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REGULARITY OF WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS NEAR THE SMOOTH BOUNDARY

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Abstract. Any weak solution u of the Navier-Stokes equations in a bounded domain satisfying the Prodi-Serrin's conditions locally near the smooth boundary cannot have singular points there. This local-up-to-the-boundary boundedness of u in space-time implies the Hölder continuity of u up-to-the-boundary in the space variables.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, let $T > 0$ and $Q_T = \Omega \times (0, T)$. We consider the Navier-Stokes initial-boundary value problem describing the evolution of the velocity $u = (u_1, u_2, u_3)$ and the pressure ϕ in Q_T:

$$\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla \phi = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.3}$$

$$u|_{t=0} = u_0, \tag{1.4}$$

where $\nu > 0$ is the viscosity coefficient. The initial data u_0 satisfy the compatibility conditions $u_0|_{\partial\Omega} = 0$ and $\nabla \cdot u_0 = 0$ and for our purposes we can suppose without loss of generality that u_0 is sufficiently smooth. The existence of a weak solution $u \in L^2(0, T; W_0^{1,2}(\Omega)^3) \cap L^\infty(0, T; L^2_\sigma(\Omega))$ of (1.1)–(1.4) is well known (see e.g. [3] or [14]). The associated pressure ϕ is a scalar function such that u and ϕ satisfy the equation (1.1) in Q_T in the sense of distributions.

Let $q > 1$. $L^q_\sigma(\Omega)$ denotes the closure of $\{\varphi \in (C_0^\infty(\Omega))^3; \nabla \cdot \varphi = 0 \text{ in } \Omega\}$ in $(L^q(\Omega))^3$. There exists a continuous projection P^q_σ from $(L^q(\Omega))^3$ onto $L^q_\sigma(\Omega)$. If Δ denotes the Laplacian then the famous Stokes operator is defined as $A_q = -P^q_\sigma \Delta$. It is known that $-A_q$ generates a bounded analytic semigroup in $L^q_\sigma(\Omega)$ (see e.g. [5]).

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In the paper we use both scalar and vector functions and for the sake of simplicity we denote by S any space S^3 of vector functions with the exception of the notation in Lemma 2.4. We use the standard notation for the Lebesgue spaces $L^p(\Omega)$ and their norms $\|\cdot\|_{p,\Omega}$. The Sobolev spaces are denoted by $W^{k,p}(\Omega)$. Sometimes we drop Ω and write only L^p, $\|\cdot\|_p$ and $W^{k,p}$. Further, if $A = B \times (t_1, t_2)$ then $L^{p,q}$ or $L^{p,q}(A)$ denote the space $L^q(t_1, t_2; L^p(B))$ with the norm $\|\cdot\|_{p,q,A}$ or simply $\|\cdot\|_{p,q}$. $L^{p,p}(A)$ is also denoted as $L^p(A)$ or L^p. $C^\beta(\overline{\Omega})$ is the space of Hölder continuous functions on Ω with the norm $\|f\|_{C^\beta(\overline{\Omega})} = \sup_{x\in\Omega} |f(x)| + \sup_{x,y\in\Omega, x\neq y} |f(x)-f(y)|/|x-y|^\beta$.

For $(x_0, t_0) \in \overline{\Omega} \times (0, T)$ and $r > 0$ we will denote $B_r = B_r(x_0)$ the ball centered at x_0 with radius r, $D_r = D_r(x_0) = B_r(x_0) \cap \Omega$, $Q_r = Q_r(x_0, t_0) = D_r(x_0) \times (t_0 - r^2, t_0 + r^2)$.

A point $(x_0, t_0) \in \overline{\Omega} \times (0, T)$ is called a regular point of a weak solution u if $u \in L^\infty(Q_r)$ for some $r > 0$. Otherwise, (x_0, t_0) is called a singular point of u.

In his famous paper (see [9]) J.Serrin proved the following interior regularity result. If $Q_r \subset Q_T$ for some $(x_0, t_0) \in Q_T$ and $r > 0$ and a weak solution u of (1.1)–(1.4) satisfies the Prodi-Serrin's conditions in Q_r, that is

$$u \in L^{p,q}(Q_r), \quad \frac{3}{p} + \frac{2}{q} < 1,\ p, q \in (1, \infty), \tag{1.5}$$

then u is necessarily a L^∞ function on compact subsets of Q_r and smooth in the space variables. This result was extended by M.Struwe in [11] for the case of $p, q \in (1, \infty)$, $3/p + 2/q \le 1$. A local version up to the boundary of the Serrin-Struwe's results was proved by S.Takahashi. He showed in [12] and [13] that if $u \in L^{p,q}(Q_r)$, where $(x_0, t_0) \in \partial\Omega \times (0, T)$, $r > 0$, $p, q \in (1, \infty)$ and $3/p + 2/q \le 1$ then $u \in L^\infty(Q_{\tilde r})$ for any $\tilde r \in (0, r)$ provided that $B_r \cap \partial\Omega$ is a part of a plane.

In this paper we improve the Takahashi's result in two directions. Firstly, we show, that $\partial\Omega$ can be an arbitrary smooth boundary, that is $B_r \cap \partial\Omega$ needn't be a part of a plane. Secondly, we show that u is locally a Hölder continuous function in the space variables up to the boundary in the neighborhood of the point x_0. Precisely, we prove the following theorem.

Theorem 1.1. *Let u be an arbitrary weak solution of* (1.1)–(1.4), $(x_0, t_0) \in \partial\Omega \times (0, T)$, $r > 0$. *We suppose that $u \in L^{p,q}(Q_r)$, where $2/q + 3/p = 1$ and $p, q \in (1, \infty)$. Then*

$$u \in L^\infty(t_0 - \tilde r^2, t_0 + \tilde r^2; C^\beta(\overline{D}_{\tilde r})) \tag{1.6}$$

for every $\beta \in (0, 1)$ and $\tilde r \in (0, r)$.

In [7] Neustupa proved a similar result. He supposed that $u \in L^q(t_1, t_2; L^p(U_r^*))$ for some $r > 0$, $0 < t_1 < t_2 < T$, $p, q \in (1, \infty)$ with $3/p + 2/q = 1$, where $U_r^* = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < r\}$. He proved under this assumption that if u is a weak solution of (1.1)–(1.4) satisfying the strong energy inequality then $u \in L^\infty(t_1 + \zeta, t_2 - \zeta; W^{2+\delta,2}(U_\rho^*))$ and $\partial u/\partial t, \nabla\phi \in L^\infty(t_1 + \zeta, t_2 - \zeta; W^{\delta,2}(U_\rho^*))$ for each $\delta \in [0, 1/2)$, $\rho \in (0, r)$ and such $\zeta > 0$ that $t_1 + \zeta < t_2 - \zeta$.

The proof of the Neustupa's result was based on the fact (see [7], Lemma 1) that $\partial u/\partial t, \nabla\phi$ and their space derivatives of an arbitrary order belong to $L^\alpha(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega_2))$ for each $\alpha \in [1, 2)$ and $\zeta \in (0, (t_2 - t_1)/2)$ if $u \in L^q(t_1, t_2; L^p(\Omega_1))$ for some $p, q \in (1, \infty)$ with $3/p + 2/q \le 1$, where Ω_1 and Ω_2 are such sub-domains of Ω that $\overline{\Omega}_2 \subset \Omega_1 \subset \Omega$. Using this result together with the cut-off function technique, it was then possible to show that the right hand side h of the localized equations and its space derivatives of an arbitrary order belong to the space $L^{\alpha}(t_1+\zeta, t_2-\zeta; L^{\infty}(\Omega))$ for each $\alpha \in [1, 2)$ (see [7, (6)]). This regularity of h produced better regularity of u near the whole boundary $\partial\Omega$ which further improved the regularity of ϕ and consequently of h. Repeating this procedure several times one of the main results of [7] presented in the preceding paragraph was obtained.

We assume in Theorem 1.1 that u satisfies the Prodi-Serrin's conditions only in a space-time neighborhood of $(x_0, t_0) \in \partial\Omega \times (0, T)$. Thus, the cut-off function technique does not produce in this case the right hand side h which is from the space $L^{\alpha}(t_1+\zeta, t_2-\zeta; L^{\infty}(\Omega_2))$, $\alpha \in [1, 2)$ and the procedure from [7] mentioned in the preceding paragraph cannot be used. Instead, we use at the beginning the regularity results of Giga,Sohr (see [5]). They lead, however, to worse regularity results for u in Theorem 1.1 in comparison with the results from [7].

The local boundary regularity of u was also studied in [2], [8] and [6]. It was proved in [2] that a suitable weak solution u is bounded locally near the boundary if $u \in L^{p,q}$, $3/p+2/q = 1$, $p, q \in (1, \infty)$ and the pressure ϕ is bounded at the boundary. Moreover, better regularity of ϕ gives better local regularity of u. G.A.Seregin presented in [8] a condition for local Hölder continuity for suitable weak solutions near the plane boundary which has the form of the famous Caffarelli-Kohn-Nirenberg condition for boundedness of suitable weak solutions in a neighborhood of an interior point of Q_T. Finally, in [6] K.Kang studied boundary regularity of weak solutions in the half-space. He proved that a weak solution u which is locally in the class $L^{p,q}$ with $3/p+2/q = 1$ and $p, q \in (1, \infty)$ near the boundary is Hölder continuous up to the boundary. The main tool in the proof of this result is a pointwise estimate for the fundamental solution of the Stokes system.

2. Auxiliary Lemmas

In this section we present a few lemmas which will be used in the proof of Theorem 1.1. We consider the Stokes problem:

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla\phi = f \quad \text{in } Q_T, \tag{2.1}$$
$$\nabla \cdot u = 0 \quad \text{in } Q_T, \tag{2.2}$$
$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{2.3}$$
$$u|_{t=0} = 0. \tag{2.4}$$

It was proved in [5, Theorem 2.8], that if $f \in L^{\beta,\beta'}$, where $\beta, \beta' \in (1, \infty)$, then there exists a unique solution (u, ϕ) of (2.1) - (2.4) such that

$$\|\frac{\partial u}{\partial t}\|_{\beta,\beta'} + \|A_\beta u\|_{\beta,\beta'} + \|\nabla\phi\|_{\beta,\beta'} \le c\|f\|_{\beta,\beta'}, \quad c = c(\beta, \beta'). \tag{2.5}$$

Lemma 2.1. *Let $\beta, \beta' \in (1, \infty)$, $\gamma \in [\beta, \infty)$, $\gamma' \in [\beta', \infty)$ and*

$$\frac{2}{\beta'} + \frac{3}{\beta} = \frac{2}{\gamma'} + \frac{3}{\gamma} + 1. \tag{2.6}$$

Then for every $f \in L^{\beta,\beta'}$ there exists a unique solution u of (2.1)–(2.4) such that $\nabla u \in L^{\gamma,\gamma'}$ and

$$\|\nabla u\|_{\gamma,\gamma'} \le c\|f\|_{\beta,\beta'}, \quad c = c(\beta, \beta', \gamma, \gamma'). \tag{2.7}$$

Proof. Equation (2.1) can be written as

$$\frac{\partial u}{\partial t} - \nu \Delta u = f - \nabla \phi,$$

where $\|f - \nabla\phi\|_{\beta,\beta'} \le c\|f\|_{\beta,\beta'}$. Lemma 2.1 now follows immediately from the following Lemma 2.2. □

Lemma 2.2. *Let the assumptions of Lemma 2.1 be satisfied. Consider the problem*

$$\frac{\partial u}{\partial t} - \nu \Delta u = f \quad in\ Q_T, \tag{2.8}$$

$$u = 0 \quad on\ \partial\Omega \times (0,T), \tag{2.9}$$

$$u|_{t=0} = 0. \tag{2.10}$$

Then for every $f \in L^{\beta,\beta'}$ there exists a unique solution u of (2.8)–(2.10) *such that*

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{\beta,\beta'} + \|\nabla^2 u\|_{\beta,\beta'} \le c\|f\|_{\beta,\beta'}, \quad c = c(\beta,\beta'). \tag{2.11}$$

Moreover, $\nabla u \in L^{\gamma,\gamma'}$ and

$$\|\nabla u\|_{\gamma,\gamma'} \le c\|f\|_{\beta,\beta'}, \quad c = c(\beta,\beta',\gamma,\gamma'). \tag{2.12}$$

Proof. The existence of a unique solution u of (2.8)–(2.10) satisfying (2.11) follows from [5, Theorem 2.1]. We will prove that u satisfies also (2.12).

Let us suppose at first that Ω is a half space, i.e. $\Omega = \mathbb{R}^3_+$, where $\mathbb{R}^3_+ = \{x = (x_1,x_2,x_3) \in \mathbb{R}^3; x_3 > 0\}$. We extend f to the whole space $\mathbb{R}^3$ in such a way that $f(x_1,x_2,x_3) = -f(x_1,x_2,-x_3)$ for any $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ and denote the extended function by f. Then the unique solution u of (2.8) - (2.10) can be written as

$$u(x,t) = \int_0^t \int_{\mathbb{R}^3} K(x-\xi,t-\tau) f(\xi,\tau)\, d\xi d\tau, \tag{2.13}$$

where

$$K(x,t) = \frac{1}{2^3 \pi^{3/2} t^{3/2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^3, t > 0.$$

It is possible to compute that

$$\|\nabla K(\cdot,t)\|_{s,\mathbb{R}^3} = ct^{-2+\frac{3}{2s}} \tag{2.14}$$

for any $s \in [1,\infty)$, where c depends only on s. Let $u_0 \in L^\beta(\mathbb{R}^3)$. If we define

$$v(x,t) = \int_{\mathbb{R}^3} K(x-y,t) u_0(y)\, dy,$$

then

$$\nabla v(x,t) = \int_{\mathbb{R}^3} \nabla K(x-y,t) u_0(y)\, dy.$$

There exists $s \in [1,\infty)$ such that $1/\gamma = 1/s + 1/\beta - 1$. According to [3, estimate (9.2), p. 85] and (2.14), we have

$$\|\nabla v(\cdot,t)\|_{\gamma,\mathbb{R}^3} \le ct^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{\beta}-\frac{1}{\gamma})} \|u_0\|_{\beta,\mathbb{R}^3}, \quad c = c(\beta,\gamma). \tag{2.15}$$

It follows from (2.13) that

$$\|\nabla u(\cdot,t)\|_{\gamma,\mathbb{R}^3} \le \int_0^t \Big\| \int_{\mathbb{R}^3} \nabla K(x-\xi,t-\tau) f(\xi,\tau)\, d\xi \Big\|_{\gamma,\mathbb{R}^3} d\tau$$

and using (2.15), we get

$$\|\nabla u(\cdot,t)\|_{\gamma,\mathbb{R}^3} \le c\int_0^t (t-\tau)^{-\frac12-\frac32\left(\frac1\beta-\frac1\gamma\right)}\|f(\cdot,\tau)\|_{\beta,\mathbb{R}^3}d\tau. \tag{2.16}$$

Applying now the Hardy-Littlewood-Sobolev inequality to (2.16) we get

$$\|\nabla u\|_{\gamma,\gamma',\mathbb{R}^3\times(0,T)} \le c\|f\|_{\beta,\beta',\mathbb{R}^3\times(0,T)}, \quad c=c(\beta,\beta',\gamma,\gamma')$$

and the inequality (2.12) for the case $\Omega=\mathbb{R}^3_+$, that is

$$\|\nabla u\|_{\gamma,\gamma',\mathbb{R}^3_+\times(0,T)} \le c\|f\|_{\beta,\beta',\mathbb{R}^3_+\times(0,T)},$$

follows immediately.

Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Let $x_0\in\partial\Omega$ be chosen arbitrarily. Let us choose a local system of coordinates with the origin at x_0 and with the axis x_3 perpendicular to $\partial\Omega$ and pointing into Ω. Thus, the axes x_1 and x_2 form the tangent plane to $\partial\Omega$ at the point x_0. Let us define for $\varepsilon>0$

$$\Omega^\varepsilon_{x_0}=\{x=(x_1,x_2,x_3)\in\mathbb{R}^3;\sqrt{x_1^2+x_2^2}<\varepsilon\wedge\varphi(x_1,x_2)<x_3<\varphi(x_1,x_2)+\varepsilon\}, \tag{2.17}$$

where the function φ describes locally the boundary $\partial\Omega$ near the point x_0. Let $\psi\in C^\infty(\overline{\Omega})$ be a cut-off function such that $\psi(x)=1$ if $x\in\Omega^{\varepsilon/2}_{x_0}$, $\psi(x)=0$ if $x\in\Omega\setminus\Omega^\varepsilon_{x_0}$ and $\psi(x)\in[0,1]$ for every $x\in\Omega$.

If we put $v=\psi u$ then v solves the system

$$\frac{\partial v}{\partial t}-\nu\Delta v=h \quad \text{in } Q_T, \tag{2.18}$$

$$v=0 \quad \text{on } \partial\Omega\times(0,T), \tag{2.19}$$

$$v|_{t=0}=0, \tag{2.20}$$

where $h=\psi f-2\nu\nabla\psi\cdot\nabla u-\nu\Delta\psi u$ and it follows from (2.11) that

$$\|h\|_{\beta,\beta'}\le c\|f\|_{\beta,\beta'}. \tag{2.21}$$

Let

$$\Phi^\varepsilon_{x_0}=\{x=(x_1,x_2,x_3)\in\mathbb{R}^3;x_1^2+x_2^2<\varepsilon\wedge 0<x_3<\varepsilon\}$$

The following equations describe the transformation between $\Phi^\varepsilon_{x_0}$ and $\Omega^\varepsilon_{x_0}$:

$$x_1'=x_1,\ x_2'=x_2,\ x_3'=x_3-\varphi(x_1,x_2). \tag{2.22}$$

If we define v' on $\Phi^\varepsilon_{x_0}$ by the equation

$$v'(x_1',x_2',x_3')=v(x_1,x_2,x_3), \tag{2.23}$$

then v' satisfies the equation

$$\begin{aligned}\frac{\partial v'}{\partial t}-\nu\Delta'v' &= h-\frac{\partial^2 v}{\partial x_3^2}\Big[\Big(\frac{\partial\varphi}{\partial x_1}\Big)^2+\Big(\frac{\partial\varphi}{\partial x_2}\Big)^2\Big]-2\frac{\partial\varphi}{\partial x_1}\frac{\partial^2 v}{\partial x_1\partial x_3}\\ &\quad -2\frac{\partial\varphi}{\partial x_2}\frac{\partial^2 v}{\partial x_2\partial x_3}-\frac{\partial v}{\partial x_3}\Big(\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}\Big) \quad \text{in } \mathbb{R}^3_+\times(0,T)\end{aligned} \tag{2.24}$$

and the boundary and initial conditions

$$v'=0 \quad \text{on } \partial\mathbb{R}^3_+\times(0,T), \tag{2.25}$$

$$v'|_{t=0}=0. \tag{2.26}$$

We denote the right hand side of (2.24) by H. v' is a solution of (2.8) - (2.10) for $\Omega = \mathbb{R}^3_+$ with the right hand side H instead of f and according to the first half of this proof (2.12) holds, that is

$$\|\nabla' v'\|_{\gamma,\gamma'} \le c\|H\|_{\beta,\beta'}. \tag{2.27}$$

Since $v \in L^{\beta'}(0,T,W^{2,\beta}(\Omega))$ and $\|v\|_{L^{\beta'}(0,T,W^{2,\beta}(\Omega))} \le c\|f\|_{\beta,\beta'}$ - see (2.11), it follows from (2.21), the smoothness of φ and the substitution theorem that

$$\|H\|_{\beta,\beta'} \le c\|f\|_{\beta,\beta'}. \tag{2.28}$$

Further, the smoothness of the transformation (2.22) gives the inequality

$$\|\nabla v\|_{\gamma,\gamma'} \le c\|\nabla' v'\|_{\gamma,\gamma'}. \tag{2.29}$$

Summing up (2.29), (2.27) and (2.28) we have

$$\|\nabla v\|_{\gamma,\gamma'} \le c\|f\|_{\beta,\beta'}$$

and thus

$$\|\nabla u\|_{\gamma,\gamma',\Omega^{\varepsilon/2}_{x_0}} \le c\|f\|_{\beta,\beta'}. \tag{2.30}$$

The estimate (2.30) can also be proved in the same way for the sets $\Omega^{\varepsilon}_I = \{x \in \Omega; \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$, where ε is an arbitrary positive number. To conclude the proof, it is now sufficient to realize, that there exist $n \in N$, points $x^i_0 \in \partial\Omega$, $i = 1,2,3,\cdots n$ and positive numbers $\varepsilon, \varepsilon_i$, $i = 1,2,3,\cdots,n$ such that

$$\Omega \subset \cup^n_{i=1}\Omega^{\varepsilon_i/2}_{x^i_0} \cup \Omega^{2\varepsilon}_I.$$

and use (2.30). $\square$

Another proof of Lemma 2.1. Let the assumptions of Lemma 2.1 be satisfied and (u,ϕ) be a unique solution of (2.1) - (2.4) satisfying the inequality (2.5). We use the integral representation of $u(\cdot,t)$ by means of the semigroup $e^{-A_\beta t}$:

$$u(\cdot,t) = \int_0^t e^{-A_\beta(t-\tau)}(u' + A_\beta u)d\tau. \tag{2.31}$$

If $\alpha \in [0,1]$ then

$$\|A^\alpha_\beta u(\cdot,t)\|_\beta \le \int_0^t \frac{1}{(t-\tau)^\alpha}(\|u'\|_\beta + \|A_\beta u\|_\beta)d\tau. \tag{2.32}$$

Let us take $\alpha \in [1/2,1]$ such that $1 + 1/\gamma' = \alpha + 1/\beta'$. The Hardy-Littlewood-Sobolev inequality gives that

$$\|A^\alpha_\beta u\|_{\beta,\gamma'} \le c(\|u'\|_{\beta,\beta'} + \|A_\beta u\|_{\beta,\beta'}). \tag{2.33}$$

It further follows from [10] that

$$\|A^\alpha_\beta u\|_\beta \ge c\|A^{1/2}_\gamma u\|_\gamma. \tag{2.34}$$

Let us show now that

the space $D(A^\eta_m)$ is continuously embedded into the space $W^{2\eta,m}$, (2.35)

if $m > 1$ and $\eta \in (0,1)$. It is known that $D(A^\eta_m) = D(B^\eta_m) \cap L^m_\sigma$, where $B_m = -\Delta$ is the Laplace operator with zero boundary condition in L^m (see [4], Theorem 3). It follows from Theorem 1.15.3. in [15], p.103, that $D(B^\eta_m)$ is the complex interpolation space $[L^m, D(B_m)]_\eta$, thus $D(B^\eta_m) = [L^m, W^{2,m} \cap W^{1,m}_0]_\eta$. Since $W^{2,m} \cap W^{1,m}_0$ is continuously embedded into $W^{2,m}$, it follows from [1], 2.4.(3), that $D(B^\eta_m)$ is continuously embedded into $[L^m, W^{2,m}]_\eta = W^{2\eta,m}$ and (2.35) is proved. The last equality follows from [15, Theorem 4.3.1/2, p.315].

The following inequality is a special case of (2.35):

$$\|A_\gamma^{1/2} u\|_\gamma \geq c\|\nabla u\|_\gamma. \tag{2.36}$$

The inequality (2.7) now follows from (2.33), (2.34), (2.36) and (2.5) and the proof of Lemma 2.1 is completed. □

Lemma 2.3. *Let $2/q+3/p = 1$, $p, q \in (1,\infty)$, $b \in L^{p,q}$, $2/\theta'+3/\theta = 3$, $p/(p-1) < \theta < 3$, $\theta' > 2$, $1/\alpha = 1/\theta - 1/3$ and $v \in L^{2,\infty}$, $\nabla v \in L^{2,2}$, $v \in L^{\alpha,\theta'}$, $\nabla v \in L^{\theta,\theta'}$. Let $r, r', l, l' \in (1,\infty)$, $1/r = 1/l - 1/p$, $1/r' = 1/l' - 1/q$, $r \geq \theta$, $r' \geq \theta'$ and $h \in L^{l,l'}$. Suppose further that the function v is a weak solution of the linearized Navier-Stokes system, that is*

$$\int_0^T \int_\Omega \big(-\frac{\partial \varphi}{\partial t} - \Delta\varphi\big)\cdot v \, dx\, dt = \int_0^T \int_\Omega (h - b\cdot\nabla v)\cdot \varphi \, dx\, dt, \tag{2.37}$$

$$\nabla\cdot v = 0 \quad \textit{in } Q_T, \tag{2.38}$$

$$v = 0 \quad \textit{on } \partial\Omega\times(0,T) \tag{2.39}$$

for every $\varphi \in C_0^\infty([0,T)\times\Omega)$, $\nabla\cdot\varphi = 0$. There exists a positive constant $\varepsilon = \varepsilon(l, l', p)$ such that if $\|b\|_{p,q} < \varepsilon$ then

$$\nabla v \in L^{r,r'} \ \textit{and} \ \|\nabla v\|_{r,r'} \leq c\|h\|_{l,l'}, \tag{2.40}$$

$$\nabla v \in L^{m,l'} \ \textit{and} \ \|\nabla v\|_{m,l'} \leq c\|h\|_{l,l'}, \ \textit{if } l \in (1,3) \ \textit{and} \ \frac{1}{m} = \frac{1}{l} - \frac{1}{3}, \tag{2.41}$$

$$\nabla\phi, \frac{\partial v}{\partial t} \in L^{l,l'} \ \textit{and} \ \|\nabla\phi\|_{l,l'}, \big\|\frac{\partial v}{\partial t}\big\|_{l,l'} \leq c\|h\|_{l,l'}, \tag{2.42}$$

where ϕ is the pressure associated to v.

Proof. This lemma was proved in [12, Proposition 4.1, Theorem 4.1] for $\Omega = \mathbb{R}^3_+$. If Ω is a bounded domain with a smooth boundary, the proof proceeds in the same way and so we present only the main steps of it.

We suppose without loss of generality that $h \in C_0^\infty(Q_T)$. Let further $b_k \in C_0^\infty(Q_T)$ such that $b_k \to b$ in $L^{p,q}$ if $k \to \infty$. By [12], Theorem 4.1 and the citation there, there exists a smooth solution (v_k, ϕ_k) of the problem

$$\frac{\partial v_k}{\partial t} - \nu\Delta_k + b_k\cdot\nabla v_k + \nabla\phi_k = h \quad \text{in } Q_T, \tag{2.43}$$

$$\nabla\cdot v_k = 0 \quad \text{in } Q_T, \tag{2.44}$$

$$v_k = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{2.45}$$

$$v_k|_{t=0} = 0. \tag{2.46}$$

If we choose l_θ and l'_θ so that $1/l_\theta = 1/\theta + 1/p$ and $1/l'_\theta = 1/\theta' + 1/q$ then $1 < l_\theta \leq l$, $1 < l'_\theta \leq l'$ and $h \in L^{l_\theta, l'_\theta}$. By the application of Lemma 2.1 to the system (2.43) - (2.46) we get

$$\|\nabla v_k\|_{\theta,\theta'} \leq c\|h - b_k\cdot\nabla v_k\|_{l_\theta,l'_\theta} \leq c(\|h\|_{l_\theta,l'_\theta} + \|b_k\|_{p,q}\|\nabla v_k\|_{\theta,\theta'}), \tag{2.47}$$

where c is independent of k. If $\|b\|_{p,q}$ is sufficiently small, we get from (2.47) that $\|\nabla v_k\|_{\theta,\theta'} \leq c\|h\|_{l_\theta,l'_\theta}$, $\|v_k\|_{\alpha,\theta'} \leq c\|h\|_{l_\theta,l'_\theta}$ and consequently, from the sequence

$\{v_k\}_{k\in N}$ we can select a subsequence which we denote again $\{v_k\}_{k\in N}$ such that

$$\nabla v_k \to \nabla\tilde{v} \quad \text{weakly in } L^{\theta,\theta'}, \tag{2.48}$$

$$v_k \to \tilde{v} \quad \text{weakly in } L^{\alpha,\theta'}. \tag{2.49}$$

Using (2.48) and (2.49) it is possible to show that $\tilde{v}$ satisfies the equations (2.37) - (2.39). It implies that $\tilde{v} = v$.

Applying again Lemma 2.1 to the system (2.43) - (2.46) we get

$$\|\nabla v_k\|_{r,r'} \le c\|h - b_k\cdot\nabla v_k\|_{l,l'} \le c(\|h\|_{l,l'} + \|b_k\|_{p,q}\|\nabla v_k\|_{r,r'})$$

and thus

$$\|\nabla v_k\|_{r,r'} \le c\|h\|_{l,l'}. \tag{2.50}$$

It follows from (2.48) and (2.50) that (again after selecting a subsequence) $\nabla v_k \to \nabla v$ weakly in $L^{r,r'}$ which gives (2.40).

v and its associated pressure ϕ satisfy the equations (2.43) - (2.46) (with b_k replaced by b) and according to [5], Theorem 2.8 and (2.40) we have

$$\|\frac{\partial v}{\partial t}\|_{l,l'} + \|\nabla^2 v\|_{l,l'} + \|\nabla\phi\|_{l,l'} \le c\|h - b\cdot\nabla v\|_{l,l'} \le c\|h\|_{l,l'}. \tag{2.51}$$

Inequality (2.42) is an immediate consequence of (2.51) and (2.41) follows from (2.51) and the fact that $\|\nabla v\|_{m,l'} \le c\|\nabla^2 v\|_{l,l'}$ if l and m are given by (2.41). The proof is complete. $\square$

For the proof of the following lemma see e.g. [3, Theorem 3.2, Chap.III.3].

Lemma 2.4. *Let D be a bounded Lipschitz domain in $\mathbb{R}^3$, Γ be an open subset of ∂D, $r \in (1,\infty)$, $j \in N\cup\{0\}$. There exists a bounded linear operator $K = K_{j,r} : W_0^{j,r}(D) \to W_0^{j+1,r}(D)^3$ such that*

- (i) $\nabla\cdot Kg = g$ *for all* $g \in W_0^{j,r}(D)$ *such that* $\int_D g\,dx = 0$
- (ii) $\|\nabla^{j+1}Kg\|_r \le c\|\nabla^j g\|_r$ *for all* $g \in W_0^{j,r}(D)$, $c = c(j,r,D)$
- (iii) $\operatorname{supp} Kg \subset D\cup\Gamma$ *if* $\operatorname{supp} g \subset D\cup\Gamma$.

In this lemma, $W_0^{j,r}(D)$ is the completion of $C_0^\infty(D)$ with respect to the standard norm of the space $W^{j,r}(D)$. It is possible to show that $K_{j,r}(g) = K_{l,s}(g)$ if $g \in W_0^{j,r}(D)\cap W_0^{l,s}(D)$, where $r,s \in (1,\infty)$ and $j,l \in N\cup\{0\}$ and so in the rest of the paper the operator $K_{j,r}$ is denoted only by K.

3. Proof of Theorem 1.1

In this section, we assume that the hypotheses of Theorem 1.1 are satisfied and ϕ is the associated pressure to u. We can suppose without loss of generality that $\|u\|_{p,q,Q_r}$ is sufficiently small - see ε from Lemma 2.3. Let $\tilde{r} \in (0,r)$. Let us localize the problem (1.1) - (1.4) in a standard way: Let $\psi \in C^\infty(\overline{Q}_T)$ be a cut-off function such that $\psi(x,t) = 0$ if $(x,t) \in Q_T\setminus\overline{Q}_{2r/3+\tilde{r}/3}$, $\psi(x,t) = 1$ if $(x,t) \in Q_{r/3+2\tilde{r}/3}$ and $\psi(x,t) \in [0,1]$ for every $(x,t) \in Q_T$. We put $w = K(\nabla\cdot(\psi u))$, $v = \psi u - w$. Then

v satisfies the following system of equations:

$$\begin{aligned}\frac{\partial v}{\partial t} - \nu\Delta v + u\cdot\nabla v + \nabla(\psi\phi) &= -\nu\Delta\psi u - 2\nu\nabla\psi\cdot\nabla u + u\cdot\nabla\psi u - \phi\nabla\psi \\ &\quad - \frac{\partial w}{\partial t} + \nu\Delta w - u\cdot\nabla w - \frac{\partial\psi}{\partial t}u \quad \text{in } Q_T,\end{aligned} \tag{3.1}$$

$$\nabla\cdot v = 0 \quad \text{in } Q_T, \tag{3.2}$$

$$v = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{3.3}$$

$$v|_{t=0} = 0. \tag{3.4}$$

We denote the right hand side of (3.1) by h and show at first that

$$h \in L^{l,l'}, \text{ for every } l' \in (1,2),\ l \in (3/2,3) \text{ such that } \frac{2}{l'} + \frac{3}{l} = 3. \tag{3.5}$$

We will use the global estimates for u and ϕ derived in [5, Theorem 3.1]:

$$\|\frac{\partial u}{\partial t}\|_{q,s} + \|\nabla^2 u\|_{q,s} + \|\nabla\phi\|_{q,s} < \infty, \quad s\in(1,2),\ q\in(1,3/2), \frac{2}{s}+\frac{3}{q} = 4, \tag{3.6}$$

$$\|\nabla u\|_{h,\rho} < \infty, \quad h\in(1,3),\ \rho\in(1,\infty), \frac{2}{\rho}+\frac{3}{h} = 3, \tag{3.7}$$

$$\|u\|_{h^*,\rho} < \infty, \quad h^*\in(3/2,\infty),\ \rho\in(1,\infty),\ \frac{2}{\rho}+\frac{3}{h^*} = 2, \tag{3.8}$$

$$\|\phi\|_{r,s} < \infty, \quad r\in(3/2,3),\ s\in(1,2), \frac{2}{s}+\frac{3}{r} = 3. \tag{3.9}$$

It is supposed in (3.9) that $\int_\Omega \phi(x,t)dx = 0$ for every $t\in(0,T)$. Thus, let l, l' satisfy the conditions from (3.5). We have immediately from (3.9) that $\phi\nabla\psi \in L^{l,l'}$. It follows further from Lemma 2.4 that

$$\|\frac{\partial w}{\partial t}\|_{l,l'} = \|\frac{\partial}{\partial t}(K(\nabla\psi\cdot u))\|_{l,l'} = \|K(\frac{\partial}{\partial t}(\nabla\psi\cdot u))\|_{l,l'} \le c\|\frac{\partial}{\partial t}(\nabla\psi\cdot u)\|_{q,l'},$$

where $1/q = 1/l + 1/3$. Since $2/l' + 3/q = 4$, we have $\partial w/\partial t \in L^{l,l'}$ by (3.6). Similarly, $\nu\Delta w \in L^{l,l'}$, as follows from Lemma 2.4 and (3.7). Finally,

$$\|u\cdot\nabla w\|_{l,l'} \le \|u\|_{p,q}\|\nabla w\|_{\frac{pl}{p-l},\frac{ql'}{q-l'}}$$

and since

$$\|\nabla w\|_{\frac{pl}{p-l}} = \|\nabla K(\nabla\psi\cdot u)\|_{\frac{pl}{p-l}} \le c\|\nabla\psi\cdot u\|_{\frac{pl}{p-l}} \le c\|u\|_{\frac{pl}{p-l}},$$

we have

$$\|u\cdot\nabla w\|_{l,l'} \le \|u\|_{p,q}\|u\|_{\frac{pl}{p-l},\frac{ql'}{q-l'}}.$$

Thus, $u\cdot\nabla w \in L^{l,l'}$ as a consequence of $3(p-l)/pl + 2(q-l')/ql' = 3/l + 2/l' - (3/p+2/q) = 2$ and (3.8). The remaining terms of h belong obviously to the space $L^{l,l'}$ and (3.5) is proved.

Lemma 3.1. *Let us consider the equations* (3.1)–(3.4)*. Let $l' \in (2q/(q+2), 2)$ and $l \in (3/2, 3p/(p+3))$, that is $m < p$ for m such that $1/m = 1/l - 1/3$. If $h \in L^{l,l'}$ and $\psi\phi \in L^{l,l'}$, then $h \in L^{m,l'}$ and $\psi\phi \in L^{m,l'}$.*

Proof. Let us define r, r' as in Lemma 2.3. Then $r > 3/2$ and $r' > 2$. There exist θ, θ' such that $p/(p-1) < \theta < 3/2$, $2 < \theta' < r'$ and $2/\theta' + 3/\theta = 3$. It follows from (3.7) that $\nabla v \in L^{\theta,\theta'}$ and $v \in L^{\alpha,\theta}$, where $1/\alpha = 1/\theta - 1/3$. Further, v is a solution of (2.37) - (2.39) with u instead of b and with h being the right hand side of (3.1). Thus, the assumptions of Lemma 2.3 are obviously satisfied and we get by (2.40)–(2.42) that

$$\|\nabla v\|_{r,r'}, \|\nabla(\psi\phi)\|_{l,l'}, \|\nabla v\|_{m,l'}, \Big\|\frac{\partial v}{\partial t}\Big\|_{l,l'} \le c\|h\|_{l,l'}. \tag{3.10}$$

Now, one can show that $h \in L^{m,l'}$ using (3.10), the assumption $\psi\phi \in L^{l,l'}$ from Lemma 3.1 and Lemma 2.4. It is possible to proceed in the same way as was done in the paragraph preceding Lemma 3.1 and during the process we can possibly diminish, if necessary, without loss of generality the support of the cut-off function ψ. □

Now, we use twice Lemma 3.1. According to (3.5) and (3.9) we have that $h, \phi \in L^{l,l'}$, where l, l' satisfy the assumptions of Lemma 3.1 and $2/l' + 3/l = 3$. By the first application of Lemma 3.1 we get that $h, \psi\phi \in L^{m,l'}$, where $1/m = 1/l - 1/3$, $m \in (3, p)$ and $2/l' + 3/m = 2$. Consequently, $h, \psi\phi \in L^{l,l'}$, where l, l' satisfy the assumptions from Lemma 3.1 and $2/l' + 3/l < 3$. By the second application of Lemma 3.1 we get that $h \in L^{m,l'}$ and $2/l' + 3/m < 2$. Lemma 2.3, (2.40) now produces that $\nabla v \in L^{r,r'}$, where $1/r = 1/m - 1/p$ and $1/r' = 1/l' - 1/q$. Consequently,

$$\|u \cdot \nabla v\|_{m,l'} \le \|u\|_{p,q} \|\nabla v\|_{\frac{pm}{p-m}, \frac{ql'}{q-l'}} \le c\|\nabla v\|_{r,r'} < \infty$$

and v satisfies the equation

$$\frac{\partial v}{\partial t} - \nu\Delta v + \nabla(\psi\phi) = h - u \cdot \nabla v \quad \text{in } Q_T \tag{3.11}$$

and equations (3.2) - (3.4), where

$$h - u \cdot \nabla v \in L^{m,l'} \text{ for every } m, l' \text{ such that } \frac{2}{l'} + \frac{3}{m} < 2,\ l' \in (1,2), m \in (3,p). \tag{3.12}$$

Using the integral representation of $v(\cdot, t)$ by means of the semigroup $e^{-A_m t}$, we have

$$v(\cdot, t) = \int_0^t e^{-A_m(t-\tau)} P_\sigma^m (h - u \cdot \nabla v) d\tau. \tag{3.13}$$

Let $\alpha < 1/2$. We can choose l' such that $\alpha l'/(l'-1) < 1$ and obtain the estimate

$$\begin{aligned} \|A_m^\alpha v(\cdot, t)\|_m &\le \int_0^t \|A_m^\alpha e^{-A_m(t-\tau)} P_\sigma^m (h - u \cdot \nabla v)\|_m d\tau \\ &\le \int_0^t \frac{\|h - u \cdot \nabla v\|_m}{(t-\tau)^\alpha} d\tau \\ &\le \Big(\int_0^t \frac{d\tau}{(t-\tau)^{\alpha l'/(l'-1)}}\Big)^{\frac{l'-1}{l'}} \|h - u \cdot \nabla v\|_{m,l'} \le c. \end{aligned} \tag{3.14}$$

The space $D(A_m^\alpha)$ is continuously embedded into the space $W^{2\alpha,m}$ - see (2.35). It further follows from in [15, Theorem 4.6.1(e), p.327] that

$$\text{the space } W^{2\alpha,m} \text{ is continuously embedded into the Hölder space } C^\beta(\overline{\Omega}), \tag{3.15}$$

if $\beta = 2\alpha - 3/m > 0$. By the suitable choice of α and m we can have β as close to $1 - 3/p$ as possible and so it follows from (2.35), (3.15), (3.14) and (3.12) that

$$v \in L^\infty(0, T, C^\beta(\overline{\Omega})) \text{ for every } \beta \in (0, 1 - 3/p). \tag{3.16}$$

Thus, $v \in L^\infty(Q_T)$ and consequently,

$$u \in L^\infty(Q_{r/3+2\tilde{r}/3}). \tag{3.17}$$

We can now use this last information on local regularity of u, go through this section once again and get that

$$h - u \cdot \nabla v \in L^{m,l'} \quad \text{for every } l' \in (1, 2), m \in (3, \infty). \tag{3.18}$$

Using (2.35), (3.15), (3.14) and (3.12) we obtain that

$$v \in L^\infty(0, T, C^\beta(\overline{\Omega})) \quad \text{for every } \beta \in (0, 1) \tag{3.19}$$

and (1.6) follows immediately. The proof of Theorem 1.1 is completed.

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