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SYMMETRY AND CONVEXITY OF LEVEL SETS OF SOLUTIONS TO THE INFINITY LAPLACE'S EQUATION

Edi Rosset

Abstract

We consider the Dirichlet problem

$$\begin{aligned} -\Delta_\infty u &= f(u) \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

where $\Delta_\infty u = u_{x_i} u_{x_j} u_{x_i x_j}$ and f is a nonnegative continuous function. We investigate whether the solutions to this equation inherit geometrical properties from the domain Ω. We obtain results concerning convexity of level sets and symmetry of solutions.

1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the following Dirichlet problem for the ∞-Laplace operator

$$\begin{aligned} -\Delta_\infty u &= f(u) \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{D_∞}$$

where $\Delta_\infty u = u_{x_i} u_{x_j} u_{x_i x_j}$ and f is a nonnegative continuous function. We investigate whether the solutions to (D_∞) inherit geometrical properties from the domain Ω.

By a solution to (D_∞) we will mean a variational solution in a sense which extends that given in [B-D-M], that is, roughly speaking, a function which is the limit of a sequence of solutions to the Dirichlet problems for the p-Laplace operator

$$\begin{aligned} -\Delta_p u &= f(u) \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{D_p}$$

as $p \to \infty$ (see Definition 2.1 below).

When Ω is a convex domain, we prove that the restriction of any solution u of (D_∞) to the convex ring $\Omega \backslash \Omega_{s_M}$, where $\Omega_{s_M} = \{x \in \Omega : d(x, \partial\Omega) > s_M\}$, has convex level sets, preserves the symmetries of Ω, and is uniquely determined (see Theorem 2.5 and Corollary 2.6). Here, the number s_M is determined by f and the maximum M of u in Ω only. If, for instance, f is strictly positive at M, then $\Omega_{s_M} = \emptyset$.

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Notice that by symmetry, we mean not only a reflection but any orthogonal transformation. When Ω is a ball B_R, any solution to (D_∞) is radially symmetric, has a very simple representation, and coincides with the distance function from $\partial\Omega$ in the annulus $\{s_M < |x| < R\}$, where, again, the number s_M only depends on f and M (see Theorem 2.7 below). Our proofs involve a variational principle for solutions to (D_∞), which is inspired by [B-D-M] (see Proposition 2.3).

Concerning problem (D_p) when Ω is a ball, let us recall that radial symmetry of solutions to (D_p) has been established when $p = 2$ and f is locally Lipschitz continuous in the famous paper by Gidas, Ni and Nirenberg ([G-N-N]), via the moving plane method. Damascelli and Pacella have recently extended the above result to any p, $1 < p < 2$, when f is locally Lipschitz continuous, via the moving plane method ([D-P]). Brock has recently proved symmetry results for the solutions to (D_p) (see [Br1], [Br2]), and, among these, radial symmetry of solutions to (D_p) for any $p \geq 1$ and any continuous nonnegative f, via continuous Steiner symmetrization ([Br2, Theorem 10.1]).

The hypothesis $f \geq 0$ plays a crucial role in deriving the variational principle (P^*_∞). On the other hand, when f is allowed to change sign, there are counterexamples to radial symmetry for the solutions to (D_p): for $p = 2$ and f Hölder continuous of any order $\alpha < 1$ (see [G-N-N], [Br-H]), and for $p > 2$ and $f \in C^1$ (see [Br-H]).

For $p > 2$ and f changing sign, Brock has established a partial form of symmetry, the so-called *local symmetry in every direction*, and symmetry results under some growth conditions on f in neighborhoods of its zero points, via continuous Steiner symmetrization (see [Br1], [Br2]).

The incompleteness of the result of Theorem 2.5 is due to the fact that a variational solution u to (D_∞) may be sensitive to the behaviour of f outside its range, through the influence of f on the sequence of solutions u_{p_k} to (D_{p_k}) converging to u. In Section 3 we provide an Example which illustrates this phenomenon.

In Section 4 we propose an alternative definition of solution which we have called a *tame variational solution* (see Definition 4.1), which prevents the occurrence of the "improper" solutions which may be introduced by the limit process described above. We show that any *tame variational solution* u has convex level sets, preserves the symmetries of the convex domain Ω and, when $\Omega = B_R$, then either $u = U$ or u is a truncation of U, where $U(x) = R - |x|$ (Theorem 4.3 and Theorem 4.4).

2. Statements and proofs

Let us recall some facts about the case $f = f(x)$, which stem from results in [B-D-M] and [J]. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a bounded nonnegative continuous function f defined in Ω, $f \not\equiv 0$, let $u_p \in W_0^{1,p}$ be the unique weak solution to

$$\begin{aligned} -\Delta_p u = f \quad &\text{in } \Omega\,, \\ u = 0 \quad &\text{on } \partial\Omega\,. \end{aligned} \tag{2.1}$$

Then there exists a unique function $u_\infty \in W^{1,\infty}(\Omega) \cap C_0(\bar{\Omega})$ such that

$$u_p \to u_\infty \quad \text{weakly in } W^{1,m}(\Omega), \forall m > 1, \text{ and uniformly in } \bar{\Omega}\,.$$

The function u_∞ obtained by this limit process is called a *variational solution* to

$$\begin{aligned} -\Delta_\infty u = f \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{2.2}$$

and is characterized by the following two conditions:

i) The function u_∞ solves the maximum problem

$$J_\infty(u_\infty) = \max_{\mathcal{K}} J_\infty, \tag{P_∞}$$

where $J_\infty(\varphi) = \int_\Omega f\varphi$, and

$$\mathcal{K} = \{\varphi \in W^{1,\infty}(\Omega) \cap C_0(\bar\Omega) \,:\, \|\nabla\varphi\|_\infty = 1\}$$

ii) The function u_∞ is a viscosity solution to

$$\Delta_\infty u = 0, \quad \text{in the interior of } \{f = 0\}. \tag{2.3}$$

Next let us consider the case $f = f(u)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nonnegative function such that the Dirichlet problem (D_p) is solvable in $W^{1,p}(\Omega)$ for p large enough, say $p \ge \bar p$. Let u_p be a solution to (D_p), for $p \ge \bar p$. Let us assume that f is bounded or, more generally, that $f(u) = O(u^s)$ as $u \to \infty$, for some $s > 0$. From the weak formulation of (D_p) and the Hölder and Poincaré inequalities, it follows easily that $\|\nabla u_p\|_m$ is bounded uniformly in p, for any $m > 1$. Therefore, one can construct a sequence $p_k \to \infty$, such that

$$u_{p_k} \to u, \quad \text{weakly in } W^{1,m}(\Omega), \forall m > 1, \text{ and uniformly in } \bar\Omega, \tag{2.4}$$

for some $u \in W^{1,\infty}(\Omega) \cap C_0(\bar\Omega)$.

In view of the above arguments, we give the following definition.

Definition 2.1. A function $u \in W^{1,\infty}(\Omega) \cap C_0(\bar\Omega)$ is called a *variational solution* to (D_∞) if there exists a sequence u_{p_k} of solutions to (D_{p_k}), with $p_k \to \infty$, such that (2.4) holds.

Let us notice that if u is a variational solution to (D_∞), then $\|u_{p_k}\|_\infty$ is uniformly bounded, so that, by the continuity of f, there exists a positive constant K such that $\|f(u_{p_k})\|_\infty \le K$. Therefore, by the Hölder and Poincaré inequalities, we have

$$\|\nabla u_{p_k}\|_m \le C^{1/(p_k-1)} K^{1/(p_k-1)} |\Omega|^{\frac{1}{m} + \frac{1}{n(p_k-1)}} \tag{2.5}$$

and

$$\|\nabla u\|_\infty = \lim_{m\to\infty} \|\nabla u\|_m \le \lim_{m\to\infty} \left(\liminf_{k\to\infty} \|\nabla u_{p_k}\|_m \right) = \lim_{m\to\infty} |\Omega|^{1/m} = 1. \tag{2.6}$$

Since $f \ge 0$, we have $u_p \ge 0$ and therefore $u \ge 0$. From (2.6) and from $u|_{\partial\Omega} \equiv 0$ it follows that u is Lipschitz continuous with Lipschitz constant $L \le 1$, and $u(x) \le d(x, \partial\Omega)$. Summarizing, we have

$$\|\nabla u\|_\infty \le 1\,, \tag{2.7}$$

$$0 \le u \le U\,, \tag{2.8}$$

where

$$U(x) = d(x, \partial\Omega). \tag{2.9}$$

Given a variational solution u to (D_∞), $u = \lim_{k\to\infty} u_{p_k}$, let us define

$$E_p = \int_\Omega |\nabla u_p|^p = \int_\Omega (f \circ u_p) u_p \,, \tag{2.10}$$

$$E_\infty = \int_\Omega (f \circ u) u = \lim_{k\to\infty} E_{p_k} \,, \tag{2.11}$$

$$f^* = f \circ u \,, \tag{2.12}$$

$$\Omega_0^* = \{x \in \Omega : u(x) \in int\{f = 0\}\} \,. \tag{2.13}$$

Lemma 2.2. *Let u be a variational solution to (D_∞). If $f^* \not\equiv 0$ then $u \not\equiv 0$ and $E_\infty > 0$.*

Proof. Let us see that $u \equiv 0$ implies $f^* \equiv 0$. If $u \equiv 0$, then there are two cases: either $f(0) = 0$ or $f(0) > 0$. In the former case $f^* \equiv 0$, whereas in the latter case, by the continuity of f, we have $f(u_{p_k}) \geq \delta$ for $k \geq \bar{k}$, for some $\bar{k} \in \mathbb{N}$, $\delta > 0$. Let v_p be the solution to

$$-\Delta_p v_p = \delta \quad \text{in } \Omega \,,$$
$$v_p = 0 \quad \text{on } \partial\Omega \,.$$

By the comparison principle for the p-Laplace operator (see [T]), we have $u_{p_k} \geq v_{p_k}$. Moreover from $i)$ it follows easily that $v_p \to v_\infty = U$ (see [B-D-M]), so that $u \geq U$, contradicting $u \equiv 0$.

Let $f^* \not\equiv 0$, so that $u \not\equiv 0$. Let us assume, by contradiction, that $0 = E_\infty = \int_{\{f^*>0\}} f^* u$. Since $u \geq 0$, we have $u \equiv 0$ in $\{f^* > 0\}$, that is: $f(u(x)) > 0$ implies $u(x) = 0$. Therefore, denoting $M = \max_\Omega u$, we have $f(t) = 0$ for every $t \in (0, M]$. From the continuity of u it follows that $f(0) = 0$, that is $f^* \equiv 0$, contradicting the hypothesis. ♢

Proposition 2.3. *Let u be a variational solution to (D_∞) such that $f^* \not\equiv 0$. Then,*

$i^*)$ *the function u solves the maximum problem*

$$J_\infty^*(u) = \max_{\mathcal{K}} J_\infty^*, \tag{P_∞^*}$$

where $J_\infty^*(\varphi) = \int_\Omega f^* \varphi$ and

$$\mathcal{K} = \{\varphi \in W^{1,\infty}(\Omega) \cap C_0(\bar{\Omega}) : \|\nabla \varphi\|_\infty = 1\}$$

and

$ii^*)$ *the function u is a viscosity solution of*

$$\Delta_\infty u = 0 \quad in\ \Omega_0^* \,. \tag{2.14}$$

Proof. From the definition of weak solution to (D_p) and from Hölder inequality, we have

$$\int_\Omega (f\circ u_p)\varphi = \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi \le E_p^{(p-1)/p}\|\nabla\varphi\|_p,$$

for any $\varphi\in W_0^{1,p}(\Omega)$. Hence, for any $\varphi\in W^{1,\infty}(\Omega)\cap C_0(\bar\Omega)$, $\varphi\not\equiv 0$, we have

$$\frac{\int_\Omega (f\circ u)\varphi}{\|\nabla\varphi\|_\infty} = \lim_{k\to\infty}\frac{\int_\Omega (f\circ u_{p_k})\varphi}{\|\nabla\varphi\|_{p_k}} \le \lim_{k\to\infty} E_{p_k}^{(p_k-1)/p_k} = E_\infty = J_\infty^*(u). \tag{2.15}$$

Substituting $\varphi = u$ in the above inequality and noting that $E_\infty > 0$ by Lemma 2.2, we have $\|\nabla u\|_\infty \ge 1$. From (2.7) it follows that $\|\nabla u\|_\infty = 1$, that is, $u\in\mathcal{K}$, and $i^*)$ follows immediately from (2.15).

In order to verify $ii^*)$, let us consider any $x\in\Omega_0^*$. Since u_{p_k} converges uniformly to u, there exist a neighborhood V of x and an index $\bar k$ such that $f\circ u_{p_k}\equiv 0$ in V for every $k\ge\bar k$. For any $p>1$, let v_p be the unique solution to

$$\begin{aligned}\Delta_p v_p &= 0 \quad \text{in } V\,,\\ v_p &= u \quad \text{on } \partial V\,.\end{aligned}$$

It is well known (see [J]) that v_p converges uniformly to the unique viscosity solution v_∞ of

$$\begin{aligned}\Delta_\infty v_\infty &= 0 \quad \text{in } V\,,\\ v_\infty &= u \quad \text{on } \partial V\,.\end{aligned}$$

On the other hand, applying the comparison principle for the p-Laplace operator (see [T]) to the functions u_{p_k}, v_{p_k} in V, we have that $\lim_{k\to\infty}\max_V |u_{p_k}-v_{p_k}| = 0$, so that $u_\infty = v_\infty$, and $ii^*)$ follows. ♢

Corollary 2.4. *In the hypotheses of Proposition 2.3, we have*

$$u(x) = U(x), \quad \forall x\in\overline{\{f^*>0\}}. \tag{2.16}$$

Proof. Substituting $U\in\mathcal{K}$ in (P_∞^*), we have

$$\int_{\{f^*>0\}} (u-U)f^* \ge 0,$$

so that (2.16) follows from (2.8). ♢

Let us introduce the following notation:

$$\begin{gathered}\Omega_t = \{x\in\Omega : d(x,\partial\Omega) > t\} = \{U>t\},\\ \Omega_{r,s} = \{x\in\Omega : r < d(x,\partial\Omega) < s\} = \Omega_r\setminus\overline{\Omega_s} = \{r<U<s\},\end{gathered}$$

with $r,s,t\in\mathbb{R}^+$, $r<s$. Given a solution $u\in W^{1,\infty}(\Omega)\cap C_0(\bar\Omega)$ to (D_∞) such that $f^*\not\equiv 0$, let

$$M = \max_\Omega u\,, \tag{2.17}$$

$$s_M = \sup\left(\{f>0\}\cap(0,M)\right)\,. \tag{2.18}$$

Then $0 < s_M \leq M$.

The possible cases to be considered are:

α) $f(M) > 0$,

β) $f(M) = 0$, $s_M = M$,

γ) $f(M) = 0$, $s_M < M$, with M not in the interior of $\{f = 0\}$,

δ) $f(M) = 0$, $s_M < M$, with M in the interior of $\{f = 0\}$.

(Note, however, that case δ) cannot occur, as proved in the Theorem below.)

Theorem 2.5 (Convexity of level sets). *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Let $u \in W^{1,\infty}(\Omega) \cap C_0(\bar{\Omega})$ be a variational solution to (D_∞) such that $f^* \not\equiv 0$. If either α) or β) occurs, then every level set of u is convex; if γ) occurs, then the level sets $\{u > t\}$ are convex for every $t \in [0, s_M)$; case δ) cannot occur. If, moreover, Ω is invariant with respect to an orthogonal transformation T, then if either α) or β) occurs, then u is symmetric with respect to T; if γ) occurs, then $u_{|\Omega \setminus \Omega_{s_M}}$ is symmetric with respect to T.*

Proof. Let $x_0 \in \Omega$ be a point where u attains its maximum M, and let s_M be as defined in (2.18). Let (c_i, d_i), $i \in I_M$, be the connected components of $\{f > 0\} \cap (0, M)$.

For any half line r having origin at x_0, let us denote $S_r = r \cap \bar{\Omega}$. We have $u(S_r) = [0, M]$. From the convexity of Ω, it follows easily that for every d, $0 \leq d < U(x_0)$, and for every half line r having origin at x_0, there is a unique $x \in S_r$ such that $U(x) = d$. Indeed, suppose on the contrary that y, $z \in S_r$, $y \neq z$, are such that $U(y) = U(z) = d$, and, for instance, let z belong to the segment joining x_0 and y. Then $B_d(y)$ and $B_{U(x_0)}(x_0)$ are contained in Ω, so that, by the convexity of Ω, we have $B_{d'}(z) \subset \Omega$ for some $d' \in (d, U(x_0))$, contradicting $U(z) = d$. Therefore, recalling (2.16), we have that for every $l \in [0, M) \cap \{f > 0\}$ there exists a unique $x \in S_r$ such that $u(x) = l = U(x)$. Since this fact holds for any half line r having origin at x_0, we have $U(x_0) \geq s_M$ and $u = U$ in Ω_{c_i,d_i} for every $i \in I_M$.

The connected components of $\Omega_{0,s_M} \setminus \cup_{i \in I_M} \overline{\Omega_{c_i,d_i}}$ are convex rings $A_j = \Omega_{a_j,b_j}$, $j \in J_M$, where $a_j < b_j$ and (a_j, b_j) are the connected components of $int(\{f = 0\}) \cap (0, s_M)$. By the continuity of u, we have $u = U$ on ∂A_j. Let us see that $u(A_j) \subset (a_j, b_j)$. From (2.8) it follows that $u < b_j$ in A_j. In order to prove that $u > a_j$ in A_j, let us introduce, for any $p > 1$, the unique solution v_p to

$$\begin{aligned} \Delta_p v_p &= 0 \quad \text{in } A_j\,, \\ v_p &= u_p \quad \text{on } \partial A_j\,, \end{aligned}$$

and the unique solution w_p to

$$\begin{aligned} \Delta_p w_p &= 0 \quad \text{in } A_j\,, \\ w_p &= U \quad \text{on } \partial A_j\,. \end{aligned}$$

From the fact that $u_{p_k} \to u = U$ on ∂A_j and from the comparison principle for the p-Laplace operator (see [T]), we see that for any $\epsilon > 0$ there exists k_ϵ such that

$$u_{p_k} \geq v_{p_k} \geq w_{p_k} - \epsilon, \qquad \text{in } A_j \tag{2.19}$$

for $k \geq k_\epsilon$. Moreover, w_p converges uniformly to the unique viscosity solution w_∞ to

$$\begin{aligned} \Delta_\infty w_\infty &= 0 \quad \text{in } A_j\,, \\ w_\infty &= U \quad \text{on } \partial A_j\,, \end{aligned}$$

(see [J]), so that the Harnack inequality for the ∞-Laplace operator (see [L-M]) implies that $w_\infty(A_j) \subset (a_j, b_j)$. Passing to the limit as $k \to \infty$ in (2.19), we have $u \geq w_\infty > a_j$ in A_j.

Let us distinguish two cases: $f(M) > 0$ and $f(M) = 0$. In the former case we have $f^*(x_0) > 0$, so that there exists a neighborhood V of x_0 where f^* is positive, and, by (2.16), $u = U$ in V. Hence x_0 has to be a point of local maximum for U, and, since Ω is convex, x_0 is a point of absolute maximum for U. Indeed, otherwise, let w be a point of absolute maximum for U and let us consider the segment L joining x_0 and w. By the convexity of Ω, we have that $U(z) > U(x_0)$, for any point $z \in L$, $z \neq x_0$, contradicting that x_0 is a point of local maximum for U. Hence $U(x_0) = R$, where

$$R = \max_{x\in\Omega} d(x, \partial\Omega) \tag{2.20}$$

is the radius of the largest ball contained in Ω, and $s_M = M = u(x_0) = U(x_0) = R$. Moreover $u = U$ in Ω_{c_i,d_i} for every $i \in I_M$ and $\Omega_0^* = \cup_{j\in J_M} A_j$.

If $f(M) = 0$, then $M \leq R$ by (2.8), $\cup_{j\in J_M} A_j \subset \Omega_0^*$, and, by the continuity of u and by (2.18), $u \equiv s_M$ on $\partial\Omega_{s_M}$ and $u \geq s_M$ in Ω_{s_M}. From the convexity of Ω it follows that Ω_t is convex for every $t \in \mathbb{R}$.

Collecting the previous results, we have: $u < a_j$ in Ω_{0,a_j} and $u > b_j$ in Ω_{b_j} for every $j \in J_M$; $u < c_i$ in Ω_{0,c_i} and $u > d_i$ in Ω_{d_i} for every $i \in I_M$ such that $d_i \neq s_M$; $u \geq s_M$ in Ω_{s_M}. It follows easily that $u > 0$ in Ω, so that the level set $\{u > 0\} = \Omega$ is convex, and that if $t \in (0, s_M) \cap \overline{\{f > 0\}}$, then $\{u > t\} = \Omega_t$ is convex. If $t \in (0, s_M) \setminus \overline{\{f > 0\}}$, then $t \in (a_j, b_j)$ for some $j \in J_M$, and, by ii^*), u is the viscosity solution in the convex ring A_j of the capacitary problem

$$\begin{aligned} \Delta_\infty u &= 0 \quad \text{in } A_j\,, \\ u &= a_j \quad \text{on } \{U = a_j\}\,, \\ u &= b_j \quad \text{on } \{U = b_j\} \end{aligned}$$

for the ∞-Laplace operator. From the previous results, $\{u > t\} = \overline{\Omega_{b_j}} \cup \{x \in A_j : u(x) > t\}$, which is convex since $u|_{\overline{A_j}}$ can be obtained as the uniform limit, as $p \to \infty$, of the solutions u_p to the p-capacitary problem (see [J])

$$\begin{aligned} \Delta_p u &= 0 \quad \text{in } A_j\,, \\ u &= a_j \quad \text{on } \{U = a_j\}\,, \\ u &= b_j \quad \text{on } \{U = b_j\}, \end{aligned}$$

for which Lewis ([L]) established convexity of level sets.

If Ω is invariant with respect to an orthogonal transformation T, the function U and the sets Ω_t are invariant with respect to T. If v is a viscosity solution to

$$\begin{aligned} \Delta_\infty v &= 0 \quad \text{in } \Omega_{r,s}, \\ v &= r \quad \text{on } \{U = r\}, \\ v &= s \quad \text{on } \{U = s\}, \end{aligned} \tag{2.21}$$

then also $v \circ T$ solves (2.21) since the ∞-Laplace operator is invariant under orthogonal transformations. By the uniqueness of the viscosity solution to (2.21), established by Jensen ([J]), it follows that $v \circ T = v$. Hence $u_{|\Omega \setminus \Omega_{s_M}}$ is invariant with respect to T.

If α) occurs, then $s_M = M = R$, so that $\Omega_{s_M} = \emptyset$, and convexity of all the level sets and symmetry of u with respect to T follow.

If β) occurs, then $u \equiv s_M$ in Ω_{s_M}, and again convexity of all the level sets and symmetry of u with respect to T follow.

Let us assume that $f(M) = 0$ and $s_M < M$, that is, that either case γ) or case δ) occurs. Let V be a neighborhood of x_0 where $u \geq \frac{M+s_M}{2}$. Let v_p, w_p be the p-harmonic functions in $\Omega_{s_M} \setminus \bar{V}$, which take the same values as u_p, u, respectively, on the boundary. From the comparison principle, we have $u_{p_k} \geq v_{p_k} \geq w_{p_k} - \epsilon$ for $k \geq k_\epsilon$, so that $u \geq w_\infty = \lim w_p$, and w_∞ is ∞-harmonic in $\Omega_{s_M} \setminus \bar{V}$. The Harnack inequality for the ∞-Laplace operator (see [L-M]) implies that either $w_\infty \equiv s_M$, or $w_\infty > s_M$ in $\Omega_{s_M} \setminus \bar{V}$. In the former case, we have a contradiction with $w_p = u > s_M$ on ∂V, whereas in the latter case we have $u > s_M$ in all of Ω_{s_M}.

Finally, if case δ) occurs, we have that $\Omega_{s_M} \subset \Omega_0^*$, and by ii^*), $\Delta_\infty u = 0$ in Ω_{s_M}. Hence $u \equiv s_M$ in Ω_{s_M}, contradicting $s_M < M$. $\diamondsuit$

From the proof of Theorem 2.5 it is clear that the values of u are uniquely determined in the convex ring $\Omega \setminus \Omega_{s_M}$, as summarized in the following Corollary.

Corollary 2.6 (Representation of the solutions). *Let the hypotheses of Theorem 2.5 be satisfied. Then $M \in \overline{\{f > 0\}}$. Moreover, the values of u are uniquely determined in $\Omega \setminus \Omega_{s_M}$. More precisely,*

$$u = U \qquad \text{in } \cup_{i \in I_M} \overline{\Omega_{c_i,d_i}}, \tag{2.22}$$

and u is the viscosity solution to

$$\begin{aligned} \Delta_\infty u = 0 \quad & \text{in } A_j\,, \\ u = a_j \quad & \text{on } \{U = a_j\}, \\ u = b_j \quad & \text{on } \{U = b_j\}, \end{aligned} \tag{2.23}$$

where (c_i, d_i), $i \in I_M$, are the connected components of $\{f > 0\} \cap (0, M)$, $A_j = \Omega_{a_j,b_j}$, $j \in J_M$, and where (a_j, b_j) are the connected components of $int(\{f = 0\}) \cap (0, s_M)$. Moreover, if β) occurs, then $u \equiv s_M$ in Ω_{s_M}.

Remark. Let us notice that if $s_M = M$, that is, if either α) or β) occurs, then u is determined in all of Ω.

Theorem 2.7 (Spherical symmetry and representation of the solutions when $\Omega = B_R$). *Let $\Omega = B_R$ and let $u \in W^{1,\infty}(B_R) \cap C_0(\bar{B}_R)$ be a variational solution to (D_∞) such that $f^* \not\equiv 0$. Then u is radially symmetric and radially non-increasing. Furthermore, $M \in \overline{\{f > 0\}}$, and case δ) cannot occur. If α) occurs, then $M = R$ and $u = U$. If β) occurs, then*

$$u(x) = \begin{cases} U(x) \equiv R - |x| & \text{if } R - s_M \leq |x| \leq R, \\ s_M = M & \text{if } |x| \leq R - s_M. \end{cases} \tag{2.24}$$

If γ) occurs, then

$$u(x) = \begin{cases} U(x) \equiv R - |x| & \text{if } R - s_M \le |x| \le R, \\ \lambda(R - s_M - |x|) + s_M & \text{if } R - s_M - \frac{M - s_M}{\lambda} \le |x| \le R - s_M, \\ M & \text{if } |x| \le R - s_M - \frac{M - s_M}{\lambda}, \end{cases} \tag{2.25}$$

for some $\lambda \in [\frac{M - s_M}{R - s_M}, 1]$. Here $U(x) = d(x, \partial\Omega) = R - |x|$, as defined in (2.9).

Proof. In view of Theorem 2.5, it only remains to prove that $u = U$ in A_j for every $j \in J_M$ and that, if γ) occurs, then (2.25) holds for some $\lambda \in [\frac{M - s_M}{R - s_M}, 1]$. Since $A_j = \{R - b_j < |x| < R - a_j\}$, $u = a_j$ on $\{|x| = R - a_j\}$, $u = b_j$ on $\{|x| = R - b_j\}$, and the Lipschitz constant of u is $L = 1$, we have that $u = U$ in A_j.

Let us assume now that case γ) occurs. Let u_{p_k} be a sequence of solutions to (D_{p_k}) such that (2.4) holds. A recent result by Brock ([Br2, Theorem 10.1]) ensures that the u_{p_k} are radially symmetric and radially non-increasing, so that, from (2.4), it follows that u is radially symmetric and radially non-increasing.

Finally, recalling that ∞-harmonic functions are absolutely minimizing Lipschitz extensions (see [J]), and that, from the proof of Theorem 2.5, $u > s_M$ in Ω_{s_M}, the representation (2.25) follows immediately. ◊

3. An Example

Let us show, by the following Example, that, when case γ) occurs, there may be nontrivial variational solutions to (D_∞) such that $f^* \equiv 0$.

Example. Let $\Omega = B_R$, $f(t) = (t - M)\chi_{(M,\infty)}$, with $0 < M < R$, where χ_S denotes the characteristic function of a set S. Let us look for a radial solution u_p to (D_p), decreasing in $r = |x|$, such that $M_p = \max_{B_R} u_p > M$, for every $p > 2$. Let $r_p \in (0, R)$ be such that $u_p(r_p) = M$. Then

$$u_p = \begin{cases} u_p^- & \text{in } r_p < |x| < R, \\ u_p^+ & \text{in } |x| < r_p, \end{cases}$$

where u_p^- is the radial solution to

$$\begin{gathered} \Delta_p u_p^- = 0 \quad \text{in } r_p < |x| < R\,, \\ u_p^- = 0 \quad \text{on } |x| = r_p\,, \\ u_p^- = M \quad \text{on } |x| = r_p\,, \end{gathered}$$

and u_p^+ is a radial solution to

$$\begin{gathered} -\Delta_p u_p^+ = u_p^+ - M \quad \text{in } B_{r_p}\,, \\ u_p^+ > M \quad \text{in } B_{r_p}\,, \\ u_p^+ = M \quad \text{on } |x| = r_p\,, \end{gathered}$$

and the following transmission condition holds

$$u_p^-{}_{,r}(r_p) = u_p^+{}_{,r}(r_p). \tag{3.1}$$

An easy calculation gives

$$u_p^- = M\left(\frac{R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}}{R^{\frac{p-n}{p-1}} - r_p^{\frac{p-n}{p-1}}}\right).$$

Let $w_p = -(u_p^+ - M)$. Then w_p is a negative radial solution to

$$\begin{aligned} -\Delta_p w_p &= w_p \quad \text{in } B_{r_p}\,, \\ w_p &= 0 \quad \text{on } |x| = r_p\,, \end{aligned}$$

or, equivalently,

$$(w_{p,r})^{p-1}{}_{,r} + \frac{n-1}{r}(w_{p,r})^{p-1} + w_p = 0\,, \tag{3.2}$$

$$w_p(r_p) = 0\,, \tag{3.3}$$

$$w_{p,r}(0) = 0\,. \tag{3.4}$$

From now on, let $n = 1$, so that the second term in (3.2) disappears and (3.2) is an autonomous nonlinear equation. Substituting $w_{p,r} = y$, thinking of y as a function of w, integrating (3.2), and imposing (3.4), we have

$$w_{p,r} = \left(\frac{p}{p-1}\right)^{1/p}\left(\frac{c_p^2 - w_p^2}{2}\right)^{1/p}, \tag{3.5}$$

where $c_p = -w_p(0) = M_p - M$. By integrating over $(0, r)$ and changing variable, we have

$$\left(\frac{p}{2(p-1)}\right)^{1/p} r = \int_0^r \frac{w_{p,r}}{(c_p^2 - w_p^2)^{1/p}}dr = \int_{-c_p}^{w_p(r)} \frac{dw}{(c_p^2 - w^2)^{1/p}}.$$

By imposing the transmission conditions (3.1) and (3.3), we easily have

$$\left(\frac{p}{p-1}\right)\frac{c_p^2}{2} = \left(\frac{M}{R-r_p}\right)^p, \tag{3.6}$$

$$\left(\frac{p}{2(p-1)}\right)^{1/p} r_p = c_p^{\frac{p-2}{p}} \int_0^1 \frac{dz}{(1-z^2)^{1/p}}\,. \tag{3.7}$$

Solving (3.6) in c_p and substituting in (3.7), we are led to find $r_p \in (0, R)$ satisfying the equation

$$g_p(x) = \gamma_p, \tag{3.8}$$

where

$$\begin{aligned} g_p(x) &= x^{1/p}(R-x)^{\frac{p-2}{2p}}\,, \\ \gamma_p &= \left(\frac{2(p-1)}{p}\right)^{1/(2p)} M^{\frac{p-2}{2p}} \left(\int_0^1 \frac{dz}{(1-z^2)^{1/p}}\right)^{1/p}. \end{aligned}$$

We have $g_p(0) = g_p(R) = 0$, $g_p'(x) = \frac{1}{p}(R-x)^{-\frac{p+2}{2p}} x^{\frac{1}{p}-1}(R - \frac{p}{2}x)$. Hence $x_p = \frac{2R}{p}$ is the unique point where g_p attains its maximum

$$g_p(x_p) = \left(\frac{2R}{p}\right)^{1/p} \left(R - \frac{2R}{p}\right)^{\frac{p-2}{2p}}$$

over the interval $[0, R]$. Notice that $\gamma_p \to \sqrt{M}$, whereas $g_p(x_p) \to \sqrt{R} > \sqrt{M}$, as $p \to \infty$. Therefore, for p sufficiently large, there are exactly two points r_p', r_p'' in $(0, R)$, with $r_p' < x_p < r_p''$ which verify (3.8).

Choosing the solution pair (r_p', c_p') to (3.6) – (3.7), we have that $r_p' \to 0$ and, by (3.6), $c_p' \to 0$, as $p \to \infty$. Let w_p be the solution to (3.5), with $c_p = c_p'$, such that $w_p(0) = -c_p'$. Then the regularity condition (3.4) and the transmission conditions (3.1) and (3.3) are satisfied. The corresponding solution u_p to (D_p) converges to the function $\bar{u}(x) = \frac{M}{R}(R - |x|)$ as $p \to \infty$.

Therefore, $\bar{u}$ is a nontrivial variational solution to (D_∞) for which $f^* \equiv 0$.

4. Tame variational solutions

The result of Theorem 2.5 is not fully satisfactory when case γ) holds. A reasonable criterion for a definition of solution to problem (D_∞) is that a solution u does not depend on the behavior of f outside the range of u. Therefore it may be convenient to select a subclass of variational solutions to the problem (D_∞), in order to prevent the anomalous phenomena which can occur when case γ) holds, as illustrated by the Example in Section 3.

To this aim, given a continuous nonnegative function $f : \mathbb{R} \to \mathbb{R}$ and a number $M > 0$, let us define

$$f_M(t) = \begin{cases} f(t) & 0 \le t \le M, \\ 0 & t \ge M, \end{cases} \tag{4.1}$$

if $f(M) = 0$, and $f_M = f$ otherwise. Let us consider the Dirichlet problem

$$\begin{aligned} -\Delta_p v = f_M(v) \quad \text{in } \Omega\,, \\ v = 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{$\tilde{D}_p$}$$

Definition 4.1. A function $u \in W^{1,\infty}(\Omega) \cap C_0(\bar{\Omega})$ such that $M = \max_\Omega u$ is called a *tame variational solution* to (D_∞) if there exists a sequence u_{p_k} of solutions to $(\tilde{D}_{p_k})$, with $p_k \to \infty$, such that (2.4) holds.

Remark. It is clear, from the preceding arguments, that tame variational solutions are variational solutions. Of course, there are either functions f for which variational solutions which are not tame do exist (see, for instance, Section 3), or functions f for which every variational solution is tame (for instance f strictly positive in some interval $[0, L)$ and vanishing outside).

Since the above definition precludes case γ), the following results follow easily from Theorem 2.5, Corollary 2.6 and Theorem 2.7.

Lemma 4.2. *Let u be a tame variational solution to (D_∞). Then $u \equiv 0$ if and only if $f^* \equiv 0$. If $u \not\equiv 0$ then $E_\infty > 0$.*

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Let $u \in W^{1,\infty}(\Omega) \cap C_0(\bar{\Omega})$ be a tame variational solution to (D_∞). Then the level sets $\{u > t\}$ are convex, $s_M = M \in \overline{\{f > 0\}}$, and u is uniquely determined in all of Ω by (2.22)–(2.23) and by $u \equiv s_M$ in Ω_{s_M}. If, moreover, Ω is invariant with respect to an orthogonal transformation T, then u is symmetric with respect to T.*

Theorem 4.4. *Let $\Omega = B_R$ and let $u \in W^{1,\infty}(B_R) \cap C_0(\bar{B}_R)$ be a nontrivial tame variational solution to (D_∞). Then u is radially symmetric and radially nonincreasing. Moreover, $s_M = M \in \overline{\{f > 0\}}$, and u is given by (2.24).*

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Edi Rosset
Dipartimento di Scienze Matematiche
Università degli Studi di Trieste
34100 Trieste, Italy
Email address: rossedi@univ.trieste.it