Electronic Journal of Differential Equations, Vol. 2005(2005), No. 23, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXTINCTION FOR FAST DIFFUSION EQUATIONS WITH NONLINEAR SOURCES

YUXIANG LI, JICHUN WU

Abstract. We establish conditions for the extinction of solutions, in finite time, of the fast diffusion problem $u_t = \Delta u^m + \lambda u^p$, $0 < m < 1$, in a bounded domain of R^N with $N > 2$. More precisely, we show that if $p > m$, the solution with small initial data vanishes in finite time, and if $p < m$, the maximal solution is positive for all $t > 0$. If $p = m$, then first eigenvalue of the Dirichlet problem plays a role.

1. Introduction

In this paper we are concerned with the porous medium equation

$$\begin{gathered} u_t = \Delta u^m + \lambda u^p, \quad x \in \Omega,\ t > 0, \\ u = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \end{gathered} \tag{1.1}$$

with $0 < m < 1$ and $p,\ \lambda > 0$, where $\Omega \subset R^N$, $N > 2$, is an open bounded domain with smooth boundary $\partial\Omega$. We are interested in the extinction of the nonnegative solution of (1.1).

The phenomena of extinction have been studied extensively for (1.1) with $\lambda \leq 0$. When $\lambda < 0$, for the case of slow diffusion, see [6, 12, 13, 14, 17, 18, 23]. For $m = 1$ we refer the reader to [16]. And for the case of fast diffusion, see [5, 9, 20, 21, 22]. When $\lambda = 0$ and $0 < m < 1$, we refer the reader to [3, 4, 8, 19, 10, 11].

For (1.1) with $p > 1$, it is well known that the solution blows up in finite time for sufficiently large initial data; see [15, 25]. In this paper we show that the solution of (1.1) vanishes in finite time for sufficiently small initial data. If $0 < m < 1$ and $p > m$, there is a maximal positive solution of (1.1). If $p < m$ and $p = m$, the first eigenvalue λ_1 of the problem below plays a crucial role:

$$-\Delta\psi(x) = \lambda\psi(x), \quad x \in \Omega; \quad \psi\big|_{\partial\Omega} = 0. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35K20, 35K55.
Key words and phrases. Extinction; fast diffusion; first eigenvalue.

Partially supported by project40272106 from the National Science Foundation of China, and by the Teaching and Research Award Program for Outstanding Young Teacher of the Ministry of Education, China.
Submitted September 15, 2004. Published February 20, 2005.

The existence and uniqueness for (1.1) have been studied in [1, 2]. To state the definition of the weak solution, we define class of nonnegative testing functions

$$\mathcal{F} = \{\xi : \xi_t, \Delta\xi, |\nabla\xi| \in L^2(\Omega_T),\ \xi \geq 0 \text{ and } \xi|_{(\partial\Omega)_T} = 0\}.$$

Definition 1.1. A function $u(x,t) \in L^\infty(\Omega_T)$ is called a subsolution (supersolution) of (1.1) in Ω_T if the following conditions hold:

(i) $u(x,0) \leq (\geq)\ u_0(x)$ in Ω,
(ii) $u(x,t) \leq (\geq)\ 0$ on $(\partial\Omega)_T$,
(iii) For every $t \in (0,T)$ and every $\xi \in \mathcal{F}$,

$$\int_\Omega u(x,t)\xi(x,t)dx \ \leq (\geq) \int_\Omega u_0(x)\xi(x,0)dx + \int_0^t \int_\Omega \{u\xi_t + u^m\Delta\xi + u^p\xi\}dx\,ds.$$

A function $u(x,t)$ is called a (local) solution of (1.1) if it is both a subsolution and a supersolution for some $T > 0$.

According to [1, Thm. 2.1] and [2, Thm. 2.1, 2.2, 2.3], if $p > m$ or $p = m$ and $\lambda \leq \lambda_1$, the nonnegative solution of (1.1) is unique. Moreover, if $u_0 \geq v_0 \geq 0$, then $u \geq v$. If $p < m$ or $p = m$ and $\lambda > \lambda_1$, then the maximal solution $U(x,t)$ of (1.1) with $u_0 \equiv 0$ has $U(x,t) \neq 0$, and $U(x,t)$ satisfies a subsolution comparison theory. Put

$$v(x,t) = g(t)\psi^{1/m}(x), \tag{1.3}$$

where $\psi(x)$ is the first eigenfunction of (1.2) with $\max\psi(x) = 1$. If $g(t)$ satisfies the ordinary differential equation

$$\begin{gathered} g'(t) = (\lambda - \lambda_1)g^m(t), \quad g(0) = 0, \\ g(t) > 0, \quad \text{for } t > 0, \end{gathered}$$

it can be verified easily that $v(x,t)$ is a subsolution of (1.1) for $p = m$ and $\lambda > \lambda_1$. If $p < m$, let $g(t)$ in (1.3) be the solution of

$$\begin{gathered} g'(t) = -\lambda_1 g^m(t) + \lambda g^p(t), \quad g(0) = 0, \\ g(t) > 0, \quad \text{for } t > 0. \end{gathered}$$

Then $v(x,t)$ is also a subsolution of (1.1). The fact that $U(x,t) > 0$ in Ω for all $t > 0$ follows from the subsolution comparison theory. From the above, we have the following statement.

Theorem 1.2. *Assume that $p < m$ or $p = m$ and $\lambda > \lambda_1$. Then for any nonnegative initial data $u_0 \in L^\infty(\Omega)$, the maximal solution $U(x,t)$ of (1.1) can't vanishes in finite time.*

For the case $p = m$ and $\lambda = \lambda_1$, $k\psi(x)$, $k > 0$, is a steady state solution of (1.1). Then for any nontrivial nonnegative initial data, the solution $u(x,t)$ of (1.1) satisfies that $u(x,t) > 0$ in Ω for $t > 0$ or $u(x,t)$ is identically zero.

In the next section we consider the case $p > m$ or $p = m$ and $\lambda \leq \lambda_1$.

2. Extinction in finite time

The regularities of the solution of (1.1) can be found in [24]. Multiplying the first equation of (1.1) by u^{s-1}, $s > 1$, and integrating over Ω, we obtain

$$\frac{1}{s}\frac{d}{dt}\int_\Omega u^s dx + \frac{4m(s-1)}{(m+s-1)^2}\int_\Omega |\nabla u^{\frac{m+s-1}{2}}|^2 dx = \lambda \int_\Omega u^{p+s-1}dx. \tag{2.1}$$

Theorem 2.1. *Assume that $0 < m < 1$ and $p > m$. Then the unique solution of* (1.1) *vanishes in finite time for small initial data.*

Proof. We consider first the case $p \le 1$. For $\frac{N-2}{N+2} \le m < 1$, let $s = 1 + m$ in (2.1). By the Hölder inequality and the embedding theorem, we have

$$\begin{aligned}\|u(\cdot,t)\|^m_{1+m,\Omega} &\le |\Omega|^{\frac{m}{1+m}-\frac{N-2}{2N}}\|u^m(\cdot,t)\|_{\frac{2N}{N-2},\Omega}\\ &\le \gamma \mid \Omega \mid^{\frac{m}{1+m}-\frac{N-2}{2N}} \|\nabla u^m(\cdot,t)\|_{2,\Omega}.\end{aligned}$$

where γ is the embedding constant. This remarks in (2.1) yields the differential inequality

$$\frac{d}{dt}\|u(\cdot,t)\|_{1+m,\Omega} + \gamma^{-2}|\Omega|^{\frac{N-2}{N}-\frac{2m}{1+m}}\|u(\cdot,t)\|^m_{1+m,\Omega} \le \lambda|\Omega|^{1-\frac{p+m}{1+m}}\|u(\cdot,t)\|^p_{1+m,\Omega}.$$

Choose

$$\|u_0\|^{p-m}_{1+m,\Omega} < \lambda^{-1}\gamma^{-2}|\Omega|^{\frac{p-m}{1+m}-\frac{2}{N}}.$$

Then

$$\frac{d}{dt}\|u(\cdot,t)\|_{1+m,\Omega} + c_1\|u(\cdot,t)\|^m_{1+m,\Omega} \le 0, \tag{2.2}$$

where

$$c_1 = \gamma^{-2}|\Omega|^{\frac{N-2}{N}-\frac{2m}{1+m}} - \lambda|\Omega|^{1-\frac{p+m}{1+m}}\|u_0\|^{p-m}_{1+m,\Omega}.$$

Integrating (2.2) gives

$$\|u(\cdot,t)\|^{1-m}_{1+m,\Omega} \le \|u_0\|^{1-m}_{1+m,\Omega} - (1-m)c_1 t,$$

as long as the right side is nonnegative. From this,

$$\|u(\cdot,t)\|_{1+m,\Omega} \le \|u_0\|_{1+m,\Omega}\big\{1 - \frac{(1-m)c_1 t}{\|u_0\|^{1-m}_{1+m,\Omega}}\big\}_+^{\frac{1}{1-m}}.$$

Next we take m in such that $0 < m < (N-2)/N$. In (2.1), let

$$s = \frac{N}{2}(1-m) > 1.$$

By the embedding theorem and the specific choice of s, we obtain

$$\|u(\cdot,t)\|^{\frac{m+s-1}{2}}_{s,\Omega} = \|u^{\frac{m+s-1}{2}}(\cdot,t)\|_{\frac{2N}{N-2},\Omega} \le \gamma\|\nabla u^{\frac{m+s-1}{2}}(\cdot,t)\|_{2,\Omega}.$$

We conclude that

$$\frac{d}{dt}\|u(\cdot,t)\|_{s,\Omega} + \gamma^{-2}\frac{4m(s-1)}{(m+s-1)^2}\|u(\cdot,t)\|^m_{s,\Omega} \le \lambda|\Omega|^{1-\frac{p+s-1}{s}}\|u(\cdot,t)\|^p_{s,\Omega}.$$

Choose

$$\|u_0\|^{p-m}_{s,\Omega} < \lambda^{-1}\gamma^{-2}\frac{4m(s-1)}{(m+s-1)^2}|\Omega|^{\frac{p+s-1}{s}-1}.$$

Then

$$\frac{d}{dt}\|u(\cdot,t)\|_{s,\Omega} + c_2\|u(\cdot,t)\|^m_{s,\Omega} \le 0,$$

where

$$c_2 = \gamma^{-2}\frac{4m(s-1)}{(m+s-1)^2} - \lambda|\Omega|^{1-\frac{p+s-1}{s}}\|u_0\|^{p-m}_{s,\Omega}.$$

By integration, we have

$$\|u(\cdot,t)\|_{s,\Omega} \le \|u_0\|_{s,\Omega}\big\{1 - \frac{(1-m)c_2 t}{\|u_0\|^{1-m}_{s,\Omega}}\big\}_+^{\frac{1}{1-m}}.$$

For the case $p > 1$, for sufficiently small $k > 0$, it can be easily verified that $k\psi^{1/m}(x)$ is a supersolution of (1.1), where $\psi(x)$ is the first eigenfunction of (1.2) with $\max \psi(x) = 1$. Then

$$u(x,t) \le k\psi^{1/m}(x), \quad t > 0,$$

by the comparison principle if $u_0(x) \le k\psi^{1/m}(x)$ in Ω. From this (2.1) can be rewritten as

$$\frac{1}{s}\frac{d}{dt}\int_\Omega u^s dx + \frac{4m(s-1)}{(m+s-1)^2}\int_\Omega |\nabla u^{\frac{m+s-1}{2}}|^2 dx \le \lambda k^{p-1}\int_\Omega u^s dx,$$

to which the above argument can be applied. The proof is completed. □

Remark 2.2. The method of the above proof is a modification of the argument in [3, Prop. 10] and [7, Prop. VII. 2.1].

Theorem 2.3. *Assume that* $0 < m = p < 1$ *and* $\lambda < \lambda_1$. *Then for any nonnegative initial data, the solution of* (1.1) *vanishes in finite time.*

Proof. First we apply the argument in the above theorem to get some results. We consider two cases: $\frac{N-2}{N+2} \le m < 1$ and $m < \frac{N-2}{N+2}$. In the first case, let $s = 1 + m$ in (2.1). Noticing that

$$\lambda_1 = \inf_{v \in H_0^1(\Omega), v \ne 0} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega |v|^2 dx},$$

we obtain

$$\frac{1}{1+m}\frac{d}{dt}\|u(\cdot,t)\|_{1+m,\Omega}^{1+m} + \big(1 - \frac{\lambda}{\lambda_1}\big)\|\nabla u^m(\cdot,t)\|_{2,\Omega}^2 \le 0.$$

Since $\lambda < \lambda_1$, as in the above proof of Theorem 2.1, there exists $T^*(u_0) < \infty$ such that $u(x,t) \equiv 0$ for all $t \ge T^*(u_0)$. In the second case, let $s = \frac{N}{2}(1-m) > 1$ in (2.1). Then we have

$$\frac{1}{s}\frac{d}{dt}\|u(\cdot,t)\|_{s,\Omega}^s + \big(\frac{4m(s-1)}{(m+s-1)^2} - \frac{\lambda}{\lambda_1}\big)\|\nabla u^{\frac{m+s-1}{2}}(\cdot,t)\|_{2,\Omega}^2 \le 0.$$

Set

$$\lambda^* = \frac{(m+s-1)^2}{4m(s-1)}\lambda > \lambda.$$

Then if $\lambda_1 > \lambda^*$, $u(x,t)$ with any initial data vanishes in finite time.

To fill the gap where $m < \frac{N-2}{N+2}$ and $\lambda < \lambda_1 < \lambda^*$, we apply a supersolution argument. In fact this supersolution argument can apply to all the case of $0 < m < 1$ and $\lambda < \lambda_1$. Denote by $\psi(x)$ the first eigenfunction of (1.2) with $\max_{x\in\Omega}\psi(x) = 1$. Let $g(t)$ be the solution of the differential equation

$$g'(t) = -(\lambda_1 - \lambda)g^m(t),$$
$$g(0) = \theta,$$

where θ is chosen that $u_0(x) \le \theta(\psi)^{1/m}(x)$ in Ω. Thus $v(x,t) = g(t)(\psi)^{1/m}(x)$ is a supersolution of (1.1). Since $0 < m < 1$, $g(t)$ vanishes in finite time. Then the theorem follows from the comparison principle. □

We note that the unique solution of the problem

$$\begin{gathered}u_t = \Delta u^m, \quad x \in \Omega,\ t > 0,\\ u = 0, \quad x \in \partial\Omega,\ t > 0,\\ u(x,0) = u_0(x) \ge 0, \quad x \in \Omega,\end{gathered} \tag{2.3}$$

$0 < m < 1$, is a subsolution of (1.1). Since the solution of the problem (2.3) is positive everywhere in Ω unless it is identically zero, by comparison we conclude that, denoting by $T^* < \infty$ the extinction time of the solution of (1.1), we have

$$u(x,T^*) \equiv 0, \quad \text{and} \quad u(x,t) > 0 \quad \text{in } \Omega,\ 0 < t < T^*.$$

In the following we consider the solution of (1.1) with negative initial energy. Define

$$\begin{gathered}\mathcal{E}(u(t)) = \frac{1}{2}\int_\Omega |\nabla u^m|^2 dx - \frac{\lambda m}{p+m}\int_\Omega u^{p+m} dx\\ \mathcal{H}(u(t)) = \frac{1}{1+m}\int_\Omega u^{1+m} dx.\end{gathered}$$

Differentiating $\mathcal{E}(u(t))$ and $\mathcal{H}(u(t))$, we obtain

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\int_\Omega u_t (u^m)_t dx = -\frac{4m}{(1+m)^2}\int_\Omega \big[(u^{\frac{1+m}{2}})_t\big]^2 dx,$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(u(t)) &= \int_\Omega u^m u_t dx\\ &= -\int_\Omega |\nabla u^m|^2 dx + \lambda \int_\Omega u^{p+m} dx\\ &= -2\mathcal{E}(u(t)) + \lambda\big(1 - \frac{2m}{p+m}\big)\int_\Omega u^{p+m} dx.\end{aligned}$$

From this, $\mathcal{E}(u(t)) \le 0$ provided that $\mathcal{E}(u_0) \le 0$. Hence, if $p > m$, we have

$$\frac{d}{dt}\mathcal{H}(u(t)) \ge \lambda\big(1 - \frac{2m}{p+m}\big)\int_\Omega u^{p+m} dx. \tag{2.4}$$

By the Hölder inequality, for $p \ge 1$,

$$\frac{d}{dt}\mathcal{H}(u(t)) \ge c_3 \mathcal{H}^{\frac{p+m}{1+m}}(u(t)),$$

where

$$c_3 = \lambda\big(1 - \frac{2m}{p+m}\big)(1+m)^{\frac{p+m}{1+m}} |\Omega|^{1-\frac{p+m}{1+m}}.$$

By integration, if $p > 1$, there exists $T^* < \infty$ such that

$$\lim_{t \to T^*} \mathcal{H}(u(t)) = \infty,$$

provided that $\mathcal{H}(u_0) > 0$. When $p = 1$, we have

$$\lim_{t \to \infty} \mathcal{H}(u(t)) = \infty,$$

if $\mathcal{H}(u_0) > 0$. For $m < p < 1$, integrating (2.4) over $(0,t)$ gives

$$\mathcal{H}(u(t)) \ge \mathcal{H}(u_0) + \lambda\big(1 - \frac{2m}{p+m}\big)\int_0^t \int_\Omega u^{p+m} dx ds.$$

Suppose on the contrary that $\|u(\cdot,t)\|_{\infty,\Omega} \le M < \infty$ for all $t > 0$. Then,

$$\frac{M^{1-p}}{1+m}\int_\Omega u^{p+m}dx \ge \mathcal{H}(u_0) + \lambda\big(1 - \frac{2m}{p+m}\big)\int_0^t\int_\Omega u^{p+m}dx\,ds\,.$$

The Gronwall inequality implies that

$$\lim_{t\to\infty}\int_\Omega u^{p+m}dx = \infty,$$

which is a contradiction. Therefore, we have the following statement.

Theorem 2.4. *Assume that $0 < m < 1$ and $p > m$. If $u_0^m \in H_0^1(\Omega)$ satisfies*

$$\mathcal{E}(u_0) \le 0, \quad \mathcal{H}(u_0) > 0,$$

then there exists $T^* \le \infty$ such that

$$\lim_{t\to T^*}\|u(\cdot,t)\|_{\infty,\Omega} = \infty.$$

References

[1] J. R. Anderson; *Local existence and uniqueness of solutions of degenerate parabolic equations*, Comm. Partial Differential Equations **16** (1991), 105-143.

[2] J. R. Anderson; *Necessary and sufficient conditions for the unique solvability of a nonlinear reaction-diffusion model*, J. Math. Anal. Appl. **228** (1998), 483-494.

[3] Ph. Benilan and M. G. Crandall; *The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$*, Indiana Univ. Math. J. **30** (1981), 161-177.

[4] J. G. Berryman and C. J. Holland; *Stability of the separable solution for fast diffusion*, Arch. Ration. Mech. Anal. **74** (1980), 379-388.

[5] M. Borelli and M. Ughi; *The fast diffusion equation with strong absorption: the instantaneous shrinking phenomenon*, Rend. Istit. Mat. Univ. Trieste **26** (1994), 109-140.

[6] X. Y. Chen, H. Matano and M. Mimura; *Finite-point extinction and continuity of interfaces in a nonlinear diffusion equation with strong absorption*, J. Reine Angew. Math. **459** (1995), 1-36.

[7] E. DiBenedetto; Degenerate parabolic equations, Springer-Verlag, New York, 1993.

[8] A. Friedman and S. Kamin; *The asymptotic behavior of gas in an n-dimensional porous medium*, Trans. Amer. Math. Soc. **262** (1980), 551-563.

[9] R. Ferreira and J. L. Vazquez; *Extinction behavior for fast diffusion equations with absorption*, Nonlinear Anal. **43** (2001), 943-985.

[10] V. A. Galaktionov and J. R. King; *Fast diffusion equation with critical Sobolev exponent in a ball*, Nonlinearity **15** (2002), 173-188.

[11] V. A. Galaktionov, L. A. Peletier and J. L. Vazquez; *Asymptotics of fast-diffusion equation with critical exponent*, SIAM J. Math. Anal. **31** (2000), 1157-1174.

[12] V. A. Galaktionov and J. L. Vazquez; *Asymptotic behavior of nonlinear parabolic equations with critical exponents. A dynamical system approach*, J. Funct. Anal. **100** (1991), 435-462.

[13] V. A. Galaktionov and J. L. Vazquez; *Extinction for a quasilinear heat equation with absorption I. Technique of intersection comparison*, Comm. Partial Differential Equations **19** (1994), 1075-1106.

[14] V. A. Galaktionov and J. L. Vazquez; *Extinction for a quasilinear heat equation with absorption II. A dynamical system approach*, Comm. Partial Differential Equations **19** (1994), 1107-1137.

[15] V. A. Galaktionov and J. L. Vazquez; *The problem of blow-up in nonlinear parabolic equations*, Discrete Contin. Dynam. Systems **8**(2) (2002), 399-433.

[16] M. A. Herrero and J. L. L. Velazquez; *On the dynamics of a semilinear heat equation with strong absorption*, J. Math. Anal. Appl. **170** (1992), 353-381.

[17] A. S. Kalashnikov; *Some problems on the qualitative theory of nonlinear degenerate parabolic second order equations*, Russian Math. Surveys **42** (1987), 169-222.

[18] R. Kersner; *Degenerate parabolic equations with general nonlinearities*, Nonlinear Anal. **4** (1980), 1043-1062.

[19] J. R. King; *Self-similar behavior for the equation of fast nonlinear diffusion*, Philos. Trans. Roy. Soc. London Ser. A **343** (1993), 337-375.

[20] G. Leoni; *A very singular solution for the porous media equation* $u_t = \Delta u^m - u^p$ *when* $0 < m < 1$, J. Differential Equations **132** (1996), 353-376.

[21] L. A. Peletier and Zhao Junning; *Large time behavior of solution of the porous media equation with absorption: The fast diffusion case*, Nonlinear Anal. **14** (1990), 107-121.

[22] L. A. Peletier and Zhao Junning; *Source-type solutions of the porous media equation with absorption: The fast diffusion case*, Nonlinear Anal. **17** (1991), 991-1009.

[23] Ph. Rosenau and S. Kamin; *Thermal waves in an absorbing and convecting medium*, Phys. D **8** (1983), 273-283.

[24] P. E. Sacks; *Continuity of solutions of a singular parabolic equation*, Nonlinear Anal. **7** (1983), 387-409.

[25] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov; *Blow-up in quasilinear parabolic equations*, Translated from the Russian by M. Grinfeld, Walter de Gruyter, Berlin, New York, 1995.

Yuxiang Li
Department of Mathematics, Southeast University, Nanjing 210096, China.
Department of Earth Sciences, Nanjing University, Nanjing 210093, China
E-mail address: `lieyuxiang@yahoo.com.cn`

Jichun Wu
Department of Earth Sciences, Nanjing University, Nanjing 210093, China
E-mail address: `jcwu@nju.edu.cn`