Electronic Journal of Differential Equations, Vol. 2007(2007), No. 27, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

PERIODICITY AND STABILITY IN NEUTRAL NONLINEAR DYNAMIC EQUATIONS WITH FUNCTIONAL DELAY ON A TIME SCALE

ERIC R. KAUFMANN, YOUSSEF N. RAFFOUL

Abstract. Let $\mathbb{T}$ be a periodic time scale. We use a fixed point theorem due to Krasnosel'skiĭ to show that the nonlinear neutral dynamic equation with delay

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + (Q(t, x(t), x(t - g(t)))))^{\Delta} + G\big(t, x(t), x(t - g(t))\big), t \in \mathbb{T},$$

has a periodic solution. Under a slightly more stringent inequality we show that the periodic solution is unique using the contraction mapping principle. Also, by the aid of the contraction mapping principle we study the asymptotic stability of the zero solution provided that $Q(t, 0, 0) = G(t, 0, 0) = 0$.

1. Introduction

We assume the reader is familiar with the notation and basic results for dynamic equations on time scales. For a review of this topic we direct the reader to the monographs [2, 3]. We begin with a few definitions.

Definition 1.1. We say that a time scale $\mathbb{T}$ is *periodic* if there exist a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the *period* of the time scale.

The above definition is due to Atici *et. al* [1] and Kaufmann and Raffoul [5].

Example 1.2. The following time scales are periodic.

1. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty}[2(i-1)h, 2ih]$, $h > 0$ has period $p = 2h$.
2. $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.
3. $\mathbb{T} = \mathbb{R}$.
4. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where,, $0 < q < 1$ has period $p = 1$.

Remark: All periodic time scales are unbounded above and below.

Definition 1.3. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f\colon \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$.

2000 *Mathematics Subject Classification.* 34K13, 34C25, 34G20.

Key words and phrases. Krasnosel'skiĭ; contraction mapping; neutral; nonlinear; Delay; time scales; periodic solution; unique solution; stability.

Submitted July 11, 2006. Published February 12, 2007.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark: If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period p.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. We will show the existence of periodic solutions for the nonlinear neutral dynamic equation with delay

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + (Q(t, x(t), x(t-g(t)))))^{\Delta} + G\big(t, x(t), x(t-g(t))\big), t \in \mathbb{T}, \quad (1.1)$$

In order for the function $x(t - g(t))$ to be well-defined over $\mathbb{T}$, we assume that $g : \mathbb{T} \to \mathbb{R}$ and that $id - g : \mathbb{T} \to \mathbb{T}$ is strictly increasing.

In the case $\mathbb{T} = \mathbb{R}$, the second author in [6] used Krasnosel'skiĭ's fixed point theorem to show the existence of a periodic solution of (1.1) when the delay is some positive continuous and periodic function $g(t)$. Also, the existence of a unique periodic solution of (1.1) was obtained by the aid of the contraction mapping principle. For $\mathbb{T} = \mathbb{R}$ and assuming $x = 0$ is a solution of (1.1), in [7], Raffoul used the notion of fixed point theory and obtained conditions that guaranteed the asymptotic stability of a zero solution.

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will sate some facts about the exponential function on a time scale as well as the Krasnosel'skiĭ fixed point theorem. We present our main results on periodicity in Section 3 and provide an example. In Section 4 we state and prove a theorem concerning the stability of the zero solution of (1.1) for a general time scale.

2. Preliminaries

We begin this section by considering some advanced topics in the theory of dynamic equations on time scales. Most of the following definitions, lemmas and theorems can be found in [2, 3]. Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales [2, Theorem 1.93].

Theorem 2.1 (Chain Rule). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(w \circ \nu)^{\Delta} = (w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t-g(t)) = f(\nu(t))$ where, $\nu(t) := t-g(t)$. Our second theorem is the substitution rule [2, Theorem 1.98].

Theorem 2.2 (Substitution). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\,\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\,\tilde{\Delta} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by
$$e_p(t,s) = \exp\Big(\int_s^t \frac{1}{\mu(z)} \operatorname{Log}(1+\mu(z)p(z))\,\Delta z\Big),$$
It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma, [2, Theorem 2.36].

Lemma 2.3. *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $\big(\frac{1}{e_p(\cdot,s)}\big)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

Lastly in this section, we state Krasnosel'skiĭ's fixed point theorem which enables us to prove the existence of a periodic solution. For its proof we refer the reader to [8].

Theorem 2.4 (Krasnosel'skiĭ). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $\big(\mathbb{B}, \|\cdot\|\big)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) *$x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,*
(ii) *A is compact and continuous,*
(iii) *B is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

3. Existence Of Periodic Solutions

We will state and prove our main result in this section as well as provide an example. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean
$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}$$
unless otherwise specified. The intervals $[a,b)$, $(a,b]$, and (a,b) are defined similarly. Define $P_T = \{\varphi \in C(\mathbb{T},R) : \varphi(t+T) = \varphi(t)\}$ where, $C(\mathbb{T},R)$ is the space of all real valued continuous functions on $\mathbb{T}$. Then P_T is a Banach space when it is endowed with the supremum norm
$$\|x\| = \sup_{t\in[0,T]} |x(t)|.$$
We will need the following lemma whose proof can be found in [5].

Lemma 3.1. *Let $x \in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

In this paper we assume that $a \in \mathcal{R}^+$ is continuous, $a(t) > 0$ for all $t \in \mathbb{T}$ and
$$a(t+T) = a(t), \quad \big(id - g\big)(t+T) = \big(id - g\big)(t), \tag{3.1}$$
where, id is the identity function on $\mathbb{T}$. We also require that $Q(t,x)$ and $G(t,x,y)$ are continuous and periodic in t and Lipschitz continuous in x and y. That is,
$$Q(t+T,x) = Q(t,x),\ G(t+T,x,y) = G(t,x,y), \tag{3.2}$$

and there are positive constants E_1, E_2, E_3 such that

$$|Q(t,x) - Q(t,y)| \le E_1\|x-y\|, \text{ for } x,y \in \mathbb{R}, \tag{3.3}$$

and

$$|G(t,x,y) - G(t,z,w)| \le E_2\|x-z\| + E_3\|y-w\|, \text{ for } x,y,z,w \in \mathbb{R}. \tag{3.4}$$

Lemma 3.2. *Suppose* (3.1), (3.2) *hold. If $x \in P_T$, then x is a solution of equation* (1.1) *if, and only if,*

$$\begin{aligned} x(t) &= Q\big(t, x(t-g(t))\big) + \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \Big[-a(s)Q^\sigma\big(s, x(s-g(s))\big) + G\big(s, x(s), x(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s. \end{aligned} \tag{3.5}$$

Proof. Let $x \in P_T$ be a solution of (1.1). First we write (1.1) as

$$\begin{aligned} \{x(t) - Q\big(t, x(t-g(t))\big)\}^\Delta &= -a(t)\{x^\sigma(t) - Q^\sigma\big(t, x(t-g(t))\big)\} \\ &\quad - a(t)Q^\sigma\big(t, x(t-g(t))\big) + G\big(t, x(t), x(t-g(t))\big). \end{aligned}$$

Multiply both sides by $e_a(t,0)$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^{t} \big[e_a(s,0)\{x(s) - Q\big(s, x(s-g(s))\big)\}\big]^\Delta \,\Delta s \\ &= \int_{t-T}^{t} \Big[-a(s)Q^\sigma\big(s, x(s-g(s))\big) + G\big(s, x(s), x(s-g(s))\big)\Big] e_a(s,0)\,\Delta s. \end{aligned}$$

Consequently, we have

$$\begin{aligned} &e_a(t,0)\Big(x(t) - a(t)Q\big(t, x(t-g(t))\big)\Big) \\ &- e_a(t-T,0)\Big(x(t-T) - a(t-T)Q\big(t-T, x(t-T-g(t-T))\big)\Big) \\ &= \int_{t-T}^{t} \big[-a(s)Q^\sigma\big(s, x(s-g(s))\big) + G\big(s, x(s), x(s-g(s))\big)\Big] e_a(s,0)\,\Delta s. \end{aligned}$$

After making use of (3.1), (3.2) and $x \in P_T$, we divide both sides of the above equation by $e_a(t,0)$ to obtain

$$\begin{aligned} x(t) &= Q\big(t, x(t-g(t))\big) + \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \Big[-a(s)Q^\sigma\big(s, x(s-g(s))\big) + G\big(s, x(s), x(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s \end{aligned}$$

where, we have used Lemma 2.3 to simplify the exponentials. Since each step is reversible, the converse follows. This completes the proof. $\square$

Define the mapping $H : P_T \to P_T$ by

$$\begin{aligned} &(H\varphi)(t) \\ &= Q\big(t, \varphi(t-g(t))\big) + \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \Big[-a(s)Q^\sigma\big(s, \varphi(s-g(s))\big) + G\big(s, \varphi(s), \varphi(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s. \end{aligned} \tag{3.6}$$

To apply Theorem 2.4 we need to construct two mappings; one map is a contraction and the other map is compact. We express equation (3.6) as

$$(H\varphi)(t) = (B\varphi)(t) + (A\varphi)(t)$$

where, A, B are given by

$$(B\varphi)(t) = Q\big(t, \varphi(t - g(t))\big) \tag{3.7}$$

and

$$\begin{aligned}(A\varphi)(t) \\ &= \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \big[- a(s) Q^{\sigma}\big(s, \varphi(s - g(s))\big) + G\big(s, \varphi(s), \varphi(s-g(s))\big)\big] e_{\ominus a}(t, s)\, \Delta s.\end{aligned} \tag{3.8}$$

Lemma 3.3. *Suppose* (3.1)–(3.4) *hold. Then* $A : P_T \to P_T$*, as defined by* (3.8)*, is compact.*

Proof. We first show that $A : P_T \to P_T$. Evaluate (3.8) at $t + T$.

$$\begin{aligned}(A\varphi)(t+T) = \big(1 - e_{\ominus a}(t+T, t)\big)^{-1} \times \int_{t}^{t+T} \Big[& - a(s) Q^{\sigma}\big(s, \varphi(s - g(s))\big) \\ & + G\big(s, \varphi(s), \varphi(s-g(s))\big)\Big] e_{\ominus a}(t+T, s)\, \Delta s.\end{aligned} \tag{3.9}$$

Use Theorem 2.2 with $u = s - T$ and conditions (3.1) – (3.2) to get

$$\begin{aligned}(A\varphi)(t+T) &= \big(1 - e_{\ominus a}(t+T, t)\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \Big[- a(u+T) Q^{\sigma}\big(u - T, \varphi(u - T - g(u-T))\big) \\ &\quad + G\big(s, \varphi(u-T), \varphi(u - T - g(u-T))\big)\Big] e_{\ominus a}(t+T, u+T)\, \Delta u.\end{aligned}$$

From (2) and Theorem 2.2, we have $e_{\ominus a}(t+T, u+T) = e_{\ominus a}(t, u)$ and $e_{\ominus a}(t+T, t) = e_{\ominus a}(t, t-T)$. Thus (3.9) becomes

$$\begin{aligned}(A\varphi)(t+T) &= \big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \times \int_{t-T}^{t} \Big[- a(u) Q^{\sigma}\big(u, \varphi(u - g(u))\big) \\ &\quad + G\big(u, \varphi(u), \varphi(u-g(u))\big)\Big] e_{\ominus a}(t+T, u)\, \Delta u \\ &= (A\varphi)(t).\end{aligned}$$

That is, $A : P_T \to P_T$.

To see that A is continuous, we let $\varphi, \psi \in P_T$ with $\|\varphi\| \le C$ and $\|\psi\| \le C$ and define

$$\eta := \max_{t \in [0,T]} \big|\big(1 - e_{\ominus a}(t, t-T)\big)^{-1}\big|, \quad \rho := \max_{t \in [0,T]} |a(t)|, \quad \gamma := \max_{u \in [t-T, t]} e_{\ominus a}(t, u). \tag{3.10}$$

Given $\varepsilon > 0$, take $\delta = \varepsilon / M$ with $M = \eta\, \gamma\, T(\rho\, E_1 + E_2 + E_3)$ where, E_1, E_2 and E_3 are given by (3.3) and (3.4) such that $\|\varphi - \psi\| < \delta$. Using (3.8) we get

$$\begin{aligned}\big\|A\varphi - A\psi\big\| &\le \eta\gamma \int_0^T \Big[\rho\, E_1 \|\varphi - \psi\| + (E_2 + E_3)\|\varphi - \psi\|\Big] \Delta u \\ &\le M \|\varphi - \psi\| < \varepsilon.\end{aligned}$$

This proves that A is continuous.

We need to show that A is compact. Consider the sequence of periodic functions $\{\varphi_n\} \subset P_T$ and assume that the sequence is uniformly bounded. Let $R > 0$ be such that $\|\varphi_n\| \le R$, for all $n \in \mathbb{N}$. In view of (3.3) and (3.4) we arrive at

$$\begin{aligned}|Q(t,x)| &= |Q(t,x) - Q(t,0) + Q(t,0)| \\ &\le |Q(t,x) - Q(t,0)| + |Q(t,0)| \\ &\le E_1\|x\| + \alpha.\end{aligned}$$

Similarly,

$$\begin{aligned}|G(t,x,y)| &= |G(t,x,y) - G(t,0,0) + G(t,0,0)| \\ &\le |G(t,x,y) - G(t,0,0)| + |G(t,0,0)| \\ &\le E_2\|x\| + E_3\|y\| + \beta,\end{aligned}$$

where, $\alpha = \sup_{t\in[0,T]} |Q(t,0)|$ and $\beta = \sup_{t\in[0,T]} |G(t,0,0)|$.

$$\begin{aligned}|(A\varphi_n)(t)| &= \Big|\big(1 - e_{\ominus a}(t, t-T)\big)^{-1} \times \int_{t-T}^{t} \Big[-a(s)Q^\sigma\big(s, \varphi(s-g(s))\big) \\ &\quad + G\big(s, \varphi(s), \varphi(s-g(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s\Big| \\ &\le \eta\gamma \int_{t-T}^{t} \big|-a(s)\big|\,\big|Q^\sigma\big(s,\varphi(s-g(s))\big)\big| + \big|G\big(s,\varphi(s),\varphi(s-g(s))\big)\big|\,\Delta s \\ &\le \eta\gamma T\big[\rho(E_1R+\alpha) + (E_2+E_3)R + \beta\big] \equiv D.\end{aligned}$$

Thus the sequence $\{A\varphi_n\}$ is uniformly bounded. Now, it can be easily checked that

$$(A\varphi_n)^\Delta(t) = -a(t)(A\varphi_n)^\sigma(t) - a(t)Q^\sigma\big(t, \varphi(t-g(t))\big) + G\big(t, \varphi(t), \varphi(t-g(t))\big).$$

Consequently,

$$|(A\varphi_n)^\Delta(t)| \le D\rho + \rho(E_1R+\alpha) + (E_2+E_3)R + \beta$$

for all n. That is $\|(A\varphi_n)^\Delta\| \le F$, for some positive constant F. Thus the sequence $\{A\varphi_n\}$ is uniformly bounded and equi-continuous. The Arzelà-Ascoli theorem implies that there is a subsequence $\{A\varphi_{n_k}\}$ which converges uniformly to a continuous T-periodic function φ^*. Thus A is compact. $\square$

Lemma 3.4. *Let B be defined by* (3.7) *and*

$$E_1 \le \zeta < 1. \tag{3.11}$$

Then $B : P_T \to P_T$ is a contraction.

Proof. Trivially, $B : P_T \to P_T$. For $\varphi, \psi \in P_T$, we have

$$\begin{aligned}\|B\varphi - B\psi\| &= \sup_{t\in[0,T]} |B\varphi(t) - B\psi(t)| \\ &\le E_1 \sup_{t\in[0,T]} |\varphi(t-g(t)) - \psi(t-g(t))| \\ &\le \zeta\|\varphi - \psi\|.\end{aligned}$$

Hence B defines a contraction mapping with contraction constant ζ. $\square$

Theorem 3.5. *Suppose the hypothesis of Lemma 3.2 hold. Suppose* (3.1)–(3.4) *hold. Let* $\alpha = \sup_{t\in[0,T]} |Q(t,0)|$ *and* $\beta = \sup_{t\in[0,T]} |G(t,0,0)|$. *Let* J *be a positive constant satisfying the inequality*

$$\alpha + E_1 J + \eta T\gamma\Big[\rho\,(E_1 J + \alpha) + (E_2 + E_3)J + \beta\Big] \leq J. \tag{3.12}$$

Let $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. *Then (1.1) has a solution in* M.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma 3.3, A is continuous and $A\mathbb{M}$ is contained in a compact set. Also, from Lemma 3.4, the mapping B is a contraction and it is clear that $B : P_T \to P_T$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $\|A\varphi + B\psi\| \leq J$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \leq J$. Then

$$\begin{aligned}
&\|A\varphi + B\psi\| \\
&\leq E_1\|\psi\| + \alpha + \eta\,\gamma \int_0^T \big[|a(u)|(\alpha + E_1\|\varphi\|) + E_2\|\varphi\| + E_3\|\varphi\| + \beta\big]\Delta u \\
&\leq \alpha + E_1 J + \eta T\gamma\,\Big[\rho(E_1 J + \alpha) + (E_2 + E_3)J + \beta\Big] \leq J.
\end{aligned}$$

We now see that all the conditions of Krasnosel'skiĭ's theorem are satisfied. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = Az + Bz$. By Lemma 3.2, this fixed point is a solution of (1.1). Hence (1.1) has a T-periodic solution. □

Theorem 3.6. *Suppose* (3.1)–(3.4) *hold. If*

$$E_1 + \eta\,\gamma\,T(\rho\,E_1 + E_2 + E_3) < 1,$$

then (1.1) has a unique T*-periodic solution.*

Proof. Let the mapping H be given by (3.6). For $\varphi, \psi \in P_T$, we have

$$\|H\varphi - H\psi\| \leq \Big(E_1 + \eta\,\gamma\,T(\rho\,E_1 + E_2 + E_3)\Big)\|\varphi - \psi\|.$$

This completes the proof by invoking the contraction mapping principle. □

It is worth noting that Theorems 3.5 and 3.6 are not applicable to functions such as

$$G\big(t, \varphi(t), \varphi(t - g(t))\big) = f_1(t)\varphi^2(t) + f_2(t)\varphi^2(t - g(t)),$$

where, $f_1(t), f_2(t)$ and $g(t) > 0$ are continuous and periodic on some applicable time scale. To accommodate such functions, we state the following corollary, in which the functions G and Q are required to satisfy local Lipschitz conditions. We note that conditions (3.2) and (3.4) require that the functions G and Q be globally Lipschitz.

Corollary 3.7. *Suppose* (3.1)–(3.4) *hold and let* α *and* β *be the constants defined in Theorem 3.5. Let* J *be a positive constant and define* $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. *Suppose there are positive constants* E_1^*, E_2^* *and* E_3^* *so that for* x, y, z *and* $w \in \mathbb{M}$ *we have*

$$\begin{gathered}
|Q(t,x) - Q(t,y)| \leq E_1^*\|x - y\|, \\
|G(t,x,y) - G(t,z,w)| \leq E_2^*\|x - z\| + E_3^*\|y - w\|,
\end{gathered}$$

and

$$\alpha + E_1^* J + \eta T\gamma\Big[\rho\,(E_1^* J + \alpha) + (E_2^* + E_3^*)J + \beta\Big] \leq J. \tag{3.13}$$

Then (1.1) has a solution in $\mathbb{M}$. *If in addition,*

$$E_1^* + \eta\,\gamma\,T(\rho\,E_1^* + E_2^* + E_3^*) < 1,$$

then the solution in $\mathbb{M}$ *is unique.*

Proof. Let $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \le J\}$. Let the mapping H be given by (3.6). Then the results follow immediately from Theorem 3.5 and Theorem 3.6. □

Now we give an example.

Example 3.8. Let $\mathbb{T}$ be a periodic time scale. For small positive ε_1 and ε_2, we consider the perturbed van der Pol equation

$$x^\Delta = -2x^\sigma(t) + \Big(\varepsilon_1 b(t)x^2(t-g(t))\Big)^\Delta + \varepsilon_2\Big(c(t) + x^2(t)\Big), \tag{3.14}$$

where

$$b(t+T) = b(t), \quad \big(id - g\big)(t+T) = \big(id - g\big)(t), \quad \text{and} \quad c(t+T) = c(t). \tag{3.15}$$

Also, we assume that the functions b, c and g are continuous with $id - g : \mathbb{T} \to \mathbb{T}$ strictly increasing. So we have

$$a(t) = 2, \quad Q(t, x(t-g(t))) = \varepsilon_1\, b(t)x^2(t-g(t))$$

and

$$G(t, x(t), x(t-g(t))) = \varepsilon_2\Big(c(t) + x^2(t)\Big).$$

Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \le J\}$, where J is a positive constant. Then for $\varphi, \psi \in \mathbb{M}$ we have

$$\eta = \Big(1 - e_{\ominus 2}(t, t-T)\Big)^{-1}, \quad \rho = 3, \quad \gamma \le 1.$$

Let

$$\iota = \sup_{t \in [0,T]} |b(t)|, \quad \kappa = \sup_{t \in [0,T]} |c(t)|.$$

Then

$$\alpha = 0, \quad \beta = \varepsilon_2\kappa, \quad E_1^* = \varepsilon_1\iota J, \quad E_3^* = 0, \quad E_2^* = \varepsilon_2 J.$$

Thus, inequality (3.13) becomes

$$\varepsilon_1\iota J^2 + \eta T\Big[3\varepsilon_1 J^2 + \varepsilon_2 J^2 + \varepsilon_2\kappa\Big] \le J$$

which is satisfied for small ε_1 and ε_2. Hence, (3.14) has a T-periodic solution, by Corollary 3.7. Moreover, if

$$\varepsilon_1\iota J + \eta\gamma T\Big[3\varepsilon_1\iota J + \varepsilon_2 J\Big] < 1$$

is satisfied for small ε_1 and ε_2, then (3.14) has a unique T-periodic solution.

4. Stability

Lyapunov functions and functionals have been successfully used to obtain boundedness, stability and the existence of periodic solutions of differential equations, differential equations with functional delays and functional differential equations. When studying differential equations with functional delays using Lyapunov functionals, many difficulties arise if the delay is unbounded or if the differential equation in question has unbounded terms. In [7], the second author studied, via fixed point theory, the asymptotic stability of the zero solution of the scalar neutral differential equation

$$x'(t) = -a(t)x(t) + c(t)x'(t-g(t)) + q\big(t, x(t), x(t-g(t))\big), \tag{4.1}$$

where, $a(t), b(t), g(t)$ and q are continuous in their respective arguments. It is clear that (1.1) is more general than (4.1).

This section is mainly concerned with the asymptotic stability of the zero solution of (1.1). We assume that the functions Q and G are continuous, as before. Also, we assume that $g(t)$ is continuous and $g(t) \geq g^* > 0$, for all $t \in \mathbb{T}$ such that $t \geq t_0$ for some $t_0 \in \mathbb{T}$ and that $Q(t,0) = G(t,0,0) = 0$ and Q and G obey the Lipschitz conditions (3.3) and (3.4). The techniques used in this section are adapted from the paper [7].

As before, we assume a time scale, $\mathbb{T}$, that is unbounded above and below and that $0 \in \mathbb{T}$. Also, we assume that $g : \mathbb{T} \to \mathbb{R}$ and that $id - g : \mathbb{T} \to \mathbb{T}$ is strictly increasing.

To arrive at the correct mapping, we rewrite (1.1) as in the proof of Lemma 3.2, multiply both sides by $e_a(t,0)$ and then integrate from 0 to t to obtain

$$\begin{aligned} x(t) &= Q(t, x(t-g(t))) + \Big[x(0) - Q\big(0, x(-g(0))\big)\Big] e_{\ominus a}(0,t) \\ &\quad + \int_0^t \Big[-a(u)Q^\sigma\big(u, x(u-g(u))\big) + G\big(u, x(u), x(u-g(u))\big)\Big] e_{\ominus a}(t,u)\, \Delta u. \end{aligned} \tag{4.2}$$

Thus, we see that x is a solution of (1.1) if and only if it satisfies (4.2).

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given Δ-differentiable bounded initial function. We say $x(t) := x(t,0,\psi)$ is a solution of (1.1) if $x(t) = \psi(t)$ for $t \leq 0$ and satisfies (1.1) for $t \geq 0$. We say the zero solution of (1.1) is stable at t_0 if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\big[\psi : (-\infty, t_0]_{\mathbb{T}} \to \mathbb{R}$ with $\|\psi\| < \delta\big]$ implies $|x(t,t_0,\psi)| < \varepsilon$.

Let $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$ be the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ and define the set S by

$$S = \{\varphi \in C_{rd} : \varphi(t) = \psi(t) \text{ if } t \leq 0,\ \varphi(t) \to 0 \text{ as } \ t \to \infty, \text{ and } \varphi \text{ is bounded}\}.$$

Then, $\big(S, \|\cdot\|\big)$ is a complete metric space where, $\|\cdot\|$ is the supremum norm. For the next theorem we impose the following conditions.

$$e_{\ominus a}(t,0) \to 0, \text{ as } t \to \infty, \tag{4.3}$$

there is an $\alpha > 0$ such that

$$E_1 + \int_0^t \Big[|a(u)|E_1 + E_2 + E_3\Big] e_{\ominus a}(t,u)\, \Delta u \leq \alpha < 1, t \geq 0, \tag{4.4}$$

$$t - g(t) \to \infty, \text{as } t \to \infty, \tag{4.5}$$

$$Q(t,0) \to 0, \text{as } t \to \infty. \tag{4.6}$$

Theorem 4.1. *If* (3.3)–(3.4) *and* (4.3)–(4.6) *hold, then every solution* $x(t,0,\psi)$ *of* (1.1) *with small continuous initial function* $\psi(t)$*, is bounded and goes to zero as* $t \to \infty$*. Moreover, the zero solution is stable at* $t_0 = 0$*.*

Proof. Define the mapping $P : S \to S$ by
$$\big(P\varphi\big)(t) = \psi(t) \quad \text{if } t \le 0$$
and
$$\begin{aligned}\big(P\varphi\big)(t) &= Q(t,\varphi(t-g(t))) + \Big(\psi(0) - Q(0,\psi(-g(0)))\Big)e_{\ominus a}(t,0)\\ &\quad + \int_0^t \Big[-a(u)Q^\sigma(u,\varphi(u-g(u)))\\ &\quad + G(u,\varphi(u),\varphi(u-g(u)))\Big]e_{\ominus a}(t,u)\,\Delta u, \quad \text{if } t \ge 0.\end{aligned}$$
It is clear that for $\varphi \in S$, $P\varphi$ is continuous. Let $\varphi \in S$ with $\|\varphi\| \le K$, for some positive constant K. Let $\psi(t)$ be a small given continuous initial function with $\|\psi\| < \delta, \delta > 0$. Then,
$$\begin{aligned}\|P\varphi\| &\le E_1K + \Big|\Big(\psi(0) - Q\big(0,\psi(-g(0))\big)\Big)\Big|e_{\ominus a}(t,0)\\ &\quad + \int_0^t \Big[|a(u)|E_1 + E_2 + E_3\Big]e_{\ominus a}(t,u)\,\Delta u\, K\\ &\le \Big(1+E_1\Big)\delta + E_1K + \int_0^t \Big[|a(u)|E_1 + E_2 + E_3\Big]e_{\ominus a}(t,u)\,\Delta u\, K\\ &\le \Big(1+E_1\Big)\delta + \alpha K,\end{aligned} \tag{4.7}$$
which implies that, $\|P\varphi\| \le K$, for the right δ. Thus, (4.7) implies that $(P\varphi)(t)$ is bounded. Next we show that $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$. The second term on the right side of $\big(P\varphi\big)(t)$ tends to zero, by condition (4.3). Also, the first term on the right side tends to zero, because of (4.5), (4.6) and the fact that $\varphi \in S$. It is left to show that the integral term goes to zero as $t \to \infty$.

Let $\varepsilon > 0$ be given and $\varphi \in S$ with $\|\varphi\| \le K$, $K > 0$. Then, there exists a $t_1 > 0$ so that for $t > t_1$, $|\varphi(t-g(t))| < \varepsilon$. Due to condition (4.3), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that $e_{\ominus a}(t,t_1) < \frac{\varepsilon}{\alpha K}$.

Thus for $t > t_2$, we have
$$\begin{aligned}&\Big|\int_0^t \Big[-a(u)Q^\sigma\big(u,\varphi(u-g(u))\big) + G\big(u,\varphi(u),\varphi(u-g(u))\big)\Big]e_{\ominus a}(t,u)\,\Delta u\Big|\\ &\le K\int_0^{t_1} \Big[|a(u)|E_1 + E_2 + E_3\Big]e_{\ominus a}(t,u)\,\Delta u\\ &\quad + \varepsilon\int_{t_1}^t \Big[|a(u)|E_1 + E_2 + E_3\Big]e_{\ominus a}(t,u)\,\Delta u\\ &\le Ke_{\ominus a}(t,t_1)\int_0^{t_1} \Big[|a(u)|E_1 + E_2 + E_3\Big]e_{\ominus a}(t_1,u)\,\Delta u + \alpha\varepsilon\\ &\le \alpha K e_{\ominus a}(t,t_1) + \alpha\varepsilon\\ &\le \varepsilon + \alpha\varepsilon.\end{aligned}$$

Hence, $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$. It remains to show that P is a contraction under the supremum norm. Let $\zeta, \eta \in S$. Then

$$\begin{aligned}\Big|(P\zeta)(t) - (P\eta)(t)\Big| &\le \Big\{E_1 + \int_0^t \Big[|a(u)|E_1 + E_2 + E_3\Big] e_{\ominus a}(t,u)\Delta u\Big\}\|\zeta - \eta\| \\ &\le \alpha\|\zeta - \eta\|.\end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (1.1), is bounded and tends to zero as t tends to infinity. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing K by ε. This completes the proof. $\square$

Example 4.2. Let

$$\mathbb{T} = -2^{\mathbb{Z}} \cup 2^{\mathbb{Z}} \cup \{0\} = \{\dots, -2, -1, -\frac{1}{2}, \dots, 0, \dots, \frac{1}{2}, 1, 2, \dots\}.$$

and let ψ be a continuous initial function, $\psi : -2^{\mathbb{Z}} \cup \{0\} \to \mathbb{R}$ with $\|\psi\| \le \delta$ for small $\delta > 0$. For small ε_1 and ε_2, we consider the nonlinear neutral dynamic equation

$$x^{\Delta}(t) = -2x^{\sigma}(t) + \varepsilon_1\big[x^2(t - (t/2))\big]^{\Delta} + \varepsilon_2\big(x^2(t) + x^2(t - (t/2))\big). \tag{4.8}$$

The function g is given by $g(t) = \frac{t}{2}$ and satisfies $g : \mathbb{T} \to \mathbb{R}$ and $id - g : \mathbb{T} \to \mathbb{T}$ is strictly increasing.

Suppose

$$0 < 4(2\varepsilon_1 + \varepsilon_2)\delta(1 + \varepsilon_1\delta) < 1. \tag{4.9}$$

Let S be defined by

$$S = \Big\{\varphi : \mathbb{T} \to \mathbb{R} |\ \varphi(t) = \psi(t) \text{ if } t \le 0, \varphi(t) \to 0 \text{ as } t \to \infty, \varphi \in C \text{ and } \|\varphi\| \le K\Big\},$$

for some positive constant K satisfying the inequality

$$\frac{1 - \sqrt{1 - 4(2\varepsilon_1 + \varepsilon_2)\delta(1 + \varepsilon_1\ \delta)}}{2(2\varepsilon_1 + \varepsilon_2)} < K < \frac{1}{2(2\varepsilon_1 + \varepsilon_2)}. \tag{4.10}$$

Then every solution $x(t, 0, \psi)$ of (4.8) is bounded and goes to 0 as $t \to \infty$.

Proof. It is clear from (4.9) that inequality (4.10) is well defined. Define

$$\big(P\varphi\big)(t) = \psi(t),\ t \le 0,$$

and for $t \ge 0$,

$$\begin{aligned}\big(P\varphi\big)(t) &= \varepsilon_1\varphi^2(t/2) + \Big(\psi(0) - \varepsilon_1\psi^2(0)\Big)e_{\ominus 2}(t, 0) \\ &\quad + \int_0^t \Big(-2\varepsilon_1\varphi^2\big(\sigma(u)/2\big) + \varepsilon_2\big(\varphi^2(u) + \varphi^2(u/2)\big)\Big)e_{\ominus 2}(t,u)\,\Delta u\,.\end{aligned}$$

Let $\varphi \in S$. Then $\|\varphi\| \le K$ and for $t \ge 0$,

$$\begin{aligned}|\big(P\varphi\big)(t)| &\le \varepsilon_1|\varphi(t/2)|^2 + \Big(|\psi(0)| + \varepsilon_1|\psi(t/2)|^2\Big)e_{\ominus 2}(t, 0) \\ &\quad + \int_0^t \Big(2\varepsilon_1\big|\varphi\big(\sigma(u)/2\big)\big|^2 + \varepsilon_2\big(|\varphi(u)|^2 + |\varphi(u/2)|^2\big)\Big)e_{\ominus 2}(t,u)\,\Delta u.\end{aligned}$$

Hence

$$\begin{aligned}\|P\varphi\| &\le \varepsilon_1 K^2 + (1 + \varepsilon_1\delta)\delta + 2(\varepsilon_1 + \varepsilon_2)K^2\int_0^t e_{\ominus 2}(t,u)\,\Delta u \\ &\le (2\varepsilon_1 + \varepsilon_2)K^2 + \delta(1 + \varepsilon_1\delta).\end{aligned} \tag{4.11}$$

From (4.11), we see that if,

$$(2\varepsilon_1 + \varepsilon_2)K^2 + \delta(1 + \varepsilon_1\, \delta) \le K, \tag{4.12}$$

then $P : S \to S$. But inequality (4.12) is satisfied for K satisfying (4.10), by noting that

$$\frac{1}{2(2\varepsilon_1 + \varepsilon_2)} < \frac{1 + \sqrt{1 - 4(2\varepsilon_1 + \varepsilon_2)\delta(1 + \varepsilon_1\, \delta)}}{2(2\varepsilon_1 + \varepsilon_2)}.$$

Thus, if $\varphi \in S$, then $\|P\varphi\| \le K$. It is obvious that conditions (4.3), (4.5) and (4.6) are satisfied.

We now show that P defines a contraction mapping on the metric space S. Let $\zeta, \eta \in S$. Then

$$\begin{aligned} \Big|(P\zeta)(t) - (P\eta)(t)\Big| &\le 2\varepsilon_1 K\|\zeta - \eta\| + \Big(4\varepsilon_1 K + 4\varepsilon_2 K\Big) \int_0^t e_{\ominus 2}(t, u)\, \Delta u\, \|\zeta - \eta\| \\ &\le \Big[2\varepsilon_1 K + 2K(\varepsilon_1 + \varepsilon_2)\Big]\|\zeta - \eta\| \\ &= 2(2\varepsilon_1 + \varepsilon_2)K\|\zeta - \eta\|. \end{aligned}$$

By condition (4.10), we have

$$\Big\|P\zeta - P\eta\Big\| \le \alpha\|\zeta - \eta\|,$$

for some $\alpha \in (0, 1)$. By Theorem 4.1, every solution $x(\cdot, 0, \psi)$ of (4.8) with small continuous initial function $\psi : -2^{\mathbb{Z}} \cup \{0\} \to \mathbb{R}$ is in S, is bounded and goes to zero as $t \to \infty$. $\square$

References

[1] F. Merdivenci Atici, G. Sh. Guseinov, and B. Kaymakcalan, Stability criteria for dynamic equations on time scales with periodic coefficients, *Proceedings of the International Confernce on Dynamic Systems and Applications, III*, **3** (1999), 43-48.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An introduction with Applications*, Birkhäuser, Boston, 2001.

[3] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[4] E. R. Kaufmann and Y. N. Raffoul, Stability in neutral nonlinear dynamic equations on a time scale with functional delay, in the press.

[5] E. R. Kaufmann and Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, *J. Math. Anal. Appl.*, 319 (2006), no. 1, 315–325.

[6] Y. N. Raffoul, Periodic solutions for neutral nonlinear differential equations with functional delay, *Electron. J. Differential Equations*, 102 (2003) 1–7.

[7] Y. N. Raffoul, Stability in neutral nonlinear differential equations with functional delays using fixed point Theory, *Math. Comput. Modelling*, 40 (2004), no. 7-8, 691–700.

[8] D. R. Smart, *Fixed points theorems*, Cambridge Univ. Press, Cambridge, 1980.

Eric R. Kaufmann
Department of Mathematics & Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA
E-mail address: erkaufmann@ualr.edu

Youssef N. Raffoul
Department of Mathematics, University of Dayton, Dayton, OH 45469-2316, USA
E-mail address: youssef.raffoul@notes.udayton.edu