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NONEXISTENCE OF GLOBAL SOLUTIONS OF EMDEN-FOWLER TYPE SEMILINEAR WAVE EQUATIONS WITH NON-POSITIVE ENERGY

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Abstract. In this article we study the blow-up phenomena of solutions to the Emden-Fowler type semilinear wave equation

$$t^2u_{tt} - u_{xx} = u^p \quad \text{in } [1, T) \times (a, b)).$$

1. Introduction

In this article we consider the nonexistence of global solutions in time of the Emden-Fowler type semilinear wave equation

$$t^2u_{tt} - u_{xx} = u^p \quad \text{in } [1, T) \times (r_1, r_2) \tag{1.1}$$

with boundary value null and initial values

$$u(0, x) = u_0(x), \quad u_0 \in H^2(r_1, r_2) \cap H_0^1(r_1, r_2),$$
$$\dot{u}(0, x) = u_1(x), \quad u_1 \in H_0^1(r_1, r_2)$$

where $p > 1$, r_1 and r_2 are real numbers. Through reviewing some properties of solutions of Emden-Fowler equations and the nonexistence of global solutions of some semi-linear wave equations with initial and boundary values problem in bounded domain solution we want to study blow-up phenomena of solutions to equation (1.1).

Review on the Emden-Fowler equation. The study of the Emden-Fowler equation originated from earlier theories concerning gaseous dynamics in astrophysics around the turn of the 20-th century. The fundamental problem in the study of stellar structure at that time was to study the equilibrium configuration of the mass of spherical clouds of gas. Under the assumption that the gaseous cloud is under convective equilibrium (first proposed in 1862 by Lord Kelvin [40]), Lane studied the equation

$$\frac{d}{dt}\Big(t^2\frac{du}{dt}\Big) + t^2u^p = 0, \tag{1.2}$$

for the cases $p = 1.5$ and 2.5. Equation (1.2) is commonly referred to as the Lane-Emden equation [8]. Astrophysicists were interested in the behavior of the solutions

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of (1.2) which satisfy the initial condition: $u(0) = 1$, $u'(0) = 0$. Special cases of (1.2), namely, when $p = 1$ the explicit solution to

$$\frac{d}{dt}\big(t^2\frac{du}{dt}\big) + t^2u = 0, \quad u(0) = 1,\ u'(0) = 0$$

is $u = \sin(t)/t$, and when $p = 5$, the explicit solution to

$$\frac{d}{dt}\big(t^2\frac{du}{dt}\big) + t^2u^5 = 0, \quad u(0) = 1,\ u'(0) = 0$$

is $u = 1/\sqrt{1 + t^2/3}$.

Many properties of solutions to the Lane-Emden equation were studied by Ritter [33] in a series of 18 papers published during 1878-1889. The publication of Emden's treatise Gaskugeln [12] marks the end of first epoch in the study of stellar configurations governed by (1.2). The mathematical foundation for the study of such an equation and also of the more general equation

$$\frac{d}{dt}\Big(t^\rho\frac{du}{dt}\Big) + t^\sigma u^\gamma = 0, \quad t \ge 0, \tag{1.3}$$

was made by Fowler [13, 14, 15, 16] in a series of four papers during 1914-1931. We refer the reader to a summary in Bellman's book [6, chap. VII]. The Emden-Fowler equation also arises in the study of gas dynamics and fluid mechanics; see, e.g., the survey article by Conti, Graffi and Sansone, the Italian contribution to the theory of nonlinear ordinary differential equations and to nonlinear mechanics during the years 1951-1961 [11]. There the solutions of physical interest are bounded non-oscillatory which possess a positive zero. The zero of such a solution corresponds to an equilibrium state in a fluid with spherical distribution of density and under mutual attraction of its particles. The Emden-Fowler equations also appear in the study of relativistic mechanics, nuclear physics and also in the study of chemically reacting systems. The Emden-Fowler equation (1.3) can be transformed into a first order nonlinear autonomous system, in fact, a quadratic system, and information concerning its solutions may be obtained from the associated quadratic systems through phase plane analysis. This approach was in fact first used by Emden in his analysis of the Lane-Emden equation (1.2). For more detailed discussions on this approach we refer to [9, 34]. Progress along Fowler's approach concerning the Emden-Fowler equation (1.3) may be found in [22, 35]. Similar analysis concerning the related Thomas-Fermi equation may be found in [31, 36]. The first serious study on the generalized Emden-Fowler equation

$$\frac{d^2u}{dt^2} + a(t)|u|^\gamma \operatorname{sgn} u = 0, \quad t \ge 0$$

was made by Atkinson [1, 2, 3, 4, 5]. For general reference, we mention the well known texts by [6, 10, 23].

Review positive solutions for the Emden-Fowler equation $t^2u'' = u^p$, $p > 1$. Consider the transformation $t = e^s$, $u(t) = v(s)$, then $v(0) = u_0$; $v_s(0) = u_1$, and the equation (1.2) can be transformed into the form

$$\begin{gathered} v_{ss}(s) - v_s(s) = v(s)^p, \quad p > 1, \\ v(0) = u_0, \quad v_s(0) = u_1. \end{gathered} \tag{1.4}$$

Thus, the existence of local solutions u for (1.2) in $(1, T)$ is equivalent to the existence of local solutions v for (1.4) in $(0, \ln T)$. In [30] we have estimated the life-span T^* of positive solutions u of (1.2) for three different cases.

(a) $u_1 = 0$, $u_0 > 0$: $T^* \le e^{k_1}$,

$$k_1 := s_0 + \frac{2(n+3)}{8-\epsilon}\frac{2}{n-1}v(s_0)^{\frac{1-p}{2}}, \quad \varepsilon \in (0,1).$$

(b) $u_1 > 0$, $u_0 > 0$:

(i) $E(0) \ge 0$, $T^* \le e^{k_2}$, $k_2 := \frac{2}{p-1}\sqrt{\frac{p+1}{2}}u_0^{\frac{1-p}{2}}$;

(ii) $E(0) < 0$, $T^* \le e^{k_3}$, $k_3 := \frac{2}{p-1}\frac{u_0}{u_1}$;

(c) $u_1 < 0$, $u_0 \in (0, (-u_1)^{1/p})$: $u(t) \le (u_0 - u_1 - u_0^p) + (u_1 + u_0^p)t - u_0^p \ln t$.

Some results on the semilinear wave equation $\Box u = u^p$ in $[0,T) \times \Omega$. We have treated the estimates for the life-span of positive solutions of the semilinear wave equation

$$\Box\, u = u^p \quad \text{in } [0,T) \times \Omega$$

with boundary value null and initial values $u(0,x) = u_0(x)$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_t(0,x) = u_1(x)$, $u_1 \in H_0^1(\Omega)$, where $p \in (1, n/n-2]$ and $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. We use the following notation:

$$\nabla := \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big), \quad Du := (u_t, \nabla u), \quad \Box := \frac{\partial^2}{\partial t^2} - \triangle,$$

$$a(t) := \int_\Omega u^2(t,x)dx, \quad E(t) := \int_\Omega (|Du|^2 - \frac{2}{p+1}u^{p+1})(t,x)dx.$$

For a Banach space X and $0 < T \le \infty$ we set $C^k(0,T,X)$ as the space of C^k functions from $[0,T) \to X$, and

$$H1 := C^1(0,T,H_0^1(\Omega)) \cap C^2(0,T,L^2(\Omega)).$$

Jörgens [18] published the first existence theorem for global solutions to the wave equation

$$\Box\, u + f(u) = 0 \quad \text{n } [0,T) \times \Omega, \tag{1.5}$$

for $\Omega = R^n$, $n = 3$ and $f(u) = g(u^2)u$, his result can be applied to the equation $\Box u + u^3 = 0$. Browder [6] generalized Jörgens's result to $n > 2$ For local Lipschitz f, Li [27] proved the non-existence of global solutions of the initial-boundary value problem of semilinear wave equation (1.5) in a bounded domain $\Omega \subset R^n$ under the assumptions

$$\bar{E}(0) = \|Du\|_2^2(0) + 2\int_\Omega f(u)(0,x)dx \le 0,$$

$$\eta f(\eta) - 2(1+2\alpha)\int_0^\eta f(r)dr \le \lambda_1 \alpha \eta^2 \quad \forall \eta \in \mathbb{R} \text{ with } \alpha > 0,$$

$$\lambda_1 := \sup\big\{\|u\|_2/\|\nabla u\|_2 : u \in H_0^1(\Omega)\big\}$$

and $a'(0) > 0$. There we have a rough estimate for the life-span

$$T \le \beta_2 := 2\big[1 - \big(1 - k_2 a(0)^{-\alpha}\big)^{1/2}\big]/(k_1 k_2),$$

with

$$k_1 := \alpha a(0)^{-\alpha-1}\sqrt{a'(0)^2 - 4\bar{E}(0)a(0)}, \quad k_2 := \big(-4\alpha^2\bar{E}(0)/k_1^2\big)^\alpha/(1+2\alpha).$$

For $n = 3$ and $f(u) = -u^3$, there exist global solutions of (1.5) for small initial data [25]; but if $E(0) < 0$ and $a'(0) > 0$. then the solutions are only local, i.e. $T < \infty$ [27]. John [19] showed the nonexistence of solutions of the initial-boundary value problem for the wave equation $\Box u = A|u|^p$, $A > 0$, $1 < p < 1 + \sqrt{2}$, $\Omega = \mathbb{R}^3$. This problem was considered by Glassey [17] in the two dimensional case $n = 2$. For $n > 3$ Sideris [39] showed the nonexistence of global solutions under the conditions $\|u_0\|_1 > 0$ and $\|u_1\|_1 > 0$. According to this result Strauss [37, p. 27] guessed that the solutions for the above mentioned wave equation are global for $p \geq p_0(n) = \lambda$ which is the positive root of the quadratic equation $(n-1)\lambda^2 - (n+1)\lambda - 2 = 0$ and $\Omega = \mathbb{R}^n$. For further information about blow up one can see [19, 27, 28, 37, 39, 32] and their references.

2. Preliminaries

Existence and uniqueness of a local solution. Under some transformations one can get the existence of solutions to the Emden-Fowler type semilinear wave equation (1.1) for suitable conditions [29]. Taking the transform $s = \ln t$, $u(t,x) = v(t,x)$, then $u_t = t^{-1}v_s$, $t^2 u_{tt} = -v_s + v_{ss}$, equation (1.1) can be transformed into

$$\begin{gathered} v_{ss} - v_{xx} = v_s + v^p \\ quad\text{in } [0, \ln T) \times (r_1, r_2), \\ v(x,0) = u_0(x), \quad v_s(x,0) = u_1(x) \end{gathered} \tag{2.1}$$

with zero boundary conditions. In this paper we focus on the nonexistence of global solutions u of (1.1). After some argumentations, we can obtain the Lemma 2.1. Let

$$\begin{gathered} v(s,x) = e^{s/2} w(s,x), \quad v_s = e^{s/2} w_s + \frac{1}{2} v, \\ v_{ss} = e^{s/2} w_{ss} + e^{s/2} w_s + \frac{1}{4} e^{s/2} w, \end{gathered}$$

then (2.1) can be rewritten as

$$w_{ss} - w_{xx} = \frac{1}{4} w + e^{(p-1)s/2} w^p. \tag{2.2}$$

Lemma 2.1. *Suppose that $w \in H1$ is a solution of the semilinear wave equation (2.2). Then for $s \geq 0$,*

$$\begin{aligned} &\frac{d}{ds} \int_{r_1}^{r_2} \Big(w_s^2 + w_x^2 - \frac{1}{4} w^2 - \frac{2}{p+1} e^{\frac{p-1}{2} s} w^{p+1} \Big)(s,x) dx \\ &= -\frac{p-1}{p+1} \int_{r_1}^{r_2} e^{\frac{p-1}{2} s} w^{p+1}(s,x) dx \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\int_{r_1}^{r_2} \Big(w_s^2 + w_x^2 - \frac{1}{4} w^2 - \frac{2}{p+1} e^{\frac{p-1}{2} s} w^{p+1} \Big)(s,x) dx \\ &= E_w(0) - \frac{p-1}{p+1} \int_0^s \int_{r_1}^{r_2} e^{\frac{p-1}{2} r} w^{p+1}(r,x)\, dx\, dr, \end{aligned} \tag{2.4}$$

where

$$E_w(0) = \int_{r_1}^{r_2} \Big(w_s^2 + w_x^2 - \frac{1}{4} w^2 - \frac{2}{p+1} w^{p+1} \Big)(0,x) dx$$

$$
\begin{aligned}
&= \int_{r_1}^{r_2} |Big(\left(u_1 - \frac{1}{2}u_0\right)^2 + (u_0')^2 - \frac{1}{4}u_0^2 - \frac{2}{p+1}u_0^{p+1}\Big)(x)dx \\
&= \int_{r_1}^{r_2} \Big(u_1^2 - u_0u_1 + (u_0')^2 - \frac{2}{p+1}u_0^{p+1}\Big)(x)dx.
\end{aligned}
$$

Proof. From(2.2) we can obtain

$$
\begin{aligned}
&\frac{d}{ds}\int_{r_1}^{r_2} \Big(w_s^2 + w_x^2 - \frac{1}{4}w^2 - \frac{2}{p+1}e^{\frac{p-1}{2}s}w^{p+1}\Big)(s,x)dx \\
&\quad + \frac{p-1}{p+1}\int_{r_1}^{r_2} e^{\frac{p-1}{2}s}w^{p+1}(s,x)dx \\
&= \int_{r_1}^{r_2} 2w_s\Big(w_{xx} + \frac{1}{4}w + e^{(p-1)s/2}w^p\Big)(s,x)dx \\
&\quad + \int_{r_1}^{r_2} \Big(2w_xw_{xs} - \frac{1}{2}ww_s - 2e^{\frac{p-1}{2}s}w^pw_s\Big)(s,x)dx \\
&= \int_{r_1}^{r_2} 2(w_sw_{xx} + w_xw_{xs})(s,x)dx = 0.
\end{aligned}
$$

Thus, assertions (2.3) and (2.4) are proved. □

3. Nonexistence of global solutions for (1.1) under null energy

After tedious computations we can obtain the nonexistence of global solutions for Emden-Fowler equation (1.1) under small amplitude initial data and also that w blows up in L^2 since at finite (1.2) and therfore u blows up in L^2 at finite $\ln S^*$. We have the following Theorem.

Theorem 3.1. *Suppose that $u \in H1$ is a positive weak solution of equation* (1.1) *with $\alpha := \int_{r_1}^{r_2} u_0u_1(x)dx > 0$,*

$$
\int_{r_1}^{r_2} (u_1^2 - u_0u_1 + (u_0')^2 - \frac{2}{p+1}u_0^{p+1})(x)dx = 0
$$

and $0 < r_2 - r_1 \le 1$. Then the life-span of u is finite. That is, there exists S_1^* so that

$$
\Big(\int_{r_1}^{r_2} u(t,x)^2dx\Big)^{-1} \to 0 \quad \text{as } t \to \ln S_1^*,
$$

where $S_1^* = \frac{2}{p-1}\frac{\|u_0\|_2}{\alpha}$.

Proof. Setting

$$
\begin{gathered}
B(s) := \frac{1}{p+1}\int_0^s\int_{r_1}^{r_2} e^{\frac{p-1}{2}r}w^{p+1}(r,x)\,dx\,dr, \\
K(s) = \int_{r_1}^{r_2} \Big(w_s^2 + w_x^2 - \frac{1}{4}w^2\Big)(s,x)dx,
\end{gathered}
$$

then (2.4) can be rewritten as

$$
K - 2B' = E_w(0) - (p-1)B, \tag{3.1}
$$

therefore,

$$
(e^{\frac{p-1}{-2}s}B)' = e^{\frac{p-1}{-2}s}\Big(B' - \frac{p-1}{2}B\Big) = \frac{1}{2}e^{\frac{p-1}{-2}s}(K - E_w(0)),
$$

$$\begin{aligned}
e^{\frac{p-1}{-2}s}B &= \frac{1}{2}\int_0^s e^{\frac{p-1}{-2}r}(K(r)-E_w(0))dr \\
&= \frac{1}{2}\int_0^s e^{\frac{p-1}{-2}r}K(r)dr - \frac{E_w(0)}{p-1}\Big(1-e^{\frac{p-1}{-2}s}\Big), \\
B &= \frac{1}{2}\int_0^s e^{\frac{p-1}{2}(s-r)}K(r)dr - \frac{E_w(0)}{p-1}\Big(e^{\frac{p-1}{2}s}-1\Big);
\end{aligned}$$

this implies

$$\begin{aligned}
&\frac{1}{p+1}\int_0^s\int_{r_1}^{r_2} e^{\frac{p-1}{2}r}w^{p+1}(r,x)\,dx\,dr \\
&= \frac{1}{2}\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}\Big(w_s^2+w_x^2-\frac{1}{4}w^2\Big)(s,x)\,dx\,dr - \frac{E_w(0)}{p-1}(e^{\frac{p-1}{2}s}-1),
\end{aligned}$$

$$\begin{aligned}
&\int_0^s\int_{r_1}^{r_2} e^{\frac{p-1}{2}r}w^{p+1}(r,x)\,dx\,dr \\
&= \frac{p+1}{2}\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}\Big(w_s^2+w_x^2-\frac{1}{4}w^2\Big)(r,x)\,dx\,dr \\
&- \frac{p+1}{p-1}E_w(0)(e^{\frac{p-1}{2}s}-1),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\int_{r_1}^{r_2} e^{\frac{p-1}{2}s}w^{p+1}(s,x)dx \\
&= \frac{p+1}{2}\int_{r_1}^{r_2}\Big(w_s^2+w_x^2-\frac{1}{4}w^2\Big)(s,x)dx - (p+1)E_w(0)e^{\frac{p-1}{2}s} \\
&\quad + \frac{p^2-1}{2}\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}\Big(w_s^2+w_x^2-\frac{1}{4}w^2\Big)(r,x)\,dx\,dr.
\end{aligned} \tag{3.3}$$

Setting $J(s) := A(s)^{-k}$, $k=\frac{p-1}{4}>0$, $A(s) := \int_{r_1}^{r_2} w^2(s,x)dx$, we have $A'(s) = 2\int_{r_1}^{r_2} ww_s(s,x)dx$,

$$\begin{aligned}
A''(s) &= 2\int_{r_1}^{r_2}(ww_{xx}+\frac{1}{4}w^2+w_s^2+e^{\frac{p-1}{2}s}w^{p+1})(s,x)dx \\
&= 2\int_{r_1}^{r_2}(-w_x^2+\frac{1}{4}w^2+w_s^2+e^{\frac{p-1}{2}s}w^{p+1})(s,x)dx.
\end{aligned}$$

By (3.3) then

$$\begin{aligned}
&A''(s) \\
&= 2\int_{r_1}^{r_2}\Big(ww_{xx}+\frac{1}{4}w^2+w_s^2+e^{\frac{p-1}{2}s}w^{p+1}\Big)(s,x)dx \\
&= 2\int_{r_1}^{r_2}\Big(-w_x^2+\frac{1}{4}w^2+w_s^2\Big)(s,x)dx + (p+1)\int_{r_1}^{r_2}\Big(w_s^2+w_x^2-\frac{1}{4}w^2\Big)(s,x)\,dx \\
&\quad + (p^2-1)\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}\Big(w_s^2+w_x^2-\frac{1}{4}w^2\Big)(r,x)\,dx\,dr \\
&\quad - 2(p+1)E_w(0)e^{\frac{p-1}{2}s} \\
&= \int_{r_1}^{r_2}\big[(p+3)w_s^2+(p-1)w_x^2-\frac{p-1}{4}w^2\big](s,x)\,dx - 2(p+1)E_w(0)e^{\frac{p-1}{2}s}
\end{aligned}$$

$$+ (p^2-1)\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}\big(w_s^2+w_x^2-\frac{1}{4}w^2\big)(r,x)\,dx\,dr\,. \tag{3.4}$$

Also $J'(s) = -kA(s)^{-k-1}A'(s)$,

$$\begin{aligned} J''(s) &= -kA(s)^{-k-2}[A(s)A''(s)-(k+1)A'(s)^2] \\ &\le -kA(s)^{-k-1}\big[A''(s)-4(k+1)\int_{r_1}^{r_2} w_s^2(s,x)dx\big]. \end{aligned} \tag{3.5}$$

Since $E_w(0) = \int_{r_1}^{r_2}(u_1^2+(u_0')^2-u_0u_1-\frac{2}{p+1}u_0^{p+1})(x)=0$, we have

$$\begin{aligned} &A''(s)-4(k+1)\int_{r_1}^{r_2} w_s^2(s,x)dx \\ &= \int_{r_1}^{r_2}[(p+3)w_s^2+(p-1)w_x^2-\frac{p-1}{4}w^2](s,x)dx \\ &\quad + (p^2-1)\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}(w_s^2+w_x^2-\frac{1}{4}w^2)(r,x)\,dx\,dr \\ &\quad - 4(k+1)\int_{r_1}^{r_2} w_s^2(s,x)dx, \end{aligned}$$

$$\begin{aligned} &A''(s)-4(k+1)\int_{r_1}^{r_2} w_s^2(s,x)dx \\ &= \int_{r_1}^{r_2}\big[(p+3)w_s^2+(p-1)w_x^2-\frac{p-1}{4}w^2\big](s,x)dx \\ &\quad + (p^2-1)\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}(w_s^2+w_x^2-\frac{1}{4}w^2)(r,x)\,dx\,dr \\ &\quad - 4(k+1)\int_{r_1}^{r_2} w_s^2(s,x)dx \\ &\ge (p-1)\int_{r_1}^{r_2}[w_x^2-\frac{1}{4}w^2](s,x)dx \\ &\quad + (p^2-1)\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}\big(w_s^2+w_x^2-\frac{1}{4}w^2\big)(r,x)\,dx\,dr \\ &\ge (p-1)\big(1-(r_2-r_1)^2\big)\Big(\int_{r_1}^{r_2} w_x^2(s,x)dx \\ &\quad + (p+1)\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}(w_s^2+w_x^2)(r,x)\,dx\,dr\Big) > 0, \end{aligned}$$

provided $r_2-r_1\le 1$. Therefore, by (3.5) we obtain that for $\int_{r_1}^{r_2} u_0u_1(x)dx > 0$, $r_2-r_1\le 1$, $J''(s)<0$ for all $s\ge 0$.

$$\begin{aligned} J'(s) \le J'(0) &= -\frac{p-1}{4}A(0)^{-\frac{p+3}{4}}A'(0) = -\frac{p-1}{2}\alpha\|u_0\|_2^{-\frac{p+3}{2}}, \\ J(s) &\le J(0)-\frac{p-1}{2}\alpha\|u_0\|_2^{-\frac{p+3}{2}}s \\ &= \|u_0\|_2^{-\frac{p-1}{2}}-\frac{p-1}{2}\alpha\|u_0\|_2^{-\frac{p+3}{2}}s \\ &= \|u_0\|_2^{-\frac{p+3}{2}}\Big(\|u_0\|_2-\frac{p-1}{2}\alpha s\Big), \end{aligned}$$

$$J(s) \to 0 \quad textass \to S^* = \frac{2}{p-1}\frac{\|u_0\|_2}{\alpha}.$$

Thus w blows up in L^2 at finite S^*, and then u blows up in L^2 at finite $\ln S^*$. □

4. Nonexistence of global solution for (1.1) under negative energy

Theorem 4.1. *Suppose that $u \in H1$ is a positive weak solution of equation* (1.1) *with* $\alpha := \int_{r_1}^{r_2} u_0u_1(x)dx > 0$, $\int_{r_1}^{r_2}(u_1^2 - u_0u_1 + (u_0')^2 - \frac{2}{p+1}u_0^{p+1})(x)dx < 0$ *and* $0 < r_2 - r_1 \le 1$. *Then the life-span of u is finite. That is, there exists S_2^* such that*

$$\Big(\int_{r_1}^{r_2} u(t,x)^2dx\Big)^{-1} \to 0 \quad \text{as } t \to \ln S_2^*.$$

Further, the life-span satisfies $\ln S_2^* < \ln S_1^*$, and we have the estimate

$$A(s) \ge A(0) - 4E_w(0)\frac{p+1}{p-1}\big[se^{\frac{p-1}{2}s} - \frac{2}{p-1}(e^{\frac{p-1}{2}s} - 1)\big],$$

where $A(s) := \int_{r_1}^{r_2} w^2(s,x)dx$.

Proof. By (3.4), $E_w(0) < 0$, $\int_{r_1}^{r_2} u_0u_1(x)dx \ge 0$ and the small width in space is $0 < r_2 - r_1 \le 1$,

$$\begin{aligned}
&J''(s)\\
&= -kA(s)^{-k-2}[A(s)A''(s) - \tfrac{p+3}{4}A'(s)^2]\\
&\le -kA(s)^{-k-1}\Big[A''(s) - (p+3)\int_{r_1}^{r_2} w_s^2(s,x)dx\Big]\\
&= -kA(s)^{-k-1}\Big[-2(p+1)E_w(0)e^{\frac{p-1}{2}s} + (p-1)\int_{r_1}^{r_2}(w_x^2 - \frac{1}{4}w^2)(s,x)dx\\
&\quad + (p^2-1)\int_0^s e^{\frac{p-1}{2}(s-r)}\int_{r_1}^{r_2}(w_s^2 + w_x^2 - \frac{1}{4}w^2)(r,x)\,dx\,dr\Big]\\
&\le 2k(p+1)E_w(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}} < 0,
\end{aligned} \tag{4.1}$$

where $k = (p-1)/4$, we can obtain the same conclusions as in Theorem 3.1.

By the inequality (4.1) and $J' < 0$ we can estimate J further,

$$\begin{gathered}
J''(s) \le 2k(p+1)E_w(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}} = \frac{1}{2}(p^2-1)E_w(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}} < 0,\\
J'(s) \le J'(0) + \frac{s}{2}(p^2-1)E_w(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}} \le \frac{s}{2}(p^2-1)E_w(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}},\\
-k\big(J(s)^{-\frac{1}{k}}\big)' = J(s)^{-1-\frac{1}{k}}J'(s) \le \frac{E_w(0)}{2}(p^2-1)se^{\frac{p-1}{2}s},\\
-k(J(s)^{-\frac{1}{k}} - J(0)^{-\frac{1}{k}}) \le \frac{E_w(0)}{2}(p^2-1)\Big(\frac{2}{p-1}se^{\frac{p-1}{2}s} - (\frac{2}{p-1})^2(e^{\frac{p-1}{2}s} - 1)\Big)\\
= E_w(0)(p+1)\big[se^{\frac{p-1}{2}s} - \frac{2}{p-1}(e^{\frac{p-1}{2}s} - 1)\big],
\end{gathered}$$

this implies

$$A(s) \ge A(0) - 4\frac{p+1}{p-1}E_w(0)\big[se^{\frac{p-1}{2}s} - \frac{2}{p-1}(e^{\frac{p-1}{2}s} - 1)\big].$$

Thus, the assertions are proved. □

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