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A LINEAR MODEL FOR THE DYNAMICS OF FISH LARVAE

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Abstract. We consider a linear model for the growth and the dispersion of fish larvae of certain species. Dispersion is modeled as entailed by the combination of transport and vertical diffusion. We generalize the work of Boushaba, Arino and Boussouar [5], [6] in the sense that horizontal velocities are uniform throughout the water column; but we deal with vertical component velocity and vertical diffusion depending on the space variables and on time, which was not the case in [5], [6]. This new vision leads us to non-autonomous problems, the aim of this work is to show the existence, uniqueness, and positivity of solutions.

1. Introduction.

In this paper, we introduce a mathematical model for the dynamics of the fish larvae of certain species. This model takes into account both the physical and biological effects. For the physical part, the model considered here stresses two main factors: 1) Transport entailed by the currents: the currents are computed using Navier-Stokes equations and are introduced in the equations of the larvae as functions of space and time with sufficient regularity to allow existence and uniqueness of stream lines. 2) Vertical diffusion induced by vertical mixing in the upper part of the water column. For the biological part the main parameters are a function which gives the instantaneous rate of progression within the stages from the egg fertilization to the end of the yolk-sac period.

The model is expressed in a generality which encompasses a large variety of situations. The motivation at the origin of this work is the study of the dynamics of the Bay of Biscay anchovy [4], that is to say, a region of the Atlantic ocean close to the French coast, bordered eastward by the continental shelf. The Bay of Biscay goes from the Northern Spanish coast up to about 46^0 in "latitude". In this region at the end of May , a thermocline establishes itself: the top of the thermocline is roughly at the same distance z_{therm} from the surface. The thermocline divides the water column into three regions: the upper part, from the surface to z_{therm} deep, the so called mixed layer. This is where the larvae grow. Below is the thermocline, a rather thin layer where the temperature loses rapidly a few degrees and the vertical mixing coefficient is negligibly small. Below the thermocline is another well mixed layer where the temperature is only slowly changing with depth. This region is of no

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concern to us for the rest of the study, which will be confined to the mathematical issues related to the above mentioned model.

The domain of study has notably been restricted to the upper layer, the so called mixed layer of the water column.

The purpose of this work is to perform a mathematical analysis of the model, notably, show existence, uniqueness and positivity of solutions. In a previous work coauthored by Boushaba, Boussouar; and Arino [6], a simplified version of the model of the phytoplankton had been investigated. It was assumed that the diffusion rate and the vertical current does not depend on time and the horizontal current is uniform throughout the water column. Under this assumption, it was possible to uncouple the vertical and the horizontal components in the following sense: the study was restricted to each of the horizontal streamlines: the restriction to such a line reduces the functions of time horizontal components to functions of time so that the full model reduces on such a line to a diffusion equation in the vertical variable coupled with a first order growth equation. Our purpose in this work is to extend this method to the more realistic situation where the diffusion rate and the vertical current depends also on time and the horizontal current is uniform throughout the water column. The idea we exploit here is the same in the first time, that is to uncouple the vertical and the horizontal components but the restriction on such a line gives equations of parabolic type with time dependent coefficients. The study of such equations takes up the main part in this work.

Time dependence is dealt with using results on time-dependent evolution equations by Acquistapace [1, 2, 3] and several other authors (Lunardi [8], Tanabe [10]). A valuable source of information of this work was a monograph by Tanabe [10]. The main result of this part, stated in theorem 4.1, ensures that, under some conditions on the coefficients of the equation, the Cauchy problem associated with the equation has a unique classical solution, which moreover is nonnegative if the initial value is non negative. The paper is organized as follows. Section 2 is devoted to recall some important theorems. Section 3 is devoted to a detailed presentation of the model. Section 4 is devoted to proving existence, uniqueness and positivity of solution of our problem.

2. Notation and preliminary results

Let Y be a Banach space and $[a, b]$ a finite interval of the real line, then we define the space

$$C([a,b];Y) = \{f : [a,b] \to Y : f \text{ is continuous}\}.$$

Note that $C([a, b]; Y)$ is a Banach space with the norm

$$\|f\|_{C([a,b];Y)} = \sup_{s\in[a,b]} \|f(s)\|_Y.$$

We also consider the space $C^1([a, b]; Y)$ consisting of functions $f \in C([a, b]; Y)$ such that f is strongly differentiable in $[a, b]$ and $f' \in C([a, b]; Y)$, with the norm

$$\|f\|_{C^1([a,b];Y)} = \|f\|_{C([a,b];Y)} + \|f'\|_{C([a,b];Y)}.$$

Let $\theta \in]0, 1[$ then we define the following Holder type spaces

$$C^\theta([a,b];Y) = \{f \in C([a,b];Y) : [f]_{C^\theta([a,b];Y)} = \sup\{\frac{\|f(s)-f(t)\|_Y}{|t-s|^\theta} \\ t,s \in [a,b], t \neq s\} < \infty\},$$

which is equipped with the norm

$$\|f\|_{C^\theta([a,b];Y)} = \|f\|_{C([a,b];Y)} + [f]_{C^\theta([a,b];Y)}\,.$$

Also let

$$C^{1,\theta}([a,b];Y) = \{f \in C^1([a,b];Y) : f' \in C^\theta([a,b];Y)\},$$

with norm

$$\|f\|_{C^{1,\theta}([a,b];Y)} = \|f\|_{C([a,b];Y)} + \|f'\|_{C^\theta([a,b];Y)}.$$

We consider the problem

$$\begin{gathered} u_t(t,x) - a(t,x)u_{xx}(t,x) - b(t,x)u_x(t,x) - c(t,x)u(t,x) = 0,\\ (t,x) \in [0,T]\times[0,1]\\ \alpha_0(t)u(t,0) - \beta_0(t)u_x(t,0) = \alpha_1(t)u(t,1) + \beta_1(t)u_x(t,1) = 0, \quad t \in [0,T],\\ u(0,x) = \Phi(x), \quad x \in [0,1], \end{gathered} \tag{2.1}$$

under the following assumptions:

$$\begin{gathered} a,b,c \in C([0,T]\times[0,1]),\\ a(.,x), b(.,x), c(.,x) \in C^{1,\delta}([0,T];\mathbb{R})\\ \text{with norms indpendent of } x \in [0,1], \text{ for some } \delta \in]0,1[,\\ a > 0, c \le 0 \quad \text{in } [0,T]\times[0,1], \end{gathered} \tag{2.2}$$

To recall some propositions, we set $E = C([0,1])$, $\|u\|_E = \sup_{x\in[0,1]}|u(x)|$, and define for each $t \in (0,T)$,

$$\begin{gathered} D(A(t)) = \{u \in C^2([0,1]) : \alpha_0(t)u(0) - \beta_0(t)u'(0) = \alpha_1(t)u(1) + \beta_1(t)u'(1) = 0,\}\\ A(t)u = a(t,.)u'' + b(t,.)u' + c(t,.)u\,. \end{gathered} \tag{2.3}$$

Proposition 2.1 ([2]). *Let a,b,c be as in* (2.1), (2.2), *and suppose that $u \in C^2([0,1])$ is a solution of*

$$\begin{gathered} \lambda u - a(t,.)u'' - b(t,.)u' - c(t,.)u = f \in C([0,1]),\\ \alpha_0(t)u(0) - \beta_0(t)u'(0) = z_0 \in \mathbb{C},\\ \alpha_1(t)u(1) + \beta_1(t)u'(1) = z_1 \in \mathbb{C}, \end{gathered} \tag{2.4}$$

where $t \in [0,T]$ is fixed and λ is a complex number lying in the sector

$$\Sigma_K := \{z \in \mathbb{C} : \operatorname{Re} z \ge 0\} \cup \{z \in \mathbb{C} : |\operatorname{Im} z| > K|\operatorname{Re} z|\} \quad (K > 0.)$$

Then there exists $M > 0$, depending on K, a, b, c, but independent of t such that

$$(1+|\lambda|)\|u\|_E + (1+|\lambda|^{1/2})\|u'\|_E + \|u''\|_E \le M(\|f\|_E + (1+|\lambda|^{1/2})(|z_0|+|z_1|)). \tag{2.5}$$

As a consequence of the above proposition we have the following result.

Proposition 2.2 ([2]). *Let a,b,c be as in* (2.1), (2.2)*; let $\{A(t)\}_{t\in[0,T]}$ be defined by (2.3). Then we have:*

(i) *$[0,\infty[\subseteq \rho(A(t))$ for all $t \in [0,T]$; where $\rho(A(t))$ is the resolvent set of $A(t)$ and $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$.*

(ii) $\Sigma_K \subseteq \rho(A(t))$ *and for each* $K > 0$ *there exists* $M(K) > 0$ *(depending also on* a, b, c*) such that*
$$\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \leq \frac{M(K)}{1+|\lambda|} \quad \forall \lambda \in \Sigma_K, \forall t \in [0,T],$$
where Σ_K *is defined above.*

Definition 2.3 ([10])**.** A classical solution of (2.1) is a function
$$u \in C([0,T],E) \cap C((0,T],D(A(t))) \cap C^1((0,T],E),$$
such that $u(0) = x$, $u'(t) - A(t)u(t) = 0$ in $(0,T]$.

Let us assume the following hypotheses:

(AT1) For each $t \in [0,T]$, $A(t) : D(A(t)) \subseteq E \to E$ is a closed linear operator and there exists $M > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that
$$\rho(A(t)) \supseteq S_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \theta\} \cup \{0\},$$
$$\|R(\lambda, A(t))\| \leq \frac{M}{1+|\lambda|} \quad \forall \lambda \in S_\theta \cup \{0\},\ \forall t \in [0,T],$$

(AT2) There exist $B > 0$ and $\delta_1, \dots, \delta_k, \nu_1, \dots, \nu_k$ with $0 \leq \nu_i < \delta_i \leq 2$ such that
$$\|A(t)R(\lambda, A(t))((A(s))^{-1} - (A(t))^{-1})\| \leq B \sum_{i=1}^{k} |t-s|^{\delta_i} |\lambda|^{\nu_i - 1},$$
for all $\lambda \in S_\theta - \{0\}$, $0 \leq s < t \leq T$.

It is obvious that Σ_K and S_θ are the same sets.

Theorem 2.4 ([10])**.** *Assume that (AT1) and (AT2) hold. Then, if* $x \in \bar{D}(A(0))$*, problem* (2.1) *has a unique classical solution.*

Remark 2.5. In general the function c in the problem (2.1)is not negative. Moreover by setting $u = ve^{\omega t}$ with $\omega \in \mathbb{R}$, the function v is solution of
$$\begin{gathered} v'(t) - (A(t) - \omega I)v(t) = 0, \quad t \in (0,T] \\ v(0) = x\,. \end{gathered} \tag{2.6}$$
Hence existence, uniqueness and positivity of solutions of problem (2.6) is equivalent to the same properties of problem (2.1).

3. The model

The domain under consideration is $\Omega = D \times (0, z^*)$, where D is an open subset of the surface, that is D is a portion of the plane and z^* is the distance from the surface to a region above the thermocline .

The state variable for the dynamics of the larvae is the density of larvae. For the part of the larval cycle which goes from fertilization to the end of stage, the density $l = l(t, s, P)$, where s denotes the position within the stages, which we take specifically of the Bay of Biscay anchovy in $[1, 12)$ [4] and $P = (x, y, z)$ represents a generic point in the physical space.

The region of observation is assimilated to the product of the horizontal plane and a vertical line. The origin is a point of the surface in the sea, the x axis is oriented westward, the y axis is oriented northward, and the z axis is oriented downward. Of course t is the chronological time. l is a density with respect to the stage and the position. The larvae are characterized by their density, that is to say, at each time $t \in [0,T]$, where T is the maximal time of observation, $l(t,s,P)$ can be thought of as the larvae biomass per unit of volume evaluated at the point P, at that time. The full model is as follows

$$\begin{gathered}
\frac{\partial l}{\partial t} + \frac{\partial (fl)}{\partial s} + div\,(Vl) - \frac{\partial}{\partial z}(h\frac{\partial l}{\partial z}) + \mu l = 0,\\
l(t,1,x,y,z) = B(t,P),\\
h\frac{\partial l}{\partial z} = 0, \quad z = 0,\\
h\frac{\partial l}{\partial z} = 0 \quad z = z^*,
\end{gathered} \tag{3.1}$$

We now discuss in detail the parameters and functions of the model.

The velocity. The velocity vector $V(t,P) = (V_1(t,P), V_2(t,P), V_3(t,P))$ describes the sea current which is supposed to be known. We assume that the sea water is incompressible, which yields:

$$\mathrm{div}(V) = 0, \tag{3.2}$$

with $V_1(t,x,y,z) = V_1(t,x,y)$, $V_2(t,x,y,z) = V_2(t,x,y)$.

The mixing coefficient. The mixing coefficient $h = h(t,P)$ gives the diffusion rate, supposed to be essentially vertical.

The growth function. The main biological parameters are functions $f(t,s)$, which gives the instantaneous rate of progression within the stages from the egg fertilization to the end of the yolk-sac period. For the principle of determination see [9, 4].

The mortality of larvae. The mortality is modelled by the expression $\mu = \mu(t,s,P)$.

Demographic boundary conditions. Demographic boundary conditions are given at $s = 1$, at any time during the spawning period, the variable s takes its values in the interval $[1,12)$, where $s = 1$ corresponds to the newly fertilized eggs, and $s = 12$, to the end of the yolk sac period.

Horizontal boundary conditions. Model (3.1) does not show any lateral boundary conditions. Choosing the right boundary in the x and y directions is a difficult issue that we mainly avoid here by assuming that the initial value has a compact support contained in the interior of the domain and we consider the solution within a time interval $[0,T]$ during which the horizontal projection of the support is contained in the interior of the domain D.

Vertical boundary conditions. Vertical boundary conditions are imposed at the surface and at z^*, here we are assuming a no flux conditions.

Initial conditions Initial conditions are given at $t = 0$ (beginning of the year). The standing assumption is that there is no larva alive at this period of the year, so that $l(0,s,x,y,z) = 0$.

Remark 3.1. What we call an initial value in the present context is not the value of the solution at a given time or rather, the only relevant information would be that at $t = 0$ (that is 1^{st} January) there is no larva in the sea. What we consider as an initial value is the distribution of newly fertilized eggs, that is the larvae at stage $s = 1$ all over the reproduction season.

4. Existence, uniqueness and positivity of the solution of the Cauchy problem

The aim of this section is to show that model (3.1) possesses a positive, unique solution. For this we use an approach by the method of characteristics to build a one dimensional time dependent parabolic equation whose solution will yield the solution of equation (3.1). We assume that

(H1) V_1, V_2 are functions in $C^1((0,T) \times D)$ and $f \in C^1((0,T) \times (1,12))$.

We introduce the flow generated by the horizontal current and the size growth, that is

$$\phi := \phi(\tau, t_0, 1, x_0, y_0),$$

and for each initial value $\tilde{\zeta} \equiv (t_0, 1, x_0, y_0)$, $\phi(\tau, \tilde{\zeta})$ is the solution of the equation

$$\Big(\frac{dt}{d\tau}, \frac{ds}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}\Big) = (1,\ f(t,s), V_1(t,x,y), V_2(t,x,y)), \tag{4.1}$$

satisfying $t(0) = t_0,\ s(0) = 1,\ x(0) = x_0,\ y(0) = y_0$, since the theory of ordinary differential equations guarantees that a unique characteristic curve passes through each point $\tilde{\zeta}$.

We denote $\bar{l}(\tau, z) \equiv \bar{l}(\tau, \tilde{\zeta}, z) = l(\phi(\tau, \tilde{\zeta}), z)$ the restriction of l along the characteristic line. The equation verified by $\bar{l}$ reads

$$\frac{\partial \bar{l}(\tau,z)}{\partial \tau} + \bar{V}_3 \frac{\partial \bar{l}(\tau,z)}{\partial z} - \frac{\partial}{\partial z}\Big(\bar{h}\frac{\partial \bar{l}(\tau,z)}{\partial z}\Big) + \bar{\gamma}\bar{l}(\tau,z) = 0,$$

where $\bar{V}_3 := \bar{V}_3(\tau, \tilde{\zeta}, z)$, $\bar{h} := \bar{h}(\tau, \tilde{\zeta}, z)$, $\bar{\gamma} := \bar{\gamma}(\tau, \tilde{\zeta}, z)$ are the restrictions of V_3, h, B, γ respectively along the characteristic line and γ is equation of order 0. So to each $\tilde{\zeta}$, we have associated the following problem

$$\begin{gathered}
\frac{\partial \bar{l}}{\partial \tau} + \bar{V}_3 \frac{\partial \bar{l}}{\partial z} - \frac{\partial}{\partial z}\Big(\bar{h}\frac{\partial \bar{l}}{\partial z}\Big) + \bar{\gamma}\bar{l} = 0, \\
\bar{l}(0,z) = \bar{B}(z), \\
\bar{h}(\tau,0)\frac{\partial \bar{l}}{\partial z}(\tau,0) = 0, \\
\bar{h}(\tau,z^*)\frac{\partial \bar{l}}{\partial z}(\tau,z^*) = 0,
\end{gathered} \tag{4.2}$$

where $\bar{B}(z)$ is the restriction of B along the characteristic line. We consider the operator $A(\tau) : D(A(\tau)) \subseteq C([0,z^*]) \to C([0,z^*])$ defined by

$$\begin{gathered}
A(\tau)u = \bar{V}_3(\tau,.)u' - (\bar{h}(\tau,.)u')' + \bar{\gamma}(\tau,.)u, \\
D(A(\tau)) = \{u \in C^2([0,z^*]),\ \ \bar{h}(\tau,0)u'(0) = \bar{h}(\tau,z^*)u'(z^*) = 0\}.
\end{gathered}$$

We now state the assumptions of this section.

(H2) $h \in C^1([0,T] \times \bar{\Omega})$, $V_3 \in C([0,T] \times \bar{\Omega})$, $\gamma \in C([0,T] \times [1,12] \times \bar{\Omega})$.
(H3) $h, \frac{\partial h}{\partial z}, V_3 \in C^{1,\delta}([0,T]; C(\bar{\Omega}))$, and $\gamma \in C^{1,\delta}([0,T]; C([1,12] \times \bar{\Omega}))$.
(H4) $h \geq c_0$ in $[0,T] \times \bar{\Omega}$ where $c_0 > 0$.

Theorem 4.1. *Assume (H2)–(H4) hold. If the positive function $\bar{B}$ is in $C([0,z^*])$, then problem (4.2) has a unique non-negative classical solution.*

Proof. Without loss of generality we can assume that $\gamma \geq 0$, otherwise we can replace γ by $\gamma + \omega \geq 0$ see Remark 2.5. The main idea is to use theorem 2.4. The first assertion (AT1) follows from the proposition 2.2. Concerning the second assertion (AT2), for $f \in C([0, z^*])$, $t, s \in G_1$, where G_1 is some neighborhood of $\tau = 0$, $\lambda \in S_\theta - \{0\}$, we set $v = (A(s))^{-1} f$ and $u = R(\lambda, A(t))(\lambda - A(s))v$, then we have to estimate the $C([0, z^*])$-norm of

$$u - v = (A(t))R(\lambda, A(t))(A(t))^{-1} - (A(s))^{-1})f.$$

Now $u - v \in C^2([0, z^*])$ and u and v solve

$$\begin{gathered} \lambda u - A(t,.)u = \lambda v - f, \\ \bar{h}(t, 0)u'(0) = \bar{h}(t, z^*)u'(z^*) = 0, \end{gathered} \tag{4.3}$$

and

$$\begin{gathered} A(s,.)v = f, \\ \bar{h}(s, 0)v'(0) = \bar{h}(s, z^*)v'(z^*) = 0, \end{gathered} \tag{4.4}$$

respectively. This shows that

$$\begin{gathered} \lambda(u - v) - A(t,.)(u - v) = (A(t,.) - A(s,.))v, \\ \bar{h}(t, 0)(u' - v')(0) = (\bar{h}(s, 0) - \bar{h}(t, 0))v'(0), \\ \bar{h}(t, z^*)(u' - v')(z^*) = (\bar{h}(s, z^*) - \bar{h}(t, z^*))v'(z^*). \end{gathered} \tag{4.5}$$

Applying Proposition 2.1 to (4.4) with $\lambda = 0$, we have

$$\|v\|_E \leq c\|f\|_E. \tag{4.6}$$

Using again the proposition 2.1 to (4.5), we get

$$\begin{aligned} |\lambda|\|u - v\|_E &\leq M(\|(A(s,.) - A(t,.))v\|_E + (1 + |\lambda|^{1/2}) \\ &\quad \times \left(|(\bar{h}(s, 0) - \bar{h}(t, 0))v'(0)| + |(\bar{h}(s, z^*) - \bar{h}(t, z^*))v'(z^*)|\right)). \end{aligned}$$

Using hypothesis (H3) and by virtue of (4.6),

$$\|u - v\|_E \leq c(|t - s||\lambda|^{-1}|t - s|^\delta|\lambda|^{-1} + |t - s||\lambda|^{-1/2})\|f\|_E.$$

Hence the hypothesis (AT2) holds.

Since $D(A(0))$ is dense in $C([0, z^*])$ see [2], then according to Theorem 2.4, for $\bar{B} \in C([0, z^*])$ we have existence and uniqueness of a classical solution of problem (4.2). It remain to see that the solution is positive which can be proved by the standard argument. If u is solution of problem (4.2), we set $u = u^+ - u^-$ where u^+ and u^- are respectively the positive and negative part of u, so multiplying the equation (4.2) by u^-, and integrating over $(0, z^*)$ we have

$$\int_0^{z^*} (\frac{\partial u}{\partial \tau}u^- + \bar{h}(\tau, z)\frac{\partial u}{\partial z}\frac{\partial u^-}{\partial z} + \bar{V}_3(\tau, z)\frac{\partial u}{\partial z}u^- + \bar{\gamma}(\tau, z)uu^-)dz = 0,$$

hence

$$-\frac{1}{2}\frac{d}{d\tau}\|u^-(\tau)\|^2_{L^2(0,z^*)} \geq c_0\int_0^{z^*} |\frac{\partial u^-}{\partial z}|^2 dz - m_3\int_0^{z^*} \frac{\partial u^-}{\partial z}u^- dz + m_4\int_0^{z^*} |u^-|^2 dz,$$

with

$$m_3 = \sup_{\tau,z} |\bar{V}_3(\tau, z)|, \quad m_4 = \inf_{\tau,z} |\bar{\gamma}(\tau, z)|.$$

Since

$$\int_0^{z^*} \frac{\partial u^-}{\partial z} u^- dz \le \int_0^{z^*} \big(\rho|\frac{\partial u^-}{\partial z}|^2 + \frac{1}{\rho}|u^-|^2\big) dz, \quad \forall \rho > 0,$$

it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|u^-(\tau)\|^2_{L^2(0,z^*)} + (c_0 - m_3\rho)\|\frac{\partial u^-(\tau)}{\partial z}\|^2_{L^2(0,z^*)} \\
&+ (m_4 - \frac{m_3}{\rho} + \omega)\|u^-(\tau)\|^2_{L^2(0,z^*)} \\
&\le \omega\|u^-(\tau)\|^2_{L^2(0,z^*)},
\end{aligned}$$

choosing ρ and ω such that

$$c_0 - m_3\rho > 0 \quad \text{and} \quad m_4 - \frac{m_3}{\rho} + \omega > 0,$$

so

$$\frac{1}{2}\frac{d}{d\tau}\|u^-(\tau)\|^2_{L^2(0,z^*)} \le \omega\|u^-(\tau)\|^2_{L^2(0,z^*)},$$

then

$$\|u^-(\tau)\|^2_{L^2(0,z^*)} \le \|u^-(0)\|^2_{L^2(0,z^*)} e^{2\omega\tau},$$

which gives $u^-(\tau) = 0$ provided $B \ge 0$, then the solution is positive. □

Recall that for $z \in [0, z^*]$ the system

$$\begin{gathered}
t = T(\tau, t_0, x_0, y_0), \quad s = S(\tau, t_0, x_0, y_0), \\
x = X(\tau, t_0, x_0, y_0), \quad y = Y(\tau, t_0, x_0, y_0),
\end{gathered}$$

is a solution of the characteristic system (4.1) emanating from the point $\tilde{\zeta}$. We have also

$$\begin{gathered}
t_0 = T(0, t_0, x_0, y_0), \quad 1 = S(0, t_0, x_0, y_0), \\
x_0 = X(0, t_0, x_0, y_0), \quad y_0 = Y(0, t_0, x_0, y_0)\,.
\end{gathered}$$

If

$$\operatorname{Jac}(T,S,X,Y) := \begin{vmatrix}
\frac{\partial T}{\partial \tau} & \frac{\partial T}{\partial t_0} & \frac{\partial T}{\partial x_0} & \frac{\partial T}{\partial y_0} \\
\frac{\partial S}{\partial \tau} & \frac{\partial S}{\partial t_0} & \frac{\partial S}{\partial x_0} & \frac{\partial S}{\partial y_0} \\
\frac{\partial X}{\partial \tau} & \frac{\partial X}{\partial t_0} & \frac{\partial X}{\partial x_0} & \frac{\partial X}{\partial y_0} \\
\frac{\partial Y}{\partial \tau} & \frac{\partial Y}{\partial t_0} & \frac{\partial Y}{\partial x_0} & \frac{\partial Y}{\partial y_0}
\end{vmatrix}_{\tau=0} \ne 0,$$

then the Jacobian does not vanish in a neighborhood of the initial curve. Therefore, the local inversion theorem guarantees that we can solve for (τ, t_0, x_0, y_0) as function of (t, s, x, y) near the initial curve; that is, there exists a neighborhood G_1 of $(0, t_0, x_0, y_0)$ and a neighborhood G of (t, s, x, y) such that

$$(T, S, X, Y) : G_1 \to G$$

is a diffeomorphism. Then

$$\begin{gathered}
\tau = \psi_1(t, s, x, y), \quad t_0 = \psi_2(t, s, x, y), \\
x_0 = \psi_3(t, s, x, y), \quad y_0 = \psi_4(t, s, x, y),
\end{gathered}$$

and for initial data

$$\begin{gathered}
0 = \psi_1(t_0, 1, x_0, y_0), \quad t_0 = \psi_2(t_0, 1, x_0, y_0), \\
x_0 = \psi_3(t_0, 1, x_0, y_0), \quad y_0 = \psi_4(t_0, 1, x_0, y_0).
\end{gathered}$$

Once problem (4.2) is solved, we have

$$l(t,s,x,y,z)=\bar{l}(\psi_1,\psi_2,\psi_3,\psi_4,z),$$

in a neighborhood of G. Indeed, by differentiation we obtain that

$$\begin{gathered}
\frac{\partial l}{\partial t}=\frac{\partial \bar{l}}{\partial \tau}\frac{\partial \psi_1}{\partial t}+\frac{\partial \bar{l}}{\partial t_0}\frac{\partial \psi_2}{\partial t}+\frac{\partial \bar{l}}{\partial x_0}\frac{\partial \psi_3}{\partial t}+\frac{\partial \bar{l}}{\partial y_0}\frac{\partial \psi_4}{\partial t},\\
f\frac{\partial l}{\partial s}=f(\frac{\partial \bar{l}}{\partial \tau}\frac{\partial \psi_1}{\partial s}+\frac{\partial \bar{l}}{\partial t_0}\frac{\partial \psi_2}{\partial s}+\frac{\partial \bar{l}}{\partial x_0}\frac{\partial \psi_3}{\partial s}+\frac{\partial \bar{l}}{\partial y_0}\frac{\partial \psi_4}{\partial s}),\\
V_1\frac{\partial l}{\partial x}=V_1(\frac{\partial \bar{l}}{\partial \tau}\frac{\partial \psi_1}{\partial x}+\frac{\partial \bar{l}}{\partial t_0}\frac{\partial \psi_2}{\partial x}+\frac{\partial \bar{l}}{\partial x_0}\frac{\partial \psi_3}{\partial x}+\frac{\partial \bar{l}}{\partial y_0}\frac{\partial \psi_4}{\partial x}),\\
V_2\frac{\partial l}{\partial y}=V_2(\frac{\partial \bar{l}}{\partial \tau}\frac{\partial \psi_1}{\partial y}+\frac{\partial \bar{l}}{\partial t_0}\frac{\partial \psi_2}{\partial y}+\frac{\partial \bar{l}}{\partial x_0}\frac{\partial \psi_3}{\partial y}+\frac{\partial \bar{l}}{\partial y_0}\frac{\partial \psi_4}{\partial y})\,.
\end{gathered}$$

Thus

$$\begin{aligned}
\frac{\partial l}{\partial t}+f\frac{\partial l}{\partial s}+V_1\frac{\partial l}{\partial x}+V_2\frac{\partial l}{\partial y}&=\frac{\partial \bar{l}}{\partial \tau}(\frac{\partial \psi_1}{\partial t}+f\frac{\partial \psi_1}{\partial s}+V_1\frac{\partial \psi_1}{\partial x}+V_2\frac{\partial \psi_1}{\partial y})\\
&\quad+\frac{\partial \bar{l}}{\partial t_0}(\frac{\partial \psi_2}{\partial t}+f\frac{\partial \psi_2}{\partial s}+V_1\frac{\partial \psi_2}{\partial x}+V_2\frac{\partial \psi_2}{\partial y})\\
&\quad+\frac{\partial \bar{l}}{\partial x_0}(\frac{\partial \psi_3}{\partial t}+f\frac{\partial \psi_3}{\partial s}+V_1\frac{\partial \psi_3}{\partial x}+V_2\frac{\partial \psi_3}{\partial y})\\
&\quad+\frac{\partial \bar{l}}{\partial y_0}(\frac{\partial \psi_4}{\partial t}+f\frac{\partial \psi_4}{\partial s}+V_1\frac{\partial \psi_4}{\partial x}+V_2\frac{\partial \psi_4}{\partial y}).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial l}{\partial t}+f\frac{\partial l}{\partial s}+V_1\frac{\partial l}{\partial x}+V_2\frac{\partial l}{\partial y}&=\frac{\partial \bar{l}}{\partial \tau}(\frac{\partial T}{\partial \tau}\frac{\partial \psi_1}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_1}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_1}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_1}{\partial y})\\
&\quad+\frac{\partial \bar{l}}{\partial t_0}(\frac{\partial T}{\partial \tau}\frac{\partial \psi_2}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_2}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_2}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_2}{\partial y})\\
&\quad+\frac{\partial \bar{l}}{\partial x_0}(\frac{\partial T}{\partial \tau}\frac{\partial \psi_3}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_3}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_3}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_3}{\partial y})\\
&\quad+\frac{\partial \bar{l}}{\partial y_0}(\frac{\partial T}{\partial \tau}\frac{\partial \psi_4}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_4}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_4}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_4}{\partial y}).
\end{aligned}$$

For $Z=(T,S,X,Y)^T$ and $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^T$, we have $(Z\circ\psi)(t,s,x,y)=(t,s,x,y)$ which implies

$$\operatorname{Jac}(Z).\operatorname{Jac}(\psi)=\operatorname{Id}_4\,. \tag{4.7}$$

By identification in (4.7), we find

$$\begin{gathered}
\frac{\partial T}{\partial \tau}\frac{\partial \psi_1}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_1}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_1}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_1}{\partial y}=1,\\
\frac{\partial T}{\partial \tau}\frac{\partial \psi_2}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_2}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_2}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_2}{\partial y}=0,\\
\frac{\partial \psi_3}{\partial t}+f\frac{\partial \psi_3}{\partial s}+V_1\frac{\partial \psi_3}{\partial x}+V_2\frac{\partial \psi_3}{\partial y}=0,\\
\frac{\partial T}{\partial \tau}\frac{\partial \psi_4}{\partial t}+\frac{\partial S}{\partial t_0}\frac{\partial \psi_4}{\partial s}+\frac{\partial X}{\partial x_0}\frac{\partial \psi_4}{\partial x}+\frac{\partial Y}{\partial y_0}\frac{\partial \psi_4}{\partial y}=0\,.
\end{gathered}$$

Therefore,
$$\frac{\partial l}{\partial t}+f\frac{\partial l}{\partial s}+V_1\frac{\partial l}{\partial x}+V_2\frac{\partial l}{\partial y}=\frac{\partial \bar{l}}{\partial \tau}\,.$$
In addition,
$$\frac{\partial l}{\partial z}=\frac{\partial \bar{l}}{\partial z},$$
for the initial data
$$\begin{aligned}
l(t,1,x,y,z)&=\bar{l}(\psi_1(t,1,x,y),\psi_2(t,1,x,y),\psi_3(t,1,x,y),\psi_4(t,1,x,y),z),\\
&=\bar{l}(0,t,x,y,z),\\
&=\bar{B}(t,x,y,z),\\
&=B(T(0,t,x,y),X(0,t,x,y),Y(0,t,x,y),z),\\
&=B(t,x,y,z)\,.
\end{aligned}$$
Then l is a unique solution of (3.1). So the solution l of (3.1) can be determined in terms of the solution $\bar{l}$ of (4.2).

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