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# GLOBAL SOLUTIONS TO A ONE-DIMENSIONAL NONLINEAR WAVE EQUATION DERIVABLE FROM A VARIATIONAL PRINCIPLE

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Abstract. This article focuses on a one-dimensional nonlinear wave equation which is the Euler-Lagrange equation of a variational principle whose Lagrangian density involves linear terms and zero term as well as quadratic terms in derivatives of the field. We establish the global existence of weak solutions to its Cauchy problem by the method of energy-dependent coordinates which allows us to rewrite the equation as a semilinear system and resolve all singularities by introducing a new set of variables related to the energy.

## 1. Introduction

The variational principle whose action is a quadratic function of the derivatives of the field with coefficients depending on the field and the independent variables takes the form [1, 19, 20]

$$\delta \int A_{\mu\nu}^{ij}(\mathbf{x},\mathbf{u})\frac{\partial u^\mu}{\partial x_i}\frac{\partial u^\nu}{\partial x_j}\,\mathrm{d}\mathbf{x}=0, \tag{1.1}$$

where the summation convention is employed. Here, $\mathbf{x}\in\mathbb{R}^{d+1}$ are the space-time independent variables and $\mathbf{u}:\mathbb{R}^{d+1}\to\mathbb{R}^n$ are the dependent variables. The coefficients $A_{\mu\nu}^{ij}:\mathbb{R}^{d+1}\times\mathbb{R}^n\to\mathbb{R}$ are smooth functions.

A particular motivation for studying (1.1) comes from the theory of nematic liquid crystals. In the regime in which inertia effects dominate viscosity, Saxton [23] modelled the propagation of the orientation waves in the director field of a nematic liquid crystal by the least action principle

$$\delta \int \Big(\frac{1}{2}\partial_t\mathbf{n}\cdot\partial_t\mathbf{n}-W_n(\mathbf{n},\nabla\mathbf{n})\Big)\mathrm{d}\mathbf{x}\mathrm{d}t=0,\quad \mathbf{n}\cdot\mathbf{n}=1, \tag{1.2}$$

where $\mathbf{n}(\mathbf{x},t)$ is the director field and $W_n(\mathbf{n},\nabla\mathbf{n})$ is the well-known Oseen-Franck potential energy density,

$$W_n(\mathbf{n},\nabla\mathbf{n})=\frac{1}{2}k_1(\nabla\cdot\mathbf{n})^2+\frac{1}{2}k_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2+\frac{1}{2}k_3|\mathbf{n}\times(\nabla\times\mathbf{n})|^2,$$

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which is a quadratic function of $\nabla\mathbf{n}$ with coefficients depending on $\mathbf{n}$. Here $k_1, k_2$ and $k_3$ are the splay, twist and bend elastic constants of the liquid crystal, respectively.

The hyperbolic systems of nonlinear wave equations derivable from the variational principle (1.2) have been widely explored since they were introduced. Hunter and Saxton [19] considered the planar deformations of the director field $\mathbf{n}$ that depend only on a single space variable $x$ with $\mathbf{n} = (\cos u(t,x), \sin u(t,x), 0)$ and derived the Euler-Lagrange equation of (1.2) given by

$$u_{tt} - c(u)[c(u)u_x]_x = 0 \tag{1.3}$$

with $c^2(u) = k_1 \sin^2 u + k_3 \cos^2 u$. For another important application, equation (1.3) describes the motion of long waves on a neutral dipole chain in the continuum limit [31]. A basic feature of (1.3) is that, even for smooth initial data, cusp-type singularities can form in finite time, see [13, 14]. In a series of papers [25, 26, 27, 28], Zhang and Zheng have studied carefully the global existence of dissipative weak solutions to the initial value problems for (1.3) and its asymptotic models. Bressan and Huang [6] proposed another way to construct a global dissipative solution to (1.3). The global existence and uniqueness of conservative weak solutions to its Cauchy problem for initial data of finite energy were established, respectively, by Bressan and Zheng [8] and Bressan, Chen and Zhang [5]. Holden and Raynaud [15] carried out a detailed construction of a global semigroup for its conservative weak solutions. A Lipschitz continuous metric of conservative weak solutions to (1.3) has been constructed recently by Bressan and Chen [4]. The generic properties of conservative solutions was studied in [3]. In [17], the first author investigated a more general nonlinear wave equation than (1.3) arising from the variational principle (1.1) and constructed a global energy-conservative weak solution to its initial value problem.

For the three-dimensional deformations depending on a single space variable $x$ and the director field $\mathbf{n}$ taking the form

$$\mathbf{n} = (\cos u(x,t), \sin u(x,t)\cos v(x,t), \sin u(x,t)\sin v(x,t)),$$

the Euler-Lagrange equations of (1.2) are

$$\begin{aligned} u_{tt} - (c_1^2(u)u_x)_x &= -c_1(u)\partial_u c_1(u)u_x^2 + a(u)\partial_u a(u)[v_t^2 - c_2^2(u)v_x^2] \\ &\quad - a^2(u)c_2(u,x)\partial_u c_2(u)v_x^2, \\ (a^2(u)v_t)_t &- [a^2(u)c_2^2(u)v_x]_x = 0 \end{aligned} \tag{1.4}$$

where $c_1^2(u) = k_1\sin^2 u + k_3\cos^2 u, c_2^2(u) = k_2 \sin^2 u + k_3 \cos^2 u$ and $a^2(u) = \sin^2 u$. System (1.4) was first derived by Ali and Hunter [2] to describe the propagation of splay and twist waves in nematic liquid crystals. They also analyzed some properties for the asymptotic equations of (1.4) in [2]. Recently, Zhang and Zheng [29, 30] demonstrated the global existence of conservative weak solutions to the Cauchy problem of (1.4) under some assumptions. The relevant results about the related nonlinear wave system derived from the variational principle of (1.1) was provided by the author [16]. We also refer the reader to [9] by Chen, Zhang and Zheng for the discussion of the nonlinear wave system obtained from (1.2) by considering the director field $\mathbf{n}$ in its natural three-component form, also see Chen and Zheng [10] for the analysis of the corresponding viscous system.

In a recent paper [18], we studied a nonlinear wave system coming from the theory of cholesteric liquid crystals, in which the Oseen-Franck potential energy density is expressed as the sum of an elastic and a chiral contribution (neglecting a constant factor)

$$W_c(\mathbf{n},\nabla\mathbf{n}) = \frac{1}{2}k_1(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}k_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2 + \frac{1}{2}k_3|\mathbf{n}\times(\nabla\times\mathbf{n})|^2 + \lambda\mathbf{n}\cdot\nabla\times\mathbf{n}, \tag{1.5}$$

where $\lambda$ is a pseudoscalar material parameter that represents molecular chirality, see e.g. [11, 12]. Compared with the nematic, linear terms in derivatives of the director field $\mathbf{n}$ must be considered in cholesteric liquid crystals. For detailed information regarding cholesteric liquid crystals, see, for example, [12, 21, 22, 24]. Considering the three-dimensional deformations and replacing $W_n(\mathbf{n},\nabla\mathbf{n})$ by $W_c(\mathbf{n},\nabla\mathbf{n})$, the variational principle (1.2) gives rise to the associated Euler-Lagrange equations

$$\begin{aligned} u_{tt} - (c_1^2(u)u_x)_x &= -c_1(u)\partial_u c_1(u)u_x^2 + a(u)\partial_u a(u)[v_t^2 - c_2^2(u)v_x^2] \\ &\quad - a^2(u)c_2(u,x)\partial_u c_2(u)v_x^2 + 2\lambda a(u)\partial_u a(u)v_x, \\ (a^2(u)v_t)_t &- [a^2(u)c_2^2(u)v_x - \lambda a^2(u)]_x = 0. \end{aligned} \tag{1.6}$$

In contrast to equations (1.4), the linear terms in (1.5) cause the total energy of solution for system (1.6) to be not conservative. The global existence of weak solutions to its Cauchy problem was established in [18] by using the method of energy-dependent coordinates and the Young measure theory.

Motivated by (1.5), we are interested in the variational principle whose Lagrangian density includes linear terms and zero term as well as quadratic terms in derivatives of the field

$$\delta\int\Big\{A^{ij}_{\mu\nu}(\mathbf{x},\mathbf{u})\frac{\partial u^\mu}{\partial x_i}\frac{\partial u^\nu}{\partial x_j} + B^i_\mu(\mathbf{x},\mathbf{u})\frac{\partial u^\mu}{\partial x_i} + F(\mathbf{x},\mathbf{u})\Big\}\,\mathrm{d}\mathbf{x} = 0, \tag{1.7}$$

where $\mathbf{x}$, $\mathbf{u}$ and $A^{ij}_{\mu\nu}$ are as in (1.1), $B^i_\mu$ and $F$ are smooth functions. The term $F$ can be regarded as the contribution from the external electrical or magnetic field. Consider $n = 1$ and $d = 1$, then the Euler-Lagrange equation for (1.7) reads that

$$\begin{aligned} &(2A^{11}u_t + (A^{12}+A^{21})u_x + B^1)_t + ((A^{12}+A^{21})u_t + 2A^{22}u_x + B^2)_x \\ &= \frac{\partial A^{11}}{\partial u}u_t^2 + \frac{\partial(A^{12}+A^{21})}{\partial u}u_tu_x + \frac{\partial A^{22}}{\partial u}u_x^2 + \frac{\partial B^1}{\partial u}u_t + \frac{\partial B^2}{\partial u}u_x + \frac{\partial F}{\partial u}. \end{aligned} \tag{1.8}$$

In this article, we consider the initial value problem for equation (1.8) with

$$(A^{ij})_{2\times2} = \frac{1}{2}\begin{pmatrix}\alpha^2 & \beta_1\\ \beta_2 & -\gamma^2\end{pmatrix}(x,u), \tag{1.9}$$

and $B^1 = \kappa(x,u)$, $B^2 = \lambda(x,u)$, $F = F(x,u)$, where $\alpha,\gamma,\kappa,\lambda,F$ and $\beta_1+\beta_2 := 2\beta$ are smooth functions, independent of $t$, satisfying

$$\begin{gathered} 0 < \alpha_1 \le \alpha(z) \le \alpha_2, \quad |\beta(z)| + |\kappa(z)| + |\lambda(z)| + |F(z)| \le \Lambda, \\ 0 < \gamma_1 \le \gamma(z) \le \gamma_2, \\ \sup_z\{|\nabla\alpha(z)|, |\nabla\beta(z)|, |\nabla\gamma(z)|, |\nabla\kappa(z)|, |\nabla\lambda(z)|, |\nabla F(z)|\} \le \Lambda, \quad \forall z\in\mathbb{R}^2 \end{gathered} \tag{1.10}$$

for positive numbers $\alpha_1,\alpha_2,\gamma_1,\gamma_2$ and $\Lambda$. Under the above assumptions, (1.8) reduces to

$$\begin{aligned} &(\alpha^2u_t + \beta u_x + \kappa)_t + (\beta u_t - \gamma^2u_x + \lambda)_x \\ &= \alpha\alpha_u u_t^2 + \beta_u u_tu_x - \gamma\gamma_u u_x^2 + \kappa_u u_t + \lambda_u u_x + F_u, \end{aligned} \tag{1.11}$$

which is strictly hyperbolic with two eigenvalues

$$\lambda_+ = \frac{\sqrt{\beta^2+\alpha^2\gamma^2}+\beta}{\alpha^2} > 0, \quad \lambda_- = -\frac{\sqrt{\beta^2+\alpha^2\gamma^2}-\beta}{\alpha^2} < 0. \tag{1.12}$$

Equation (1.11) is a second order quasilinear hyperbolic equation whose main difficulty arises from the possible cusp-type singularities of solutions in finite time.

The purpose of this article is to establish the global existence of weak solutions to the initial value problem (1.11) with the initial data

$$u(0,x) = u_0(x) \in H^1, \quad u_t(0,x) = u_1(x) \in L^2 \tag{1.13}$$

under the assumption (1.10). The approach we used here follows the method of energy-dependent coordinates proposed by Bressan, Zhang and Zheng [7, 8] to deal with (1.3) and its related asymptotic model. However, in contrast to equation (1.3), the energy of solution for (1.11), denoted by

$$\mathcal{E}(t) := \frac{1}{2}\int \Big\{\alpha^2(x,u)u_t^2 + \gamma^2(x,u)u_x^2\Big\}\mathrm{d}x,$$

is not conservative. In spite of this, we can still establish a priori estimates of solutions for the equivalent semilinear system in the energy-dependent coordinates. By returning the solution in terms of the original variables, we thus recover a global weak solution to (1.11).

Before stating the main results, let us first give the definition of weak solutions to problem (1.11) (1.13).

**Definition 1.1** (Weak solution)**.** A function $u(t,x)$ with $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ is a *weak solution* to the Cauchy problem (1.11) (1.13) if the following hold:
(i) In the $t$-$x$ plane, the function $u$ is locally Hölder continuous with exponent $1/2$. The function $t \mapsto u(t,\cdot)$ is continuously differentiable as a map with values in $L^\theta_{\rm loc}$ for all $1 \le \theta < 2$. Moreover, for any $T > 0$, it satisfies the Lipschitz continuity property

$$\|u(t,\cdot) - u(s,\cdot)\|_{L^2_{\rm loc}} \le L|t-s|, \quad \forall t,s \in (0,T] \tag{1.14}$$

for some constant $L$ depending on $T$ with $L = O(\sqrt{T})$.
(ii) The function $u(t,x)$ takes on the initial condition in (1.13) pointwise, while its temporal derivative holds in $L^\theta_{\rm loc}$ for $\theta \in [1,2)$.
(iii) Equation (1.11) is satisfied in the distributional sense, that is

$$\begin{aligned}\iint_{\mathbb{R}^+\times\mathbb{R}} &\Big\{\varphi_t(\alpha^2 u_t + \beta u_x + \kappa) + \varphi_x(\beta u_t - \gamma^2 u_x + \lambda) \\ &+ \varphi\Big(\alpha\alpha_u u_t^2 + \beta_u u_t u_x - \gamma\gamma_u u_x^2 + \kappa_u u_t + \lambda_u u_x + F_u\Big)\Big\}\mathrm{d}x\mathrm{d}t = 0\end{aligned} \tag{1.15}$$

for all test functions $\varphi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$.

The conclusions of this paper are as follows.

**Theorem 1.2** (Existence)**.** *Let condition* (1.10) *be satisfied. Then the Cauchy problem* (1.11) (1.13) *admits a global weak solution defined for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.

The continuous dependence of the solution upon the initial data follows directly from the constructive procedure.

**Theorem 1.3** (Continuous dependence). *Suppose the assumptions in Theorem 1.2 hold. For the Cauchy problem* (1.11) (1.13)*, let a sequence of initial conditions satisfy*

$$\|(u_0^\nu)_x - (u_0)_x\|_{L^2} \to 0, \quad \|u_1^\nu - u_1\|_{L^2} \to 0,$$

*and $u_0^\nu \to u_0$ uniformly on compact sets, as $\nu \to \infty$. Then one has the convergence of the corresponding solutions $u^\nu \to u$, uniformly on bounded subsets of the $(t,x)$-plane with $t > 0$.*

The article is organized as follows. In Section 2, we introduce a new set of dependent and independent variables, and derive an equivalent semilinear system of (1.11) for smooth solutions. Section 3 presents the existence and continuous dependence results for solutions to the equivalent semilinear system. We show the Hölder continuity of solutions $(u, v)$ in terms of the original independent variables $t, x$, and verify that the integral equation (1.15) is satisfied in Section 4. Finally, in Section 5, we establish the Lipschitz continuity property (1.14) and the continuity of the maps $t \mapsto u_t(t,\cdot)$, $t \mapsto u_x(t,\cdot)$ as functions with values in $L^\theta_{\rm loc}(1 \le \theta < 2)$, which complete the proof of Theorems 1.2.

## 2. New formulation in energy-dependent coordinates

In this section, we derive an equivalent system of (1.11) for smooth solutions by introducing a new set of variables to replace the original variables $u, t, x$.

2.1. **Energy-dependent coordinates.** Denote $c_1 := \alpha\lambda_- < 0$ and $c_2 := \alpha\lambda_+ > 0$, and

$$R := \alpha u_t + c_2 u_x, \quad S := \alpha u_t + c_1 u_x. \tag{2.1}$$

Then (1.11) can be rewritten as

$$\begin{gathered}
\alpha(x,u)R_t + c_1(x,u)R_x = a_1 R^2 - (a_1 + a_2)RS + a_2 S^2 + c_2 bS - d_1 R - \lambda_x + F_u,\\
\alpha(x,u)S_t + c_2(x,u)S_x = -a_1 R^2 + (a_1 + a_2)RS - a_2 S^2 + c_1 bR - d_2 S - \lambda_x + F_u,\\
\alpha(x,u)u_t + c_1(x,u)u_x = S,
\end{gathered} \tag{2.2}$$

where

$$\begin{gathered}
a_i = \frac{c_i\partial_u\alpha - \alpha\partial_u c_i}{2\alpha(c_2 - c_1)}, \quad b = \frac{\alpha\partial_x(c_1 - c_2) + (c_1 - c_2)\partial_x\alpha}{2\alpha(c_2 - c_1)}\\
d_i = \frac{c_2\partial_x c_1 - c_1\partial_x c_2}{2(c_2 - c_1)} + \frac{\alpha\partial_x c_i - c_i\partial_x\alpha}{2\alpha}, \quad (i = 1,2)
\end{gathered} \tag{2.3}$$

and $\partial_x$ and $\partial_u$ denote, respectively, partial derivatives with respect to the arguments $x$ and $u$. System (2.2) is equivalent to equation (1.6) for smooth solutions if we supplement it with initial restriction at $t = 0$,

$$u_x = \frac{R - S}{c_2(x,u) - c_1(x,u)}. \tag{2.4}$$

For convenience to deal with possibly unbounded values of $R$ and $S$, we introduce a new set of dependent variables

$$\ell := \frac{R}{1+R^2}, \quad h := \frac{1}{1+R^2}, \quad m := \frac{S}{1+S^2}, \quad g := \frac{1}{1+S^2}, \tag{2.5}$$

from which one easily checks that

$$\ell^2 + h^2 = h, \quad m^2 + g^2 = g. \tag{2.6}$$

Define the forward and backward characteristics as follows:

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}s}x^{\pm}(s;t,x)=\lambda_{\pm}(x^{\pm}(s;t,x),u(s;x^{\pm}(s;t,x))),\\ x^{\pm}|_{s=t}=x,\end{gathered}\tag{2.7}$$

where $\lambda_{\pm}$ are given in (1.12). Now we define the coordinate transformation $(t,x)\to(X,Y)$ where

$$X:=\int_0^{x^-(0;t,x)}\Big(1+R^2(0,\xi)\Big)\mathrm{d}\xi,\quad Y:=\int_{x^+(0;t,x)}^0\Big(1+S^2(0,\xi)\Big)\mathrm{d}\xi,\tag{2.8}$$

which indicates that

$$\alpha(x,u)X_t+c_1(x,u)X_x=0,\quad \alpha(x,u)Y_t+c_2(x,u)Y_x=0,\tag{2.9}$$

from which, it turns out that

$$f_t=-\frac{c_1}{\alpha}X_xf_X-\frac{c_2}{\alpha}Y_xf_Y,\quad f_x=X_xf_X+Y_xf_Y\tag{2.10}$$

for any smooth function $f(t,x)$. Moreover, we introduce the new variables

$$p:=\frac{1+R^2}{X_x},\quad q:=\frac{1+S^2}{-Y_x}.\tag{2.11}$$

Then we can obtain a semilinear hyperbolic system with smooth coefficients for the variables $g,h,\ell,m,p,q,u,x$ in $(X,Y)$ coordinates as follows:

$$\begin{aligned}\ell_Y&=\frac{q(2h-1)}{c_2-c_1}\Big\{a_1g+a_2h-(a_1+a_2)(gh+m\ell)+c_2bhm\\&\quad-d_1g\ell+(F_u-\lambda_x)gh\Big\},\\ m_X&=\frac{p(2g-1)}{c_2-c_1}\Big\{-a_1g-a_2h+(a_1+a_2)(gh+m\ell)+c_1bg\ell\\&\quad-d_2hm+(F_u-\lambda_x)gh\Big\},\\ u_X&=\frac{1}{c_2-c_1}p\ell\quad\Big(\text{or } u_Y=\frac{1}{c_2-c_1}qm\Big),\\ x_X&=\frac{c_2}{c_2-c_1}ph\quad\Big(\text{or } x_Y=\frac{c_1}{c_2-c_1}qg\Big),\end{aligned}\tag{2.12}$$

$$\begin{aligned}h_Y&=-\frac{2q\ell}{c_2-c_1}\Big\{a_1g+a_2h-(a_1+a_2)(gh+m\ell)+c_2bhm\\&\quad-d_1g\ell+(F_u-\lambda_x)gh\Big\},\\ g_X&=-\frac{2pm}{c_2-c_1}\Big\{-a_1g-a_2h+(a_1+a_2)(gh+m\ell)+c_1bg\ell\\&\quad-d_2hm+(F_u-\lambda_x)gh\Big\},\end{aligned}\tag{2.13}$$

and

$$
\begin{aligned}
p_Y &= \frac{2pq}{c_2-c_1}\Big\{a_2(\ell-m)+(a_1+a_2)(hm-g\ell)+c_2bm\ell+d_1gh \\
&\quad + \frac{c_1\partial_x c_2-c_2\partial_x c_1}{2(c_2-c_1)}g+(F_u-\lambda_x)g\ell\Big\}, \\
q_X &= \frac{2pq}{c_2-c_1}\Big\{a_1(\ell-m)+(a_1+a_2)(hm-g\ell)+c_1bm\ell+d_2gh \\
&\quad + \frac{c_1\partial_x c_2-c_2\partial_x c_1}{2(c_2-c_1)}h+(F_u-\lambda_x)mh\Big\}.
\end{aligned} \tag{2.14}
$$

We here point out that

$$
\Big(\frac{qm}{c_2-c_1}\Big)_X=\Big(\frac{p\ell}{c_2-c_1}\Big)_Y,\quad \Big(\frac{c_1qg}{c_2-c_1}\Big)_X=\Big(\frac{c_2ph}{c_2-c_1}\Big)_Y, \tag{2.15}
$$

which imply that $u_{XY}=u_{YX}$ and $x_{XY}=x_{YX}$, so we may use either $u_X,x_X$ or $u_Y,x_Y$ in (2.12).

2.2. **Initial data conversion.** We now consider the boundary conditions of system (2.12)-(2.14) in the energy-dependent coordinates $(X,Y)$, corresponding to (1.13) in the original coordinates $(t,x)$.

It is easily known by (1.10) and (1.13) that

$$
\begin{aligned}
R(0,x)&=\alpha(x,u_0(x))u_1(x)+c_2(x,u_0(x))u_0'(x)\in L^2,\\
S(0,x)&=\alpha(x,u_0(x))u_1(x)+c_1(x,u_0(x))u_0'(x)\in L^2.
\end{aligned}
$$

The initial line $t=0$ in the $(t,x)$ plane is transformed to a curve $\Gamma_0: Y=\phi(X)$ defined through a parametric $x\in\mathbb{R}$

$$
X=\int_0^x[1+R^2(0,\xi)]\mathrm{d}\xi,\quad Y=\int_x^0[1+S^2(0,\xi)]\mathrm{d}\xi, \tag{2.16}
$$

which, clearly, is non-characteristic. We see by (1.10) and (1.12) that the two functions $X=X(x)$, $Y=Y(x)$ are well defined and absolutely continuous. Moreover, $X$ is strictly increasing while $Y$ is strictly decreasing. Hence the map $X\mapsto\phi(X)$ is continuous and strictly decreasing. In addition, applying (1.10) and (1.12) again arrives at

$$
\begin{aligned}
\frac{1}{\underline{M}} &:= \frac{\alpha_1^2\gamma_1^2}{2(\Lambda^2+\alpha_2^2\gamma_2^2+\Lambda\sqrt{\Lambda^2+\alpha_2^2\gamma_2^2})}\\
&\le \Big|\frac{c_i}{c_2-c_1}\Big|\\
&\le \frac{\Lambda+\sqrt{\Lambda^2+\alpha_2^2\gamma_2^2}}{2\alpha_1\gamma_1}:=\overline{M}
\end{aligned} \tag{2.17}
$$

for $i=1,2$. Therefore, the map $X\mapsto\phi(X)$ satisfies

$$
\begin{aligned}
|X+\phi(X)| &\le \int_{\mathbb{R}}\Big(R^2(0,\xi)+S^2(0,\xi)\Big)\mathrm{d}\xi\\
&=\underline{M}\int_{\mathbb{R}}\Big(\alpha^2(\xi,u_0(\xi))u_1^2(\xi)+\gamma^2(\xi,u_0(\xi))(u_0'(\xi))^2\Big)\mathrm{d}\xi\\
&=: 2\underline{M}\mathcal{E}_0,
\end{aligned} \tag{2.18}
$$

which is a finite number. The coordinate transformation maps the domain $[0,\infty)\times\mathbb{R}$ in the $(t,x)$ plane into the set

$$\Omega^+ := \{(X,Y); Y \ge \phi(X)\} \tag{2.19}$$

in the $(X,Y)$ plane. Along the curve $\Gamma_0$ parameterized by $x \mapsto (X(x),Y(x))$, we can thus assign the boundary data $(\bar h,\bar g,\bar\ell,\bar m,\bar p,\bar q,\bar u) \in L^\infty$ defined by

$$\begin{gathered}
\bar h = \frac{1}{1+R^2(0,x)}, \quad \bar g = \frac{1}{1+S^2(0,x)},\\
\bar\ell = R(0,x)\bar h, \quad \bar m = S(0,x)\bar g,\\
\bar p = 1, \quad \bar q = 1,\\
\bar u = u_0(x), \quad \bar x = x.
\end{gathered} \tag{2.20}$$

It is easily checked that $h^2+\ell^2-h=0$ and $g^2+m^2-g=0$ on $\Gamma_0$.

## 3. Solutions to the equivalent system

This section is devoted to establishing the existence of a unique global solution for system (2.12)-(2.14) with boundary data (2.20) in the energy coordinates $(X,Y)$. The method follows from Bressan and Zheng [8], and also see Zhang and Zheng [29] and Hu [17].

We first derive several identities from system (2.12)-(2.14), which are useful to derive the desired a priori estimates for the solutions.

(i) Consistency:

$$\partial_Y(h^2+\ell^2-h)=0, \quad \partial_X(g^2+m^2-g)=0. \tag{3.1}$$

*Proof.* The proof is directly from (2.12) and (2.13). □

Thanks to (3.1) and the boundary conditions (2.20), we find that

$$h^2+\ell^2=h, \quad g^2+m^2=g, \quad \forall\,(X,Y)\in\Omega^+. \tag{3.2}$$

(ii) An identity:

$$\Big(\frac{c_2q(1-g)}{2(c_2-c_1)}+\frac{\lambda qm}{c_2-c_1}+\frac{c_1Fqg}{c_2-c_1}\Big)_X-\Big(\frac{c_1p(1-h)}{2(c_2-c_1)}+\frac{\lambda p\ell}{c_2-c_1}+\frac{c_2Fph}{c_2-c_1}\Big)_Y=0. \tag{3.3}$$

*Proof.* It first follows from (2.15) that

$$\begin{gathered}
\Big(\frac{\lambda qm}{c_2-c_1}\Big)_X-\Big(\frac{\lambda p\ell}{c_2-c_1}\Big)_Y=\frac{-\lambda_x pq}{(c_2-c_1)^2}(c_1g\ell-c_2mh),\\
\Big(\frac{c_1Fqg}{c_2-c_1}\Big)_X-\Big(\frac{c_2Fph}{c_2-c_1}\Big)_Y=\frac{F_upq}{(c_2-c_1)^2}(c_1g\ell-c_2mh),
\end{gathered}$$

from which one deduces

$$\begin{aligned}
&\Big(\frac{\lambda qm}{c_2-c_1}+\frac{c_1Fqg}{c_2-c_1}\Big)_X-\Big(\frac{\lambda p\ell}{c_2-c_1}+\frac{c_2Fph}{c_2-c_1}\Big)_Y\\
&=\frac{pq}{(c_2-c_1)^2}(F_u-\lambda_x)(c_1g\ell-c_2mh).
\end{aligned} \tag{3.4}$$

On the other hand, by (2.10) and (2.5) we compute

$$\Big(\frac{c_2}{c_2-c_1}\Big)_X=\frac{p}{(c_2-c_1)^2}\Big\{\frac{c_2\partial_uc_1-c_1\partial_uc_2}{c_2-c_1}\ell+\frac{c_2(c_2\partial_xc_1-c_1\partial_xc_2)}{c_2-c_1}h\Big\}. \tag{3.5}$$

According to (2.13) and (2.14) and employing (3.2) yields

$$\begin{aligned}\Big(q(1-g)\Big)_X = \frac{2pq}{c_2-c_1}\Big\{ & a_1(\ell - m - g\ell + hm) + c_1 bm\ell \\ & + \frac{c_1\partial_x c_2 - c_2\partial_x c_1}{2(c_2-c_1)}(1-g)h + (F_u - \lambda_x)mh\Big\}.\end{aligned} \tag{3.6}$$

We combine (3.5) and (3.6) and use (2.3) to deduce

$$\begin{aligned}\Big(\frac{c_2 q(1-g)}{2(c_2-c_1)}\Big)_X = \frac{pq}{(c_2-c_1)^2}\Big\{ & c_1 a_2(1-g)\ell - c_2 a_1(1-h)m \\ & + c_1c_2 bm\ell + (F_u - \lambda_x)c_2 mh\Big\}.\end{aligned} \tag{3.7}$$

Similarly, one finds

$$\begin{aligned}\Big(\frac{c_1 p(1-h)}{2(c_2-c_1)}\Big)_Y = \frac{pq}{(c_2-c_1)^2}\Big\{ & c_1 a_2(1-g)\ell - c_2 a_1(1-h)m \\ & + c_1c_2 bm\ell + (F_u - \lambda_x)c_1 g\ell\Big\}.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) leads to

$$\Big(\frac{c_2 q(1-g)}{2(c_2-c_1)}\Big)_X - \Big(\frac{c_1 p(1-h)}{2(c_2-c_1)}\Big)_Y = \frac{pq}{(c_2-c_1)^2}(F_u-\lambda_x)(c_2 mh - c_1 g\ell),$$

which together with (3.4) gives (3.3). □

We now establish a priori estimates for solutions to the semilinear hyperbolic system (2.12)-(2.14) in $\Omega^+$. Obviously, it turns out by (3.2) that

$$0 \le h \le 1, \quad 0 \le g \le 1, \quad |\ell| \le \frac{1}{2}, \quad |m| \le \frac{1}{2}. \tag{3.9}$$

Based on (3.3) and (2.14), We next estimate the functions $p$ and $q$. It is easy to see from (2.14) and the initial condition $\bar p = \bar q = 1$ that $p$ and $q$ are positive in $\Omega^+$. In view of (2.15) the differential form

$$\frac{c_2 ph}{c_2-c_1}\mathrm{d}X + \frac{c_1 qg}{c_2-c_1}\mathrm{d}Y \tag{3.10}$$

has zero integral along every closed curve contained in $\Omega^+$. Then, for every $(X,Y) \in \Omega^+$, we construct the closed curve $S$ composed of the following three parts: the vertical segment with the endpoints $(X,\phi(X))$ and $(X,Y)$, the horizontal segment with the endpoints $(X,Y)$ and $(\phi^{-1}(Y),Y)$, and the boundary curve $\Gamma$ with the endpoints $(\phi^{-1}(Y),Y)$ and $(X,\phi(X))$. Here $\phi^{-1}$ denotes the inverse of $\phi$. We integrate (3.10) along the closed curve $S$ and use (2.17) and the boundary data (2.20) to obtain

$$\begin{aligned}&\int_{\phi^{-1}(Y)}^{X} \frac{c_2}{c_2-c_1} ph(X',Y)\mathrm{d}X' + \int_{\phi(X)}^{Y} \frac{-c_1}{c_2-c_1} qg(X,Y')\mathrm{d}Y' \\ &= \int_{\phi^{-1}(Y)}^{X} \frac{c_2\bar h}{c_2-c_1}(X',\phi(X'))\mathrm{d}X' + \int_{\phi(X)}^{Y} \frac{-c_1\bar g}{c_2-c_1}(\phi^{-1}(Y'),Y')\mathrm{d}Y' \\ &\le \overline{M}\Big(X - \phi^{-1}(Y) + Y - \phi(X)\Big) \\ &\le 2\overline{M}(|X| + |Y| + 2\underline{M}\mathcal{E}_0),\end{aligned} \tag{3.11}$$

from which, we employ (2.17) again to arrive at

$$\int_{\phi^{-1}(Y)}^{X} ph(X',Y)\mathrm{d}X' + \int_{\phi(X)}^{Y} qg(X,Y')\mathrm{d}Y' \le 2\underline{M}\overline{M}(|X|+|Y|+2\underline{M}\mathcal{E}_0). \tag{3.12}$$

On the other hand, making use of (1.10), (2.17) and the boundary data (2.20) yields

$$\begin{aligned}
&\int_{\phi^{-1}(Y)}^{X} \Big[\frac{-c_1}{2(c_2-c_1)}p(1-h) - \frac{\lambda}{c_2-c_1}p\ell - \frac{c_2F}{c_2-c_1}ph\Big](X',\phi(X'))\mathrm{d}X' \\
&+ \int_{\phi(X)}^{Y} \Big[\frac{c_2}{2(c_2-c_1)}q(1-g) + \frac{\lambda}{c_2-c_1}qm + \frac{c_1F}{c_2-c_1}qg\Big](\phi^{-1}(Y'),Y')\mathrm{d}Y' \\
&\le \Big(\frac{1}{2}\overline{M} + \frac{\Lambda}{4\gamma_1} + \Lambda\overline{M}\Big)\cdot\Big(X-\phi^{-1}(Y)+Y-\phi(X)\Big) \\
&\le \Big(\overline{M} + \frac{\Lambda}{2\gamma_1} + 2\Lambda\overline{M}\Big)(|X|+|Y|+2\underline{M}\mathcal{E}_0).
\end{aligned}$$

Then, integrating (3.3) along the closed curve $S$ obtains

$$\begin{aligned}
&\int_{\phi^{-1}(Y)}^{X} \frac{-c_1}{2(c_2-c_1)}p(X',Y)\mathrm{d}X' + \int_{\phi(X)}^{Y} \frac{c_2}{2(c_2-c_1)}q(X,Y')\mathrm{d}Y' \\
&\le \int_{\phi^{-1}(Y)}^{X} \frac{c_1+2c_2F}{2(c_2-c_1)}ph(X',Y)\mathrm{d}X' + \int_{\phi(X)}^{Y} \frac{-c_2-2c_1F}{2(c_2-c_1)}qg(X,Y')\mathrm{d}Y' \\
&\quad + \int_{\phi^{-1}(Y)}^{X} \frac{\lambda}{c_2-c_1}p\ell(X',Y)\mathrm{d}X' + \int_{\phi(X)}^{Y} \frac{-\lambda}{2(c_2-c_1)}qm(X,Y')\mathrm{d}Y' \\
&\quad + \Big(\overline{M} + \frac{\Lambda}{2\gamma_1} + 2\Lambda\overline{M}\Big)(|X|+|Y|+2\underline{M}\mathcal{E}_0),
\end{aligned}$$

from which and the following inequalities

$$\Big|\frac{\lambda p\ell}{c_2-c_1}\Big| \le \frac{-c_1}{4(c_2-c_1)}p + \frac{\overline{M}\Lambda^2}{\gamma_1^2}ph, \quad \Big|\frac{\lambda qm}{c_2-c_1}\Big| \le \frac{c_2}{4(c_2-c_1)}q + \frac{\overline{M}\Lambda^2}{\gamma_1^2}qg,$$

one has

$$\begin{aligned}
&\int_{\phi^{-1}(Y)}^{X} \frac{-c_1}{4(c_2-c_1)}p(X',Y)\mathrm{d}X' + \int_{\phi(X)}^{Y} \frac{c_2}{4(c_2-c_1)}q(X,Y')\mathrm{d}Y' \\
&\le \overline{M}\Big(\frac{1}{2}+\Lambda+\frac{\Lambda^2}{\gamma_1^2}\Big)\Big\{\int_{\phi^{-1}(Y)}^{X} ph(X',Y)\mathrm{d}X' + \int_{\phi(X)}^{Y} qg(X,Y')\mathrm{d}Y'\Big\} \\
&\quad + \Big(\overline{M} + \frac{\Lambda}{2\gamma_1} + 2\Lambda\overline{M}\Big)(|X|+|Y|+2\underline{M}\mathcal{E}_0),
\end{aligned}$$

which combined with (3.12) and (2.17) concludes

$$\begin{aligned}
&\int_{\phi^{-1}(Y)}^{X} p(X',Y)\mathrm{d}X' + \int_{\phi(X)}^{Y} q(X,Y')\mathrm{d}Y' \\
&\le 4\underline{M}\Big[\underline{M}\overline{M}^2\Big(1+2\Lambda+\frac{2\Lambda^2}{\gamma_1^2}\Big) + \overline{M} + \frac{\Lambda}{2\gamma_1} + 2\Lambda\overline{M}\Big](|X|+|Y|+2\underline{M}\mathcal{E}_0).
\end{aligned} \tag{3.13}$$

For any $(X,Y)\in\Omega^+$, we now integrate the first equation of (2.14) vertically and apply (3.13) to have

$$\exp\Big\{-\widetilde{C}(|X|+|Y|+2\underline{M}\mathcal{E}_0)\Big\} \le p(X,Y) \le \exp\Big\{\widetilde{C}(|X|+|Y|+2\underline{M}\mathcal{E}_0)\Big\} \tag{3.14}$$

for some constant $\widetilde{C}$ depending only on $\Lambda, \alpha_1, \alpha_2, \gamma_1$ and $\gamma_2$. A similar inequality also holds for $q$. To estimate the function $u$, we integrate the equation for $u$ in (2.12) horizontally and use (3.13) to obtain

$$\begin{aligned} |u(X,Y)| &\le |u_0| + \frac{1}{4\gamma_1}\int_{\varphi^{-1}(Y)}^{X} p(X',Y)\mathrm{d}X' \\ &\le |u_0| + \frac{M}{\gamma_1}\Big[\underline{M}\overline{M}^2\Big(1+2\Lambda+\frac{2\Lambda^2}{\gamma_1^2}\Big)+\overline{M} \\ &\quad + \frac{\Lambda}{2\gamma_1} + 2\Lambda\overline{M}\Big](|X|+|Y|+2\underline{M}\mathcal{E}_0). \end{aligned} \tag{3.15}$$

Integrating the equation for $x$ in (2.12) horizontally, it suggests by (3.11) that

$$\begin{aligned} |x(X,Y)| &\le |x(\varphi^{-1}(Y),Y)| + \int_{\varphi^{-1}(Y)}^{X} \frac{c_2}{c_2-c_1} ph(X',Y)\mathrm{d}X' \\ &\le |x(\varphi^{-1}(Y),Y)| + 2\overline{M}(|X|+|Y|+2\underline{M}\mathcal{E}_0) \\ &\le (2\overline{M}+1)(|X|+|Y|+2\underline{M}\mathcal{E}_0). \end{aligned} \tag{3.16}$$

Since all right-hand side functions in system (2.12)-(2.14) are locally Lipschitz continuous, then the local existence of solutions follows straightforward from the fixed point method. from the a priori estimates (3.9), (3.14)-(3.15) and (3.16), we can extend this local solution to the entire domain $\Omega^+$ by using the technique in Bressan and Zheng [8]. Thus we have the global existence theorem.

**Theorem 3.1.** *Let* (1.10) *and* (1.13) *be satisfied. Then problem* (2.12)-(2.14) *with boundary data* (2.20) *has a unique global solution defined for all* $(X,Y) \in \Omega^+$.

The above construction leads directly to a useful consequence.

**Corollary 3.2.** *Suppose that* (1.10) *holds. If the initial data* $(u_0, u_1)$ *are smooth, then the solution of* (2.12)-(2.14) (2.20) *is a smooth function of the variables* $(X,Y)$. *Moreover, assume that a sequence of smooth functions* $(u_0^\nu, u_1^\nu)_{\nu\ge1}$ *satisfies*

$$u_0^\nu \to u_0, \quad (u_0^\nu)_x \to (u_0)_x, \quad u_1^\nu \to u_1,$$

*uniformly on compact subsets of* $\mathbb{R}$. *Then it has the following convergence properties:*

$$(u^\nu, h^\nu, g^\nu, \ell^\nu, m^\nu, p^\nu, q^\nu) \to (u, h, g, \ell, m, p, q),$$

*uniformly on bounded subsets of the* $X$-$Y$ *plane.*

## 4. Solutions in the original variables

This section is devoted to returning the solution in the $X$-$Y$ plane to the original variables $(t,x)$. The Hölder continuous of solution and the integral equation (1.15) are also verified in this section.

We first examine the regularity of the solution constructed in the previous section. Since the initial data $(u_0)_x$ and $u_1$ are assumed only to be in $L^2$, we see that, on bounded subsets of the $X$-$Y$ plane,

– The functions $h$, $\ell$ and $p$ are Lipschitz continuous with respect to $Y$, measurable with respect to $X$,

– The functions $g, m$ and $q$ are Lipschitz continuous with respect to $X$, measurable with respect to $Y$,

– The functions $u$ and $x$ are Lipschitz continuous with respect to both $X$ and $Y$.

To return the solution $u$ to the original variables $(t,x)$, we need the inverse functions $X = X(t,x)$, $Y = Y(t,x)$. The function $x = x(X,Y)$ can be obtained by solving problem (2.12)-(2.14) (2.20), so it suffices to construct the function $t = t(X,Y)$. Owning to (2.10), it gives

$$t_X = \frac{\alpha ph}{c_2 - c_1}, \quad t_Y = \frac{\alpha qg}{c_2 - c_1}. \tag{4.1}$$

It is not difficult to show that $t_{XY} = t_{YX}$, which indicates that we may integrate one of the equations in (4.1) to obtain the function $t = t(X,Y)$. Note that the map $(X,Y) \mapsto (t,x)$ may not be one-to-one mapping, which, however, does not cause any real difficulty due to the following assertion: for any fixed $(t,x)$, the values of $u$ do not depend on the choice of $(X,Y)$. We omit the proof of this assertion since it is completely analogous to Bressan and Zheng[8]. Then, for each given point $(t^*,x^*)$, we can choose an arbitrary point $(X^*,Y^*)$ satisfying $t(X^*,Y^*) = t^*$, $x(X^*,Y^*) = x^*$, and define $u(t^*,x^*) := u(X^*,Y^*)$.

We now show that the function $u(t,x)$, obtained as above, is Hölder continuous on bounded sets. In fact, integrating along any forward characteristic $t \mapsto x^+(t)$ and noting $Y = const.$ on this kind of characteristics achieves

$$\begin{aligned} \int_0^\tau [\alpha(x,u)u_t + c_2(x,u)u_x]^2 \mathrm{d}t &= \int_{X_0}^{X_\tau} [(c_2 - c_1)u_X X_x]^2 t_X \mathrm{d}X \\ &= \int_{X_0}^{X_\tau} \frac{\alpha(1-h)p}{c_2 - c_1} \mathrm{d}X \le C_\tau \end{aligned} \tag{4.2}$$

for some constant $C_\tau$ depending only on $\tau$. Analogously, one has

$$\int_0^\tau [\alpha(x,u)u_t + c_1(x,u)u_x]^2 \mathrm{d}t \le C_\tau, \tag{4.3}$$

which together with (4.2) and (1.10) means that the function $u = u(t,x)$ is Hölder continuous with exponent 1/2. Moreover, it leads by (2.7) to the fact that all characteristic curves are $C^1$ with Hölder continuous derivative. In addition, by (4.2) and (4.3), the functions $R$ and $S$ at (2.1) are square integrable on bounded subsets of the $t$-$x$ plane. From the identity

$$\alpha(x,u)u_t + c_2(x,u)u_x = (c_2 - c_1)u_X X_x = \frac{\ell}{h} = R,$$

the function $R$ is indeed the same as recovered from (2.5). It is also true for $S$.

We next demonstrate that the function $u = u(t,x)$ satisfies (1.11) in the distributional sense. For any test function $\varphi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$, it suggests by (2.10) that

$$\begin{aligned} &\varphi_t(\alpha^2 u_t + \beta u_x + \kappa) + \varphi_x(\beta u_t - \gamma^2 u_x + \lambda) \\ &= \Big(\frac{c_2}{\alpha qg}\varphi_Y - \frac{c_1}{\alpha ph}\varphi_X\Big) \cdot \Big[\frac{\alpha(c_2 S - c_1 R)}{c_2 - c_1} + \frac{\beta(R-S)}{c_2 - c_1} + \kappa\Big] \\ &\quad + \Big(-\frac{1}{qg}\varphi_Y + \frac{1}{ph}\varphi_X\Big) \cdot \Big[\frac{\beta(c_2 S - c_1 R)}{\alpha(c_2 - c_1)} - \frac{\gamma^2(R-S)}{c_2 - c_1} + \lambda\Big] \\ &= \frac{1}{qg}\Big\{\frac{c_2 - c_1}{2}R + \frac{c_2}{\alpha}\kappa - \lambda\Big\}\varphi_Y + \frac{1}{ph}\Big\{\frac{c_2 - c_1}{2}S - \frac{c_1}{\alpha}\kappa + \lambda\Big\}\varphi_X \\ &= \frac{1}{qgh}\Big\{\frac{c_2 - c_1}{2}\ell + \Big(\frac{c_2}{\alpha}\kappa - \lambda\Big)h\Big\}\varphi_Y + \frac{1}{pgh}\Big\{\frac{c_2 - c_1}{2}m - \Big(\frac{c_1}{\alpha}\kappa - \lambda\Big)g\Big\}\varphi_X, \end{aligned} \tag{4.4}$$

which combined with the Jacobian

$$\frac{\partial(x,t)}{\partial(X,Y)} = \frac{\alpha pqgh}{c_2 - c_1}, \tag{4.5}$$

ensures

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}} \{\varphi_t(\alpha^2 u_t + \beta u_x + \kappa) + \varphi_x(\beta u_t - \gamma^2 u_x + \lambda)\} \mathrm{d}x\mathrm{d}t \\
&= \iint_{(X,Y)\in\Omega^+} \Big\{\Big[\frac{\alpha}{2} qm + \frac{\alpha\lambda - c_1\kappa}{c_2 - c_1} qg\Big]\varphi_X + \Big[\frac{\alpha}{2} p\ell - \frac{\alpha\lambda - c_2\kappa}{c_2 - c_1} ph\Big]\varphi_Y\Big\} \mathrm{d}X\mathrm{d}Y \\
&= -\iint_{(X,Y)\in\Omega^+} \varphi\Big\{\Big[\frac{\alpha}{2} qm + \frac{\alpha\lambda - c_1\kappa}{c_2 - c_1} qg\Big]_X + \Big[\frac{\alpha}{2} p\ell - \frac{\alpha\lambda - c_2\kappa}{c_2 - c_1} ph\Big]_Y\Big\} \mathrm{d}X\mathrm{d}Y.
\end{aligned} \tag{4.6}$$

A straightforward computation yields

$$\begin{aligned}
&\Big(\frac{\alpha}{2} qm\Big)_X + \Big(\frac{\alpha}{2} p\ell\Big)_Y \\
&= \frac{\alpha pqgh}{c_2 - c_1}\Big\{a_2 \frac{1-g}{g} - a_1 \frac{1-h}{h} + \Big(a_1 - a_2 + \frac{\alpha_u}{\alpha}\Big)\frac{m\ell}{gh} + (F_u - \lambda_x)\Big\},
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\Big\{\frac{\alpha\lambda - c_1\kappa}{c_2 - c_1} qg\Big\}_X - \Big\{\frac{\alpha\lambda - c_2\kappa}{c_2 - c_1} ph\Big\}_Y \\
&= \frac{\alpha pqgh}{c_2 - c_1}\Big\{\frac{\lambda_u}{c_2 - c_1} \cdot \frac{g\ell - mh}{gh} + \frac{\kappa_u}{\alpha(c_2 - c_1)} \cdot \frac{c_2 mh - c_1 g\ell}{gh} + \lambda_x\Big\}.
\end{aligned} \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.6) and using the Jacobian (4.5) again leads to

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}} \{\varphi_t(\alpha^2 u_t + \beta u_x + \kappa) + \varphi_x(\beta u_t - \gamma^2 u_x + \lambda)\} \mathrm{d}x\mathrm{d}t \\
&= -\iint_{\mathbb{R}^+\times\mathbb{R}} \varphi\Big\{a_2 \frac{1-g}{g} - a_1 \frac{1-h}{h} + \Big(a_1 - a_2 + \frac{\alpha_u}{\alpha}\Big)\frac{m\ell}{gh} + F_u \\
&\quad + \frac{\lambda_u}{c_2 - c_1} \frac{g\ell - mh}{gh} + \frac{\kappa_u}{\alpha(c_2 - c_1)} \frac{c_2 mh - c_1 g\ell}{gh}\Big\} \mathrm{d}x\mathrm{d}t \\
&= -\iint_{\mathbb{R}^+\times\mathbb{R}} \varphi\Big\{a_2 S^2 - a_1 R^2 + \Big(a_1 - a_2 + \frac{\alpha_u}{\alpha}\Big) RS + F_u \\
&\quad + \frac{\lambda_u}{c_2 - c_1}(R - S) + \frac{\kappa_u}{\alpha(c_2 - c_1)}(c_2 S - c_1 R)\Big\} \mathrm{d}x\mathrm{d}t \\
&= -\iint_{\mathbb{R}^+\times\mathbb{R}} \varphi\Big\{\alpha\alpha_u u_t^2 + \beta_u u_t u_x - \gamma\gamma_u u_x^2 + \kappa_u u_t + \lambda_u u_x + F_u\Big\} \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

which finishes the proof of (1.15).

## 5. Regularity of trajectories

In this section, we complete the proof of Theorem 1.2. For $M > 0$, denote

$$\mathcal{E}_M(t) = \frac{1}{2}\int_{-M}^{M} \Big\{\alpha^2(x,u)u_t^2 + \gamma^2(x,u)u_x^2\Big\} \mathrm{d}x.$$

We first show the following lemma

**Lemma 5.1.** *For any $M > 0$ and $t > 0$, the solution constructed in previous section satisfies*

$$\mathcal{E}_M(t) \leq \widehat{C}(\mathcal{E}_0 + M + t) \tag{5.1}$$

*for some positive constant $\widehat{C}$ independent of $M$ and $t$. Here $\mathcal{E}_0$ is defined as in* (2.18).

*Proof.* Fix any $\tau > 0$, we denote $\Gamma_\tau := \{(X, Y) : t(X, Y) = \tau\}$. Let $A_1$ and $A_2$ on $\Gamma_\tau$ be any two corresponding points of the points $(\tau, -M)$ and $(\tau, M)$ in $t$-$x$ plane, respectively. Then we draw the horizontal and vertical lines from $A_1$ and $A_2$ up to $\Gamma_0$ at points $A_4$ and $A_3$, respectively. Consider the region $D$ bounded by $\Gamma_0$, $\Gamma_\tau$, $A_1A_4$ and $A_2A_3$, see Figure 1.

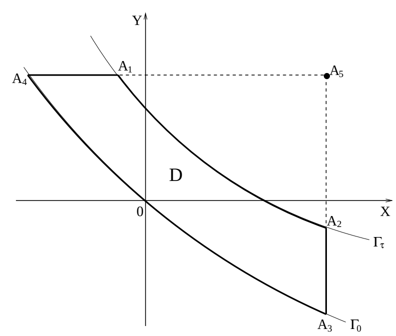


FIGURE 1. Region $D$.

We integrate (3.3) along the boundary of $D$ to obtain

$$\begin{aligned} &\int_{A_1A_2} \frac{-c_1p(1-h)}{2(c_2-c_1)}\mathrm{d}X - \frac{c_2q(1-g)}{2(c_2-c_1)}\mathrm{d}Y \\ &= \int_{A_4A_3} \frac{-c_1p(1-h)}{2(c_2-c_1)}\mathrm{d}X - \frac{c_2q(1-g)}{2(c_2-c_1)}\mathrm{d}Y + I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} I_1 &= \int_{A_4A_3} \Big(\frac{-\lambda p\ell}{c_2-c_1} + \frac{-c_2Fph}{c_2-c_1}\Big)\mathrm{d}X - \Big(\frac{\lambda qm}{c_2-c_1} + \frac{c_1Fqg}{c_2-c_1}\Big)\mathrm{d}Y, \\ I_2 &= \int_{A_1A_2} \Big(\frac{\lambda p\ell}{c_2-c_1} + \frac{c_2Fph}{c_2-c_1}\Big)\mathrm{d}X + \Big(\frac{\lambda qm}{c_2-c_1} + \frac{c_1Fqg}{c_2-c_1}\Big)\mathrm{d}Y, \\ I_3 &= -\int_{A_3A_2} \Big\{\frac{c_2q(1-g)}{2(c_2-c_1)} + \frac{\lambda qm}{c_2-c_1} + \frac{c_1Fqg}{c_2-c_1}\Big\}\mathrm{d}Y, \\ I_4 &= -\int_{A_4A_1} \Big\{\frac{-c_1p(1-h)}{2(c_2-c_1)} - \frac{\lambda p\ell}{c_2-c_1} - \frac{c_2Fph}{c_2-c_1}\Big\}\mathrm{d}X. \end{aligned}$$

Notice that the length of the segment corresponding to $A_4A_3$ in the initial line $t=0$ is less or equal to $2(M+\max\{|\lambda_\pm|\}\tau)$, then we have

$$
\begin{aligned}
|I_1| &\le \int_{A_4A_3} \Big(\frac{\Lambda}{2(c_2-c_1)} + \frac{c_2\Lambda}{c_2-c_1}\Big)\mathrm{d}X - \Big(\frac{\Lambda}{2(c_2-c_1)} + \frac{-c_1\Lambda}{c_2-c_1}\Big)\mathrm{d}Y \\
&\le \Lambda\Big(\frac{1}{\gamma_1} + 4\overline{M}\Big)\Big(M+\max\{|\lambda_\pm|\}\tau\Big) \\
&\le \Lambda\Big(\frac{1}{\gamma_1} + 4\overline{M}\Big)\Big(M+\frac{2\gamma_1\overline{M}}{\alpha_1}\tau\Big).
\end{aligned} \tag{5.3}
$$

From the inequalities

$$
\big|\frac{\lambda p\ell}{c_2-c_1}\big| \le \frac{\Lambda^2 ph}{-c_1(c_2-c_1)} + \frac{-c_1p(1-h)}{4(c_2-c_1)}, \quad \big|\frac{\lambda qm}{c_2-c_1}\big| \le \frac{\Lambda^2 qg}{c_2(c_2-c_1)} + \frac{c_2q(1-g)}{4(c_2-c_1)},
$$

one gets

$$
\begin{aligned}
|I_2| \le{}& \frac{1}{2}\int_{A_1A_2} \frac{-c_1p(1-h)}{2(c_2-c_1)}\mathrm{d}X + \frac{c_2q(1-g)}{2(c_2-c_1)}\mathrm{d}Y \\
&+ \Big(\frac{4\Lambda^2\overline{M}}{\gamma_1^2} + \Lambda\overline{M}\Big)\int_{A_1A_2} ph\mathrm{d}X - qg\mathrm{d}Y.
\end{aligned} \tag{5.4}
$$

Using the above inequalities yields

$$
\begin{aligned}
|I_3| &\le \int_{A_3A_2} \Big\{\frac{-c_2q(1-g)}{4(c_2-c_1)} + \Big(\frac{\Lambda^2}{c_2(c_2-c_1)} + \Lambda\overline{M}\Big)qg\Big\}\mathrm{d}Y, \\
&\le \Big(\frac{4\Lambda^2\overline{M}}{\gamma_1^2} + \Lambda\overline{M}\Big)\int_{A_3A_2} qg\mathrm{d}Y,
\end{aligned} \tag{5.5}
$$

and

$$
\begin{aligned}
|I_4| &\le \int_{A_4A_1} \Big\{\frac{c_1q(1-g)}{4(c_2-c_1)} + \Big(\frac{\Lambda^2}{-c_1(c_2-c_1)} + \Lambda\overline{M}\Big)ph\Big\}\mathrm{d}X, \\
&\le \Big(\frac{4\Lambda^2\overline{M}}{\gamma_1^2} + \Lambda\overline{M}\Big)\int_{A_4A_1} ph\mathrm{d}X.
\end{aligned} \tag{5.6}
$$

Putting (5.3)–(5.6) in (5.2) gives

$$
\begin{aligned}
&\int_{A_1A_2} \frac{-c_1p(1-h)}{2(c_2-c_1)}\mathrm{d}X - \frac{c_2q(1-g)}{2(c_2-c_1)}\mathrm{d}Y \\
&\le 2\mathcal{E}_0 + 2\Lambda\Big(\frac{1}{\gamma_1} + 4\overline{M}\Big)\Big(M+\frac{2\gamma_1\overline{M}}{\alpha_1}\tau\Big) + \Big(\frac{8\Lambda^2\overline{M}}{\gamma_1^2} + 2\Lambda\overline{M}\Big) \\
&\quad\times\Big\{\int_{A_1A_2} ph\mathrm{d}X - qg\mathrm{d}Y + \int_{A_4A_1} ph\mathrm{d}X + \int_{A_3A_2} qg\mathrm{d}Y\Big\},
\end{aligned} \tag{5.7}
$$

where we used the fact that the variables $h, g$ never assume the value zero at the initial time. Moreover, integrating (3.10) along the boundary of $D$ arrives at

$$
\begin{aligned}
&\int_{A_1A_2} \frac{c_2ph}{c_2-c_1}\mathrm{d}X - \frac{-c_1qg}{c_2-c_1}\mathrm{d}Y \\
&= \int_{A_4A_3} \frac{c_2ph}{c_2-c_1}\mathrm{d}X - \frac{-c_1qg}{c_2-c_1}\mathrm{d}Y - \int_{A_4A_1} \frac{c_2ph}{c_2-c_1}\mathrm{d}X - \int_{A_3A_2} \frac{-c_1qg}{c_2-c_1}\mathrm{d}Y \\
&\le 4\overline{M}\Big(M+\frac{2\gamma_1\overline{M}}{\alpha_1}\tau\Big),
\end{aligned}
$$

which along with (2.17) leads to

$$\int_{A_1A_2} ph\mathrm{d}X - qg\mathrm{d}Y \le 4\underline{M}\overline{M}\Big(M + \frac{2\gamma_1\overline{M}}{\alpha_1}\tau\Big). \tag{5.8}$$

To estimate the last two terms in (5.7), we extend the segments $A_4A_1$ and $A_3A_2$ such that they intersect a point $A_5$ and integrate (3.10) along the closed curve composed of $A_4A_5, A_5A_3$ and $A_3A_4$ to find

$$\begin{aligned}\int_{A_4A_5} \frac{c_2ph}{c_2-c_1}\mathrm{d}X + \int_{A_3A_5} \frac{-c_1qg}{c_2-c_1}\mathrm{d}Y &= \int_{A_4A_3} \frac{c_2ph}{c_2-c_1}\mathrm{d}X - \frac{-c_1qg}{c_2-c_1}\mathrm{d}Y\\ &\le 4\overline{M}\Big(M + \frac{2\gamma_1\overline{M}}{\alpha_1}\tau\Big),\end{aligned}$$

from which, one has

$$\begin{aligned}\int_{A_4A_1} ph\mathrm{d}X + \int_{A_3A_2} qg\mathrm{d}Y &\le \underline{M}\Big\{\int_{A_4A_5} \frac{c_2ph}{c_2-c_1}\mathrm{d}X + \int_{A_3A_5} \frac{-c_1qg}{c_2-c_1}\mathrm{d}Y\Big\}\\ &\le 4\underline{M}\overline{M}\Big(M + \frac{2\gamma_1\overline{M}}{\alpha_1}\tau\Big).\end{aligned} \tag{5.9}$$

Inserting (5.8) and (5.9) into (5.7), we conclude that

$$\begin{aligned}&\int_{A_1A_2} \frac{-c_1p(1-h)}{2(c_2-c_1)}\mathrm{d}X - \frac{c_2q(1-g)}{2(c_2-c_1)}\mathrm{d}Y\\ &\le 2\mathcal{E}_0 + 2\Lambda\Big\{\frac{1}{\gamma_1} + 4\overline{M} + 8\underline{M}\overline{M}^2\Big(\frac{4\Lambda}{\gamma_1^2}+1\Big)\Big\}\Big(M + \frac{2\gamma_1\overline{M}}{\alpha_1}\tau\Big)\\ &\le \widehat{C}(\mathcal{E}_0 + M + \tau)\end{aligned} \tag{5.10}$$

for some constant $\widehat{C}$ independent of $M$ and $\tau$. On the other hand, it follows that

$$\begin{aligned}&\int_{-M}^{M} \frac{1}{2}\Big\{\alpha^2(x,u(\tau,x))u_t^2(\tau,x) + \gamma^2(x,u(\tau,x))u_x^2(\tau,x)\Big\}\mathrm{d}x\\ &= \int_{A_1A_2\cap\{h\neq 0\}} \frac{-c_1p(1-h)}{2(c_2-c_1)}\mathrm{d}X - \int_{A_1A_2\cap\{g\neq 0\}} \frac{c_2q(1-g)}{2(c_2-c_1)}\mathrm{d}Y,\end{aligned}$$

which together with (5.10) concludes (5.1). □

We now use (5.1) to prove (1.14). For any $t, s \in \mathbb{R}^+$, we see that

$$\begin{aligned}\|u(t,x) - u(s,x)\|_{L^2([-M,M])} &\le |t-s| \int_0^1 \|u_t(s+\xi(t-s),\cdot)\|_{L^2([-M,M])}\mathrm{d}\xi\\ &\le \sqrt{\widehat{C}(\mathcal{E}_0 + M + t + s)}|t-s|\end{aligned} \tag{5.11}$$

for any $M > 0$, where the constant $\widehat{C}$ is independent of $t, s$ and $M$. This proves (1.14).

We next prove that, for any $M > 0$, the functions $t \mapsto u_t(t,\cdot)$ and $t \mapsto u_x(t,\cdot)$ are continuous with values in $L^\theta([-M,M])$ $(1 \le \theta < 2)$, which will complete the proof of Theorem 1.2. Let us first consider the arguments for smooth initial data with compact support, in which, the solution $u = u(X,Y)$ remains smooth on the entire region $\Omega^+$. For a fixed time $\tau$ and any fixed $M > 0$, we assert that,

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,\cdot)|_{t=\tau} = u_t(\tau,\cdot) \tag{5.12}$$

in interval $[-M, M]$, where

$$u_t(\tau, x) := u_X X_t + u_Y Y_t = \frac{-c_1}{\alpha(c_2 - c_1)} \cdot \frac{\ell}{h} + \frac{c_2}{\alpha(c_2 - c_1)} \cdot \frac{m}{g}, \tag{5.13}$$

which defines the value of $u_t(\tau, \cdot)$ at almost every point $x \in [-M, M]$ by (5.1).

To verify the assertion (5.12), we consider the curve segment $A_1A_2$ as before. For any $\theta \in [1, 2)$, let $\sigma := 2/(2 - \theta)$ be the conjugate exponent of $2/\theta$ and denote $\widetilde{M} := \widehat{C}(\mathcal{E}_0 + M + 2\tau)$. Given any $\varepsilon > 0$, it is clear that there exist finitely many disjoint intervals $[a_i, b_i] \subset [-M, M]$, $i = 1, 2, \dots, N$, such that

$$\min\{h(P), g(P)\} < \frac{2\varepsilon}{(\widetilde{M} + 1)^\sigma} \tag{5.14}$$

for every point $P = (X(x_P, \tau), Y(x_P, \tau))$ and

$$h(Q) > \frac{\varepsilon}{(\widetilde{M} + 1)^\sigma}, \quad g(Q) > \frac{\varepsilon}{(\widetilde{M} + 1)^\sigma} \tag{5.15}$$

for every point $Q = (X(x_Q, \tau), Y(x_Q, \tau))$, where $x_P \in J := \cup_{i=1}^N [a_i, b_i]$ and $x_Q \in J' := [-M, M] \setminus J$. Obviously, the function $u = u(t, x)$ is smooth in a neighborhood of the set $\{\tau\} \times J'$ by the construction of $J'$. By employing Minkowski's inequality, we find that

$$\begin{aligned} &\lim_{\rho \to 0} \frac{1}{\rho} \Big[ \int_{-M}^{M} |u(\tau + \rho, x) - u(\tau, x) - \rho u_t(\tau, x)|^\theta \mathrm{d}x \Big]^{1/\theta} \\ &\le \lim_{\rho \to 0} \frac{1}{\rho} \Big[ \int_J |u(\tau + \rho, x) - u(\tau, x)|^\theta \mathrm{d}x \Big]^{1/\theta} + \Big[ \int_J |u_t(\tau, x)|^\theta \mathrm{d}x \Big]^{1/\theta}. \end{aligned} \tag{5.16}$$

We use (5.14) and (5.10) to estimate the measure of the "bad" set $J$,

$$\begin{aligned} \mathrm{meas}(J) &= \int_J \mathrm{d}x = \sum_{i=1}^N \int_{(X_{a_i}, Y_{a_i})}^{(X_{b_i}, Y_{b_i})} \frac{c_2 p h}{c_2 - c_1} \mathrm{d}X + \frac{c_1 q g}{c_2 - c_1} \mathrm{d}Y \\ &\le \underline{M}\overline{M} \sum_{i=1}^N \int_{(X_{a_i}, Y_{a_i})}^{(X_{b_i}, Y_{b_i})} \frac{-c_1 p h}{c_2 - c_1} \mathrm{d}X - \frac{c_2 q g}{c_2 - c_1} \mathrm{d}Y \\ &\le \frac{\underline{M}\overline{M} \frac{4\varepsilon}{(\widetilde{M}+1)^\sigma}}{1 - \frac{2\varepsilon}{(\widetilde{M}+1)^\sigma}} \sum_{i=1}^N \int_{(X_{a_i}, Y_{a_i})}^{(X_{b_i}, Y_{b_i})} \frac{-c_1 p(1 - h)}{2(c_2 - c_1)} \mathrm{d}X - \frac{c_2 q(1 - g)}{2(c_2 - c_1)} \mathrm{d}Y \\ &\le \frac{4\underline{M}\overline{M}\varepsilon}{(1 - 2\varepsilon)(\widetilde{M} + 1)^\sigma} \int_{A_1A_2} \frac{-c_1 p(1 - h)}{2(c_2 - c_1)} \mathrm{d}X - \frac{c_2 q(1 - g)}{2(c_2 - c_1)} \mathrm{d}Y \\ &\le \frac{4\underline{M}\overline{M}\widetilde{M}\varepsilon}{(1 - 2\varepsilon)(\widetilde{M} + 1)^\sigma}, \end{aligned} \tag{5.17}$$

where $(X_{a_i}, Y_{a_i}) = (X(a_i, \tau), Y(a_i, \tau))$ and $(X_{b_i}, Y_{b_i}) = (X(b_i, \tau), Y(b_i, \tau))$. Applying Hölder's inequality and recalling (5.11) yields

$$\begin{aligned} &\int_J |u(\tau + \rho, x) - u(\tau, x)|^\theta \mathrm{d}x \\ &\le \mathrm{meas}(J)^{1/\sigma} \Big( \int_J |u(\tau + \rho, x) - u(\tau, x)|^2 \mathrm{d}x \Big)^{\theta/2} \end{aligned}$$

$$\begin{aligned}&\le \Big(\frac{4\underline{M}\overline{M}\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\Big)^{1/\sigma}\|u(\tau+\rho,\cdot)-u(\tau,\cdot)\|^\theta_{L^2([-M,M])}\\ &\le \Big(\frac{4\underline{M}\overline{M}\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\Big)^{1/\sigma}(\widetilde{M}+\widehat{C}\rho)^{\theta/2}\rho^\theta,\end{aligned}$$

from which, we have

$$\begin{aligned}\limsup_{\rho\to 0}\frac{1}{\rho}\Big(\int_J|u(\tau+\rho,x)-u(\tau,x)|^\theta \mathrm{d}x\Big)^{1/\theta} &\le \sqrt{\widetilde{M}}\Big(\frac{4\underline{M}\overline{M}\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\Big)^{\frac{1}{\sigma\theta}}\\ &\le \Big(\frac{4\underline{M}\overline{M}\varepsilon}{1-2\varepsilon}\Big)^{\frac{1}{\sigma\theta}}.\end{aligned}\tag{5.18}$$

Analogously, one has

$$\begin{aligned}\Big(\int_J|u_t(\tau,x)|^\theta\mathrm{d}x\Big)^{1/\theta} &\le \mathrm{meas}(J)^{\frac{1}{\sigma\theta}}\Big(\int_J|u_t(\tau,x)|^2\mathrm{d}x\Big)^{1/2}\\ &\le \Big(\frac{4\underline{M}\overline{M}\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\Big)^{\frac{1}{\sigma\theta}}\|u_t(\tau,\cdot)\|_{L^2([-M,M])}\\ &\le \sqrt{2\widetilde{M}}\Big(\frac{4\underline{M}\overline{M}\widetilde{M}\varepsilon}{(1-2\varepsilon)(\widetilde{M}+1)^\sigma}\Big)^{\frac{1}{\sigma\theta}}\\ &\le \sqrt{2}\Big(\frac{4\underline{M}\overline{M}\varepsilon}{1-2\varepsilon}\Big)^{\frac{1}{\sigma\theta}}.\end{aligned}\tag{5.19}$$

Combining with (5.16), (5.18) and (5.19), it follows by the arbitrariness of $\varepsilon>0$ that

$$\lim_{\rho\to 0}\frac{1}{\rho}\Big(\int_{-M}^M|u(\tau+\rho,x)-u(\tau,x)-\rho u_t(\tau,x)|^\theta\mathrm{d}x\Big)^{1/\sigma}=0. \tag{5.20}$$

Based on the same method, we can establish the continuity of the function $t\mapsto u_t(t,\cdot)$.

To extend the result to general initial data $(u_0),u_1\in L^2$, we let $\{(u_0^\nu)_x\},\{u_1^\nu\}\in C_c^\infty$ be a sequence of smooth initial data such that $u_0^\nu\to u_0$ uniformly, $(u_0^\nu)_x\to (u_0)_x$ almost everywhere and in $L^2$, $u_1^\nu\to u_1$ almost everywhere and in $L^2$. The proof is concluded by Corollary 3.2. The continuity of the function $t\to u_x(t,\cdot)$ as a map with values in $L^\theta([-M,M]), 1\le\theta<2$ can be verified in an entirely similar way, so we omit it here.

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