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CONVERGENCE BEHAVIOUR OF SOLUTIONS TO DELAY CELLULAR NEURAL NETWORKS WITH NON-PERIODIC COEFFICIENTS

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Abstract. In this note we studied delay neural networks without periodic coefficients. Sufficient conditions are established to ensure that all solutions of the networks converge to a periodic function. An example is given to illustrate our results.

1. Introduction

Let n be the number of units in a neural network, $x_i(t)$ be the state vector of the i-th unit at time t, $a_{ij}(t)$ be the strength of the j-th unit on the i-th unit at time t, $b_{ij}(t)$ be the strength of the j-th unit on the i-th unit at time $t-\tau_{ij}(t)$, and $\tau_{ij}(t) \geq 0$ denote the transmission delay of the i-th unit along the axon of the j-th unit at the time t. It is well known that the delayed cellular neural networks are described by the differential equations

$$x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i(t), \quad (1.1)$$

for $i = 1, 2, \dots, n$, for any activation functions of signal transmission f_j and g_j. Here $I_i(t)$ denotes the external bias on the i-th unit at the time t, $c_i(t)$ represents the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t.

Since the cellular neural networks (CNNs) were introduced by Chua and Yang [2] in 1990, they have been successfully applied to signal and image processing, pattern recognition and optimization. Hence, CNNs have been the object of intensive analysis by numerous authors in recent years. In particular, extensive results on the problem of the existence and stability of periodic solutions for system (1.1) are given out in many literatures. We refer the reader to [3, 5, 4, 6, 7, 8, 9] and the references cited therein. Suppose that the following condition holds:

(H0) $c_i, I_i, a_{ij}, b_{ij},: \mathbb{R} \to \mathbb{R}$ are continuous periodic functions, where $i, j = 1, 2, \dots, n$.

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Under this assumption, most of reference above obtain that all solutions of system (1.1) converge to a periodic function. However, to the best of our knowledge, few authors have considered the convergence behavior for solutions of (1.1) without assuming (H0). Thus, it is worth while to consider the convergence behavior for solutions of (1.1) in this case.

The main purpose of this paper is to give the new criteria for the convergence behavior for all solutions of (1.1). By applying mathematical analysis techniques, without assuming (H0), we derive some sufficient conditions ensuring that all solutions of system (1.1) converge to a periodic function. These results are new and complement of previously known results. An example is provided to illustrate our results.

Consider the delay cellular neural networks

$$x_i'(t) = -c_i^*(t)x_i(t) + \sum_{j=1}^{n} a_{ij}^*(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}^*(t)g_j(x_j(t-\tau_{ij}(t))) + I_i^*(t), \quad (1.2)$$

where $i = 1, 2, \dots, n$. For the rest of this paper this paper, for $i, j = 1, 2, \dots, n$, it will be assumed that $c_i^*, I_i^*, a_{ij}^*, b_{ij}^*, \tau_{ij} : \mathbb{R} \to \mathbb{R}$ are continuous ω-periodic functions. Then, we can choose a constant τ such that

$$\tau = \max_{1\le i,j\le n} \big\{ \max_{t\in[0,\omega]} \tau_{ij}(t) \big\}. \quad (1.3)$$

We also use the following conditions:

(H1) For each $j \in \{1, 2, \dots, n\}$, there exist nonnegative constants $\tilde{L}_j$ and L_j such that

$$|f_j(u) - f_j(v)| \le \tilde{L}_j|u - v|, \quad |g_j(u) - g_j(v)| \le L_j|u - v|, \quad \forall u, v \in \mathbb{R}. \quad (1.4)$$

(H2) There exist constants $\eta > 0$, $\lambda > 0$ and $\xi_i > 0$, $i = 1, 2, \dots, n$, such that for all $t > 0$ and $i = 1, 2, \dots, n$,

$$-[c_i^*(t) - \lambda]\xi_i + \sum_{j=1}^{n} |a_{ij}^*(t)|\tilde{L}_j\xi_j + \sum_{j=1}^{n} |b_{ij}^*(t)|e^{\lambda\tau}L_j\xi_j < -\eta < 0,$$

(H3) For $i, j = 1, 2, \dots, n$, $c_i, I_i, a_{ij}, b_{ij} : \mathbb{R} \to \mathbb{R}$ are continuous functions, and

$$\lim_{t\to+\infty}(c_i(t) - c_i^*(t)) = 0, \quad \lim_{t\to+\infty}(I_i(t) - I_i^*(t)) = 0,$$
$$\lim_{t\to+\infty}(a_{ij}(t) - a_{ij}^*(t)) = 0, \lim_{t\to+\infty}(b_{ij}(t) - b_{ij}^*(t)) = 0.$$

The following lemma will be useful to prove our main results in Section 2.

Lemma 1.1 ([7]). *Let (H1) and (H2) hold. Then system* (1.2) *has exactly one* ω*-periodic solution.*

As usual, we introduce the phase space $C([-\tau, 0]; \mathbb{R}^n)$ as a Banach space of continuous mappings from $[-\tau, 0]$ to $\mathbb{R}^n$ equipped with the supremum norm,

$$\|\varphi\| = \max_{1\le i\le n} \sup_{-\tau\le t\le 0} |\varphi_i(t)|$$

for all $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in C([-\tau, 0]; \mathbb{R}^n)$.

The initial conditions associated with system (1.1) are of the form

$$x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad i = 1, 2, \dots, n, \quad (1.5)$$

where $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in C([-\tau, 0]; \mathbb{R}^n)$.

For $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, we define the norm

$$\|Z(t)\|_\xi = \max_{i=1,2,\dots,n} |\xi_i^{-1} x_i(t)|.$$

The remaining part of this paper is organized as follows. In Section 2, we present some new sufficient conditions to ensure that all solutions of system (1.1) converge to a periodic function. In Section 3, we shall give some examples and remarks to illustrate our results obtained in the previous sections.

2. Main Results

Theorem 2.1. *Assume (H1)–(H3) and that* $Z^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ *is the* ω*-periodic solution of* (1.2)*. Then every solution*

$$Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$$

of (1.1) *with initial value* $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in C([-\tau, 0]; \mathbb{R}^n)$*, satisfies*

$$\lim_{t\to+\infty} |x_i(t) - x_i^*(t)| = 0, \quad i = 1, 2, \dots, n.$$

Proof. Set

$$\begin{aligned}\delta_i(t) = -[c_i(t) - c_i^*(t)]x_i^*(t) &+ \sum_{j=1}^n [a_{ij}(t) - a_{ij}^*(t)] f_j(x_j^*(t)) \\ &+ \sum_{j=1}^n [b_{ij}(t) - b_{ij}^*(t)] g_j(x_j^*(t - \tau_{ij}(t))) + [I_i(t) - I_i^*(t)],\end{aligned}$$

where $i = 1, 2, \dots, n$. Since $Z^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ is ω-periodic, together with (H_2) and (H3), then for all $\epsilon > 0$, we can choose a sufficient large constant $T > 0$ such that

$$|\delta_i(t)| < \frac{1}{4}\eta\epsilon, \quad \text{for all } t \ge T, \tag{2.1}$$

and

$$-[c_i(t) - \lambda]\xi_i + \sum_{j=1}^n |a_{ij}(t)| \tilde{L}_j \xi_j + \sum_{j=1}^n |b_{ij}(t)| e^{\lambda\tau} L_j \xi_j < -\frac{1}{2}\eta < 0, \tag{2.2}$$

for all $t \ge T$, $i = 1, 2, \dots, n$. Let $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be a solution of (1.1) with initial value $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in C([-\tau, 0]; \mathbb{R}^n)$. Define

$$u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T = Z(t) - Z^*(t).$$

Then for $i = 1, 2, \dots, n$,

$$\begin{aligned}u_i'(t) = -c_i(t)u_i(t) &+ \sum_{j=1}^n a_{ij}(t)[f_j(x_j(t)) - f_j(x_j^*(t))] \\ &+ \sum_{j=1}^n b_{ij}(t)[g_j(x_j(t - \tau_{ij}(t))) - g_j(x_j^*(t - \tau_{ij}(t)))] + \delta_i(t)\,.\end{aligned} \tag{2.3}$$

Let i_t be an index such that

$$\xi_{i_t}^{-1} |u_{i_t}(t)| = \|u(t)\|_\xi. \tag{2.4}$$

Calculating the upper right derivative of $e^{\lambda s}|u_{i_s}(s)|$ along (2.3), in view of (2.1) and (H1), we have

$$\begin{aligned} & D^+(e^{\lambda s}|u_{i_s}(s)|)\Big|_{s=t} \\ &= \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\operatorname{sign}(u_{i_t}(t))\Big\{ - c_{i_t}(t)u_{i_t}(t) + \sum_{j=1}^{n} a_{i_t j}(t)[f_j(x_j(t)) - f_j(x_j^*(t))] \\ &\quad + \sum_{j=1}^{n} b_{i_t j}(t)[g_j(x_j(t-\tau_{i_t j}(t))) - g_j(x_j^*(t-\tau_{i_t j}(t)))] + \delta_{i_t}(t)\Big\} \\ &\le e^{\lambda t}\Big\{ - [c_{i_t}(t) - \lambda]|u_{i_t}(t)|\xi_{i_t}^{-1}\xi_{i_t} + \sum_{j=1}^{n} a_{i_t j}(t)\tilde{L}_j|u_j(t)|\xi_j^{-1}\xi_j \\ &\quad + \sum_{j=1}^{n} b_{i_t j}(t)L_j|u_j(t-\tau_{i_t j}(t))|\xi_j^{-1}\xi_j\Big\} + \frac{1}{4}\eta\epsilon e^{\lambda t}. \end{aligned} \tag{2.5}$$

Let

$$M(t) = \max_{-\tau \le s \le t}\{e^{\lambda s}\|u(s)\|_\xi\}. \tag{2.6}$$

It is obvious that $e^{\lambda t}\|u(t)\|_\xi \le M(t)$, and $M(t)$ is non-decreasing. Now, we consider two cases.

Case (i). If

$$M(t) > e^{\lambda t}\|u(t)\|_\xi \quad \text{for all } t \ge T. \tag{2.7}$$

Then, we claim that

$$M(t) \equiv M(T) \tag{2.8}$$

which is a constant for all $t \ge T$. By way of contradiction, assume that (2.8) does not hold. Consequently, there exists $t_1 > T$ such that $M(t_1) > M(T)$. Since

$$e^{\lambda t}\|u(t)\|_\xi \le M(T) \quad \text{for all } -\tau \le t \le T.$$

There must exist $\beta \in (T, t_1)$ such that

$$e^{\lambda\beta}\|u(\beta)\|_\xi = M(t_1) \ge M(\beta),$$

which contradicts (2.7). This contradiction implies (2.8). It follows that there exists $t_2 > T$ such that

$$\|u(t)\|_\xi < e^{-\lambda t}M(t) = e^{-\lambda t}M(T) < \epsilon \quad \text{for all } t \ge t_2. \tag{2.9}$$

Case (ii). If there is a point $t_0 \geq T$ such that $M(t_0) = e^{\lambda t_0}\|u(t_0)\|_\xi$, Then, using (2.1), (2.2) and (2.5), we get

$$
\begin{aligned}
&D^+(e^{\lambda s}|u_{i_s}(s)|)\Big|_{s=t_0}\\
&\leq \Big\{ -[c_{i_{t_0}}(t_0)-\lambda]e^{\lambda t_0}|u_{i_{t_0}}(t_0)|\xi_{i_{t_0}}^{-1}\xi_{i_{t_0}} + \sum_{j=1}^n a_{i_{t_0}j}(t_0)\tilde{L}_j e^{\lambda t_0}|u_j(t_0)|\xi_j^{-1}\xi_j\\
&\quad + \sum_{j=1}^n b_{i_{t_0}j}(t_0)L_j e^{\lambda(t_0-\tau_{i_{t_0}j}(t_0))}|u_j(t_0-\tau_{i_{t_0}j}(t_0))|\xi_j^{-1}e^{\lambda\tau_{i_{t_0}j}(t_0)}\xi_j\Big\} + \frac{1}{4}\eta\epsilon e^{\lambda t_0}\\
&\leq \Big\{ -[c_{i_{t_0}}(t_0)-\lambda]\xi_{i_{t_0}} + \sum_{j=1}^n a_{i_{t_0}j}(t_0)\tilde{L}_j\xi_j + \sum_{j=1}^n b_{i_{t_0}j}(t_0)e^{\lambda\tau}L_j\xi_j\Big\}M(t_0) + \frac{1}{4}\eta\epsilon e^{\lambda t_0}\\
&< -\frac{1}{2}\eta M(t_0) + \frac{1}{2}\eta\epsilon e^{\lambda t_0}.
\end{aligned}
$$

In addition, if $M(t_0) \geq \epsilon e^{\lambda t_0}$, then $M(t)$ is strictly decreasing in a small neighborhood $(t_0, t_0+\delta_0)$. This contradicts that $M(t)$ is non-decreasing. Hence,

$$
e^{\lambda t_0}\|u(t_0)\|_\xi = M(t_0) < \epsilon e^{\lambda t_0}, \quad \text{and} \quad \|u(t_0)\|_\xi < \epsilon. \tag{2.10}
$$

Furthermore, for any $t > t_0$, by the same approach used in the proof of (2.10), we have

$$
e^{\lambda t}\|u(t)\|_\xi < \epsilon e^{\lambda t}, \quad \text{and} \quad \|u(t)\|_\xi < \epsilon, \quad \text{if} \quad M(t) = e^{\lambda t}\|u(t)\|_\xi.
$$

On the other hand, if $M(t) > e^{\lambda t}\|u(t)\|_\xi, t > t_0$. We can choose $t_0 \leq t_3 < t$ such that

$$
M(t_3) = e^{\lambda t_3}\|u(t_3)\|_\xi, \quad \|u(t_3)\|_\xi < \epsilon, \quad M(s) > e^{\lambda s}\|u(s)\|_\xi \quad \text{for all } s \in (t_3, t].
$$

Using a similar argument as in the proof of **Case (i)**, we can show that

$$
M(s) \equiv M(t_3)
$$

which is a constant for all $s \in (t_3, t]$ which implies that

$$
\|u(t)\|_\xi < e^{-\lambda t}M(t) = e^{-\lambda t}M(t_3) = \|u(t_3)\|_\xi e^{-\lambda(t-t_3)} < \epsilon.
$$

In summary, there must exist $N > 0$ such that $\|u(t)\|_\xi \leq \epsilon$ holds for all $t > N$. This completes the proof of Theorem 2.1. $\square$

Remark 2.2. Without assumption (H3), we suppose that

(H1*) For each $j \in \{1, 2, \dots, n\}$, there exist nonnegative constants L_j and $\tilde{L}_j$ such that
$$
|f_j(u)| \leq \tilde{L}_j|u|, \quad |g_j(u)| \leq L_j|u|, \quad \text{for all } u \in \mathbb{R}
$$

(H2*) There exist constants $K > 0$, $\eta > 0$, $\lambda > 0$ and $\xi_i > 0$, $i = 1, 2, \dots, n$, such that for all $t > K$, there holds
$$
-[c_i(t)-\lambda]\xi_i + \sum_{j=1}^n |a_{ij}(t)|\tilde{L}_j\xi_j + \sum_{j=1}^n |b_{ij}(t)|e^{\lambda\tau}L_j\xi_j < -\eta < 0,
$$
are satisfied for $i = 1, 2, \dots, n$.

Moreover, assume that $I_i(t)$ converge to zero as $t \to \infty$. Then, applying the similar mathematical analysis techniques in this paper, we find that the sufficient conditions can be established to ensure the convergence to zero of all solutions of (1.1).

Remark 2.3. If the original system has asymptotically periodic coefficients, there may not be sufficient conditions ensuring the existence and the convergence of periodic solutions, which is the case in the example $x'(t) = -x(t) + \frac{1}{t^2+1}\sin^2 t$.

3. An Example

In this section, we illustrate the results obtained in previous sections. Consider the cellular neural networks with time-varying delays:

$$
\begin{aligned}
x_1'(t) &= -(1 - \frac{2}{1+|t|})x_1(t) + (\frac{1}{4} + \frac{t}{1+t^2})f_1(x_1(t)) + (\frac{1}{36} + \frac{2t}{1+t^2})f_2(x_2(t)) \\
&\quad + (\frac{1}{4} + \frac{t}{2+t^2})g_1(x_1(t-\sin^2 t)) + (\frac{1}{36} + \frac{4t}{1+t^2})g_2(x_2(t-2\sin^2 t)) \\
&\quad + (\cos t + \frac{t}{1+t^2}), \\
x_2'(t) &= -(1 - \frac{4}{1+2|t|})x_2(t) + (1 + \frac{t}{1+t^2})f_1(x_1(t)) + (\frac{1}{4} + \frac{5t}{1+t^2})f(x_2(t)) \\
&\quad + (1 + \frac{t}{1+6t^2})g_1(x_1(t-5\sin^2 t)) + (\frac{1}{4} + \frac{t}{8+t^2})g_2(x_2(t-\sin^4 t)) \\
&\quad + (\sin t + \frac{t}{1+t^6}),
\end{aligned}
\tag{3.1}
$$

where $f_1(x) = f_2(x) = g_1(x) = g_2(x) = \arctan x$.

Noting the following cellular neural networks

$$
\begin{aligned}
x_1'(t) &= -x_1(t) + \frac{1}{4}f_1(x_1(t)) + \frac{1}{36}f_2(x_2(t)) + \frac{1}{4}g_1(x_1(t-\sin^2 t)) \\
&\quad + \frac{1}{36}g_2(x_2(t-2\sin^2 t)) + \cos t, \\
x_2'(t) &= -x_2(t) + f_1(x_1(t)) + \frac{1}{4}f(x_2(t)) + g_1(x_1(t-5\sin^2 t)) \\
&\quad + \frac{1}{4}g_2(x_2(t-\sin^4 t)) + \sin t,
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{gathered}
c_1^*(t) = c_2^*(t) = L_1 = L_2 = \tilde{L}_1 = \tilde{L}_2 = 1, \\
a_{11}^*(t) = b_{11}^*(t) = \frac{1}{4}, \quad a_{12}^*(t) = b_{12}^*(t) = \frac{1}{36}, \\
a_{21}^*(t) = b_{21}^*(t) = 1, \quad a_{22}^*(t) = b_{22}^*(t) = \frac{1}{4}, \quad \tau = 5.
\end{gathered}
$$

Then

$$
\begin{gathered}
d_{ij} = \frac{1}{c_i^*(t)}(a_{ij}^*(t)\tilde{L}_j + b_{ij}^*(t)L_j) \quad i,j = 1,2, \\
D = (d_{ij})_{2\times 2} = \begin{pmatrix} 1/2 & 1/18 \\ 2 & 1/2 \end{pmatrix}.
\end{gathered}
$$

Hence, $\rho(D) = 5/6 < 1$. It follows from the theory of M-matrix in [9] that there exist constants $\bar{\eta} > 0$ and $\xi_i > 0$, $i = 1, 2$, such that for all $t > 0$, there holds

$$
-c_i^*(t)\xi_i + \sum_{j=1}^n |a_{ij}^*(t)|\tilde{L}_j\xi_j + \sum_{j=1}^n |b_{ij}^*(t)|L_j\xi_j < -\bar{\eta} < 0, \quad i = 1, 2.
$$

Then, we can choose constants $\eta > 0$ and $0 < \lambda < 1$ such that

$$-[c_i^*(t) - \lambda]\xi_i + \sum_{j=1}^{n} |a_{ij}^*(t)|\tilde{L}_j\xi_j + \sum_{j=1}^{n} |b_{ij}^*(t)|e^{\lambda\tau} L_j\xi_j < -\eta < 0, \quad i = 1, 2, \ \forall t > 0,$$

which implies that systems (3.1) and (3.2) satisfy (H1)–(H3). Hence, from Lemma 1.1 and Theorem 2.1, system (3.2) has exactly one 2π-periodic solution. Moreover, all solutions of (3.1) converge to the periodic solution of (3.2).

Remark 3.1. Since CNNs (3.1) is a delayed neural networks without periodic coefficients, the results in [3, 5, 4, 6, 7, 8, 9] and the references therein can not be applied to prove that all solutions converge to a periodic function. This implies that the results of this paper are essentially new nd they complement previously known results.

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