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ENTROPY SOLUTIONS OF EXTERIOR PROBLEMS FOR NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH NONHOMOGENEOUS BOUNDARY CONDITION

LI ZHANG, NING SU

Abstract. In this article, we consider the exterior problem for the nonlinear degenerate parabolic equation

$$u_t - \Delta b(u) + \nabla \cdot \Phi(u) = F(u), \quad (t,x) \in (0,T) \times \Omega,$$

Ω is the exterior domain of Ω_0 (a closed bounded domain in $\mathbb{R}^N$ with its boundary $\Gamma \in \mathcal{C}^{1,1}$), b is non-decreasing and Lipschitz continuous, $\Phi = (\phi_1, \dots, \phi_N)$ is vectorial continuous, and F is Lipschitz continuous. In the nonhomogeneous boundary condition where $b(u) = b(a)$ on $(0,T) \times \Gamma$, we establish the comparison and uniqueness, the existence using penalized method.

1. Introduction

Let $N \geq 3$. Let Ω be the exterior domain of Ω_0, where $\Omega_0 \subset \mathbb{R}^N$ is a bounded closed domain with its boundary $\Gamma = \partial\Omega_0 \in C^{1,1}$. Without loss of generality, we assume $\Omega_0 \subset \{x \in \mathbb{R}^N : |x| \leq r_0\}$ with $0 < r_0 < 1$. Denote $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$, $T > 0$. Consider the exterior problem

$$\begin{gathered} u_t - \Delta b(u) + \operatorname{div} \Phi(u) = F(u) \quad (t,x) \in Q, \\ b(u) = b(a) \quad (t,x) \in \Sigma, \\ u(0,x) = u_0(x) \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where $b : \mathbb{R} \to \mathbb{R}$ is nondecreasing and Lipschitz continuous, $\Phi = (\phi_1, \dots, \phi_N) : \mathbb{R} \to \mathbb{R}^N$ is continuous, F is Lipschitz continuous with constant L, and a will be defined in Section 2.

For the case $b' \equiv 0$, Kruzkov [7] considered the Cauchy problem and proved the existence and uniqueness in the case where Φ is continuously differentiable. Then, Kruzkov and other authors [3, 9, 10] proved uniqueness of entropy solutions in the case where Φ satisfies some Osgood's type conditions or local Hölder continuity of order $\alpha = 1 - \frac{1}{N}$. In particular, Kruzkov and Panov [9] gave an counter-example to explain that the condition Φ is locally Hölder continuous is sharp in a definite case. In 1999, Su [14] studied the problem in one-dimensional space and proved comparison principle of entropy solutions in the case where F is continuous in u.

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For the degenerate parabolic problem with given source $f(t,x)$, the Cauchy problem and the Dirichlet problem have been investigated by many people. For the initial value problems, the existence, comparison and uniqueness was established in [2, 13]. For a more complicated case, the initial value problem in one-dimensional space, the problem in a bounded domain with homogeneous boundary condition (that is, $b(u)=0$ on Σ), and the problem with nonhomogeneous condition were considered respectively by Liu and Wang [11], Carrillo[4] and Ammar[1]. For the Dirichlet problem, Carrillo [4] also gave a brief proof of the comparison and uniqueness, where $F\in\mathcal{C}(\mathbb{R})$ is a nondecreasing function vanishing at zero. For the Cauchy problem, Karlsen and Risebro [5] established uniqueness and stability under the conditions that b, ϕ, and f are all locally Lipschitz continuous.

This article is organized as follows. In Section 2, we will give the definition of entropy solutions, basic assumptions and main results. Combining the techniques of Ammar[1], Andreianov and Maliki[2], the comparison and uniqueness is established in Section 3. In Section 4, a problem with homogeneous condition is investigated at the beginning, and the existence is given by the penalized method (see [1]).

2. Basic assumptions and statement of main results

Now we recall some notation from [1, 4]. For any $s_1,s_2\in\mathbb{R}$, for almost all $x\in\partial\Omega$, define

$$\omega^+(x,s_1,s_2)=\max_{s_1\le r,s\le s_1\vee s_2}|(\Phi(r)-\Phi(s))\cdot\eta(x)|, \tag{2.1}$$

$$\omega^-(x,s_1,s_2)=\max_{s_1\wedge s_2\le r,s\le s_2}|(\Phi(r)-\Phi(s))\cdot\eta(x)|, \tag{2.2}$$

$$\omega(x,s_1,s_2)=\omega^+(x,s_1,s_2)+\omega^-(x,s_1,s_2), \tag{2.3}$$

where $\eta(x)$ is the outward unit normal vector to $\partial\Omega$ at x.

For any $s\in\mathbb{R}$, define

$$H_0(s)=\begin{cases}1 & \text{if } s>0,\\ 0 & \text{if } s\le 0,\end{cases}\quad H(s)=\begin{cases}1 & \text{if } s>0,\\ [0,1] & \text{if } s=0,\\ 0 & \text{if } s<0.\end{cases}$$

Motivated by [1, 6, 13], we give the definition of the entropy solutions of (1.1).

Definition 2.1. A measurable function $u\in L^\infty(Q)$ is called an entropy subsolution of (1.1), if $b(u)\in L^2(0,T;H^1_{\rm loc}(\mathbb{R}^N))$, $b(u)\le b(a)$ a.e. on Σ, and for all $(s,\xi)\in\mathbb{R}\times\mathcal{D}([0,T)\times\mathbb{R}^N)$ such that $\xi\ge 0$ and $H(b(a)-b(s))\xi=0$ a.e. on Σ,

$$\begin{aligned}&-\int_\Sigma \omega^+(x,s,a)\xi\,dx\,dt\\ &\le\int_Q H_0(u-s)\{(u-s)\xi_t-\nabla b(u)\cdot\nabla\xi+(\Phi(u)-\Phi(s))\cdot\nabla\xi+F(u)\xi\}\,dx\,dt\\ &\quad+\int_\Omega (u_0-s)^+\xi(0)\,dx.\end{aligned} \tag{2.4}$$

A measurable function $u\in L^\infty(Q)$ is called an entropy super-solution of (1.1), if $b(u)\in L^2(0,T;H^1_{\rm loc}(\mathbb{R}^N))$, $b(u)\ge b(a)$ a.e. on Σ, and for all $(s,\xi)\in\mathbb{R}\times\mathcal{D}([0,T)\times\mathbb{R}^N)$ such that $\xi\ge0$ and $H(b(s)-b(a))\xi=0$ a.e. on Σ,

$$-\int_\Sigma \omega^-(x,s,a)\xi\,dx\,dt$$

$$\begin{aligned}&\le \int_Q H_0(s-u)\{(s-u)\xi_t - \nabla b(u)\cdot\nabla\xi + (\Phi(u)-\Phi(s))\cdot\nabla\xi + F(u)\xi\}\,dx\,dt\\&\quad + \int_\Omega (s-u_0)^+\xi(0)\,dx.\end{aligned} \tag{2.5}$$

A measurable function $u \in L^\infty(Q)$ is called an entropy solution of (1.1), if u is both an entropy sub-solution and an entropy super-solution.

In this article, the basic assumptions are as follows.

- (H1) $a \in \mathcal{C}(\Sigma)$ is the trace of $\tilde a \in (Q)$, where $b(\tilde a) \in L^2(0,T;H^1_{\rm loc}(\Omega))$, $\Delta b(\tilde a) \in L^1(0,T;L^1_{\rm loc}(\Omega))$, and $\tilde a_t \in L^1(0,T;L^1_{\rm loc}(\Omega))$.
- (H2) $b:\mathbb{R}\to\mathbb{R}$ is nondecreasing and Lipschitz continuous with $b(0)=0$.
- (H3) $\Phi=(\phi_1,\dots,\phi_N):\mathbb{R}\to\mathbb{R}^N$ is continuous with $\phi_i(0)=0$, $i=1,\dots N$.
- (H4) F is Lipschitz continuous with constant L, and $F(0)=0$.

Remark 2.2. (H1) and (H2) are introduced by Ammar[1], in which he investigated an initial-boundary value problem of parabolic-hyperbolic type.

In some works, we assume that

- (H5) $\Phi=(\phi_1,\dots,\phi_N):\mathbb{R}\to\mathbb{R}^N$ is Hölder continuous of order $1-\frac{1}{N}$, and $\phi_i(0)=0$, $i=1,\dots N$.

Remark 2.3. (H5) is necessary because (H3) is not enough to establish the existence, comparison and uniqueness for entropy solutions (see [2, 8]).

Our main results read as follows:

Theorem 2.4. Assume (H2), (H4), (H5). For all $u_0 \in L^\infty(\Omega)$ and $a \in \mathcal{C}(\Sigma)$ satisfying (H1), there exists an entropy solution of (1.1).

Theorem 2.5. Assume (H2), (H3), (H4). Assume that $u_{0i} \in L^\infty(\Omega)$, and $a_i \in \mathcal{C}(\Sigma)$ satisfies (H1), $i=1,2$. Let u_1 be an entropy sub-solution of (1.1), and u_2 be an entropy super-solution. Whenever $(u_{01}-u_{02})^+ \in L^1(\Omega)$ and $b(a_1)\le b(a_2)$,

$$\begin{aligned}&-\int_\Sigma \omega^-(x,a_1,a_2)\,dx\,dt\\&\le \int_Q \Big\{(u_1-u_2)^+\xi_t - \nabla(b(u_1)-b(u_2))^+\cdot\nabla\xi\\&\quad + H_0(u_1-u_2)\big(\Phi(u_1)-\Phi(u_2)\big)\cdot\nabla\xi\Big\}\,dx\,dt\\&\quad + \int_Q (F(u_1)-F(u_2))^+\xi\,dx\,dt + \int_\Omega (u_{01}-u_{02})^+\xi(0)\,dx\end{aligned} \tag{2.6}$$

for all $0\le\xi\in\mathcal{D}([0,T)\times\mathbb{R}^N)$. Moreover, if Φ satisfies (H5),

$$\begin{aligned}&\int_\Omega (u_1(t)-u_2(t))^+\,dx\\&\le (1+Lte^{Lt})\Big(\int_\Omega (u_{01}-u_{02})^+\,dx + \int_0^t\int_\Gamma \omega^-(x,a_1,a_2)\,d\tau\Big).\end{aligned} \tag{2.7}$$

In particular, if $u_{01}\le u_{02}$ and $\omega^-(x,a_1,a_2)=0$, then $u_1\le u_2$ a.e. on Q.

3. Comparison and uniqueness

Consider the nonlinear parabolic problem

$$\begin{gathered} u_t - \Delta b(u) + \operatorname{div}\Phi(u) = f(t,x) \quad (t,x)\in Q,\\ b(u) = b(a) \quad (t,x)\in\Sigma,\\ u(0,x) = u_0(x) \quad x\in\Omega. \end{gathered} \tag{3.1}$$

We can prove the comparison and uniqueness for entropy solutions of (3.1) by combining the techniques in [1, 2, 13].

Proposition 3.1. *Assume* (H1)–(H3), $u_{0i}\in L^\infty(\Omega)$, $f_i\in L^\infty(Q)$, $i=1,2$. *If* u_1 *is an entropy sub-solution of* (3.1), *and* u_2 *is an entropy super-solution, whenever* $(u_{01}-u_{02})^+\in L^1(\Omega)$, $(f_1-f_2)^+\in L^1(Q)$, *and* $b(a_1)\le b(a_2)$ *a.e. on* Σ, *there exists* $\kappa\in H(u_1-u_2)$ *such that*

$$\begin{aligned} -\int_\Sigma \omega^-(x,a_1,a_2)\,dx\,dt \le \int_Q &\{(u_1-u_2)^+\xi_t - \nabla(b(u_1)-b(u_2))^+\cdot\nabla\xi\\ &+ H_0(u_1-u_2)(\Phi(u_1)-\Phi(u_2))\cdot\nabla\xi\}\,dx\,dt\\ &+\int_Q \kappa(f_1-f_2)^+\xi\,dx\,dt + \int_\Omega (u_{01}-u_{02})^+\xi(0)\,dx \end{aligned}\tag{3.2}$$

for all $0\le\xi\in\mathcal{D}([0,T)\times\mathbb{R}^N)$. *Moreover, if* Φ *satisfies* (H5), *then*

$$\begin{aligned} \int_\Omega (u_1(t)-u_2(t))^+\,dx \le \int_\Omega (u_{01}-u_{02})^+\,dx &+ \int_0^t\int_\Omega \kappa(f_1-f_2)^+\,dx\,d\tau\\ &+\int_0^t\int_\Gamma \omega^-(x,a_1,a_2)\,d\tau. \end{aligned}\tag{3.3}$$

In particular, if $u_{01}\le u_{02}$ *a.e. in* Ω, $f_1\le f_2$ *a.e. in* Q, *and*

$$\omega^-(x,a_1,a_2) = 0 \quad \text{a.e. on } \Sigma, \tag{3.4}$$

then $u_1\le u_2$ *a.e. in* Q.

Remark 3.2. Condition (3.4) can be satisfied as follows. From the definition of ω^-, we have

$$\omega^-(x,a_1,a_2)\begin{cases} >0 & \text{if } a_1(t,x)>a_2(t,x), \text{ and } \exists r,s\in[a_2(t,x),a_1(t,x)]\\ & \text{such that } (\Phi(r)-\Phi(s))\cdot\eta(x)\ne 0,\\ =0 & \text{otherwise.}\end{cases}$$

Therefore, for any a_1,a_2 such that $b(a_1)\le b(a_2)$, the equality $\omega^-(x,a_1,a_2)=0$ holds for almost all $(t,x)\in\Sigma$ whenever a_1 and a_2 satisfies that either $a_1(t,x)\le a_2(t,x)$, or $(\Phi(r)-\Phi(s))\cdot\eta(x)=0$ for all $r,s\in[a_2(t,x),a_1(t,x)]$.

Proof. For $\Gamma\in\mathcal{C}^{1,1}$, there exists a finite open cover of Ω, denoted by $\{\Omega_i, i=0,1,\dots,m\}$, such that $\Omega_0\Subset\Omega$, and for any $1\le i\le m$, either $\Omega_i\cap\Gamma=\emptyset$, or else there exists Ω_i' satisfying $\Omega_i\Subset\Omega_i'$ and $\Omega_i'\cap\Gamma$ is a part of $\partial\Gamma$. Let $\{\eta_i, i=0,\dots,m\}$ be a partition of unity subordinate to the covering $\{\Omega_i, 0\le i\le m\}$.

For any Ω_i, if $\Omega_i \cap \Sigma = \emptyset$, then arguing as in [2, Theorem 2], we have

$$\begin{aligned}
&\int_Q \{(u_1 - u_2)^+ \xi_t \eta_i - \nabla(b(u_1) - b(u_2))^+ \cdot \nabla(\xi\eta_i) \\
&+ H_0(u_1 - u_2)(\Phi(u_1) - \Phi(u_2)) \cdot \nabla(\xi\eta_i)\}\,dx\,dt \\
&+ \int_Q \kappa(f_1 - f_2)^+ \xi\eta_i\,dx\,dt + \int_\Omega (u_{01} - u_{02})^+ \xi(0)\eta_i\,dx \geq 0.
\end{aligned} \tag{3.5}$$

Otherwise, if $\Omega_i \cap \Gamma \neq \emptyset$, then from the continuity of a_1 and a_2, for all $(t,x) \in [0,T] \times (\Omega_i \cap \Gamma)$, for all $\varepsilon > 0$, there exists $\delta > 0$, whenever $d((t,x),(s,y)) < \delta$, we have

$$|a_1(t,x) - a_1(s,y)| < \varepsilon, \quad |a_2(t,x) - a_2(s,y)| < \varepsilon. \tag{3.6}$$

For $\Gamma \in \mathcal{C}^{1,1}$, there exists a finite open cover of Q, denoted by $\{B_j^\varepsilon,\ j = 0,\dots,m_\varepsilon\}$, where $B_0^\varepsilon \subset\subset Q$, $B_j^\varepsilon = B((t_j,x_j),\delta)$, $j = 1,\dots,m_\varepsilon$. Let $\{\eta_j^\varepsilon, j = 0,\dots,m_\varepsilon\}$ be a partition of unity subordinate to the covering $\{B_j^\varepsilon, j = 0,\dots,m_\varepsilon\}$.

For any $0 \leq \xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$, take $\xi_{i,j} = \xi\eta_i\eta_j^\delta$. Arguing as in [1, Theorem 2.3], we have

$$\begin{aligned}
&\int_Q \{(u_1 - u_2)^+ \xi_t \eta_i - \nabla(b(u_1) - b(u_2))^+ \cdot \nabla(\xi\eta_i) \\
&+ H_0(u_1 - u_2)(\Phi(u_1) - \Phi(u_2)) \cdot \nabla(\xi\eta_i)\}\,dx\,dt \\
&+ \int_Q \kappa(f_1 - f_2)^+ \xi\eta_i\,dx\,dt + \int_\Omega (u_{01} - u_{02})^+ \xi(0)\eta_i\,dx \\
&\geq -\sum_{j=1}^{m_\delta} \int_0^T \int_{\Gamma\cap\Omega_i} \omega^-(x, a_1 + \varepsilon, a_2 - \varepsilon)\xi\eta_i\eta_j^\delta \\
&\geq -\int_0^T \int_{\Gamma\cap\Omega_i} \omega^-(x, a_1 + \varepsilon, a_2 - \varepsilon)\xi\eta_i.
\end{aligned}$$

Form the arbitrary choose of ε and the continuity of ω^-, we deduce that

$$\begin{aligned}
&\int_Q \{(u_1 - u_2)^+ \xi_t \eta_i - \nabla(b(u_1) - b(u_2))^+ \cdot \nabla(\xi\eta_i) \\
&+ H_0(u_1 - u_2)(\Phi(u_1) - \Phi(u_2)) \cdot \nabla(\xi\eta_i)\}\,dx\,dt \\
&+ \int_Q \kappa(f_1 - f_2)^+ \xi\eta_i\,dx\,dt + \int_\Omega (u_{01} - u_{02})^+ \xi(0)\eta_i\,dx \\
&\geq -\int_0^T \int_{\Gamma\cap\Omega_i} \omega^-(x, a_1, a_2)\xi\eta_i\,dx\,dt.
\end{aligned} \tag{3.7}$$

Eventually, we get the inequality (3.2) from (3.5) and (3.7) by summing up over i .

If Φ is Hölder continuous of order $1 - \frac{1}{N}$, arguing as [2, Theorem 2], we deduce the following conclusion from (3.2),

$$\begin{aligned}
\int_Q \{(u_1 - u_2)^+(-\mu_t) &\leq \int_\Omega (u_{01} - u_{02})^+ \mu(0)\,dx + \int_Q \kappa(f_1 - f_2)^+ \mu\,dx\,dt \\
&\quad + \int_0^T \int_\Gamma \omega^-(x, a_1, a_2)\mu\,dx\,dt
\end{aligned}$$

for all $\mu \in \mathcal{D}([0,T))$. Then (3.3) is obtained by applying the Gronwall's inequality. $\square$

Corollary 3.3. Assume (H2) and (H5). Let u_i be an entropy solution of (3.1) for data (u_{0i}, a_i, f_i), where $u_{0i} \in L^\infty(\Omega)$, $f_i \in L^\infty(Q)$, and $a_i \in \mathcal{C}(\Sigma)$ satisfies (H1). Then there exists $\kappa \in H(u_1 - u_2)$ such that

$$\begin{aligned} &\|u_1(t) - u_2(t)\|_{L^1(\Omega)} \\ &\leq \|u_{01} - u_{02}\|_{L^1(\Omega)} + \int_0^t \int_\Omega \kappa \|f_1 - f_2\|_{L^1(\Omega)} d\tau + \int_0^t \|\omega(x, a_1, a_2)\|_{L^1(\Gamma)} d\tau, \end{aligned} \tag{3.8}$$

whenever $u_{01} - u_{02} \in L^1(\Omega)$, $f_1 - f_2 \in L^1(Q)$, and $\omega(x, a_1, a_2) \in L^1(\Sigma)$.

In particular, if $u_{01} = u_{02}$, $f_1 = f_2$, $b(a_1) = b(a_2)$, $\omega(x, a_1, a_2) = 0$, then $u_1 = u_2$ a.e. in Q.

Remark 3.4. Arguing as in Remark 3.2, if $b(a_1) = b(a_2)$ a.e. on Σ, then for almost all $(t, x) \in \Sigma$, $\omega(x, a_1, a_2) = 0$ holds whenever a_1 and a_2 satisfies that either $a_1 = a_2$ or $(\Phi(r) - \Phi(s)) \cdot \eta(x) = 0$ for all $r, s \in [m, M]$, where $m = \min\{a_1(t, x), a_2(t, x)\}$, $M = \max\{a_1(t, x), a_2(t, x)\}$.

Remark 3.5. From Remark 3.4, the boundary condition $b(u) = b(a)$ does not mean that $u = a$. In fact, if and only if b is non-degenerate at $a(t, x)$, $b(u(t, x)) = b(a(t, x))$ implies $u(t, x) = a(t, x)$. If b is degenerate at $a(t, x)$, then we can only claim that $u(t, x)$ is located in $E_{(t,x)} = \{r \in \mathbb{R} | b(r) = a(t, x)\}$ and $\Phi(s) \cdot \eta(x)$ is constant for all $s \in E_{(t,x)}$.

Corollary 3.6. *Assume* (H2) *and* (H5). *For any* $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$, *and* $a \in L^\infty(\Sigma)$ *satisfying (H1), if* u *is an entropy solution of* (3.1), *then*

$$\|u\|_{L^\infty(Q)} \leq \|u_0\|_{L^\infty(\Omega)} + T(\|f\|_{L^\infty(Q)} + \|a\|_{L^\infty(\Sigma)}). \tag{3.9}$$

Proof. Take $\overline{u} = M_1 + M_2 t$, $\underline{u} = -M_1 - M_2 t$, where $M_1 = \|u_0\|_{L^\infty(\Omega)}$, $M_2 = \|f\|_{L^\infty(Q)} + \|a\|_{L^\infty(\Sigma)}$, then $\overline{u}$ is an entropy solution of the nonlinear problem

$$\begin{gathered} u_t - \Delta b(u) + \operatorname{div} \Phi(u) = M_2 \quad (t, x) \in Q, \\ b(u) = b(M_1 + M_2 t) \quad (t, x) \in \Sigma, \\ u(0, x) = M_1 \quad x \in \Omega. \end{gathered} \tag{3.10}$$

Hence, applying Proposition 3.1, we have

$$u(t, x) \leq \overline{u} \quad \text{a.e. on } Q. \tag{3.11}$$

Arguing as above, we have $u(t, x) \geq \underline{u}$ a.e. on Q. □

Proof of Theorem2.5. From Proposition 3.1, we can prove theorem 2.5 by using Gronwall's inequality. From (3.2), there exists $\kappa \in H(u_1 - u_2)$ such that

$$\begin{aligned} &- \int_\Sigma \omega^-(x, a_1, a_2)\, dx\, dt \\ &\leq \int_Q \{(u_1 - u_2)^+ \xi_t - \nabla(b(u_1) - b(u_2))^+ \cdot \nabla \xi \\ &\quad + H_0(u_1 - u_2)(\Phi(u_1) - \Phi(u_2)) \cdot \nabla \xi\}\, dx\, dt \\ &\quad + \int_Q \kappa (F(u_1) - F(u_2))^+ \xi\, dx\, dt + \int_\Omega (u_{01} - u_{02})^+ \xi(0)\, dx \end{aligned}$$

for $0 \le \xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$. Since F is Lipschitz continuous with constant L, we have

$$\begin{aligned}\int_Q \kappa(F(u_1) - F(u_2))^+ \xi \, dx \, dt &= \int_Q (F(u_1) - F(u_2))^+ \xi \, dx \, dt \\ &\le L \int_Q (u_1 - u_2)^+ \xi \, dx \, dt\end{aligned}$$

for all $0 \le \xi \in \mathcal{D}([0,T) \times \mathbb{R}^N)$. Therefore, we obtain (2.6) and deduce (2.7) by applying Gronwall's inequality. □

Applying Theorem 2.5 and arguing as above, we have the following corollaries.

Corollary 3.7. *Assume* (H2), (H4), (H5). *Assume that* $u_{0i} \in L^\infty(\Omega)$, $a_i \in \mathcal{C}(\Sigma)$ *satisfies* (H1), *and* u_i *is an entropy solution of* (1.1), $i = 1, 2$. *Whenever* $u_{01} - u_{02} \in L^1(\Omega)$, $b(a_1) = b(a_2)$, *and* $\omega(x, a_1, a_2) \in L^1(\Sigma)$, *we have*

$$\|u_1(t) - u_2(t)\|_{L^1(\Omega)} \le (1 + Lte^{Lt})\big(\|u_{01} - u_{02}\|_{L^1(\Omega)} + \|\omega(x, a_1, a_2)\|_{L^1(\Sigma)}\big).$$

In particular, if $u_{01} = u_{02}$, *and* $\omega(x, a_1, a_2) = 0$,

$$u_1 = u_2 \quad \text{a.e. in } Q.$$

Corollary 3.8. *Assume* (H2), (H4), (H5). *For any* $u_0 \in L^\infty(\Omega)$ *and* $a \in L^\infty(\Sigma)$ *satisfying* (H1), *if* u *is an entropy solution of* (3.1), *then*

$$\|u\|_{L^\infty(Q)} \le e^{LT}(\|u_0\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Sigma)}).$$

4. Existence of solutions

In this section, we will prove the existence of solutions by using penalized method.

4.1. Homogeneous condition.

Consider the following problem with homogeneous boundary condition,

$$\begin{gathered} u_t - \Delta b(u) + \operatorname{div} \Phi(u) = f(t,x) \quad (t,x) \in Q, \\ b(u) = 0 \quad (t,x) \in \Sigma, \\ u(0,x) = u_0(x) \quad x \in \Omega. \end{gathered} \tag{4.1}$$

Motivated by [1, 4, 13], we define entropy solutions of (4.1) as follows.

Definition 4.1. A measurable function $u \in L^\infty(Q)$ is an entropy sub-solution of (4.1), if $b(u) \in L^2(0,T; H^1_{\rm loc}(\mathbb{R}^N))$, $b(u) \le 0$, and

$$\begin{aligned} 0 \le \int_Q & H_0(u - s)\{(u - s)\xi_t - \nabla b(u) \nabla \xi + (\Phi(u) - \Phi(s)) \nabla \xi \\ &+ f(t,x)\xi\} \, dx \, dt + \int_\Omega (u_0 - s)^+ \xi(0) \, dx \end{aligned} \tag{4.2}$$

for all $(s, \xi) \in \mathbb{R} \times \mathcal{D}([0,T) \times \mathbb{R}^N)$ such that $\xi \ge 0$ and $H(-b(s))\xi = 0$ a.e. on Σ.

A measurable function $u \in L^\infty(Q)$ is an entropy super-solution of (4.1), if $b(u) \in L^2(0,T; H^1_{\rm loc}(\mathbb{R}^N))$, $b(u) \ge 0$, and

$$\begin{aligned} 0 \le \int_Q & H_0(s - u)\{(s - u)\xi_t - \nabla b(u) \nabla \xi + (\Phi(u) - \Phi(s)) \nabla \xi \\ &+ f(t,x)\xi\} \, dx \, dt + \int_\Omega (s - u_0)^+ \xi(0) \, dx \end{aligned} \tag{4.3}$$

for all $(s,\xi) \in \mathbb{R} \times \mathcal{D}([0,T)\times\mathbb{R}^N)$ such that $\xi \geq 0$ and $H(b(s))\xi = 0$ a.e. on Σ.

A measurable function $u \in L^\infty(Q)$ is an entropy solution of (4.1), if u is both an entropy sub-solution and an entropy super-solution.

Remark 4.2. Definition 4.1 is not equivalent to Definition 2.1 for the case where $a = 0$. In fact, we can only claim that the entropy sub-solution (resp. super-solution) of (4.1) is definitely an entropy sub-solution (resp. super-solution) of (3.1), since ω^+ and ω^- are nonnegative.

Denote $B(0,R) = \{x \in \Omega : |x| < R\}$, and consider the stationary problem

$$\begin{gathered} u - \Delta b(u) + \operatorname{div}\Phi(u) = g(x) \quad x \in B(0,n), \\ b(u) = 0 \quad x \in \partial B(0,n). \end{gathered} \tag{4.4}$$

Applying [4, Theorem 7], for all $g(x) \in L^\infty(\Omega)$, there exists an entropy solution u_n of (4.4). Arguing as in [13, Theorem 3.7], we can deduce the existence for entropy solutions of (4.1) (see [13] for details).

Proposition 4.3. *Assume* (H2), (H5)*. For any $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q)$, there exists an entropy solution of* (4.1)*.*

Combining the techniques in [4, 2], we prove the comparison and L^1-contraction for entropy solutions of (4.1).

Proposition 4.4. *Assume* (H2), (H3)*. Assume that $u_{0i} \in L^\infty(\Omega)$, $f_i \in L^\infty(Q)$, and u_1 (resp. u_2) is an entropy sub-solution (resp. super-solution) of* (4.1)*. Whenever $(u_{01} - u_{02})^+ \in L^1(\Omega)$ and $(f_1 - f_2)^+ \in L^1(Q)$, there exists $\kappa \in H(u_1 - u_2)$ such that*

$$\begin{aligned} &\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi - H_0(u_1 - u_2)(\Phi(u_1) - \Phi(u_2)) \cdot \nabla\xi \\ &- (u_1 - u_2)^+ \xi_t\}\,dx\,dt - \int_\Omega (u_{01} - u_{02})^+ \xi(0)\,dx \\ &\leq \int_Q \kappa (f_1 - f_2)^+ \xi\,dx\,dt \end{aligned} \tag{4.5}$$

for all $0 \leq \xi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$. Moreover, if Φ satisfies (H5)*, then*

$$\int_\Omega (u_1(t) - u_2(t))^+\,dx \leq \int_\Omega (u_{01} - u_{02})^+\,dx + \int_0^t \int_\Omega \kappa (f_1 - f_2)^+\,dx\,d\tau. \tag{4.6}$$

In particular, if $u_{01} \leq u_{02}$, and $f_1 \leq f_2$, then $u_1 \leq u_2$.

Proof. Using the same notation as in Proposition 3.1. Let $\{\Omega_i, i = 0,\dots,m\}$ a finite open covering of Ω, and $\{\eta_i, i = 0,\dots,m\}$ be a partition of unity subordinate to $\{\Omega_i, i = 0,\dots,m\}$.

For any $0 \leq \xi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$, take $\xi_i = \xi\eta_i$, $0 \leq i \leq m$. If $\Omega_i \cap \Gamma = \emptyset$, arguing as in [13, Theorem 3.9], there exists $\kappa \in H(u_1 - u_2)$ such that

$$\begin{aligned} &\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi_i - H_0(u_1 - u_2)(\Phi(u_1) - \Phi(u_2)) \cdot \nabla\xi_i \\ &- (u_1 - u_2)^+ \xi_{it}\}\,dx\,dt - \int_\Omega (u_{01} - u_{02})^+ \xi_i(0)\,dx \\ &\leq \int_Q \kappa (f_1 - f_2)^+ \xi_i\,dx\,dt. \end{aligned} \tag{4.7}$$

Otherwise, if $\Omega_i \cap \Gamma \neq \emptyset$, arguing as in [4, Theorem 14], (4.7) is still satisfied. Summing up over i from 0 to m, we deduce (4.5).

Moreover, if Φ is locally Hölder continuous of order $1-\frac{1}{N}$, starting from (4.5) and arguing as in [2, Theorem 2], there exists $\kappa \in H(u_1-u_2)$ such that

$$\int_Q (u_1-u_2)^+(-\mu_t)\,dx\,dt \le \int_\Omega (u_{01}-u_{02})^+\mu(0)\,dx + \int_{Q_T} \kappa(f_1-f_2)^+\mu\,dx\,dt \tag{4.8}$$

for all $0 \le \mu \in \mathcal{D}([0,T))$.

Then (4.6) is obtained by applying Gronwall's inequality. $\square$

Corollary 4.5. *Assume* (H2), (H5). *Assume that* $u_{0i} \in L^\infty(\Omega)$, $f_i \in L^\infty(Q)$, *and* u_i *is an entropy solution of* (4.1), $i=1,2$. *Whenever* $u_{01}-u_{02} \in L^1(\Omega)$ *and* $f_1-f_2 \in L^1(Q)$,

$$\|u_1(t)-u_2(t)\|_{L^1(\Omega)} \le \|u_{01}-u_{02}\|_{L^1(\Omega)} + \int_0^t \|f_1-f_2\|_{L^1(\Omega)}\,d\tau. \tag{4.9}$$

In particular, if $u_{01}=u_{02}$ *and* $f_1=f_2$, *then* $u_1=u_2$.

4.2. **Proof of Theorem 2.4.** Based on the results in Section 4.1, we prove the existence for entropy solutions of (3.1) by penalized method.

Proposition 4.6. *Assume* (H2), (H5). *For any* $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q)$, *and* $a \in \mathcal{C}(\Sigma)$ *satisfying (H1), there exists an entropy solution of* (3.1).

Proof. For any closed surface $\tilde\Gamma \in \mathcal{C}^{1,1}$ located in the inner domain of Γ, denote the exterior domain of $\tilde\Gamma$ by $\tilde\Omega$, and $\tilde Q = (0,T)\times\tilde\Omega$, $\tilde\Sigma = (0,T)\times\tilde\Gamma$. Since $\Omega \subset\subset \tilde\Omega$, we extend a to $\tilde a \in \mathcal{C}(\tilde Q)$ with $b(\tilde a) \in L^2(0,T;H^1_{\rm loc}(\tilde\Omega))$, $b(\tilde a)|_{\tilde\Gamma} = 0$, and $\Delta a_t \in L^1(0,T;L^1_{\rm loc}(\tilde\Omega))$.

Define the penalized function $\beta_{m,n}$ as follows (see [1]): for all $r \in \mathbb{R}^N$,

$$\beta_{m,n}(t,x,r) = \chi_{\tilde Q\setminus Q}(m(r-\tilde a(x))^+ - n(\tilde a(x)-r)^+) \quad \text{a.e. in } \tilde Q. \tag{4.10}$$

It is clear that $\beta_{m,n}$ is Lipschitz continuous.

Extend u_0 and f as follows:

$$\tilde u_0 = \begin{cases} u_0 & x \in \Omega, \\ 0 & x \in \tilde\Omega\setminus\Omega, \end{cases} \qquad \tilde f(t,x) = \begin{cases} f(t,x) & (t,x) \in Q, \\ 0 & (t,x) \in \tilde Q\setminus Q. \end{cases}$$

Consider the penalized problem

$$\begin{gathered} u_t - \Delta b(u) + \operatorname{div}\Phi(u) + \beta_{m,n}(u) = \tilde f(t,x), \quad (t,x) \in \tilde Q, \\ b(u) = 0, \quad (t,x) \in \tilde\Sigma, \\ u(0,x) = \tilde u_0(x), \quad x \in \tilde\Omega. \end{gathered} \tag{4.11}$$

We claim that there exists a unique entropy solution of (4.11), denoted by $u^{m,n}$.

In fact, since $\beta_{m,n}$ is Lipschitz continuous, for any $u_0 \in L^1(\tilde\Omega)\cap L^\infty(\tilde\Omega)$, the existence of entropy solution in $L^1(\tilde Q)\cap L^\infty(\tilde Q)$ can be deduced from Proposition 4.3 by applying Banach's contraction principle.

For any $u_0 \in L^\infty$, define $u_0^{l,k}$ as follows,

$$u_0^{l,k}(x) = u_0^+\chi_{B(0,l)}(x) - u_0^-(x)\chi_{B(0,k)}(x), \quad x \in \Omega.$$

For $u_0^{l,k} \in\in L^1(\Omega)\cap L^\infty(\Omega)$, there exists an entropy solution of (1.1), denoted by $u^{l,k}$. Using the technique in [12], we can prove that there exists $u^{m,n} \in$

$\mathcal{C}(0,T;L^1_{\rm loc}(\Omega))$ and a subsequence of $u^{l,k}$, denoted by $u^{l(k),k}$, such that $u^{l(k),k} \to u^{m,n}$ as $k \to \infty$, and $u^{m,n}$ is indeed an entropy solution of (4.11).

Applying Proposition 4.4, for any $m \le m'$, there exists $\kappa \in H(u^{m',n} - u^{m,n})$ such that

$$\begin{aligned} &\int_{\tilde\Omega} (u^{m',n}(t) - u^{m,n}(t))^+ \,dx \\ &\le \int_Q \kappa(-\beta_{m',n}(u^{m',n}) + \beta_{m,n}(u^{m,n}))^+ \,dx\,dt \le 0. \end{aligned} \tag{4.12}$$

Thus, $u^{m',n} \le u^{m,n}$. Combining Remark 4.2 and Corollary 3.6, $u^{m,n}$ is uniformly bounded in $L^\infty(\tilde Q)$, and

$$\|u\|_{L^\infty(\tilde Q)} \le \|u_0\|_{L^\infty(\Omega)} + T(\|f\|_{L^\infty(Q)} + \|a\|_{L^\infty(\Sigma)}). \tag{4.13}$$

And hence, there exists an subsequence of $u^{m(n),n}$ and $u \in L^\infty(\tilde Q)$, such that $u^{m(n),n} \to u$ strongly in $\mathcal{C}(0,T;L^1_{\rm loc}(\tilde Q))$.

Arguing as in [1, Proposition 4.1], we have $u = \tilde a$ a.e. on $\tilde Q \setminus Q$. Then from the convergence of $u^{m(n),n}$, we deduce that $b^{m(n),n} \to b(u)$ strongly in $L^2(0,T;H^1_{\rm loc}(\Omega))$, and the trace of $b(u)$ on Σ is equal to $b(a)$. In the end, from the continuity of a, we prove that u is an entropy solution of (3.1) by passing the limit $n \to \infty$ in the entropy inequalities of $u^{m(n),n}$. $\square$

Proof of Theorem 2.4. Based on the results above, we give a brief proof of Theorem 2.4. For $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$, since F is Lipschitz continuous, we can deduce the existence for entropy solutions of (1.1) from Proposition 3.1 and Proposition 4.6 by Banach's contraction principle.

For any $u_0 \in L^\infty(\Omega)$, define $u_0^{m,n}$ as follows

$$u_0^{m,n}(x) = u_0^+ \chi_{B(0,m)}(x) - u_0^-(x)\chi_{B(0,n)}(x), \quad x \in \Omega.$$

For $u_0^{m,n} \in\in L^1(\Omega) \cap L^\infty(\Omega)$, there exists an entropy solution of (1.1), denoted by $u^{m,n}$. Applying Proposition 3.1 and using the method in [12], we can prove that there exists $u \in \mathcal{C}(0,T;L^1_{\rm loc}(\Omega))$ and a subsequence of $u^{m,n}$, denoted by $u^{m(n),n}$, such that $u^{m(n),n} \to u$ as $n \to \infty$, and u is indeed an entropy solution of (1.1). $\square$

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Li Zhang
Management School, Hangzhou Dianzi University, Hangzhou 310018, China
E-mail address: `zhli25@163.com`

Ning Su
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
E-mail address: `nsu@math.tsinghua.edu.cn`