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A DOUBLE EIGENVALUE PROBLEM FOR SCHRÖDINGER EQUATIONS INVOLVING SUBLINEAR NONLINEARITIES AT INFINITY

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ABSTRACT. We present some multiplicity results concerning parameterized Schrödinger type equations which involve nonlinearities with sublinear growth at infinity. Some stability properties of solutions with respect to the parameters are also established in an appropriate Sobolev space.

1. INTRODUCTION

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function, $V : \mathbb{R}^N \to \mathbb{R}$ be a positive potential, and consider the problem

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N). \tag{1.1}$$

The study of this problem is motivated by mathematical physics; it is well-known that certain kinds of solitary waves in nonlinear Klein-Gordon or Schrödinger equations are solutions of (1.1), see Rabinowitz [16], Strauss [21]. Due to its importance, many papers are concerned with the existence and multiplicity of solutions of (1.1); without seek of completeness, we refer the reader to the works of Bartsch-Wang [4, 5], Strauss [21], Willem [23]; for a non-smooth approach, where f is allowed to be discontinuous, see Gazzola-Rădulescu [9]. The aforementioned papers have two common features: the nonlinearity $s \mapsto f(x, s)$ is *subcritical* and *superlinear at infinity.* To be more precise, most of the authors use the well-known Ambrosetti-Rabinowitz type condition:

(AR) There is a $\eta > 1$ such that

$$0 < (\eta + 1)F(x, s) \le f(x, s)s \quad \text{for each } x \in \mathbb{R}^N,\ s \in \mathbb{R} \setminus \{0\},$$

where $F(x, s) = \int_0^s f(x, t)dt$.

As we know, this condition implies the *superlinearity* of the function $s \mapsto f(x, s)$ at infinity, i.e., there exist some numbers $C > 0, s_0 > 0$ such that

$$|f(x, s)| \ge C|s|^\eta \quad \text{for each } x \in \mathbb{R}^N,\ |s| \ge s_0.$$

The main objective of this paper is to consider (1.1) when $s \mapsto f(x, s)$ has a *sublinear* growth at infinity. More precisely, we assume that

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(F1) There exist $W \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $W \not\equiv 0$, and $q \in (0,1)$ such that
$$|f(x,s)| \le W(x)|s|^q \quad \text{for each } (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

In such a case the energy functional associated with (1.1) is coercive and bounded from below; thus the existence of at least *one* solution is always expected. However, one may happen that (1.1) has only the trivial solution, even if the nonlinearity fulfills (F1). Indeed, if we consider for instance $V = 1$, $f(x,s) = \lambda W(x)\sin^2 s$ with $W \not\equiv 0$ as above, and $0 < \lambda < (2\|W\|_{L^\infty})^{-1}$, then (1.1) has only the trivial solution. Thus, this fact motivates the study of an *eigenvalue problem* rather than problem (1.1). On account of this statement, we shall investigate the following general double eigenvalue problem
$$-\Delta u + V(x)u = \lambda(f(x,u) + \mu g(x,u)), \quad x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N), \tag{1.2}$$
where the functions $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions verifying the above sublinearity hypothesis, while $\lambda, \mu \in \mathbb{R}$ are some parameters. Briefly speaking - under further assumptions on V, f and g, which will be specified later - our main results can be formulated as follows:

> If we fix $\mu \in \mathbb{R}$ (resp. $\lambda \in \mathbb{R}$) of certain range, we guarantee the existence of a non-degenerate interval $\Lambda_\mu \subset \mathbb{R}$ (resp. $\Pi_\lambda \subset \mathbb{R}$) such that (1.2) has at least two nontrivial, weak solutions whenever $\lambda \in \Lambda_\mu$ (resp. $\mu \in \Pi_\lambda$).

Before stating our results precisely, we mention that the potential $V : \mathbb{R}^N \to \mathbb{R}$ also has an important role concerning the existence and behaviour of solutions of (1.2). When $V(x)$ is a positive constant, or V is radially symmetric, it is natural to look for radially symmetric solutions of (1.2), see e.g., [11, 21, 23]. Motivated by the work of Rabinowitz [16] (where $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V > 0$, and $V(x) \to +\infty$ as $|x| \to +\infty$), Bartsch-Wang [5] considered a new class of potentials, namely:

(BW) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{\mathbb{R}^N} V > 0$, and for any $M > 0$,
$$\mu(\{x \in \mathbb{R}^N : V(x) \le M\}) < +\infty,$$
where μ denotes the Lebesgue measure in $\mathbb{R}^N$.

Under (BW), Bartsch-Wang [5] proved the existence of infinitely many solutions of (1.2) (for any fixed $\lambda > 0$) when f is subcritical, odd and verifies (AR). Furtado-Maia-Silva [8] studied (1.2) in the case when F (defined in (AR)) has some sort of resonance with a local nonquadraticity condition at infinity, while the potential V verifies (BW). Gazzola-Rădulescu [9] studied (1.2) when V verifies (BW), f is not necessarily continuous and satisfies an appropriate non-smooth (AR) condition.

For the rest of this paper, we will assume that the potential V satisfies:

(V1) $V \in L^\infty_{loc}(\mathbb{R}^N)$, $\quad \operatorname{essinf}_{\mathbb{R}^N} V > 0$; and

(V2) One of the following conditions is satisfied:
- (V2A) $N \ge 2$ and $V : \mathbb{R}^N \to \mathbb{R}$ is radially symmetric;
- (V2B) For any $M > 0$ and any $r > 0$ there holds:
$$\mu(\{x \in B(y,r) : V(x) \le M\}) \to 0 \quad \text{as } |y| \to +\infty,$$
where $B(y,r)$ denotes the open ball in $\mathbb{R}^N$ with center y and radius $r > 0$.

Note that hypotheses (V1)–(V2B) are weaker conditions than (BW), see Bartsch-Pankov-Wang [3]. Requiring (V1)–(V2B), Bartsch-Liu-Weth [2] proved recently the existence of three solutions of (1.2) for any fixed $\lambda > 0$, with f subcritical and verifying (AR).

Problems involving only sublinear terms (in certain papers this type of problem is referred to one involving *pure concave nonlinearities*) have been also studied by several authors, see for instance Yu [24], Chabrowski-doÓ [7]. Recently, Ricceri [18], [19] established a new variational principle, ensuring the existence of at least *three* distinct critical points of a coercive functional acting on a reflexive Banach space. By means of Ricceri's principle, several nonlinear elliptic eigenvalue problems have been treated (involving only concave nonlinearities), see for instance Ricceri [19], [20], Motreanu-Marano [12] (on bounded domains); Kristály [10] (on strip-like domains). By using a version of the Mountain Pass theorem, Marano-Motreanu [13] established a counterpart of the main result of [18]. Other multiplicity results for elliptic eigenvalue problems involving pure concave nonlinearities on bounded domains can be found in Perera [15] (*three* or *four* nontrivial solutions), as well as in Ambrosetti-Badiale [1] and Wang [22] where *infinitely many* solutions are obtained, assuming the oddness of the involved nonlinearities.

In the next section we will give the precise form of our main results; Section 3 contains some auxiliary result while in Sections 4 and 5 we prove our results.

2. Main results and remarks

In view of (V1), we introduce the Hilbert space

$$E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < +\infty\}$$

which will be endowed with the inner product

$$(u, v)_E = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv) \quad u, v \in E,$$

and with the induced norm $\|\cdot\|_E$. Note that solutions of (1.2) are being sought in E which can be continuously embedded into $L^p(\mathbb{R}^N)$ whenever $2 \le p < 2^*$. Here, 2^* denotes the critical Sobolev exponent, i.e., $2^* = 2N/(N-2)$ for $N \ge 3$ and $2^* = +\infty$ for $N = 1, 2$.

Let $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be two continuous functions and let $F(x, s) = \int_0^s f(x, t)dt$ and $G(x, s) = \int_0^s g(x, t)dt$, respectively. We assume:

(FG1) There exist $W \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $W \not\equiv 0$, and $q \in (0, 1)$ such that
$$\max\{|f(x, s)|, |g(x, s)|\} \le W(x)|s|^q \quad \text{for each } (x, s) \in \mathbb{R}^N \times \mathbb{R}.$$

(FG2) $\lim_{s\to 0} \frac{f(x,s)}{s} = \lim_{s\to 0} \frac{g(x,s)}{s} = 0$ uniformly for each $x \in \mathbb{R}^N$.

(FG3) $f(\cdot, s)$ and $g(\cdot, s)$ are radially symmetric functions for each $s \in \mathbb{R}$.

(F4) There exist $R_0 > 0$ and $s_0 \in \mathbb{R}$ such that $\min_{|x|\le R_0} F(x, s_0) > 0$.

We introduce the functions $\mathcal{F}, \mathcal{G} : E \to \mathbb{R}$, defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(x, u(x))dx, \quad \mathcal{G}(u) = \int_{\mathbb{R}^N} G(x, u(x))dx.$$

A standard argument, which is based on the facts that $W \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, f, g satisfy (FG1), and the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous $(2 \le p < 2^*)$, show that the functionals $\mathcal{F},\mathcal{G}:E\to\mathbb{R}$ are well defined, are of class $\mathcal{C}^1$, and

$$\mathcal{F}'(u)(v)=\int_{\mathbb{R}^N}f(x,u(x))v(x)dx,\quad \mathcal{G}'(u)(v)=\int_{\mathbb{R}^N}g(x,u(x))v(x)dx \tag{2.1}$$

for each $u,v\in E$ (see, for instance [23, Lemma 3.10]). Finally, let $\mathcal{H}_{\lambda,\mu}:E\to\mathbb{R}$ be the energy functional associated to (1.2), defined by

$$\mathcal{H}_{\lambda,\mu}(u)=\frac{1}{2}\|u\|_E^2-\lambda\mathcal{F}(u)-\lambda\mu\mathcal{G}(u).$$

Our results can be stated as follows.

Theorem 2.1. *Let $f,g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ be two continuous functions and $V:\mathbb{R}^N\to\mathbb{R}$ be a potential which satisfy* (FG1)–(FG2), (F4)*, and* (V1)–(V2)*, respectively. Assume moreover that* (FG3) *is verified whenever* (V2A) *holds. Then, there exists $\mu_0>0$ such that to every $\mu\in[-\mu_0,\mu_0]$ it corresponds a nonempty open interval $\Lambda_\mu\subset(0,\infty)$ and a number $\sigma_\mu>0$ for which* (1.2) *has at least two distinct, nontrivial weak solutions $v_{\lambda\mu}$ and $w_{\lambda\mu}$ with the property that*

$$\max\{\|v_{\lambda\mu}\|_E,\|w_{\lambda\mu}\|_E\}\le\sigma_\mu$$

whenever $\lambda\in\Lambda_\mu$. Moreover, $v_{\lambda\mu}$ and $w_{\lambda\mu}$ are radially symmetric whenever (V2A) *holds.*

From the point of view of the eigenvalues, the counterpart of Theorem 2.1 is the following:

Theorem 2.2. *Under the assumptions of Theorem 2.1, there exists $\lambda_0>0$ such that to every $\lambda\in(\lambda_0,\infty)$ it corresponds a nonempty open interval $\Pi_\lambda\subset\mathbb{R}$ and a number $\sigma_\lambda>0$ for which* (1.2) *has at least two distinct, nontrivial weak solutions $v_{\lambda\mu}$ and $w_{\lambda\mu}$ with $\mathcal{H}_{\lambda,\mu}(v_{\lambda\mu})<0<\mathcal{H}_{\lambda,\mu}(w_{\lambda\mu})$ and $\max\{\|v_{\lambda\mu}\|_E,\|w_{\lambda\mu}\|_E\}\le\sigma_\lambda$ whenever $\mu\in\Pi_\lambda$. Moreover, $v_{\lambda\mu}$ and $w_{\lambda\mu}$ are radially symmetric whenever* (V2A) *holds.*

Although the two theorems above are completely independent, as a simple by-product of Theorem 2.2 we obtain the following result whose conclusion partially goes back to Theorem 2.1.

Theorem 2.3. *Under the assumptions of Theorem 2.1, there exists $\overline{\mu}>0$ such that for every $\mu\in[-\overline{\mu},\overline{\mu}]$ the set*

$$\{\lambda>0:(1.2)\ \textit{has at least two distinct, nontrivial weak solutions}\}$$

contains an interval.

Our approach is variational. In the proof of Theorem 2.1 we use a recent abstract critical point result of Ricceri [18] while the proof of Theorem 2.2 is based on the well-known Mountain Pass theorem. We mention that Theorems 2.1-2.3 extend [18, Theorem 1.1] and [13, Theorem 1] to the whole $\mathbb{R}^N$, respectively. Now, we will conclude this section with some remarks.

Remark 2.4. (a) If we omit (F4) in the above results, then we may assume $f=g=0$, thus (1.2) has only the trivial solution.
(b) The above theorems do not work in general for every $\mu\in\mathbb{R}$. Indeed, if one consider a function f which verifies the hypotheses of our results, and we set $g=-f$, then for $\mu=1$, the problem (1.2) has only the trivial solution.

(c) The above theorems do not work in general for every $\lambda \in \mathbb{R}$ (see also the Introduction). To see this, consider the nonlinearities $f(x,s) = W(x)h(s)$ and $g(x,s) = W(x)k(s)$ which fulfill the hypotheses of the above theorems where $W : \mathbb{R}^N \to \mathbb{R}$ is from (FG1), while h and k are Lipschitz continuous functions with Lipschitz constants $L_h > 0$ and $L_k > 0$, respectively. For instance, one can consider $W(x) = (1+|x|^\alpha)^{-\beta}$ with $\alpha, \beta > 0$ such that $\alpha\beta > N \geq 3$, and $h(s) = \sin^2 s$, $k(s) = \arctan^2 s$. Fix $\mu \in \mathbb{R}$ arbitrarily. Then, if $0 < \lambda < (\|W\|_{L^\infty})^{-1}(L_h + |\mu| L_k)^{-1}$, the problem (1.2) has only the zero solution. Indeed, let us observe the critical points of $\mathcal{H}_{\lambda,\mu}$ are precisely the fixed points of the operator $A_{\lambda,\mu} = \lambda(\mathcal{F}' + \mu\mathcal{G}')$. Since $A_{\lambda,\mu}$ is a contraction for λ's specified above, then $A_{\lambda,\mu}$ has a unique fixed point. Since $A_{\lambda,\mu}(0) = 0$ (due to (FG1)), then 0 will be the unique solution of (1.2).

For the rest of this article, we suppose that assumptions of Theorem 2.1 are fulfilled.

3. Auxiliary results

Lemma 3.1. *Let $p \in (2, 2^*)$. For each $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that*
- (i) $\max\{|f(x,s)|, |g(x,s)|\} \leq \varepsilon|s| + c(\varepsilon)|s|^{p-1}$ *for every* $(x,s) \in \mathbb{R}^N \times \mathbb{R}$;
- (ii) $\max\{|F(x,s)|, |G(x,s)|\} \leq \varepsilon s^2 + c(\varepsilon)|s|^p$ *for every* $(x,s) \in \mathbb{R}^N \times \mathbb{R}$.

Proof. Taking into account (FG1), (FG2) and the facts that $W \in L^\infty(\mathbb{R}^N)$, and $q + 1 < 2 < p$, the claims easily follow. $\square$

Due to this result, one can show in a standard way (see for instance [23, Lemma 3.10]) that the energy functional $\mathcal{H}_{\lambda,\mu} : E \to \mathbb{R}$ is well-defined and of class $\mathcal{C}^1$. Moreover,

$$\mathcal{H}'_{\lambda,\mu}(u)(v) = (u,v)_E - \lambda \int_{\mathbb{R}^N} f(x,u)v - \lambda\mu \int_{\mathbb{R}^N} g(x,u)v, \quad \forall\, u, v \in E,$$

thus the critical points of $\mathcal{H}_{\lambda,\mu}$ are exactly the weak solutions of (1.2).

The embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is not compact for any $p \in [2, 2^*]$. However, when (V1)-(V2B) hold, the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact when $2 \leq p < 2^*$, cf. Bartsch-Pankov-Wang [3]. On the other hand, when (V1)-(V2A) hold, in general, the space E cannot be compactly embedded into $L^p(\mathbb{R}^N)$. In this case, introducing the subspace of radially symmetric functions of E, i.e.

$$E_r = \{u \in E : u(gx) = u(x) \quad \text{for each} \quad g \in O(N), \quad \text{a.e. } x \in \mathbb{R}^N\},$$

the embedding $E_r \hookrightarrow L^p(\mathbb{R}^N)$ is compact whenever $N \geq 2$ and $2 < p < 2^*$, cf. Strauss [21]. Taking into account (FG3), the functional $\mathcal{H}_{\lambda,\mu}$ is $O(N)$-invariant, thus the critical points of the functional $\mathcal{H}_{\lambda,\mu}$ restricted to the space E_r, i.e. $\mathcal{H}^r_{\lambda,\mu} = \mathcal{H}_{\lambda,\mu}|_{E_r}$, are critical points of $\mathcal{H}_{\lambda,\mu}$, cf. Palais [14]. In order to consider simultaneously the two cases, we introduce some new notations. Let

$$X = \begin{cases} E_r, & \text{if (V2A) holds,} \\ E, & \text{if (V2B) holds,} \end{cases} \quad \text{and} \quad \mathcal{E}_{\lambda,\mu} = \begin{cases} \mathcal{H}^r_{\lambda,\mu}, & \text{if (V2A) holds,} \\ \mathcal{H}_{\lambda,\mu}, & \text{if (V2B) holds.} \end{cases}$$

On account of these notation, it is sufficient to find critical points of $\mathcal{E}_{\lambda,\mu}$ on X. We further denote by $(\cdot,\cdot)_X$, $\|\cdot\|_X$, $\mathcal{F}_X$, and $\mathcal{G}_X$, the restriction of $(\cdot,\cdot)_E$, $\|\cdot\|_E$, $\mathcal{F}$, and $\mathcal{G}$ to the space X, respectively.

In the sequel, we denote by $S_p > 0$ the best Sobolev embedding constant of $X \hookrightarrow L^p(\mathbb{R}^N)$, $p \in [2, 2^*)$. The usual norm on $L^p(\mathbb{R}^N)$ is $\|\cdot\|_p$, $p \in [1, \infty]$.

Lemma 3.2. *There exists $u_0 \in X$ such that $\mathcal{F}_X(u_0) > 0$.*

Proof. Let $R_0 > 0$ and $s_0 \in \mathbb{R}$ from (F4) and fix $\varepsilon \in (0, R_0/2)$. Fix also a radially symmetric function $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $u_\varepsilon(x) = 0$ for $|x| \geq R_0$, $u_\varepsilon(x) = s_0$ for $|x| \leq R_0 - \varepsilon$, and $\|u_\varepsilon\|_\infty \leq |s_0|$. Due to (V1), u_ε belongs to X in both cases, i.e. either (V2A) or (V2B) hold. Denoting by $A_0 = \min_{|x|\leq R_0} F(x, s_0) > 0$ and by $\mathrm{Vol}(B_r)$ the volume of the ball $B(0, r)$, we have

$$\begin{aligned}\mathcal{F}_X(u_\varepsilon) &= \int_{|x|\leq R_0-\varepsilon} F(x, u_\varepsilon(x)) + \int_{|x|\geq R_0} F(x, u_\varepsilon(x)) + \int_{R_0-\varepsilon<|x|<R_0} F(x, u_\varepsilon(x)) \\ &\geq A_0 \,\mathrm{Vol}(B_{R_0/2}) - \int_{R_0-\varepsilon<|x|<R_0} |F(x, u_\varepsilon(x))| \\ &\geq A_0 \,\mathrm{Vol}(B_{R_0/2}) - \max_{|x|\in[R_0/2,R_0]\,,|s|\leq|s_0|} |F(x, s)|(\mathrm{Vol}(B_{R_0}) - \mathrm{Vol}(B_{R_0-\varepsilon})).\end{aligned}$$

If $\varepsilon \to 0^+$, the last term of the above expression becomes as small as we want. Thus, setting $\varepsilon = \varepsilon_0$ small enough, $u_0 = u_{\varepsilon_0} \in X$ verifies the requirement. □

Lemma 3.3. *Let $\lambda, \mu \in \mathbb{R}$ be arbitrary fixed. Then every bounded sequence $\{u_n\} \subset X$ such that $\|\mathcal{E}'_{\lambda,\mu}(u_n)\|_{X^*} \to 0$, contains a strongly convergent subsequence.*

Proof. Taking a subsequence if necessary, we may assume that $\{u_n\}$ converges to u, weakly in X, and strongly in $L^p(\mathbb{R}^N)$ for some $p \in (2, 2^*)$ arbitrarily fixed. Therefore,

$$\begin{aligned}\|u - u_n\|_X^2 &= (u, u - u_n)_X + \mathcal{E}'_{\lambda,\mu}(u_n)(u_n - u) \\ &\quad - \lambda \int_{\mathbb{R}^N} f(x, u_n)(u - u_n) - \lambda\mu \int_{\mathbb{R}^N} g(x, u_n)(u - u_n) \\ &\leq (u, u - u_n)_X + \|\mathcal{E}'_{\lambda,\mu}(u_n)\|_{X^*}\|u_n - u\|_X \\ &\quad + |\lambda|(1 + |\mu|)\|W\|_{p/(p-q-1)}\|u_n\|_p^q\|u - u_n\|_p.\end{aligned}$$

Thus, $\|u - u_n\|_X \to 0$ as $n \to \infty$. □

Lemma 3.4. *For every $\lambda, \mu \in \mathbb{R}$, $\mathcal{E}_{\lambda,\mu}$ is coercive and bounded from below on X. In particular, $\mathcal{E}_{\lambda,\mu}$ satisfies the Palais-Smale condition.*

Proof. By (FG1) we have for every $u \in X$ that

$$\mathcal{E}_{\lambda,\mu}(u) \geq \frac{1}{2}\|u\|_X^2 - |\lambda|(1 + |\mu|)\|W\|_{2/(1-q)} S_2^{q+1}\|u\|_X^{q+1}. \tag{3.1}$$

Since $q < 1$, the first assertion holds. Now, consider a sequence $\{u_n\} \subset X$ such that $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded and $\|\mathcal{E}'_{\lambda,\mu}(u_n)\|_{X^*} \to 0$ as $n \to \infty$. In particular, $\{u_n\} \subset X$ is bounded. On account of Lemma 3.3, $\mathcal{E}_{\lambda,\mu}$ satisfies the Palais-Smale condition. □

4. Proof of Theorem 2.1

Lemma 4.1. *For every $\mu \in \mathbb{R}$,*

$$\lim_{\rho\to 0^+} \frac{\sup\{\mathcal{F}_X(u) + \mu\mathcal{G}_X(u) : \|u\|_X < \sqrt{2\rho}\}}{\rho} = 0.$$

Proof. Fix arbitrarily $\varepsilon > 0$ and $p \in (2, 2^*)$. Due to Lemma 3.1, for every $u \in X$, one has

$$\mathcal{F}_X(u) + \mu\mathcal{G}_X(u) \leq (1 + |\mu|)(\varepsilon S_2^2\|u\|_X^2 + c(\varepsilon)S_p^p\|u\|_X^p).$$

Therefore, for every $\rho > 0$,

$$0 \le \frac{\sup\{\mathcal{F}_X(u) + \mu\mathcal{G}_X(u) : \|u\|_X < \sqrt{2\rho}\}}{\rho} \le (1 + |\mu|)(2\varepsilon S_2^2 + c(\varepsilon)2^{\frac{p}{2}} S_p^p \rho^{\frac{p}{2}-1}).$$

When $\rho \to 0^+$, the right hand side of the above inequality tends to 0, due to the arbitrariness of $\varepsilon > 0$, which concludes the proof. □

Lemma 4.2. *For every $\lambda, \mu \in \mathbb{R}$, the functional $\mathcal{E}_{\lambda,\mu}$ is sequentially weakly lower semicontinuous.*

Proof. Since the function $u \mapsto \|u\|_X^2$ is sequentially weakly lower semicontinuous, it is enough to prove that $\mathcal{F}_X + \mu\mathcal{G}_X$ is sequentially weakly continuous for every $\mu \in \mathbb{R}$. Let us assume the contrary, i.e., let $\{u_n\}$ be a sequence in X which converges weakly to $u \in X$ and the sequence $\{(\mathcal{F}_X + \mu\mathcal{G}_X)(u_n)\}$ does not converge to $(\mathcal{F}_X + \mu\mathcal{G}_X)(u)$ as $n \to \infty$. Therefore, there exist $\varepsilon_0 > 0$ and a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that

$$0 < \varepsilon_0 \le |(\mathcal{F}_X + \mu\mathcal{G}_X)(u_n) - (\mathcal{F}_X + \mu\mathcal{G}_X)(u)|$$

for every $n \in \mathbb{N}$, and $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$ for some $p \in (2, 2^*)$. (Note that the embedding $X \hookrightarrow L^p(\mathbb{R}^N)$ is compact.) By the mean value theorem, (2.1) and (FG1), for some $\theta_n \in (0, 1)$, the above inequality gives

$$\begin{aligned} 0 < \varepsilon_0 &\le |(\mathcal{F}_X' + +\mu\mathcal{G}_X')(u + \theta_n(u_n - u))(u_n - u)| \\ &\le (1 + |\mu|) \int_{\mathbb{R}^N} W(x)|u + \theta_n(u_n - u)|^q |u_n - u| \\ &\le (1 + |\mu|)\|W\|_{p/(p-q-1)}(\|u\|_p + \|u_n - u\|_p)^q \|u_n - u\|_p. \end{aligned}$$

The last term tends to 0, which is a contradiction. □

The main ingredient in the proof of Theorem 2.1 is a Ricceri-type critical point theorem, see Ricceri [18, 19]. Here, we recall a refinement of this result, established by Bonanno [6].

Theorem 4.3. *Let Y be a separable and reflexive real Banach space, and let $\Phi, J : Y \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in Y$ such that $\Phi(x_0) = J(x_0) = 0$ and $\Phi(x) \ge 0$ for every $x \in Y$ and that there exists $x_1 \in Y$, $\rho > 0$ such that*

(i) $\rho < \Phi(x_1)$ *and* $\sup_{\Phi(x)<\rho} J(x) < \rho\frac{J(x_1)}{\Phi(x_1)}$. *Further, put*

$$\overline{a} = \frac{\zeta\rho}{\rho\frac{J(x_1)}{\Phi(x_1)} - \sup_{\Phi(x)<\rho} J(x)},$$

with $\zeta > 1$, assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition; and

(ii) $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$, *for every $\lambda \in [0, \overline{a}]$.*

Then there is an open interval $\Lambda \subset [0, \overline{a}]$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda$, the equation $\Phi'(x) - \lambda J'(x) = 0$ admits at least three distinct solutions in Y having norm less than σ.

Proof of Theorem 2.1. Let $u_0 \in X$ the element from Lemma 3.2, and fix

$$\mu_0 = \frac{\mathcal{F}_X(u_0)}{|\mathcal{G}_X(u_0)| + 1}.$$

Now, we apply Theorem 4.3 by choosing $Y = X$, $\Phi = \frac{1}{2}\|\cdot\|_X^2$, and $J = J_\mu = \mathcal{F}_X + \mu\mathcal{G}_X$ for $\mu \in [-\mu_0, \mu_0]$. Note that $\mathcal{E}_{\lambda,\mu} = \Phi - \lambda J_\mu$.

A simple calculation shows that for every $\mu \in [-\mu_0, \mu_0]$ we have

$$J_\mu(u_0) = \mathcal{F}_X(u_0) + \mu\mathcal{G}_X(u_0) \geq \mu_0 > 0. \tag{4.1}$$

On account of the above inequality and Lemma 4.1, for every $\mu \in [-\mu_0, \mu_0]$ one can choose $\rho_\mu > 0$ so small that

$$\rho_\mu < \min\{1, \frac{\|u_0\|_X^2}{2}\}; \tag{4.2}$$

$$\frac{\sup\{J_\mu(u) : \ \|u\|_X < \sqrt{2\rho_\mu}\}}{\rho_\mu} < \frac{J_\mu(u_0)}{\|u_0\|_X^2}. \tag{4.3}$$

Now, choosing $x_1 = u_0$, $x_0 = 0$, $\zeta = 1 + \rho_\mu$ and

$$\overline{a} = \overline{a}_\mu = \frac{1 + \rho_\mu}{2J_\mu(u_0)\|u_0\|_X^{-2} - \sup\{J_\mu(u) : \ \|u\|_X < \sqrt{2\rho_\mu}\}\rho_\mu^{-1}},$$

all the assumptions of Theorem 4.3 are verified, cf. Lemmas 4.2 and 3.4, respectively.

Then there is an open interval $\Lambda_\mu \subset [0, \overline{a}_\mu]$ and a number $\sigma_\mu > 0$ such that for any $\lambda \in \Lambda_\mu$, the functional $\mathcal{E}_{\lambda,\mu} = \Phi - \lambda J_\mu$ admits at least three distinct critical points $u^i_{\lambda,\mu} \in X$ ($i \in \{1, 2, 3\}$), having norm less than σ_μ, concluding the proof of Theorem 2.1. □

Remark 4.4. On account of (4.2), (4.3) and (4.1), for every $\mu \in [-\mu_0, \mu_0]$ one has

$$\overline{a}_\mu < \frac{2\|u_0\|_X^2}{J_\mu(u_0)} \leq \frac{2\|u_0\|_X^2}{\mu_0} = \frac{2\|u_0\|_X^2}{\mathcal{F}_X(u_0)}(1 + |\mathcal{G}_X(u_0)|).$$

Since the right hand side does not depend on $\mu \in \mathbb{R}$, we have a uniform estimation of Λ_μ, i.e., for every $\mu \in [-\mu_0, \mu_0]$,

$$\Lambda_\mu \subset [0, \frac{2\|u_0\|_X^2}{\mathcal{F}_X(u_0)}(1 + |\mathcal{G}_X(u_0)|)].$$

5. Proof of Theorems 2.2 and 2.3

Let us define

$$c_{\mathcal{G}} = \int_{\mathbb{R}^N} |G(x, u_0(x))|dx \quad \text{and} \quad \lambda_0 = \frac{\|u_0\|_X^2}{2\mathcal{F}_X(u_0)},$$

where $u_0 \in X$ is from Lemma 3.2. For every $\lambda > \lambda_0$ set finally

$$\mu^*_\lambda = \frac{1}{1 + c_{\mathcal{G}}}\big(1 - \frac{\lambda_0}{\lambda}\big)\mathcal{F}_X(u_0). \tag{5.1}$$

Lemma 5.1. *Let $\lambda > \lambda_0$ and $|\mu| < \mu^*_\lambda$. Then $\inf_{u\in X} \mathcal{E}_{\lambda,\mu}(u) < 0$.*

Proof. It is sufficient to show that $\mathcal{E}_{\lambda,\mu}(u_0) < 0$ whenever $\lambda > \lambda_0$ and $|\mu| < \mu^*_\lambda$. Due to the choice of λ_0 and μ^*_λ, one has

$$\begin{aligned}\mathcal{E}_{\lambda,\mu}(u_0) &= \frac{1}{2}\|u_0\|_X^2 - \lambda\mathcal{F}_X(u_0) - \lambda\mu\mathcal{G}_X(u_0)\\ &\leq (\lambda_0 - \lambda)\mathcal{F}_X(u_0) + \lambda|\mu|c_{\mathcal{G}}\\ &= -\lambda(1 + c_{\mathcal{G}})\mu^*_\lambda + \lambda|\mu|c_{\mathcal{G}} < 0.\end{aligned}$$

□

Lemma 5.2. *For every $\lambda > \lambda_0$ and $\mu \in \mathbb{R}$ which complies with $|\mu| < \mu_\lambda^*$, the functional $\mathcal{E}_{\lambda,\mu}$ has the Mountain Pass geometry.*

Proof. Fix $p \in (2, 2^*)$ arbitrary. By Lemma 3.1 one has for every $\varepsilon > 0$ that there exists $c(\varepsilon) > 0$ such that

$$\max\{|\mathcal{F}_X(u)|, |\mathcal{G}_X(u)|\} \leq \varepsilon S_2^2 \|u\|_X^2 + c(\varepsilon) S_p^p \|u\|_X^p \quad \text{for every } u \in X.$$

Thus, for every $u \in X$ one has

$$\begin{aligned}\mathcal{E}_{\lambda,\mu}(u) &\geq \frac{1}{2}\|u\|_X^2 - \lambda|\mathcal{F}_X(u)| - \lambda|\mu||\mathcal{G}_X(u)| \\ &\geq \Big(\frac{1}{2} - \lambda(1+|\mu|)\varepsilon S_2^2\Big)\|u\|_X^2 - \lambda(1+|\mu|)c(\varepsilon)S_p^p\|u\|_X^p.\end{aligned}$$

Let $\varepsilon = (4\lambda(1+|\mu|)S_2^2)^{-1}$. Then

$$\mathcal{E}_{\lambda,\mu}(u) \geq \Big(\frac{1}{4} - \lambda(1+|\mu|)c(\lambda,\mu)S_p^p\rho^{p-2}\Big)\rho^2 \equiv \eta(\lambda,\mu) > 0$$

if $\|u\|_X = \rho < \min\{(4\lambda(1+|\mu|)c(\lambda,\mu)S_p^p)^{\frac{1}{2-p}}, \|u_0\|_X\}$. Moreover, by construction, $\rho < \|u_0\|_X$ and by the proof of Lemma 5.1 we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$. Thus, $\mathcal{E}_{\lambda,\mu}$ has indeed the Mountain Pass geometry. □

Proof of Theorem 2.2. Fix $\lambda > \lambda_0$ and $\mu \in (-\mu_\lambda^*, \mu_\lambda^*) \equiv \Pi_\lambda$. Lemma 3.4 ensures in particular that there exists an element $v_{\lambda\mu} \in X$ such that $\mathcal{E}_{\lambda,\mu}(v_{\lambda\mu}) = \inf_{u\in X)} \mathcal{E}_{\lambda,\mu}(u)$. By using Lemma 5.1, $\mathcal{E}_{\lambda,\mu}(v_{\lambda\mu}) < 0$.

On the other hand, by Lemma 5.2, we find $w_{\lambda\mu} \in X$ such that $\mathcal{E}'_{\lambda,\mu}(w_{\lambda\mu}) = 0$ and $\mathcal{E}_{\lambda,\mu}(w_{\lambda\mu}) \geq \eta(\lambda,\mu) > 0$ (see for instance [17, Theorem 2.2]). The mountain pass level $\mathcal{E}_{\lambda,\mu}(w_{\lambda\mu})$ is characterized as

$$\mathcal{E}_{\lambda,\mu}(w_{\lambda\mu}) = \inf_{g\in\Gamma}\max_{t\in[0,1]} \mathcal{E}_{\lambda,\mu}(g(t)), \tag{5.2}$$

where $\Gamma = \{g \in C([0,1]; X) : g(0) = 0,\ g(1) = u_0\}$. Let $g_0 : [0,1] \to X$, defined by $g_0(t) = tu_0$. Since $g_0 \in \Gamma$, by using (5.2), one has for every $\mu \in \Pi_\lambda$

$$\begin{aligned}\mathcal{E}_{\lambda,\mu}(w_{\lambda\mu}) &\leq \max_{t\in[0,1]} \mathcal{E}_{\lambda,\mu}(tu_0) \\ &\leq \frac{1}{2}\|u_0\|_X^2 + \lambda \max_{t\in[0,1]} (|\mathcal{F}_X(tu_0)| + \mu_\lambda^*|\mathcal{G}_X(tu_0)|) \equiv C_\lambda.\end{aligned}$$

By using (3.1), for every $\mu \in \Pi_\lambda$ we have

$$\|w_{\lambda\mu}\|_X^2 \leq 2\lambda(1+\mu_\lambda^*)\|W\|_{2/(1-q)}S_2^{q+1}\|w_{\lambda\mu}\|_X^{q+1} + 2C_\lambda.$$

Since $q + 1 < 2$ we have at once that there exits a number $\sigma_\lambda^1 > 0$ such that $\|w_{\lambda\mu}\|_X \leq \sigma_\lambda^1$ for every $\mu \in \Pi_\lambda$. Moreover, since $\mathcal{E}_{\lambda,\mu}(v_{\lambda\mu}) < 0$ for every $\mu \in \Pi_\lambda$, a similar argument as above (put formally $C_\lambda = 0$) shows the existence of $\sigma_\lambda^2 > 0$ such that $\|v_{\lambda\mu}\|_X \leq \sigma_\lambda^2$ for every $\mu \in \Pi_\lambda$. Thus, letting $\sigma_\lambda = \max\{\sigma_\lambda^1, \sigma_\lambda^2\}$, the proof is completed. □

Remark 5.3. On account of (5.1), for every $\lambda > \lambda_0$ one has

$$\mu_\lambda^* < \frac{\mathcal{F}_X(u_0)}{1 + c_{\mathcal{G}}}.$$

Since the right hand side does not depend on $\lambda \in \mathbb{R}$, we have a uniform estimation of Π_λ, i.e., for every $\lambda > \lambda_0$,

$$\Pi_\lambda \subset \Big[-\frac{\mathcal{F}_X(u_0)}{1+c_{\mathcal{G}}}, \frac{\mathcal{F}_X(u_0)}{1+c_{\mathcal{G}}} \Big].$$

Proof of Theorem 2.3. Let us fix $\overline{\lambda} > \lambda_0$, $\gamma \in (0, \overline{\lambda} - \lambda_0)$ and define

$$\overline{\mu} = \mu^*_{\overline{\lambda}+\gamma} \frac{\overline{\lambda} - \lambda_0 - \gamma}{\overline{\lambda} - \lambda_0 + \gamma},$$

where $\mu^*_{(\cdot)}$ is defined in (5.1).

Now, fix $\mu \in \mathbb{R}$ such that $|\mu| \leq \overline{\mu}$. It is easy to verify that the inequality $\overline{\mu} < \mu^*_\lambda$ holds for every $\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma)$; thus, for every $\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma)$ one has $\mu \in (-\mu^*_\lambda, \mu^*_\lambda) = \Pi_\lambda$. By applying Theorem 2.2 for each $\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma)$, we conclude that (1.2) has at least two distinct, nontrivial, weak solutions. Consequently, the interval $(\overline{\lambda} - \gamma, \overline{\lambda} + \gamma)$ is included in the set defined in the statement of Theorem 2.3, which completes the proof. $\square$

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