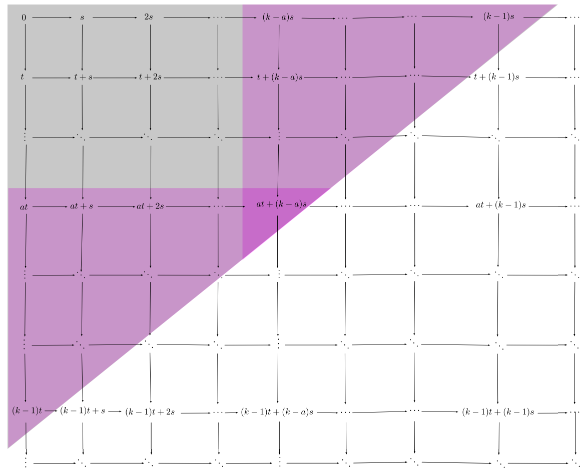


Figure 36: Triangles $T(S')$, $T(T')$, $T(T^a(T'))$, and $T(T^a(S'))$

Suppose that for a non-negative integer a, such that $a \leq k$, $at + (k - a + 1)s = rt + qs$ for $0 \leq r < a$ and $0 \leq q < k - a$. In this case,

$$at + (k - a + 1)s - rt - qs = 0$$

$$(a - r)t + (k - a + 1 - q)s = 0.$$

Notice that,

$$0 < k - (q + a) \leq k - a < k - a + r.$$

So, we have

$$k + 1 - (q + a) \leq k - a + r.$$

This means that $(a-r)t+(k-a+1-q)s$ is in $T(S)$. Since $a-r > 0$ and $(k-a+1-q) > 0$, then

$$at + (k - a + 1)s - rt - qs \neq 0.$$

Thus, it must be that $at + (k - a + 1)s$ is not in $T(S')$.

A similar contradiction occurs if we suppose that $(a + 1)t + (k - a)s$ is in $T(S')$. Therefore, $B(S') = T(S')$, and $B(S') \neq V(D)$, concluding that S' is not a zero forcing path.

Up to this point, we have shown that S is a minimum zero forcing path. We will next show that for any set of vertices R where R is not a path, if $|R| < |S|$, then R is not a zero forcing set. This would mean that S is a minimum zero forcing set. This would complete the proof.

Claim: Let P_1 and P_2 be paths such that $|P_1| + |P_2| = k$, then $B(P_1 \cup P_2) \neq V(D)$.

We may suppose that

$$P_1 = \{0, s, \ldots, k_1 s\} \text{ and } P_2 = \{it + js, it + (j+1)s, \ldots, it + (j+k_2)s\}.$$

Let $S' = \{0, s, \ldots, (k-1)s\}$. Note that none of S', P_1, and P_2 are zero forcing sets of D. Also,

$$T(S') \geq |T(P_1)| + |T(P_2)|.$$

So, for $P_1 \cup P_2$ to force beyond $T(P_1) \cup T(P_2)$, then $N^+(T(P_1)) \cap T(P_2) \neq \varnothing$. Well, the longest paths in $T(P_1)$ and $T(P_2)$ have lengths $|P_1|$ and $|P_2|$ respectively. So, the longest path in $N^+(T(P_1)) \cup T(P_2)$ has length $|P_1| + |P_2| = k$. Thus,

$$B(P_1 \cup P_2) \subseteq B(S') \neq V(D).$$

Therefore, S is a minimum zero forcing set of D.

$\square$

REFERENCES

[1] A. Aazami, *Hardness results and approximation algorithms for some problems on graphs,* Ph.D. thesis, University of Waterloo (2008), `https://uwspace.uwaterloo.ca/handle/10012/4147?show=full`.

[2] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness), *Zero forcing sets and the minimum rank of graphs*, Linear Algebra Appl. **428** (2008), 1628–1648.

[3] K. F. Benson, D. Ferrero, M. Flagg, V. Furst, L. Hogben, V. Vasilevska, and B. Wissman, *Zero forcing and power domination for graph products,* Australas. J. Combin. **70** (2018), 221–235.

[4] A. Berman, S. Friedland, L. Hogben, U. G. Rothblum, and B. Shader, *An upper bound for the minimum rank of a graph,* Linear Algebra Appl. **429** (2008), 1629–1638.

[5] F. Boesch and R. Tindell, *Circulants and their connectivities*, J. Graph Theory **8** (1984), 487–499.

[6] D. Burgarth and V. Giovannetti, *Full control by locally induced relaxation*, Phys. Rev. Lett. **99** (2007), 100501.

[7] A. Chang and Q. Huang, *Circulant digraphs determined by their spectra,* Discrete Math. **240** (2001), 261–270.

[8] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs*, Textbooks in Mathematics, Vol 39, CRC Press, Boca Raton, Florida, 2016.

[9] K. B. Chilakamarri, N. Dean, C. X. Kang, and E. Yi, *Iteration Index of a Zero Forcing Set in a Graph,* Bull. Inst. Combin. Appl. **64** (2018), 57–72.

[10] P. J. Davis, *Circulant Matrices*, American Mathematical Soc., Providence, Rhode Island, 2012.

[11] B. Elspas and J. Turner, *Graphs with circulant adjacency matrices*, J. Combin. Theory **9** (1970), 297–307.

[12] S. Gravier, M. Kovše, and A. Parreau, *Generalized Sierpiński graphs*, EuroComb'11, Budapest, 2011. `http://www.renyi.hu/conferences/ec11/posters/parreau.pdf`.

[13] A. M. Hinz, S. Klavžar, U. Milutinović, and C. Petr, *The Tower of Hanoi–myths and maths*, Springer Basel, New York–London, 2013.

[14] L. Hogben, *Minimum rank problems*, Linear Algebra Appl. **432** (2010), 1961–1974.

[15] A. W. Ingleton, *The rank of circulant matrices,* J. Lond. Math. Soc. **1** (1956), 445–460.

[16] S. Klavžar and U. Milutinović, *Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem,* Czechoslovak Math. J. **47** (1997), 95–104.

[17] S. A. Meyer, *Zero forcing sets and bipartite circulants*, Linear Algebra Appl. **436** (2012), 888–900.

[18] E. A. Van Doorn, *Connectivity of circulant digraphs*, J. Graph Theory **10** (1986), 9–14.

[19] B. Yang, *Fast-mixed searching and related problems on graphs,* Theoret. Comput. Sci. **507** (2013), 100–113.

[20] AIM Minimum rank graph catalog, `admin.aimath.org/structuredlists_testing/graph-invariants/minimumrankoffamilies/#/prodcomps`