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STRUCTURAL STABILITY OF POLYNOMIAL SECOND ORDER DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract. This work characterizes the structurally stable second order differential equations of the form $x'' = {}_{i=0}^{n} a_i(x)(x')^i$ where $a_i : \Re \to \Re$ are C^r periodic functions. These equations have naturally the cylinder $M = S^1 \times \Re$ as the phase space and are associated to the vector fields $X(f) = y\frac{\partial}{\partial x} + f(x,y)\frac{\partial}{\partial y}$, where $f(x,y) = {}_{i=0}^{n} a_i(x)y^i\frac{\partial}{\partial y}$. We apply a compactification to M as well as to $X(f)$ to study the behavior at infinity. For $n \geq 1$, we define a set Σ^n of $X(f)$ that is open and dense and characterizes the class of structural differential equations as above.

1. Introduction

We denote by $\mathcal{E}^{n,r}$ the space of vector fields

$$X(f) = y\frac{\partial}{\partial x} + \sum_{i=0}^{n} a_i(x)y^i\frac{\partial}{\partial y}$$

defined on $M = S^1 \times \Re$ where $a_i(x)$ are C^r periodic functions, $r \geq 1$ and $n \geq 1$.

A vector field $X(f)$ is associated naturally, it is in fact equivalent, to the second order differential equation

$$E_f : x'' = f(x,x') \quad \text{where} \quad f(x,y) = \sum_{i=0}^{n} a_i(x)y^i.$$

We endow $\mathcal{E}^{n,r}$ with the structure in which $X(f)$ is identified with the $n+1$-tuple $(a_0(x),\dots,a_n(x))$ of its coefficient functions and the norm is defined by

$$||X(f)|| = \sup_{1\leq k\leq r}\{|\frac{d^k}{dx^k}a_i(x)| : x \in S^1, 0 \leq i \leq n\}.$$

The aim of this paper is to characterize the vector fields $X(f) \in \mathcal{E}^{n,r}$ (therefore, E_f) that are structurally stable under small perturbations in the space $\mathcal{E}^{n,r}$. See precise definition below.

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We establish the structural stability of $X(f) \in \mathcal{E}^{n,r}$ on the open surface M using a compactification of the type $u = x$ and $v = \arctan(y)$ (cylindrical compactification). We denote the compactifications of $X(f)$ and M by $\widetilde{X}(f)$ and $\widetilde{M}$ respectively. In section 2, we find the following expressions:
For $n = 1,2,3$, $\widetilde{X}(f) = \sin(v)\frac{\partial}{\partial u} + \sum_{i=0}^{n} a_i(u)\cos^{3-i}(v)\sin^i(v)\frac{\partial}{\partial v}$;
for $n \geq 4$, $\widetilde{X}(f) = \sin(v)\cos^{n-3}(v)\frac{\partial}{\partial u} + \sum_{i=0}^{n} a_i(u)\sin^i(v)\cos^{n-i}(v)\frac{\partial}{\partial v}$.
This allows us to understand the behavior of $X(f)$ at infinity (i.e at the ends) of M by studying $\widetilde{X}(f)$ near the boundary of $\widetilde{M}$. Thus we find that on $\partial\widetilde{M}$, $\widetilde{X}(f)$ has: periodic orbits when $n = 1,2$; tangency points when $n = 3$; hyperbolic singularities when $n = 4$; and semi-hyperbolic or nilpotent singularities when $n > 4$. See section 3.

The characterization of structurally stable vector fields $X(f)$ on M is expressed in terms of $\widetilde{X}(f)$ on $\widetilde{M}$. For that, we give the following definition:

Definition 1.1. A vector field $X(f)$ is structurally stable in M if there is a neighborhood $\mathcal{U}$ in $\mathcal{E}^{n,r}$ such that $\forall X(g) \in \mathcal{U}$, there exists a homeomorphism $h_g : \widetilde{M} \to \widetilde{M}$ which maps trajectories of $\widetilde{X}(f)$ onto trajectories of $\widetilde{X}(g)$, preserving orientation and $\partial\widetilde{M}$.

In section 4, we define for each $n \geq 1$ a subset Σ^n of $\mathcal{E}^{n,r}$ such that if $X(f) \in \Sigma^n$, its compactification $\widetilde{X}(f)$ has generic properties with respect to singularities, to periodic orbits and to connections of singularity separatrix on $\widetilde{M}$. In this way, we extend the conditions of Peixoto M.M and Peixoto M.C (see [13], also [17]) that characterize the C^1-structurally stable systems on closed surfaces, with singularities on the boundary. We recall that these conditions include tangencies at the boundary of a closed surface which insure the C^1 structural stability.

The main results of this paper can be formulated as follows:

Theorem 1.2 (Genericity)**.** *Σ^n is open and dense in $\mathcal{E}^{n,r}$ where $r \geq 2$ for $n = 1$ or $n \geq 5$ and $r \geq 1$ for $n = 2,3$ or 4.*

Theorem 1.3 (Characterization)**.** *$X(f) \in \mathcal{E}^{n,r}$ is structurally stable if and only if $X(f) \in \Sigma^n$ where $r \geq 2$ for $n = 1$ or $n \geq 5$ and $r \geq 1$ for $n = 2,3$ or 4.*

We prove Theorems 1.2 and 1.3 in sections 5 and 6 respectively. In section 7, we present a discussion of the sources that motivate this work. Our results make a link between the works of Sotomayor [18] and Barreto [2] and Shahshahani [16] dedicated to C^r-structurally stable second order differential equations. The first author considered E_f with the uniform topology on compact regions of $\Re^2$ and M; the second two authors considered E_f with the Whitney topology on the whole M.

2. Compactification

In this section, we define a vector field on the cylinder $\widetilde{M} = S^1 \times [\frac{\pi}{2}, \frac{\pi}{2}]$ induced by $X(f) \in \mathcal{E}^{n,r}$, where $f(x,y) = \sum_{i=0}^{n} a_i(x)y^i$, and we describe $X(f)$ in coordinate neighborhoods of the infinity of M. We denote by $\widetilde{M}^\circ = S^1 \times (\frac{\pi}{2}, \frac{\pi}{2})$.

Let $\mathcal{C} : M \to \widetilde{M}^\circ$ be a diffeomorphism defined by

$$\mathcal{C}(x,y) = (x, \arctan(y)).$$

The ends of M are transformed into the circles $C_{\pm\frac{\pi}{2}} = S^1 \times \{\pm\frac{\pi}{2}\}$.

Now, we induce the vector field $\mathcal{C}_*(X(f))$ on $\widetilde{M}^\circ = S^1 \times (\frac{\pi}{2}, \frac{\pi}{2})$ by $X(f)$ as follows

$$\mathcal{C}_*(X(f))(u,v) = D\mathcal{C}(x,y) \cdot X(f)(x,y)$$

where $(u,v) = \mathcal{C}(x,y)$ and $D\mathcal{C}(x,y)$ is the derivative of $\mathcal{C}$ at (x,y).

Thus we obtain

$$\mathcal{C}_*(X(f))(u,v) = \frac{\sin(v)}{\cos(v)}\frac{\partial}{\partial u} + \sum_{i=0}^{n} a_i(u)\sin^i(v)\cos^{-i+2}(v)\frac{\partial}{\partial v}.$$

Then the following vector field

$$\widetilde{X}(f) = \begin{cases} \cos(v) \cdot \mathcal{C}_*(X(f)) & \text{for } n = 1,2,3 \\ \cos(v)^{n-2} \cdot \mathcal{C}_*(X(f)) & \text{for } n \geq 4 \end{cases}$$

can be extended to the whole $\widetilde{M}$.

We call $\widetilde{X}(f)$ the cylindrical compactification of $X(f)$. The explicit expressions of $\widetilde{X}(f)$ are:
for $n = 1,2,3$,

$$\widetilde{X}(f) = \sin(v)\frac{\partial}{\partial u} + \sum_{i=0}^{n} a_i(u)\cos^{3-i}(v)\sin^i(v)\frac{\partial}{\partial v} \tag{2.1}$$

and for $n \geq 4$,

$$\widetilde{X}(f) = \sin(v)\cos^{n-3}(v)\frac{\partial}{\partial u} + \sum_{i=0}^{n} a_i(u)\sin^i(v)\cos^{n-i}(v)\frac{\partial}{\partial v}. \tag{2.2}$$

In the sequel, we write $X(f)$ in coordinate neighborhoods of the ends of M. Under the map $\Upsilon : U^\pm \to \widetilde{U}^\pm$ defined by $\Upsilon(x,y) = (x, \frac{1}{y})$ for $y \neq 0$, the sets

$$U^+ = \{(x,y) \in M : x \in S^1, y > y_0\},$$
$$U^- = \{(x,y) \in M : x \in S^1, y < -y_0\}$$

(where $y_0 \in \Re^+$) are transformed into

$$\widetilde{U}^+ = \{(x,y) \in M : x \in S^1, 0 \leq y < y_0^{-1}\},$$
$$\widetilde{U}^- = \{(x,y) \in M : x \in S^1, -y_0^{-1} < y \leq 0\}$$

respectively. The ends of M are represented by the circle $\widetilde{C}_0 = S^1 \times \{0\} \subset \widetilde{U}^\pm$.

Now, we induce the vector field $\Upsilon_*(X(f))$ on $\widetilde{U}^\pm \setminus \widetilde{C}_0$ by $X(f)$ as follows

$$\Upsilon_*(X(f))(u,v) = D\Upsilon(x,y) \cdot X(f)(x,y)$$

where $(u,v) = \Upsilon(x,y)$. Then the vector field

$$\widetilde{X}_1(f) = \begin{cases} v \cdot \Upsilon_*(X(f)) & \text{for } n = 1,2,3 \\ v^{n-2} \cdot \Upsilon_*(X(f)) & \text{for } n \geq 4 \end{cases}$$

can be extended to the whole $\widetilde{U}^\pm$. The explicit expressions of $\widetilde{X}_1(f)$ are

$$\text{for } n = 1,2,3, \quad \widetilde{X}_1(f) = \frac{\partial}{\partial u} - \sum_{i=0}^{n} a_i(u)v^{3-i}\frac{\partial}{\partial v} \tag{2.3}$$

$$\text{for } n \geq 4, \quad \widetilde{X}_1(f) = v^{n-3}\frac{\partial}{\partial u} - \sum_{i=0}^{n} a_i(u)v^{n-i}\frac{\partial}{\partial v}. \tag{2.4}$$

3. Behavior of $X(f)$ at infinity

In this section, we study the behavior of $X(f)$ near infinity by means of its cylindrical compactification $\widetilde{X}(f)$.

Proposition 3.1. *Let $X(f) \in \mathcal{E}^{n,r}(M)$, where $f(x,y) = \sum_{i=0}^{n} a_i(x)y^i$, $r \geq 2$ and $n = 1, 2$. Then*

(a) *For $n = 1$, $C_{\pm\frac{\pi}{2}}$ are periodic orbits of $\widetilde{X}(f)$ with the first and second derivatives of the Poincaré map equal to 1 and $\pm 2\int_0^\tau a_1(s)ds$ respectively.*

(b) *For $n = 2$, $C_{\pm\frac{\pi}{2}}$ are periodic orbits of $\widetilde{X}(f)$ with the first derivative of the Poincaré map equal to $\exp(\mp\int_0^\tau a_2(s)ds)$.*

Proof. (a) Let $\gamma_\pm(u) = (\pm u, \pm\frac{\pi}{2})$ be the periodic orbits of $\widetilde{X}(f)$ on $C_{\pm\frac{\pi}{2}}$ where $u \in [0,\tau]$ and τ is the period of $a_i(u)$. Taking the change of variables $u = s$ and $v = \pm\frac{\pi}{2} - \eta$, $\widetilde{X}(f)$ is associated to the differential equations

$$\begin{aligned} \frac{ds}{dt} &= \pm\cos(\eta) \\ \frac{d\eta}{dt} &= a_0(s)\sin^3(\eta) \pm a_1(s)\sin^2(\eta)\cos(\eta) \end{aligned} \tag{3.1}$$

Dividing the last equation by the first, we obtain

$$\frac{d\eta}{ds} = R(s,\eta) \tag{3.2}$$

where $R(s,\eta) = \pm a_0(s)\frac{\sin^3(\eta)}{\cos(\eta)} + a_1(s)\sin^2(\eta)$ and $R(s,0) = 0$.

Let $\eta = f(s; s_0, \eta_0)$ be a solution of (3.2) with the initial condition $f(s_0; s_0, \eta_0) = \eta_0$. Without loss of generality we can consider $s_0 = 0$.

Now, we consider the function $d(\eta_0) = \phi(\eta_0) - \eta_0$ where $\phi(\eta_0) = f(\tau; 0, \eta_0)$ is the first return map, which is defined on an arc normal to $\gamma_\pm$. The integral expressions for $d'(\eta_0)$ and $d''(\eta_0)$ can be found in [1] page 252. Their calculation is included here for the sake of completeness. From the relation $\frac{d\eta}{ds} = R(s, f(s;0,\eta_0)) = \frac{d}{ds}f(s;0,\eta_0)$, we obtain

$$\frac{d}{ds}\Big(\frac{\partial f}{\partial \eta_0}\Big) = \frac{\partial}{\partial \eta}R(s, f(s;0,\eta_0)) \cdot \frac{\partial f}{\partial \eta_0}(s;0,\eta_0) \tag{3.3}$$

$$\begin{aligned} \frac{d}{ds}\Big(\frac{\partial^2 f}{\partial \eta_0^2}\Big) &= \frac{\partial^2}{\partial \eta^2}R(s, f(s;0,\eta_0)) \cdot \Big(\frac{\partial f}{\partial \eta_0}(s;0,\eta_0)\Big)^2 \\ &\quad + \frac{\partial}{\partial \eta}R(s, f(s;0,\eta_0))\frac{\partial^2 f}{\partial \eta_0^2}(s;0,\eta_0) \end{aligned} \tag{3.4}$$

The solutions of (3.3) and (3.4) are given by

$$\frac{\partial f}{\partial \eta_0}(s;0,\eta_0) = \exp\Big(\int_0^s \frac{\partial}{\partial \eta}R(t, f(t;0,\eta_0))dt\Big) \tag{3.5}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial \eta_0^2}(s;0,\eta_0) &= \exp\Big(\int_0^s \frac{\partial}{\partial \eta}R(t, f(t;0,\eta_0))dt\Big) \cdot \int_0^s \frac{\partial^2}{\partial \eta^2}R(t, f(t;0,\eta_0)) \\ &\quad \cdot \Big(\frac{\partial f}{\partial \eta_0}(t;0,\eta_0)\Big)^2 \cdot \exp\Big(-\int_0^t \frac{\partial}{\partial \eta}R(\tilde{t}, f(t;0,\eta_0))d\tilde{t}\Big)dt \end{aligned} \tag{3.6}$$

The partial derivatives of $R(s,\eta)$ with respect to η are:

$$\frac{\partial R}{\partial \eta}(s,\eta) = \pm a_0(s)(3\sin^2(\eta) + \frac{\sin^4(\eta)}{\cos^2(\eta)}) + 2a_1(s)\sin(\eta)\cos(\eta) \tag{3.7}$$

$$\begin{aligned} \frac{\partial^2 R}{\partial \eta^2}(s,\eta) = \pm a_0(s)(6\frac{\sin(\eta)}{\cos^3(\eta)} + 4\frac{\sin^3(\eta)}{\cos(\eta)} + 2\frac{\sin^5(\eta)}{\cos^3(\eta)}) \\ + 2a_1(s)(\cos^2(\eta) - \sin(\eta)). \end{aligned} \tag{3.8}$$

Since $\eta_0 = 0$ and $f(s;0,\eta_0) = \eta_0$, it follows from (3.5)-(3.8),

$$\begin{gathered} d(\eta_0) = f(\tau;0,\eta_0) - \eta = 0 \\ d'(\eta_0) = 0 \\ d''(\eta_0) = \pm 2\int_0^\tau a_1(s)ds \end{gathered}$$

(b) It follows from a straightforward computation of the divergence of

$$\begin{aligned} \widetilde{X}(f) = \sin(v)\frac{\partial}{\partial u} + (a_0(u)\cos^3(v) + a_1(u)\sin(v)\cos^2(v) \\ + a_2(u)\sin^2(v)\cos(v))\frac{\partial}{\partial v} \end{aligned}$$

i.e. for a periodic orbit $\gamma_\pm = (\pm u, \pm\frac{\pi}{2}))$, $\operatorname{div} D\widetilde{X}(\gamma_\pm) = \mp a_2(u)$. □

We say that $\widetilde{X}(f)$ is *transversal to* $\partial\widetilde{M}$ at (u,v) if

$$\widetilde{X}(f)\cdot b(u,v) \doteq \langle \widetilde{X}(f)(u,v), \nabla b(u,v)\rangle \neq 0,$$

where $b(u,v)$ is a C^2 function such that $b(u,v) = 0$ on $\partial\widetilde{M}$, $b(u,v) > 0$ on $\widetilde{M}^\circ$ (interior of $\widetilde{M}$) and $\nabla b(u,v) \neq 0$ at $(u,v) \in \partial\widetilde{M}$.

When $\widetilde{X}(f)\cdot b(u,v) = 0$, we say that $\widetilde{X}(f)$ is tangent to $\partial\widetilde{M}$. Moreover, we say that the tangency is *parabolic* if

$$\widetilde{X}^2(f)\cdot b(u,v) \doteq \widetilde{X}(f)\cdot(\widetilde{X}(f)\cdot b(u,v)) \neq 0.$$

Proposition 3.2. *Let $\widetilde{X}(f) \in \mathcal{E}^{3,r}(\widetilde{M})$, where $f(x,y) = \sum_{i=0}^3 a_i(x)y^i$ and $r \geq 1$. Then the trajectories of $\widetilde{X}(f)$ are transversal to the circles $C_{\pm\frac{\pi}{2}}$, except at points $(u_*, \pm\frac{\pi}{2})$ in which they are tangent, with $a_3(u_*) = 0$. The tangency is parabolic if $a_3'(u_*) \neq 0$. The tangency is external (resp. internal) in $C_{+\frac{\pi}{2}}$ and internal (resp. external) in $C_{-\frac{\pi}{2}}$ when $a_3'(u_*) > 0$ (resp. $a_3'(u_*) < 0$).*

Proof. We consider the function $b : \widetilde{M} \to \Re$ defined by $b(u,v) = \cos(v)$. It satisfies $b(u,v) > 0$ on $\widetilde{M}^\circ$ and $b(u,v) = 0$ for $(u,v) \in C_{\pm\frac{\pi}{2}}$.

Here, the condition of the transversality of $\widetilde{X}(f)$ to $C_{\pm\frac{\pi}{2}}$ is given by

$$\widetilde{X}(f)\cdot b(u,v) = -\sum_{i=0}^{3} a_i(u)\sin^{i+1}(v)\cos^{3-i}(v) = -a_3(u) \neq 0$$

for $(u,v) \in C_{\pm\frac{\pi}{2}}$.

The condition of parabolic tangency is given by $\widetilde{X}(f)\cdot b(u,v) = 0$ and $\widetilde{X}^2(f)\cdot b(u,\pm\frac{\pi}{2}) = \mp a_3'(u) \mp a_3(u)a_2(u) \neq 0$. Hence we have a parabolic tangency if $a_3(u_*) = 0$ and $a_3'(u_*) \neq 0$, where u_* is a root of $a_3(u)$. □

We say that a singularity, (u,v), of $\widetilde{X}(f)$ is *semi-hyperbolic* if $D\widetilde{X}(f)(u,v)$ has exactly one zero eigenvalue. Also we say that a singularity, (u,v) of $\widetilde{X}(f)$ is *nilpotent* if $D\widetilde{X}(f)(u,v)$ is nilpotent.

Proposition 3.3. *Let $X(f) \in \mathcal{E}^{n,r}(M)$, where $f(x,y) = \sum_{i=0}^{n} a_i(x)y^i$, $r \geq 1$ and $n \geq 4$. Then*

(1) *For $n = 4$, $\widetilde{X}(f)$ has singularities, $(u_*, \pm\frac{\pi}{2})$, on $C_{\pm\frac{\pi}{2}}$ where u_* is root of $a_4(u)$. The singularities are hyperbolic if $a_4'(u_*) \neq 0$ and $a_3(u_*) \neq 0$, semi-hyperbolic if $a_4'(u_*) = 0$ and $a_3(u_*) \neq 0$, and nilpotent if $a_4'(u_*) = 0$ and $a_3(u_*) = 0$.*

(2) *For $n > 4$, $\widetilde{X}(f)$ has singularities, $(u_*, \pm\frac{\pi}{2})$, on $C_{\pm\frac{\pi}{2}}$ where u_* is a root of $a_n(u)$. The singularities are semi-hyperbolic if $a_{n-1}(u_*) \neq 0$, and nilpotent if $a_{n-1}(u_*) = 0$ and $a_n'(u_*) \neq 0$.*

Proof. (1) For $n = 4$, the Jacobian matrix of

$$\widetilde{X}(f) = \sin(v)\cos(v)\frac{\partial}{\partial u} + \sum_{i=0}^{4} a_i(u)\sin^i(v)\cos^{4-i}(v)\frac{\partial}{\partial v}$$

at a singularity $(u_*, \pm\frac{\pi}{2})$ is

$$D\widetilde{X}(f)(u_*, \pm\frac{\pi}{2}) = \begin{pmatrix} 0 & -1 \\ a_4'(u_*) & -a_3(u_*) \end{pmatrix}$$

The proof is finished by analyzing $det(D\widetilde{X}(f)) = a_4'(u_*)$ and $\operatorname{trace}(D\widetilde{X}(f)) = -a_3(u_*)$.

(2). For $n > 4$, $\widetilde{X}(f) = \sin(v)\cos^{n-3}(v)\frac{\partial}{\partial u} + \sum\limits_{i=0}^{n} a_i(u)\sin^i(v)\cos^{n-i}(v)\frac{\partial}{\partial v}$ has the Jacobian matrix at $(u_*, \pm\frac{\pi}{2})$

$$D\widetilde{X}(f)(u_*, \pm\frac{\pi}{2}) = \begin{pmatrix} 0 & 0 \\ (\pm1)^n a_n'(u_*) & (\pm1)^n a_{n-1}(u_*) \end{pmatrix}$$

Hence this singularity is semi-hyperbolic when $a_{n-1}(u_*) \neq 0$ and nilpotent when $a_{n-1}(u_*) = 0$ and $a_n'(u_*) \neq 0$. $\square$

Remark 3.4. The Propositions 3.1-3.3 also hold if we consider $\widetilde{X}_1(f)$ given by (2.3) and (2.4) instead of $\widetilde{X}(f)$. In this case, the singularities or periodic orbits of $\widetilde{X}_1(f)$ lie on $\widetilde{C}_0$.

For simplicity, in the following propositions, we use $\widetilde{X}_1(f)$ as (2.4) to describe the phase portrait of the semi-hyperbolic singularities of $X(f)$ at infinity when $n \geq 4$.

Proposition 3.5. *Let $X(f) \in \mathcal{E}^{4,r}$, where $f(x,y) = \sum_{i=0}^{4} a_i(x)y^i$, $r \geq 2$. Let $(u_*, 0)$ be a semi-hyperbolic singularity of $\widetilde{X}_1(f)$ in $\widetilde{C}_0$ and let $k \in N$, $k \geq 2$, such that*

$$a_4^{(k)}(u_*) \neq 0 \text{ and } a_4^{(j)}(u_*) = 0, \text{ for } j < k.$$

Then $(u_*, 0)$ is one of the following topological types:

(a) *a node, if k is odd and $a_4^{(k)}(u_*) > 0$, figure 1 (a);*
(b) *a saddle, if k is odd and $a_4^{(k)}(u_*) < 0$, figure 1 (b);*
(c) *a saddle-node, if k is even, figure 2.*

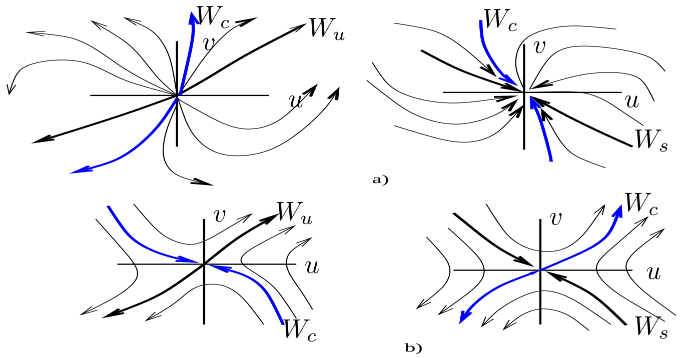

FIGURE 1. Phase portraits of semi-hyperbolic singularities in $\widetilde{C}_0$ for $n = 4$. (a) Node , (b) Saddle.

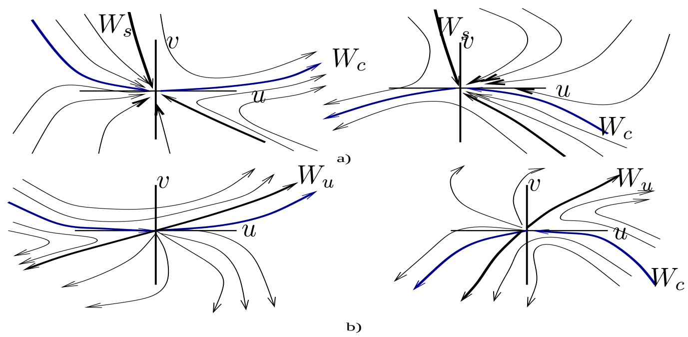

FIGURE 2. Phase portraits of semi-hyperbolic singularities saddle-node in $\widetilde{C}_0$ for $n = 4$: (a) $a_3(u_*) > 0$, (b) $a_3(u_*) < 0$.

Proof. We can suppose that $u_* = 0$. We calculate the restriction of

$$\widetilde{X}_1(f) = v\frac{\partial}{\partial u} - \sum_{i=0}^{4} a_i(u)v^{4-i}\frac{\partial}{\partial v}$$

to the center manifold W_c in $(u_*, 0)$ when $a_4'(u_*) = 0$ and $a_3(u_*) \neq 0$.

The center manifold of $\widetilde{X}_1(f)$ is tangent to eigenspace T_c associated to null eigenvalue and is spanned by the vector $(1, 0)$. Then W_c is the graph of a C^r-function $h : \Re \to \Re$,

$$W_c = \{(u, v) \in \Re^2 : v = h(u)\}.$$

By the condition of tangency of W_c, $h(u_*) = h'(u_*) = 0$.

The restriction of $\widetilde{X}_1(f)$ to the center manifold is of the form

$$u' = h(u) \tag{3.9}$$

Replacing $v = h(u)$ in the second component of $\widetilde{X}_1(f)$, we obtain $\phi(u) = h'(u)h(u) + \sum_{i=0}^{4} a_i(u)h^{4-i}(u) = 0$ for all u.

Let $k \in \mathcal{N}$, $k \geq 2$ such that $a_4^{(k)}(u_*) \neq 0$ and $a_4^{(j)}(u_*) = 0$ as $j < k$. Writing $h(u) = h_2 u^2 + \cdots + h_k u^k + \cdots$, we have

$$\begin{aligned}
\phi(u_*) = 0, \quad \phi'(u_*) = 0, \quad \phi''(u_*) = 2a_3(u_*)h_2 = 0,\\
\phi'''(u_*) = 3!(2h_2^2 + a_3(u_*)h_3) = 0, \ \dots,\\
\phi^{(k)}(u_*) = k!(a_4^{(k)}(u_* + a_3(u_*)h_k) + a_3'(u_*)h_{k-1} + \cdots + a_3^{(k-2)}(u_*)h_2)\\
+ A_{3k}(h_2, \cdots, h_{k-1}) + A_{2k}(h_2, \cdots, h_{k-2}) + A_{1k}(h_2, \cdots, h_{k-3})\\
+ A_{0k}(h_2, \dots, h_{k-4}) = 0
\end{aligned}$$

where $A_{ik}(h_2, \dots) = \frac{d^k}{du^k}(a_i \cdot h^{4-i})(u_*)$ as $i = 0, \dots, 3$.

We solve these equations with respect to h_i as follows

$$h_2 = h_3 = \cdots = h_{k-1} = 0 \ \text{ and } \ h_k = -\frac{a_4^{(k)}}{a_3(u_*)}.$$

Then $h(u) \equiv \alpha u^k + O(u^{k+1})$, where $\alpha = -\frac{a_4^{(k)}(u_*)}{a_3(u_*)}$. Hence, (3.9) is of the form

$$u' = \alpha u^k + O(u^{k+1}).$$

The proposition follows by analyzing the sign of α and the orientation of the hyperbolic manifold (unstable W_u or stable W_s) which is tangent to $v = -a_3(u_*)u$. $\square$

Proposition 3.6. *Let $X(f) \in \mathcal{E}^{n,r}$, where $f(x,y) = \sum_{i=0}^n a_i(x)y^i$, $r \geq 2$ and $n > 4$. Let $(u_*, 0)$ be a semi-hyperbolic singularity of $\widetilde{X}_1(f)$ and let $k \in N$, $k \geq 1$, such that*

$$a_n^{(k)}(u_*) \neq 0 \ \text{ and } a_n^{(j)}(u_*) = 0 \ \text{ for } j < k.$$

Then $(u_*, 0)$ is one of the following topological types:

(a) *a node, if n is even, k is odd and $a_n^{(k)}(u_*) > 0$, figures 3 (a) and 3 (b);*
(b) *a saddle, if n is even, k is odd and $a_n^{(k)}(u_*) < 0$, figures 3 (c) and 3 d);*
(c) *a saddle-node, if $(n-3) \cdot k$ is even, figures 4 and 5.*

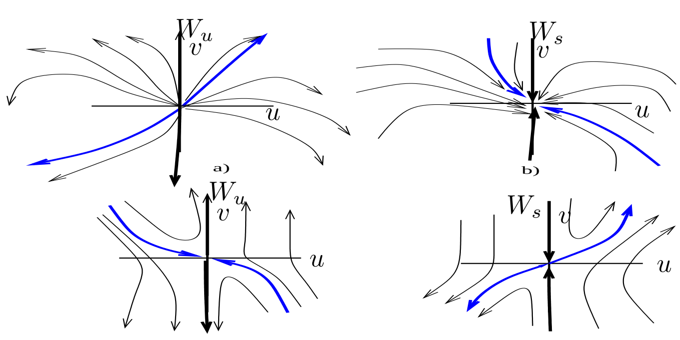

FIGURE 3. Phase portraits of semi-hyperbolic singularities saddle and node in $\widetilde{C}_0$ for $n \geq 5$

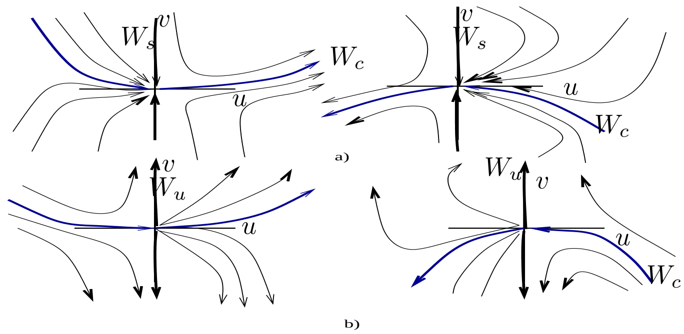

FIGURE 4. Phase portraits of semi-hyperbolic singularities saddle-node for in $\widetilde{C}_0$ $n \geq 5$ and $k \geq 2$

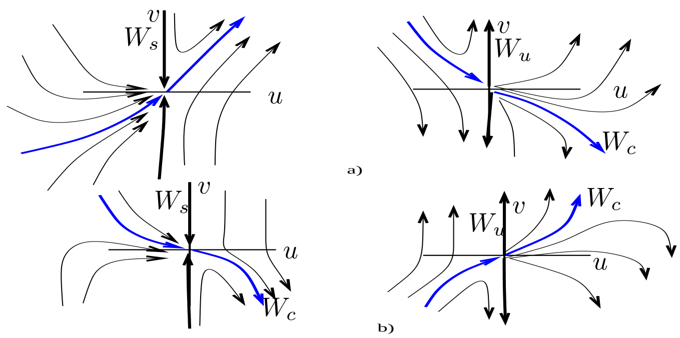

FIGURE 5. Phase portraits of semi-hyperbolic singularities saddle-node in $\widetilde{C}_0$ for $n \geq 5$ and $k = 1$

Proof. We use the method of the center manifold as in Proposition 3.5. We find that the restriction of $\widetilde{X}_1(f)$ to the center manifold, W_c, is of the form

$$g(u) = \alpha^{n-3} u^{k(n-3)} + O(u^{k+1(n-2)})$$

where $\alpha = -\frac{a_n^{(k)}(u_*)}{a_{n-1}(u_*)}$.

The proof is finished by analyzing the sign of α and the orientation of the hyperbolic manifold (unstable W_u or stable W_s) which is tangent to v-axis. $\square$

4. DEFINITION OF THE SETS Σ^n

According to section 3, the behaviors of $\widetilde{X}(f)$ on $C_{\pm\frac{\pi}{2}}$, under non-degeneracy conditions on the periodic orbits, singularities and tangencies, split in the following cases.

- **A:** $C_{\pm\frac{\pi}{2}}$ are periodic orbits of $\widetilde{X}(f)$ with the first and second derivatives of the Poincaré map equal 1 and $\pm\int_0^\tau a_1(s)ds$ respectively. This occurs when $n = 1$.
- **B:** $C_{\pm\frac{\pi}{2}}$ are hyperbolic periodic orbits of $\widetilde{X}(f)$ with second derivative of the Poincaré map equals $\pm 2\int_0^\tau a_2(s)ds$. This occurs when $n = 2$.
- **C:** the trajectories of $\widetilde{X}(f)$ are either transversal or tangent to $C_{\pm\frac{\pi}{2}}$. This case occurs when $n = 3$.

D: $\widetilde{X}(f)$ has hyperbolic singularities in $C_{\pm\frac{\pi}{2}}$. This cases occurs when $n = 4$.

E: $\widetilde{X}(f)$ has semi-hyperbolic singularities in $C_{\pm\frac{\pi}{2}}$. This occurs when $n \geq 5$.

With these cases in mind, in the subsections 4.1-4.5, we define for the corresponding n the set $\Sigma^n \subset \mathcal{E}^{n,r}$ and we prove its density in the space $\mathcal{E}^{n,r}$.

Throughout this section, we denote by $\widetilde{X}(f)$ and $\widetilde{M} = S^1 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ the cylindrical compactification of $X(f)$ and M respectively. Also we consider the following notation:

$C_{\pm\frac{\pi}{2}} = S^1 \times \{\pm\frac{\pi}{2}\}$, $C_0 = S^1 \times 0$,

$\Delta\widetilde{X}(f) = \det D\widetilde{X}(f)$ and $\sigma\widetilde{X}(f) = \operatorname{trace} D\widetilde{X}(f)$ where $D\widetilde{X}(f)$ is the Jacobian matrix of $\widetilde{X}(f)$.

In the first subsection, we present several lemmas which hold for all cases.

Preliminary lemmas. The first lemma is a particular and easy case of Sard's Theorem [15].

Lemma 4.1. *Let $h : I \to \Re$ be a C^1 function. The set of critical values of h, given by $Crit(h) = \{h(x) : h'(x) = 0\}$, has zero Lebesgue measure in $\Re$.*

Lemma 4.2. *Let $X(f) \in \mathcal{E}^{n,r}(M)$ with $r \geq 2$ and $n \geq 1$. Then the set*

$$\begin{aligned} B_1^n(X(f)) = \{\mu_0 \in \Re : \widetilde{X}(f+\mu_0) \textit{ has some singularity} \\ (u_*, 0) \in C_0 \textit{ with } \Delta\widetilde{X}(f+\mu_0) = 0\} \end{aligned}$$

has Lebesgue measure zero in $\Re$.

Proof. Let $f(x,y) = \sum_{i=0}^n a_i(x)y^i$. The set of critical value of $-a_0$ is given by

$$\operatorname{Crit}(-a_0) = \{\mu_0 \in \Re : \exists x_{\mu_0} \text{ such that } -a_0(x_{\mu_0}) = \mu_0 \text{ e } a_0'(x_{\mu_0}) = 0\}.$$

By Lemma 4.1, $Crit(-a_0)$ has zero Lebesgue measure in $\Re$. On the other hand, we can write

$$\begin{aligned} Crit(-a_0) = \quad & \{\mu_0 \in \Re : \exists (x_{\mu_0}, 0) \in C_0 \text{ such that } \widetilde{X}(f+\mu_0)(x_{\mu_0}, 0) = (0,0) \\ & \text{with } \Delta\widetilde{X}(f+\mu_0)(x_{\mu_0}, 0) = 0\}. \end{aligned}$$

It follows that $B_1^n(X(f)) = Crit(-a_0)$. $\square$

Lemma 4.3. *Let $X(f) \in \mathcal{E}^{n,r}(M)$, with $r \geq 1$, $n \geq 1$, and $\mu_0 \notin B_1^n(X(f))$. Then the set*

$$\begin{aligned} B_2^n(X(f);\mu_0) = & \{\mu_1 \in \Re : \widetilde{X}(f+\mu_0+\mu_1 y) \textit{ has some} \\ & \textit{non-hyperbolic singularity}\} \end{aligned}$$

has zero Lebesgue measure in $\Re$.

Proof. Let $f(x,y) = \sum_{i=0}^n a_i(x)y^i$. For $\mu_0 \notin B_1^n(X(f))$, all singularities of $\widetilde{X}(f+\mu_0)$ in C_0, say $(u_*, 0)$, satisfy $\Delta\widetilde{X}(f+\mu_0)(u_*, 0) \neq 0$.

Thus $\widetilde{X}(f+\mu_0+\mu_1 y)$ has in C_0 a finite number of singularities. For each one, $(u_*, 0)$, there is a single value $\mu_1^* \in \Re$ such that $\sigma\widetilde{X}(f+\mu_0+\mu_1^* y) = 0$, since $\sigma\widetilde{X}(f+\mu_0+\mu_1 y) = a_1(u) + \mu_1$. Hence, $(u_*, 0)$ is a non-hyperbolic singularity of $\widetilde{X}(f+\mu_0+\mu_1^* y)$.

Then $B_2^n(X(f);\mu_0)$ can be written of the form

$$B_2^n(X(f);\mu_0) = \{\mu_1 \in \Re : a_1(x) + \mu_1 = 0, a_0(x) + \mu_0 = 0 \text{ e } a_0'(x) > 0\}$$

It follows that $B_2^n(X(f);\mu_0)$ is a finite set and has therefore zero measure in $\Re$. $\square$

Lemma 4.4. *Let $X(f) \in \mathcal{E}^{n,r}(M)$, $n \geq 1$. Then for $\mu_0 \in \Re$ and $r \geq 1$, the set*

$$B_3^n(X(f);\mu_0) = \{\mu_1 \in \Re : \widetilde{X}(f+\mu_0+\mu_1 y) \text{ has some non-hyperbolic periodic orbit contained in } M\}$$

has zero Lebesgue measure in $\Re$.

Proof. A period orbit γ of $\widetilde{X}(f+\mu_0+\mu_1 y)$ is one of the following types:

(a) homotopic to zero. It contains in its interior a singularity and cuts the x-axis transversally.
(b) non-homotopic to zero. It circles the cylinder without intercepting the x-axis.

For each type we find expressions for the derivatives of the first return map with respect to a parameter.

We consider $X(f)$ with $f(x,y) = \sum_{i=0}^n a_i(x)y^i$ and $n \geq 1$.

Case (a). For $\mu_0 \in \Re$, we consider a periodic orbit of $\widetilde{X}(f+\mu_0+\mu_1 y)$, $\gamma(t,p,\mu_1) = (\varphi(t,p,\mu_1), \psi(t,p,\mu_1))$ through $p=(x_0,0)$ with period $\tau = \tau(x_0,\mu_1)$.

Let $\pi(x,\mu_1)$ be the Poincaré map defined in an interval $I \subset C_0$, where $x_0 \in I$, and associated to $X(f+\mu_0+\mu_1 y)$.

The derivatives of $\pi(x,\mu_1)$ with respect to x and μ_1 were calculated in [1] by Andronov *et al.* Here, we write them in terms of the coefficients of $f(x,y)+\mu_0+\mu_1 y$ as follows

$$\frac{\partial \pi}{\partial x}(x_0,\mu_1) = \prod_{i=1}^n \exp\Big(\int_0^{\tau(x_0,\mu_1)} (ia_i(\varphi(s))\psi^{i-1}(s)+\mu_1)ds\Big)$$

$$\frac{\partial \pi}{\partial \mu_1}(x_0,\mu_1) = \frac{\frac{\partial \pi}{\partial x}(x_0,\mu_1)}{-|\gamma'(0,x_0,\mu_1)|}\int_0^{\tau(x_0,\mu_1)} \prod_{i=1}^n \exp\Big(-\int_0^s (ia_i(\varphi(t))\psi^{i-1}(t) + \mu_1)dt\Big)\psi^2(s)ds.$$

Since $\frac{\partial \pi}{\partial \mu_1}(x_0,\mu_1) \neq 0$ (and by the Implicit Function Theorem) there is a function $\mu_1(x)$ defined in a neighborhood I_{x_0} of x_0 such that $\pi(x,\mu_1(x)) - x = 0$ for $\forall x \in I_{x_0}$. The derivative of the last equation with respect to x is

$$\frac{\partial \pi}{\partial x}(x_0,\mu_1) + \frac{\partial \pi}{\partial \mu_1}(x_0,\mu_1)\frac{\partial \mu_1}{\partial x} - 1 = 0.$$

Thus γ is non-hyperbolic if and only if $\frac{\partial \mu_1}{\partial x} = 0$.

We remark that for each $x \in I_{x_0}$, $\mu_1(x)$ is a value of the parameter such that $X(f+\mu_0+\mu_1(x)y)$ has a periodic orbit through $(x,0)$. This periodic orbit is non-hyperbolic when $\mu_1(x)$ is a critical value.

Then, for each $\mu_0 \in \Re$ fixed, the set of critical values of $\mu_1(x)$ is written as

$$O_1 = \{\mu_1(x) \in \Re : X(f+\mu_0+\mu_1(x)y) \text{ has a homotopic to zero periodic orbit non-hyperbolic through } (x,0)\}.$$

Case (b). Let $\gamma(t) = (\varphi(t), \psi(t))$ be a non-homotopic to zero periodic orbit, with period τ, through $(0,y_0)$ where $y_0 \neq 0$. The Poincaré map, $\pi(y,\mu_1)$, is defined by

$$\pi(y,\mu_1) = \Psi(\tau,y,\mu_1) \tag{4.1}$$

where $\Psi(x,y,\mu_1)$ is a solution of differential equation

$$\frac{d\Psi}{dx}(x,y,\mu_1)=\frac{1}{\Psi}\left(\sum_{i=0}^{n}a_i(x)\Psi^i+\mu_1\Psi\right)=F(x,y,\mu_1)$$

with the initial condition $\Psi(0,y,\mu_1)=y$.

The derivative of $\Psi(x,y,\mu_1)$ with respect to μ_1 is the solution of linear equation

$$\frac{d}{dx}\Big(\frac{\partial\Psi}{\partial\mu_1}\Big)=D_2F(x,\Psi(x,y,\mu_1),\mu_1)\frac{\partial\Psi}{\partial\mu_1}(x,y,\mu_1)+D_3F(x,\Psi(x,y,\mu_1),\mu_1)$$

where $D_2F(x,y,\mu_1)=-\frac{a_0(x)}{y^2}$ and $D_3F(x,y,\mu_1)=1$.

By equation (4.1) the derivative of $\pi(y,\mu_1)$ with respect to μ_1 is given by

$$\frac{\partial\pi}{\partial\mu_1}(y,\mu_1)=\exp(\mathcal{I}(\tau,y,\mu_1))\int_0^\tau\exp(-\mathcal{I}(s,y,\mu_1))ds$$

where $\mathcal{I}(t,y,\mu_1)=\int_0^t D_2F(s,\Psi(s,y,\mu_1),\mu_1)ds$

Similarly to (a), we find a function $\mu_1:I_{y_0}\to\Re$ where I_{y_0} is an interval of y-axis containing y_0 and without intersection with the x-axis such that $\pi(y,\mu_1(y))-y=0$. Hence, γ is non-hyperbolic if and only if $\frac{\partial\mu_1}{\partial y}=0$.

The critical value set of $\mu_1(y)$ is given by

$$\begin{aligned}O_2=&\{\mu_1(\xi)\in\Re: X(f+\mu_0+\mu_1(\xi)y)\text{ has a non-hyperbolic periodic orbit}\\&\quad\text{circling the cylinder through }(0,\xi)\}\end{aligned}$$

The proof of i) is complete by observing that

$$B_3^1(X(f);\mu_0)=O_1\cup O_2$$

and by applying the Sard Lemma to sets O_1 e O_2. □

4.1. **Case A.** In this case, $X_\mu(f)$ denotes the vector field $X(f+\mu_0+\mu_1y)$ where $\mu=(\mu_0,\mu_1)\in\Re^2$.

Definition 4.5. Let Σ^1 be the set of C^r-vector fields $X(f)\in\mathcal{E}^{1,r}$ with $r\geq 2$ such that $\widetilde{X}(f)$ satisfies:

(1) the singularities are hyperbolic and contained in C_0.
(2) the periodic orbits in the $\widetilde{M}^\circ$ are hyperbolic and the periodic orbits in $C_{\pm\frac{\pi}{2}}$ are semi-stable.
(3) no saddle connection.

Next, we give the measure of the complementary set of Σ^1 in the parameter space $\Re^2$.

Theorem 4.6. *Let $X(f)\in\mathcal{E}^{1,r}$, with $r\geq 2$. Then the set*

$$B^1(X(f))=\{\mu\in\Re^2: X_\mu(f)\notin\Sigma^1\}$$

has zero Lebesgue measure in $\Re^2$.

We have divided the proof in a sequence of lemmas.

Lemma 4.7. *Let $X(f)\in\mathcal{E}^{1,r}$. Then for $r\geq 2$, the set*

$$\begin{aligned}B_4^1(X(f))=\{\mu_1\in\Re:\ &\widetilde{X}(f+\mu_1y)\text{ has some non semi-hyperbolic}\\&\text{periodic orbit in }C_{\pm\frac{\pi}{2}}\}\end{aligned}$$

has zero Lebesgue measure in $\Re$.

Proof. We observe that the periodic orbits of $\widetilde{X}(f+\mu_0+\mu_1 y)$ in $C_{\pm\frac{\pi}{2}}$ do not depend of μ_0. Moreover, the second derivative of the Poincaré map of these periodic orbits is $\pi'' = \pm 2\int_0^\tau a_1(s)ds + 2\mu_1\tau$ (see Proposition 3.1).

Thus $\pi'' = 0$ only for a finite number of μ_1. Then $B_4^1(X(f))$ has zero Lebesgue measure in $\Re$. $\square$

Lemma 4.8. *Let $X(f) \in \mathcal{E}^{1,r}$, $r \geq 1$ and $\mu_0 \notin B_1^1(X(f))$. Then the set*

$$B_5^1(X(f);\mu_0) = \{\mu_1 \in \Re : \widetilde{X}(f+\mu_0+\mu_1 y) \textit{ has not the condition } 3 \textit{ of } \Sigma^1\}$$

has zero Lebesgue measure in $\Re$.

Proof. Fix $\mu_0 \notin B_1(X(f))$. The number of singularities of $\widetilde{X}(f+\mu_0)$ is finite. Thus the saddle connections of $\widetilde{X}_\mu(f)$ also are finite in number for values of μ_1.

We claim that all connections can be broken with perturbations of the form $\widetilde{X}(f+\mu_0+\mu_1 y)$. Suppose that for μ_1^*, $\widetilde{X}(f+\mu_0+\mu_1^* y)$ has a trajectory $\widehat{pq}$ in the superior part of $\widetilde{M}$ connecting the saddles p and q in C_0. We denote by $S_E(q)$ the stable separatrix of q and by $S_I(p)$ the unstable separatrix of p. See figure 6.

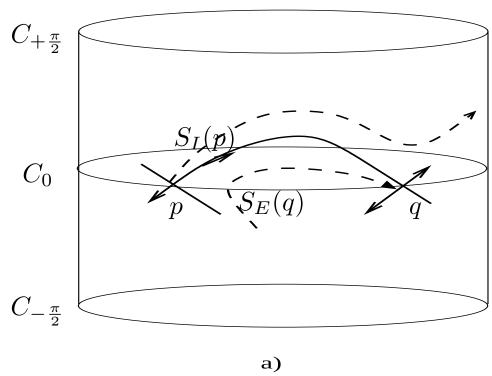

Figure 6. Breaking connection of saddles p and q for $\mu_1 > \mu_1^*$

Let $\mathrm{sep}(\mu_1)$ be the separation function of $S_I(p,\widetilde{X}(f+\mu_0+\mu_1 y))$ and $S_E(q,\widetilde{X}(f+\mu_0+\mu_1 y))$. It is defined on a transversal section to the trajectory that links the saddles. The derivative of $\mathrm{sep}(\mu_1)$ with respect to μ_1 is of the form

$$\begin{aligned}\underset{\mu_1}{\mathrm{sep}}(\mu_1^*) = \int_{-\infty}^{+\infty} &\exp(-\int_0^t \mathrm{div}(\widetilde{X}(f+\mu_0+\mu_1^* y)(u(s),v(s)))ds) \\ &\cdot \widetilde{X}(f+\mu_0+\mu_1^* y) \wedge \frac{d}{d\mu_1}\widetilde{X}(f+\mu_0+\mu_1^* y)(u(s),v(s))dt\end{aligned}$$

where $\wedge$ is the vectorial product defined by $(v_1,v_2)\wedge(w_1,w_2) = -\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2\end{pmatrix}$ and $(u(s),v(s))$ is an orbit connecting the saddles p and q. For a treatment of the integral formula of $\mathrm{sep}_{\mu_1}(\cdot)$ we refer the reader to Guckeheimer-Holmes [9] and Chicone [4].

In our case,

$$\begin{aligned} \operatorname*{sep}_{\mu_1}(\mu_1^*) = -\int_{-\infty}^{+\infty} & \exp(-\int_0^t \operatorname{div}(\widetilde{X}(f+\mu_0+\mu_1^* y)(u(s),v(s)))ds) \\ & \times \sin^2(v(t))\cos^2(v(t))dt. \end{aligned} \tag{4.2}$$

Since $v(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the integrand in (4.2) is a non-negative function. Therefore $\operatorname{sep}_{\mu_1}(\mu_1^*) < 0$. Then the connection of p and q saddles is broken, without another connection to arise. Thus $B_5^1(X(f);\mu_0)$ is a discrete set. This ends the proof. $\square$

Proof of Theorem 4.6. The set $B^1(X(f))$ is the union of the following sets

$$\mathcal{B}_1 = B_1^1(X(f)) \times \Re, \quad \mathcal{B}_2 = \bigcup_{\mu_0 \in \Re - B_1^1(X(f))} \{\mu_0\} \times B_2^1(X(f),\mu_0),$$
$$\mathcal{B}_3 = \bigcup_{\mu_0 \in \Re} \{\mu_0\} \times B_3^1(X(f);\mu_0), \quad \mathcal{B}_4 = \Re \times B_4^1(X(f)),$$
$$\mathcal{B}_5 = \bigcup_{\mu_0 \in \Re - B_1^1(X(f))} \{\mu_0\} \times B_5^1(X(f),\mu_0),$$

where $B_i^1(X(f),\cdot)$, $i = 1,\dots,5$ are given by Lemmas 4.2-4.8 respectively.

Each set $\mathcal{B}_i$:

- contains parameters $(\mu_0,\mu_1) \in \Re^2$ such that $X(f+\mu_0+\mu_1 y)$ violates at least a condition of Σ^1.
- is measurable. Because its complement in $\Re^2$ is open.
- has measure zero in $\Re^2$. Because $\mathcal{B}_1$ and $\mathcal{B}_4$ are products of $\Re$ times a zero measure set in $\Re$. To calculate the measure of $\mathcal{B}_i$ for $i = 2, 3$ and 5, we use Fubini's Theorem [14]:

  $$\int_{\Re^2} \chi(\mathcal{B}_i) d\mu_0 d\mu_1 = \int (\int \chi(\cdot, B_i^1(X(f),\mu_0)) d\mu_1) d\mu_0 = 0$$

  where $\chi(\cdot)$ is the characteristic function of sets in $\Re^2$.

Then $B^1(X(f))$ is measurable with zero Lebesgue measure in $\Re^2$. $\square$

4.2. **Case B.** In this case, $X_\mu(f)$ denotes the vector field $X(f+\mu_0+\mu_1 y+\mu_2 y^2)$ where $\mu = (\mu_0,\mu_1,\mu_2) \in \Re^3$.

Definition 4.9. Let Σ^2 be the set of C^r-vector fields $X(f) \in \mathcal{E}^{2,r}$ with $r \geq 1$ such that $\widetilde{X}(f)$ satisfies:

(1) the singularities are hyperbolic and contained in C_0.
(2) the periodic orbits are hyperbolic and contained in $\widetilde{M}$.
(3) no saddle connection.

We now give the measure of the complement of Σ^2 in the parameter space $\Re^3$.

Theorem 4.10. *Let $X(f) \in \mathcal{E}^{2,r}$, with $r \geq 1$. Then the set*

$$B^2(X(f)) = \{\mu \in \Re^3 : X_\mu(f) \notin \Sigma^2\}$$

has zero Lebesgue measure in $\Re^3$.

We have divided the proof in a sequence of lemmas.

Lemma 4.11. *Let $X(f) \in \mathcal{E}^{2,r}$ with $r \geq 1$. Then, the set*

$$B_4^2(X(f)) = \{\mu_2 \in \Re : \widetilde{X}(f + \mu_2 y^2) \text{ has some non semi-hyperbolic periodic orbit in } C_{\pm\frac{\pi}{2}}\}$$

has zero Lebesgue measure in $\Re$.

Proof. Let $f(x,y) = a_0(x) + a_1(x)y + a_2(x)y^2$. By Proposition 3.1, $\widetilde{X}(f)$ has two periodic orbits of period τ (the same period of function $a_i(u)$) on $C_{\pm\frac{\pi}{2}}$ with first derivative of the Poincaré maps equals to $\exp(\mp \int_0^\tau a_2(u)du)$. Then, $\widetilde{X}(f + \mu_2 y^2)$ has a periodic orbit on $C_{\pm\frac{\pi}{2}}$ non-hyperbolic if and only if $\mu_2 = \pm\frac{1}{\tau}\int_0^\tau a_2(u)du$. It follows that $B_4^2(X(f))$ is discrete and has therefore zero measure in $\Re$. □

We remark that for every $(\mu_0, \mu_1) \in \Re^2$, $B_4^2(X(f + \mu_0 + \mu_1 y)) = B_4^2(X(f))$.

Lemma 4.12. *Let $X(f) \in \mathcal{E}^{2,r}(M)$, where $r \geq 1$ and let $\mu_0 \notin B_1^2(X(f))$. Then the set*

$$B_5^2(X(f);\mu_0) = \{\mu_1 \in \Re : \widetilde{X}(f + \mu_0 + \mu_1 y) \text{ has not the condition 3 of } \Sigma^2\}$$

has zero Lebesgue measure in $\Re$.

For the proof of this lemma; see proof of Lemma 4.8.

Proof of Theorem 4.10. The set $B^2(X(f))$ is union of following sets:

$$\mathcal{B}_1 = B_1^2(X(f)) \times \Re^2, \quad \mathcal{B}_2 = \bigcup_{\mu_0 \in \Re - B_1^2(X(f))} \{\mu_0\} \times B_2^2(X(f),\mu_0) \times \Re,$$
$$\mathcal{B}_3 = \bigcup_{\mu_0 \in \Re} \{\mu_0\} \times B_3^2(X(f);\mu_0) \times \Re, \quad \mathcal{B}_4 = \Re^2 \times B_4^2(X(f)),$$
$$\mathcal{B}_5 = \bigcup_{\mu_0 \in \Re - B_1^2(X(f))} \{\mu_0\} \times B_5^2(X(f),\mu_0) \times \Re,$$

where $B_i^2(X(f),\cdot)$, $i = 1,\dots,5$, are given by Lemmas 4.2-4.4, 4.11 and 4.12 respectively.

Each $\mathcal{B}_i$

- contains parameters $(\mu_0,\mu_1,\mu_2) \in \Re^3$ such that $X(f + \mu_0 + \mu_1 y + \mu_2 y^2)$ violates at least a condition of Σ^2.
- is measurable. Because its complement in $\Re^3$ is open.
- has zero measure in $\Re^3$. Because $\mathcal{B}_1$ and $\mathcal{B}_4$ are product of $\Re^2$ times a zero measure set in $\Re$. To measure $\mathcal{B}_i$ for $i = 2, 3$ and 5, we apply the Fubini Theorem as follows

  $$\int_{\Re^3} \chi(\mathcal{B}_i) d\mu_0 d\mu_1 d\mu_2 = \int_{\Re^2} (\int \chi(\cdot, B_i^2(X(f),\mu_0),\cdot) d\mu_1) d\mu_0 d\mu_2 = 0$$

  where $\chi(\cdot)$ is characteristic function in $\Re^3$.

Then $B^2(X(f))$ has zero measure in $\Re^3$. □

4.3. **Case C.** Here $X_\mu(f)$ denotes the vector field $X(f+\mu_0+\mu_1 y+\mu_2 y^3)$ where $\mu=(\mu_0,\mu_1,\mu_2)\in\Re^3$.

Definition 4.13. *Let Σ^3 be the set of $X(f)\in\mathcal{E}^{3,r}$, $r\geq 1$ such that $\widetilde{X}(f)$ satisfies:*

1. *the singularities are hyperbolic and contained in C_0.*
2. *the periodic orbits are hyperbolic and contained in $\widetilde{M}^\circ$.*
3. *the tangency points of $\widetilde{X}(f)$ with $C_{\pm\frac{\pi}{2}}$ are parabolic.*
4. (a) *there are no saddle connection of $\widetilde{X}(f)$.*
   (b) *the separatrix of the singularity points in C_0 can be only transversal to $C_{\pm\frac{\pi}{2}}$.*
   (c) *The trajectories are tangent to $C_{\pm\frac{\pi}{2}}$ at most one point.*

Theorem 4.14. *Let $X(f)\in\mathcal{E}^{3,r}$ with $r\geq 1$. Then the set*

$$B^3(X(f))=\{(\mu_0,\mu_1,\mu_2)\in\Re^3 : X(f+\mu_0+\mu_1 y+\mu_2 y^3)\notin\Sigma^3\}$$

has Lebesgue measure zero in $\Re^3$.

The proof of this Theorem needs the following lemmas.

Lemma 4.15. *Let $X(f)\in\mathcal{E}^{3,r}$ with $r\geq 1$. Then, the set*

$$B_4^3(X(f))=\{\mu_2\in\Re : \widetilde{X}(f+\mu_2 y^3) \text{ has not the condition 3 of } \Sigma^3\}$$

has Lebesgue measure zero in $\Re$.

Proof. Let $X(f)$ with $f(x,y)=\sum_{i=0}^3 a_i(x)y^i$. By Proposition 3.2, $(u_*,\pm\frac{\pi}{2})\in C_{\pm\frac{\pi}{2}}$ is a tangency point of $\widetilde{X}(f+\mu_2 y^3)$ if $a_3(u_*)+\mu_2=0$. Moreover the tangency is parabolic when $a_3'(u_*)\neq 0$.

The set of critical values of $-a_3(u)$,

$$Crit(-a_3(u))=\{\mu_2\in\Re : \exists\,(u_*,0) \text{ such that } a_3(u_*)+\mu_2=0 \text{ e } a_3'(u_*)=0\},$$

describes the set of $\mu_2\in\Re$ such that $X(f+\mu_0+\mu_1 y+\mu_2 y^3)$ does not satisfy the condition 3 of Σ^3.

By Sard's Lemma, $B_4^3(X(f);\mu_0,\mu_1)$ has Lebesgue measure zero in $\Re$. □

Lemma 4.16. *Let $X(f)\in\mathcal{E}^{3,r}$, $r\geq 1$, $\mu_0\notin B_1^3(X(f))$ and $\mu_2\notin B_4^3(X(f))$. Then the set*

$$B_5^3(X(f);\mu_0,\mu_2)=\{\mu_1\in\Re : \widetilde{X}(f+\mu_0+\mu_1 y+\mu_2 y^3) \text{ has not the condition 4 of } \Sigma^3\}$$

has zero Lebesgue measure in $\Re$.

Proof. Fix $\mu_0\notin B_1^3(X(f))$ and $\mu_2\notin B_3^3(X(f))$ as in Lemmas 4.2 and 4.15.

There are three types of connections (see figure 7).

(a) Connections of saddles in C_0.
(b) A saddle separatrix that is tangent to $C_{+\frac{\pi}{2}}$ or $C_{-\frac{\pi}{2}}$ with parabolic tangency.
(c) A trajectory with two parabolic tangencies to $C_{\pm\frac{\pi}{2}}$.

The saddle connections of $\widetilde{X}(f+\mu_0+\mu_1 y+\mu_2 y^3)$ happen in a number finite of μ_1, since the number of singularities is finite.

The Lemma 4.7 applies straightforwardly to (a), (b) and (c). In fact, we can break these connections with perturbations of the form $\widetilde{X}(f+\mu_0+\mu_1 y+\mu_2 y^3)$

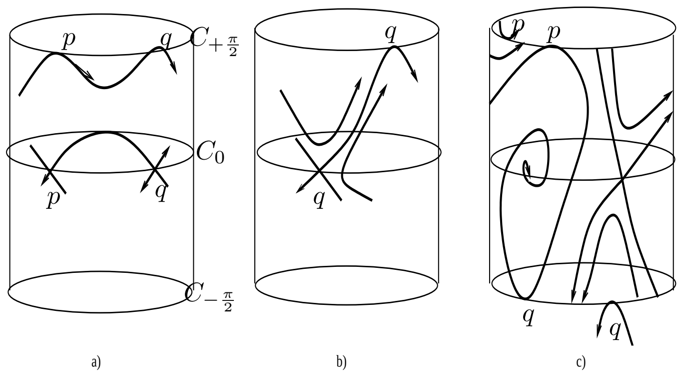

Figure 7. Saddle connections in C_0 and tangency points in $C_{\pm\frac{\pi}{2}}$.

where μ_0 and μ_2 are fixed. The derivative of the separation function of the stable and unstable manifolds is

$$\begin{aligned}\operatorname*{sep}_{\mu_1}(\mu_1^*) = -\int_{-\infty}^{+\infty} &\exp(-\int_0^t \operatorname{div}(\widetilde{X}(f+\mu_0+\mu_1^* y+\mu_2 y^3)(u(s),v(s)))ds) \\ &\times \sin^2(v(t))\cos^2(v(t))dt.\end{aligned} \tag{4.3}$$

For (b) and (c) we must consider the time T taken by a trajectory from $q \in \widetilde{M}^\circ$ to a tangency point $p \in C_{\pm\frac{\pi}{2}}$. Thus we have

$$\begin{aligned}\operatorname*{sep}_{\mu_1}(\mu_1^*) = -\frac{1}{|\widetilde{X}(f)(q)|}\int_0^T &\exp(-\int_0^t \operatorname{div}(\widetilde{X}(f+\mu_0+\mu_1^* y+\mu_2 y^3)(u(s),v(s)))ds) \\ &\times \sin^2(v(t))\cos^2(v(t))dt.\end{aligned} \tag{4.4}$$

The integrands in (4.3) and (4.4) are non-negative functions for $v(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Since $\operatorname{sep}_{\mu_1}(\mu_1^*) < 0$, the connections are broken and no other one can arise. Then $B_4(X(f); \mu_0, \mu_2)$ is a discrete subset of $\Re$ and has zero Lebesgue measure. □

Proof of Theorem 4.14. The set $B^3(X(f))$ is union of

$$\mathcal{B}_1 = B_1^3(X(f)) \times \Re^2, \quad \mathcal{B}_2 = \bigcup_{\mu_0 \in \Re - B_1^3(X(f))} \{\mu_0\} \times B_2^3(X(f), \mu_0) \times \Re,$$
$$\mathcal{B}_3 = \bigcup_{\mu_0 \in \Re} \{\mu_0\} \times B_3^3(X(f); \mu_0) \times \Re, \quad \mathcal{B}_4 = \Re^2 \times B_4^3(X(f)),$$
$$\mathcal{B}_5 = \bigcup_{(\mu_0,\mu_2) \in S} \{\mu_0\} \times B_5^3(X(f), \mu_0, \mu_2) \times \{\mu_2\},$$

where $S = \Re - B_1^3(X(f)) \times \Re - B_4^3(X(f))$ and $B_i^3(X(f), \cdot)$, $i = 1, \dots, 5$, are given by the Lemmas 4.2-4.4 and 4.15-4.16 respectively.

Each $\mathcal{B}_i$:

- contains $(\mu_0, \mu_1, \mu_2) \in \Re^3$ such that the $X(f+\mu_0+\mu_1 y+\mu_2 y^3)$ violates at least some condition of Σ^3.
- is mensurable. Because its complement in $\Re^3$ is open.

- has zero measure in $\Re^3$. Because $\mathcal{B}_1$ and $\mathcal{B}_4$ are products of $\Re^2$ times a set of zero measure in $\Re$. For $\mathcal{B}_i$ with $i = 2, 3$ e 5, we apply the Fubini Theorem.

This completes the proof. $\square$

4.4. **Case D.** In this case, $X_\mu(f)$ denotes the vector field $X(f + \mu_0 + \mu_1 y + \mu_2 y^3 + \mu_3 y^4)$ where $\mu = (\mu_0, \mu_1, \mu_2, \mu_3) \in \Re^4$.

Definition 4.17. Let Σ^4 be the set of $X(f) \in \mathcal{E}^{4,r}$ with $r \geq 1$ such that $\widetilde{X}(f)$ satisfies:

(1) the singularities $(u_*, \cdot) \in C_0 \cup C_{\pm\frac{\pi}{2}}$ are hyperbolic and the eigenvalues associated to singularities in $C_{\pm\frac{\pi}{2}}$ are distinct.
(2) the periodic orbits are hyperbolic and contained in $\widetilde{M}^\circ$.
(3) there are no connection of singularity separatrix. More specifically
   (a) no connection of saddle that belongs to C_0 or $C_{\pm\frac{\pi}{2}}$,
   (b) separatrices of saddle in C_0 or in $C_{\pm\frac{\pi}{2}}$ are not weak manifolds of a node in $C_{\pm\frac{\pi}{2}}$.

Theorem 4.18. *Let $X(f) \in \mathcal{E}^{4,r}$, where $r \geq 1$. Then*
$$B^4(X(f)) = \{\mu \in \Re^4 : \ X(f + \mu_0 + \mu_1 y + \mu_2 y^3 + \mu_3 y^4) \notin \Sigma^4\}$$
has Lebesgue measure zero in $\Re^4$.

The proof of this Theorem needs the following lemmas.

Lemma 4.19. *Let $X(f) = X(\sum_{i=0}^4 a_i(x) y^i) \in \mathcal{E}^{4,r}$ with $r \geq 1$. Then, the set*
$$B_4^4(X(f)) = \{\mu_3 \in \Re : \widetilde{X}(f + \mu_3 y^4) \text{ has some singularity in } C_{\pm\frac{\pi}{2}} \text{ with } \Delta\widetilde{X}(f + \mu_3 y^4) = 0\}$$
has Lebesgue measure zero in $\Re$.

Proof. The set $Crit(-a_4) = \{\mu_3 = -a_4(x) : a_4'(x) = 0\}$ determines the set of μ_3 such that $\widetilde{X}(f + \mu_3 y^4)$ has singularities in $C_{\pm\frac{\pi}{2}}$ with $\Delta\widetilde{X}(f + \mu_3 y^4) = 0$. By Sard's theorem, this set has Lebesgue measure zero in $\Re$. $\square$

Lemma 4.20. *Let $X(f) = X(\sum_{i=0}^4 a_i(x) y^i) \in \mathcal{E}^{4,r}$ with $r \geq 1$, and let $\mu_0 \notin B_1^4(X(f))$ and $\mu_3 \notin B_2^4(X(f))$. Then the set*
$$B_5^4(X(f); \mu_3) = \{\mu_2 \in \Re : \widetilde{X}(f + \mu_2 y^3 + \mu_3 y^4) \text{ has some hyperbolic, with equal eigenvalues, or non-hyperbolic singularity in } C_{\pm\frac{\pi}{2}}\}$$
has Lebesgue measure zero in $\Re$.

Proof. Let $\mu_3 \notin B_4^4(X(f))$. Since
$$a_3(u_*) + \mu_2 = 2\sqrt{a_4'(u_*)}, \tag{4.5}$$
$\widetilde{X}(f + \mu_2 y^3 + \mu_3 y^4)$ has a hyperbolic singularity $(u_*, \pm\frac{\pi}{2})$ with equal eigenvalues. Then there exists a single value μ_2^* that satisfies the condition (4.5).

On the other hand, for each non-hyperbolic singularity $(u_*, \pm\frac{\pi}{2})$ of $\widetilde{X}(f + \mu_2 y^3 + \mu_3 y^4)$, there exists a unique value μ_2^* such that $\sigma(\widetilde{X}(f + \mu_2^* y^3 + \mu_3 y^4)) = -(a_3(u_*) + \mu_2) = 0$. It follows that $B_5^4(X(f); \mu_3)$ is discrete in $\Re$ and has therefore zero Lebesgue measure. $\square$

Lemma 4.21. *Let $X(f) \in \mathcal{E}^{4,r}$ with $r \geq 1$, $\mu_0 \notin B_1^4(X(f))$ and $\mu_3 \notin B_4^4(X(f))$. Then, the set*

$$B_6^4(X(f);\mu_0,\mu_3) = \{\mu_1 \in \Re : \widetilde{X}(f+\mu_0+\mu_1 y+\mu_3 y^4) \textit{ has not the condition 3 of } \Sigma^4\}$$

has Lebesgue measure zero in $\Re$.

Proof. All singularity connections of $\widetilde{X}(f)$ can be broken with perturbations of the form $\widetilde{X}(f+\mu_0+\mu_1 y+\mu_3 y^4)$. The derivative of the separation function is of the form

$$\begin{aligned} \operatorname*{sep}_{\mu_1}(\mu_1^*) = -\int_{-\infty}^{+\infty} \exp(-\int_0^t \operatorname{div}(\widetilde{X}(f+\mu_0+\mu_1^* y+\mu_3 y^4))ds) \\ \times \sin^2(v(t))\cos^3(v(t))dt. \end{aligned} \tag{4.6}$$

The integrand in (4.6) is non-negative for $v(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $\operatorname{sep}_{\mu_1}(0) < 0$. This ends the proof. $\square$

Proof of Theorem 4.18. The set $B^4(X(f))$ is union of

$$\mathcal{B}_1 = B_1^4(X(f)) \times \Re^3, \quad \mathcal{B}_2 = \bigcup_{\mu_0 \in \Re - B_2^4(X(f))} \{\mu_0\} \times B_3^4(X(f),\mu_0) \times \Re^2,$$

$$\mathcal{B}_3 = \bigcup_{\mu_0 \in \Re} \{\mu_0\} \times B_3^4(X(f);\mu_0) \times \Re^2,$$

$$\mathcal{B}_4 = \Re^3 \times B_4^4(X(f)), \quad \mathcal{B}_5 = \Re^2 \times B_5^4(X(f),\mu_3) \times \bigcup_{\mu_3 \in \Re - B_4^4(X(f))} \{\mu_3\},$$

$$\mathcal{B}_6 = \bigcup_{(\mu_0,\mu_3) \in S} \{\mu_0\} \times B_6^4(X(f),\mu_0,\mu_3) \times \Re \times \{\mu_3\},$$

where $S = \Re - B_1^4(X(f)) \times \Re - B_4^4(X(f))$ and $B_i^4(X(f),\cdot)$, $i = 1,\dots,6$, are given by the Lemmas 4.2-4.4 and 4.19-4.21.

Each $\mathcal{B}_i$

- contains $(\mu_0,\mu_1,\mu_2,\mu_3) \in \Re^4$ such that $X(f+\mu_0+\mu_1 y+\mu_2 y^3+\mu_3 y^4)$ violates at least a condition of Σ^4.
- is measurable. Because its complement in $\Re^4$ is open.
- has measure zero in $\Re^4$. Because $\mathcal{B}_1$ and $\mathcal{B}_4$ are products of $\Re^3$ times a set of measure zero in $\Re$. For other $\mathcal{B}_i$, we apply the Fubini Theorem.

This completes the proof. $\square$

4.5. **Case E.** In this case, $X_\mu(f)$ denotes the vector field $X(f+\mu_0+\mu_1 y+\mu_2 y^{n-1})$, where $\mu = (\mu_0,\mu_1,\mu_2) \in \Re^3$ and $n \geq 5$.

Definition 4.22. Let Σ^n be the set of $X(f) \in \mathcal{E}^{n,r}$ for $n \geq 5$ and $r \geq 2$ such that $\widetilde{X}(f)$ satisfies:

1. the singularities $(u_*,0) \in C_0$ are hyperbolic and the singularities $(u_*,\pm\frac{\pi}{2}) \in C_{\pm\frac{\pi}{2}}$ are semi-hyperbolic.
2. the periodic orbits are hyperbolic and contained in $\widetilde{M}$.
3. no connections of singularity separatrix. More specifically
   (a) no connection of saddles that belong to C_0,

(b) saddle separatrices in C_0 are not invariant manifolds of singularities in $C_{\pm\frac{\pi}{2}}$.
(c) no connection between singularities that belong to $C_{\pm\frac{\pi}{2}}$ by invariant manifolds.

Theorem 4.23. *Let $X(f) \in \mathcal{E}^{n,r}$ with $n \geq 5$ and $r \geq 2$. Then the set*
$$B^n(X(f)) = \{(\mu_0, \mu_1, \mu_2) \in \Re^3 : X(f + \mu_0 + \mu_1 y + \mu_2 y^{n-1}) \notin \Sigma^n\}$$
has zero Lebesgue measure in $\Re^3$.

Lemma 4.24. *Let $X(f) = X(\sum_{i=0}^n a_i(x)y^i) \in \mathcal{E}^{n,r}$ with $n \geq 5$, $r \geq 1$. Then the set*
$$\begin{gathered} B_4^n(X(f)) = \{\mu_2 \in \Re : \widetilde{X}(f + \mu_2 y^{n-1}) \text{ has some singularity in} \\ C_{\pm\frac{\pi}{2}} \text{ with } \sigma\widetilde{X}(f + \mu_2 y^{n-1}) = 0\} \end{gathered}$$
has zero Lebesgue measure in $\Re$.

Proof. Each singularity $(u, \pm\frac{\pi}{2})$ of $\widetilde{X}(f + \mu_2 y^{n-1})$ that satisfies $a_{n-1}(u) + \mu_2 = 0$, determines a non semi-hyperbolic singularity. Then, $B_4^n(X(f))$ must be a discrete set with measure zero in $\Re$. □

Lemma 4.25. *Let $X(f) \in \mathcal{E}^{n,r}$ with $n \geq 5$, $r \geq 2$, $\mu_0 \notin B_1^n(X(f))$ and $\mu_2 \notin B_2^n(X(f))$. Then the set*
$$\begin{aligned} B_5^n(X(f); \mu_0, \mu_2) = \big\{&\mu_1 \in \Re : \widetilde{X}(f + \mu_0 + \mu_1 y + \mu_2 y^{n-1}) \\ &\text{does not satisfying property 3 of } \Sigma^n\big\} \end{aligned}$$
has zero Lebesgue measure in $\Re$.

Proof. It follows by the same way as in Lemma 4.21. In this case, the derivative of the separation function of the stable and unstable manifolds is of the form
$$\begin{aligned} \operatorname{sep}_{\mu_1}(\mu_1^*) = -\int_{-\infty}^{+\infty} &\exp\Big(-\int_0^t \operatorname{div}(\widetilde{X}(f + \mu_0 + \mu_1^* y + \mu_2 y^{n-1}))ds\Big) \\ &\times \sin^2(v(t)) \cos^{n-1}(v(t))dt. \end{aligned} \tag{4.7}$$
The integrand in (4.7) is a non-negative function. □

Proof of Theorem 4.23. The set $B^n(X(f))$ is union of:
$$\mathcal{B}_1 = B_1^n(X(f)) \times \Re^2, \quad \mathcal{B}_2 = \bigcup_{\mu_0 \in \Re - B_1^n(X(f))} \{\mu_0\} \times B_2^n(X(f), \mu_0) \times \Re,$$
$$\mathcal{B}_3 = \bigcup_{\mu_0 \in \Re} \{\mu_0\} \times B_3^n(X(f); \mu_0) \times \Re, \quad \mathcal{B}_4 = \Re^2 \times B_4^n(X(f)),$$
$$\mathcal{B}_5 = \bigcup_{(\mu_0, \mu_2) \in S} \{\mu_0\} \times B_5^n(X(f); \mu_0, \mu_2) \times \{\mu_2\}$$
where $S = \Re - B_1^n(X(f)) \times \Re - B_4^n(X(f))$ and $B_i^n(X(f), \cdot)$, $i = 1, \dots, 5$, are given by Lemmas 4.2-4.4 and 4.24-4.25.

Each $\mathcal{B}_i$

- contains $(\mu_0, \mu_1, \mu_2) \in \Re^3$ such that $X(f + \mu_0 + \mu_1 y + \mu_2 y^{n-1})$ violates at least a condition of Σ^n.
- is measurable. Because its complement in $\Re^3$ is open.

- has measure zero in $\Re^3$. (As in the proof of Theorem 4.18 by the Fubini Theorem).

This completes the proof. $\square$

Remark 4.26. Theorems 4.6 to 4.23 express the measure of the complementary set of Σ^n for $n \geq 1$. They may be summarized by stating that

$$B^n(X(f)) = \{(\mu_0, \mu_1, \mu_{n-1}, \mu_n) : X(f + \mu_0 + \mu_1 y + \mu_{n-1} y^{n-1} + \mu_n y^n) \notin \Sigma^n\}$$

has zero Lebesgue measure. However, we observe that for $n > 4$ three parameters only are sufficient. In the present work, the proof has been divided in cases **A**-**E** to make the presentation more accessible.

5. Genericity of Σ^n.

In this section, we prove the genericity of Σ^n.

Proof of Theorem 1.2. The density of Σ^n follows straightforwardly from Theorems 4.6-4.23 for corresponding n. In fact, for $X(f) \in \mathcal{E}^{n,r}$ and $\forall\, \epsilon > 0$ we can find $X_\mu(f) \in \Sigma^n$ such that $|\widetilde{X}_\mu(f) - \widetilde{X}(f)| < \epsilon$. $\mu \in \Re^k$ where $k = 2$ for $n = 1$; $k = 3$ for $n = 2, 3$ and ≥ 5; and $k = 4$ for $n = 4$.

To see that Σ^n is open in $\mathcal{E}^{n,r}$ we write $X(f) = y\frac{\partial}{\partial x} + \sum_{i=0}^n a_i(x) y^i \frac{\partial}{\partial y}$. In C_0, a singularity of $\widetilde{X}(f)$, say $(u_*, 0)$, satisfies $a_0(u_*) = 0$ and $a_0'(u_*) \neq 0$. In $C_{\pm\frac{\pi}{2}}$, we must consider the following cases:

1. $n = 3$, since $\widetilde{X}(f)$ has a parabolic tangency at $(u_*, \cdot)$, it satisfies $a_3(u_*) = 0$ and $a_3'(u_*) \neq 0$.
2. $n = 4$, a singularity of $\widetilde{X}(f)$, say $(u_*, \cdot)$, satisfies $a_4(u_*) = 0$ and $a_4'(u_*) \neq 0$, and, also, $a_3(u_*) \neq 2\sqrt{a_4'(u_*)}$ if $a_4'(u_*) > 0$.
3. $n \geq 5$, a singularity of $\widetilde{X}(f)$, say $(u_*, \cdot)$, satisfies $a_n(u_*) = 0$ and $a_{n-1}(u_*) \neq 0$.

In all those cases, we can choose μ_0, μ_1, μ_2 and μ_3 sufficiently small such that $X(\sum_{i=0}^n b_i(x) y^i)$, where $b_0 = a_0 + \mu_0$, $b_1 = a_1 + \mu_1$, $b_{n-1} = a_{n-1} + \mu_2$ and $b_n = a_n + \mu_3$, has singularities or tangencies of the same type than $X(f)$.

If $X(f)$ has a hyperbolic periodic orbit, $X(g)$ with $b_1 = a_1(u) + \mu_1$ will also have a hyperbolic periodic orbit for small values of μ_1. For $n = 1$ and 2, we obtain the same type of periodic orbits in $C_{\pm\frac{\pi}{2}}$, taking $b_1 = a_1 + \mu_1$ and $b_2 = a_2 + \mu_2$. $\square$

6. Characterization

We will prove the necessity of Theorem 1.3 in detail. For the sufficiency, we will only touch on some aspects with respect to new canonical regions and to the building of homeomorphism.

6.1. **Necessity.**

There is a neighborhood $\mathcal{U}$ of $X(f)$ in $\mathcal{E}^{n,r}$ such that for $X(g) \in \mathcal{U}$ there is a homeomorphism $h_g : \widetilde{M} \to \widetilde{M}$ that transforms trajectories of $\widetilde{X}(f)$ in trajectories of $\widetilde{X}(g)$.

By density of Σ^n, there is $X(\sum_{i=0}^n b_i(x) y^i) \in \Sigma^n \cap \mathcal{U}$ such that it is topologically equivalent to $X(f)$. Then $X(f)$ inherits the following properties of $X(\sum_{i=0}^n b_i(x) y^i)$:

- In C_0, $X(f)$ has a finite number of singularities and all topologically equivalent to saddles, focus or nodes.

- In $C_{\pm\frac{\pi}{2}}$, $X(f)$ has a finite number of tangencies (when $n = 3$) or of singularities (when $n \geq 4$). The tangencies are internal or external.
- The periodic orbits are finite in number, all are attractors or repellors and are contained in the interior of $\widetilde{M}$.
- $X(f)$ has no connection of singularity separatrix, nor separatrix of saddle tangent to $C_{\pm\frac{\pi}{2}}$.

The following properties of $X(f)$ remain to be proved.
(a) The singularities in C_0 are all hyperbolic. In fact, if $\Delta\widetilde{X}(f) = 0$ at some singularity, say $(u_*, 0)$, we can consider $\widetilde{X}(f + \epsilon\beta(u)(u - u_*))$ where β is a non-negative periodic function on a neighborhood V of u_* and with the compact support contained in V, $\beta(u_*) = 1$ and V does not contain other singularities. Consequently, $\Delta\widetilde{X}(f)(u_*, 0) = -\epsilon\beta(u_*)$. Then, by choosing the adequate sign for ϵ, $(u_*, 0)$ is also a singularity of $\widetilde{X}(f + \epsilon\beta(u)(u - u_*))$, but it has the different index as singularity of $\widetilde{X}(f)$. This contradicts the structural stability of $X(f)$.

(b) In $C_{\pm\frac{\pi}{2}}$, depending on n, we have hyperbolic and semi-hyperbolic singularities, or parabolic tangency points.

For $n = 3$, suppose that $(u_*, \frac{\pi}{2})$ is a point of non-parabolic tangency but it is internal or external. That is, $(u_*, \frac{\pi}{2})$ satisfies $a_3(u_*) = 0$ and $a_3'(u_*) = 0$.

Let V be a neighborhood of u_* in $\Re$ such that it contains no other tangency points of $\widetilde{X}(f)$. For ϵ to be small, $\widetilde{X}(f + \epsilon(x - u_*)\beta(x)y^3)$ is sufficiently close to $X(f)$, where $\beta(x)$ is a non-negative C^r periodic function with support contained in V. Since $\widetilde{X}(f+\epsilon(x-u_*)\beta(x)y^3)\cdot b(u_*, \frac{\pi}{2}) = 0$, $(u_*, \frac{\pi}{2})$ is a tangency point. Moreover the tangency is parabolic, since $\widetilde{X}^2(f + \epsilon(x - u_*)\beta(x)y^3) \cdot b(u_*, \frac{\pi}{2}) = \epsilon \neq 0$, where $b : \widetilde{M} \to \Re$ is defined by $b(u, v) = \cos(v)$, see the Proposition 3.2.

Then, if $(u_*, \frac{\pi}{2})$ is an external (respectively internal) tangency point of $\widetilde{X}(f)$, we take ϵ positive (respectively, negative) such that $\widetilde{X}(f + \epsilon(x - u_*)\beta(x)y^3)$ has at $(u_*, \frac{\pi}{2})$ an internal (respectively, external) tangency point, contradicting the structural stability of $X(f)$.

For $n = 4$, the hyperbolicity of all singularities of $\widetilde{X}(f)$ in $C_{\pm\frac{\pi}{2}}$ can be shown in the same way that (a), considering the system $\widetilde{X}(f + \epsilon(x - u_*)\beta(x)y^4)$ close to $\widetilde{X}(f)$ and $(u_*, \pm\frac{\pi}{2})$ is a singularity of $\widetilde{X}(f)$.

We need to treat here only the case that $D\widetilde{X}(f)$ at a singularity of $\widetilde{X}(f)$ has equal eigenvectors. In the fact, if $(u_*, \pm\frac{\pi}{2})$ is one such singularity then $a_4'(u_*) > 0$ and $a_3(u_*) = 2\sqrt{a_4'(u_*)}$. Thus taking $X(f + \epsilon(x - u_*)\beta(x)y^3)$ sufficiently near to $\widetilde{X}(f)$, where $\beta(x)$ is a non-negative periodic function with support in a neighborhood of u_* and without any other singularity, we find that $(u_*, \pm\frac{\pi}{2})$ is also a singularity of $X(f + \epsilon(x - u_*)\beta(x)y^3)$, but with non-equal eigenvalues. More specifically, if $\epsilon < 2\sqrt{a_4'(u_*)} - a_3(u_*)$, the singularity is a focus and if $\epsilon > 2\sqrt{a_4'(u_*)} - a_3(u_*)$ the singularity is a hyperbolic node. Therefore, $X(f + \epsilon(x - u_*)\beta(x)y^3)$ and $X(f)$ can not be topologically equivalent.

For $n \geq 5$, if $(u_*, \pm\frac{\pi}{2})$ is a singularity of $\widetilde{X}(f)$ non-semi-hyperbolic then $a_n(u_*) = 0$ and $a_{n-1}(u_*) = 0$. Now, $X(f + \epsilon(x - u_*)\beta(x)y^{n-1})$ sufficiently close to $\widetilde{X}(f)$ has $(u_*, \pm\frac{\pi}{2})$ as a semi-hyperbolic singularity. This contradicts the structural stability of $X(f)$.

(c) The periodic orbits in $\widetilde{M}^\circ$ are all hyperbolic. In fact, we suppose that $\gamma(s) = (\varphi(s), \xi(s))$ is a periodic orbit of $X(f)$ with period τ such that $\chi(\gamma, X(f)) = 0$. We will obtain a field $X(f+g)$ sufficiently near to $X(f)$ in which γ is a periodic orbit but $\chi(\gamma, X(f+g)) \neq 0$.

Then, we search a function $g(x,y) = b_0(x) + b_1(x)y$, where b_0 and b_1 are C^r-periodic such that

- (i) $X(f+g)$ is near to $X(f)$, we say at a distance $|\epsilon| \neq 0$.
- (ii) $X(f+g)(\gamma) = X(f)(\gamma)$ (i.e. γ is a periodic orbit of both vector fields).
- (iii) $\chi(\gamma, X(f+g)) \neq 0$

The conditions $i), ii)$, and $iii)$ determine:

- $*$ $\sup |b_i^{(k)}(x)| < |\epsilon|$ for $i = 1, 2, 0 \leq k \leq r$ and $x \in S^1$.
- $*$ $b_0(\varphi(s)) + b_1(\varphi(s))\xi(s) = 0$.
- $*$ $b_1(\varphi(s))$ is not identically null in $\Re$.

The task is now to find a function $b_1 : \Re \to \Re$ which is C^r-periodic and not identically null. Thus we can choose $b_0 : \Re \to \Re$ C^r-periodic such that $b_0(\varphi(s)) = -b_1(\varphi(s))\xi(s)$ for $s \in \Re$.

We should consider the following three situations:

(1) b_0 can not be null. Because, in otherwise, $\xi(s) = 0$ for $s \in \Re$ and C_0 should be a periodic orbit of $X(f)$, contradicting the fact that all periodic orbits only can intercept C_0 at isolated points.

(2) If $\xi(s)$ is not a non-null constant, say ξ_0, then we can put $b_1(x) = \epsilon \neq 0$ and $b_0(x) = -\epsilon\xi_0$ both constant. Thus, $g(x,y) = -\epsilon\xi_0 + \epsilon y$ defines a perturbation $|\epsilon|$-close to $X(f)$ such that $\chi(\gamma, X(f+g)) = \epsilon$.

(3) If $\xi(s)$ is not constant, we can choose some $s_0 \in (0, \tau)$ such that $\xi(s_0) \neq 0$ and $\varphi(s_0) \neq 0$. Let $W = \varphi([0,\tau]) \subset \Re$. Then there is an open neighborhood U of s_0 contained in $[0, \tau]$ such that $\varphi_{|U}$ has differentiable inverse (since $\varphi'(s_0) = \xi(s_0) \neq 0$).

We suppose that $\xi(s_0) > 0$. We denote $\widetilde{U} = \varphi(U) \subset W$ and let $V = (\alpha, \beta)$ be a neighborhood of $\varphi(s_0)$ such that $V \subset \overline{V} \subset \widetilde{U}$. (See figure 8).

Now, we can take a function $b_1 : \Re \to \Re$ with the following properties:
- $*$ $b_1(x) > 0$ for $x \in V$ and $b_1(x) = 0$ for $x \in W - V$,
- $*$ C^r with $r \geq 1$,
- $*$ periodically extended to whole $\Re$.

Next, we define $b_0 : \Re \to \Re$ by $b_0(x) = -b_1(x)\varphi'(\varphi^{-1}(x))$ for $x \in V$ and $b_0 = 0$ for $x \in W - V$. b_0 can be extended periodically to all $\Re$ and of C^r class. It follows that $g(x,y) = \epsilon b_0(x) + \epsilon b_1(x) y$ is a perturbation of $f(x,y)$, as we wanted. In fact, $X(f+g)$ has γ as a period orbit with characteristic index

$$\chi(\gamma, X(f+g)) = \epsilon \int_{\varphi^{-1}(\alpha)}^{\varphi^{-1}(\beta)} b_1(\varphi(s))ds \neq 0.$$

It follows from (1)-(3) that if γ is attractor for $X(f)$ then by taking $\epsilon > 0$, $X(f+g)$ has at least two periodic orbits more than $X(f)$. This contradicts the structural stability of $X(f)$. The proof of (c) is complete. Finally (a), (b) and (c) together with the properties described above show that $X(f) \in \Sigma^n$.

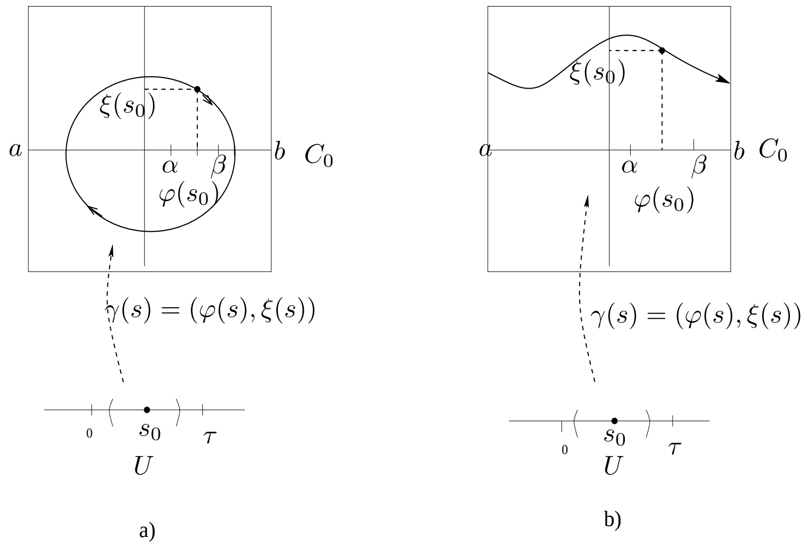

FIGURE 8. (a) Homotopic to zero periodic orbit. (b) Periodic orbit circling the cylinder. The points a and b are identified

6.2. **Sufficiency.** The proof of the sufficiency of Theorem 1.3 is essentially the same given by Peixoto-Peixoto in [13] and later by Sotomayor in [17]. Here we indicate some important aspects of this Theorem.

Every $X(f) \in \Sigma^n$ determines a decomposition of $\widetilde{M}$ in connected components, called *canonical regions*, whose boundary are formed by separatrices (i.e. an arc of trajectory, a singularity, a limit cycle, a saddle separatrix and a portion of $\partial\widetilde{M}$).

An *attractor* (resp. a *source*) of $\widetilde{X}(f)$ associated to a canonical region $R \subset \widetilde{M}$ is a node, a focus or a limit cycle of $\widetilde{X}(f)$ or is an arc of ∂R where the trajectories in R tend as $t \to +\infty$ (resp. $t \to -\infty$). A *critical region of a singularity* p of $\widetilde{X}(f)$ is a neighborhood D of p such that the systems sufficiently near to $\widetilde{X}(f)$ have a single singularity and of the same type as p. A *critical region of a limit cycle* Γ of $\widetilde{X}(f)$ is a ring A contained Γ such that the systems sufficiently close to $\widetilde{X}(f)$ have a limit cycle and of the same stability that Γ.

Every canonical region has in its boundary only a source and an attractor. This property plays a very important role in the classification of canonical regions. See [13]. We denote by α, ω and γ a source, an attractor and a separatrix of $\widetilde{X}(f)$ respectively. A separatrix can be a point (e.g. a singularity or an external tangency).

Then we classify a canonical region R by the number and type of separatrices contained in ∂R, as follows (see figure 9).

- *Type i.* $\partial R = \{\alpha, \omega\}$.
- *Type ii.* $\partial R = \{\alpha, \omega, \gamma_1\}$ where γ_1 is a trajectory that is transversal to ∂M, and α and ω belong to ∂M.
- *Type iii.* $\partial R = \{\alpha, \omega, \gamma_1, p_1\}$ where γ_1 is a trajectory with internal tangency to ∂M and p_1 is a point of external tangency.
- *Type iv.* $\partial R = \{\alpha, \omega, \gamma_1, \gamma_2\}$ where γ_1 and γ_2 are arcs of trajectories.

- *Type v.* $\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2\}$ where p_1 is a saddle, γ_1 and γ_2 are trajectories tending to p_1.
- *Type vi.* $\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2, \gamma_3\}$ where p_1 is a saddle, γ_1, γ_2 are trajectories tending to p_1 and γ_3 is a trajectory that does not tend to p_1.
- *Type vii.* $\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2, \gamma_3\}$ where p_1 is a saddle, γ_1, γ_2 and γ_3 are trajectories tending to p_1.
- *Type viii.* $\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2, p_2, \gamma_3\}$ where p_1 and p_2 are saddles, γ_1, γ_2 trajectories tending to p_1 and γ_3 trajectory tending to p_2.
- *Type ix.* $\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2, p_2, \gamma_3, \gamma_4\}$ where p_1 and p_2 are saddles, γ_1, γ_2 trajectories tending to p_1, γ_3 and γ_4 are trajectories tending to p_2.

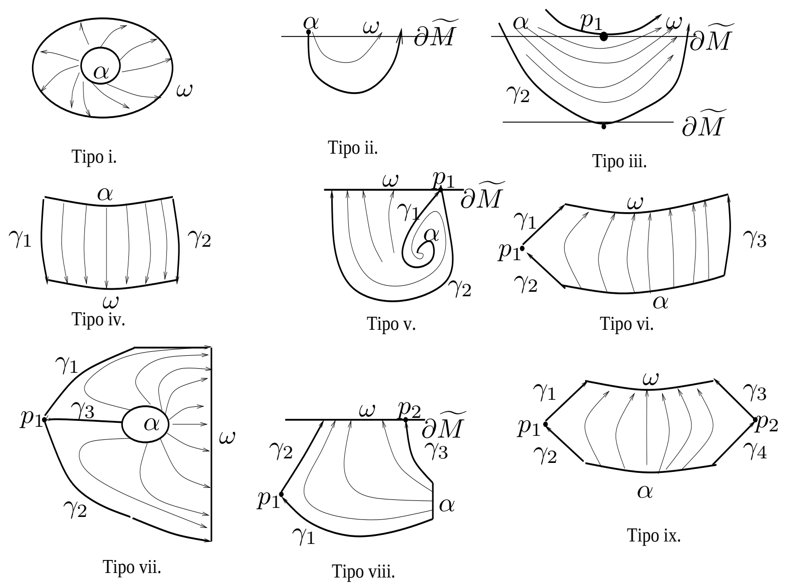

FIGURE 9. Canonical regions classified by the number and the type of separatrices.

In [13], Peixoto and Peixoto showed five types of canonical regions of C^1 vector fields in the plane. Later, Sotomayor in [17] extended to seven regions considering external and internal tangencies of trajectories (that are not saddle separatrices) with the boundary.

We remark that the types *ii*, *v* and *viii* are new ones, which arise by considering singularities on the boundary of $\widetilde{M}$. In Table 1 we compare the classifications of Peixoto-Peixoto, Sotomayor and ours.

Now we define a homeomorphism between same type canonical regions of two vector fields of Σ^n which are sufficiently close to each other. We present details of homeomorphism between two canonical regions R and $\widetilde{R}$ of the type v. For that we consider the function $Z_{AB} : AB \to [0,1]$ defined by $Z_{AB}(m) = \frac{l(Am)}{l(AB)}$ where $l(AB)$ is the length of arc AB that links the points A and B, and $m \in AB$ (see figure 10).

Let D (respectively $\widetilde{D}$) be a critical region of $\alpha \in R$ (resp. $\widetilde{\alpha} \in \widetilde{R}$) and let $B_0 = \gamma_1 \cap \partial D$ (resp. $\widetilde{B}_0 = \widetilde{\gamma}_1 \cap \partial \widetilde{D}$) and $A_0 = \gamma_2 \cap \omega$ (resp. $\widetilde{A}_0 = \widetilde{\gamma}_2 \cap \widetilde{\omega}$). Hence, the boundary of R-D° (respectively $\widetilde{R}$-$\widetilde{D}^\circ$) is formed by the singularity p_1 (resp.

	Type	Peixoto-Peixoto	Sotomayor
i.	$\partial R = \{\alpha, \omega\}$	I	1
ii.	$\partial R = \{\alpha, \omega, \gamma_1\}$		
iii.	$\partial R = \{\alpha, \omega, \gamma_1, p_1\}$		6
iv.	$\partial R = \{\alpha, \omega, \gamma_1, \gamma_2\}$	II	7
v.	$\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2\}$		
vi.	$\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2, \gamma_3\}$	III	4, 5
vii.	$\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2, \gamma_3\}$	IV	3
viii.	$\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2, p_2, \gamma_3\}$		
ix.	$\partial R = \{\alpha, \omega, p_1, \gamma_1, \gamma_2, p_2, \gamma_3, \gamma_4\}$	V	2

TABLE 1. Classification of canonical regions with the notation given by Peixoto-Peixoto (*I-V*), by Sotomayor (1-7) and by us (*i-ix*).

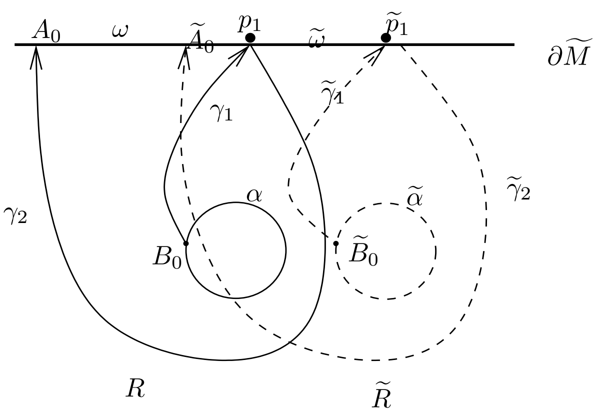

FIGURE 10. Canonical regions of the type v.

$\widetilde{p_1}$), the arcs of trajectory p_1A_0 (resp. $\widetilde{p_1}\widetilde{A_0}$) and B_0p_1 (resp. $\widetilde{B}_0\widetilde{p_1}$) and the arcs B_0B_0 (i.e. ∂D) and A_0p_1 (resp. $\widetilde{B}_0\widetilde{B}_0$ and $\widetilde{A}_0\widetilde{p_1}$).

Each arc of $\partial(R\text{-}D^\circ)$ is mapped to the corresponding one of $\partial(\widetilde{R}\text{-}\widetilde{D}^\circ)$ by

$$\varphi(A) = \begin{cases} Z^{-1}_{A_0p_1} \circ Z_{A_0p_1}(A) & \text{if } A \in A_0p_1 \\ Z^{-1}_{p_1A_0} \circ Z_{p_1A_0}(A) & \text{if } A \in p_1A_0 \\ Z^{-1}_{B_0p_1} \circ Z_{B_0p_1}(A) & \text{if } A \in B_0p_1 \end{cases}$$

We extend φ to interior of R-D° as follows. For $P \in (R\text{-}D^\circ)^\circ$, there is a trajectory BA through P where $B \in B_0B_0$ and $A \in A_0p_1$. Hence,

$$\varphi(P) = Z^{-1}_{BA} \circ Z_{BA}(P).$$

The function $\varphi : R\text{-}D^\circ \to \widetilde{R}\text{-}\widetilde{D}^\circ$ is bijective. φ^{-1} is defined in the same way. The continuity of φ follows from that of solutions with respect to the initial values.

On the other hand, the homeomorphism of the critical region D, say ψ, also is built using the function Z_{AB}. See [13, 17] for more details. We observe that φ and ψ coincide in ∂D, since both are defined in the same way. This finishes the building of the homeomorphism.

7. Concluding remarks

A study of the structural stability of second order differential equation E_f on $M = S^1 \times \Re$ for C^1 functions $f(x, y)$ periodic in x and with the Whitney topology was carried out by Barreto in [2] (see also Shahshahani [16]). Below we review the conditions for the structurally stable equations E_f proposed by Barreto:

(1) All singularities are hyperbolic and, therefore, finite in number
(2) If a trajectory λ has a saddle as its α-limit (resp. ω-limit), then every trajectory in some tubular neighborhood of λ has the same ω-limit (resp. α-limit). In particular, no trajectory joins saddle points.
(3) All periodic orbits are hyperbolic and, therefore, countable in number. Only a finite number of these intersect C_0.
(4) The α and ω-limit set of any trajectory can be only singularities, periodic orbits or infinite.

Among the conditions above, (2) is the most difficult, if not impossible, to verify in concrete cases. It corresponds to the asymptotic behavior of trajectory near infinity. In this context, Camacho *et al.* in [3] and Kotus *et al.* in [11] and [10] formulated analogous conditions of those by Peixoto for C^r vector fields on open surfaces with finite genus and a countable space of ends. They distinguish a behavior at infinity of the type "saddle at infinity" formed by two unbounded semi-trajectories with prolongational limit sets contained in the space of ends.

On compact regions of $\Re^2$ and M, in [18], Sotomayor established a characterization theorem for C^1-structurally stable second order differential equations, using the uniform topology in the space of equations and the tangency conditions between the orbits and the boundary of the regions.

The purpose of our work is to establish a link between [2] and [18], obtaining conditions that allow verification in a simple calculatory fashion, when restricted to the class $\mathcal{E}^{n,r}$. This is done by applying a compactification to the open surface M and to the equation E_f, as explained in section 2. This compactification allows us to obtain a large class of stable equations E_f that have behaviors at infinity not exhibited by the conditions studied in [2], and also to obtain some patterns of behavior at infinity described in terms of the tangency conditions between the orbits and the boundary of the compactification of M, as given in [18].

Other compactifications present discouraging results. For example, by applying a two-point compactification, we would be able to induce a field with two singularities at the infinite. In order to know the topological type of those singularities, we would need to apply a blow-up [7]. We would expect the same topological types already calculated in section 3.

Another example is the known Poincaré compactification that is applied successfully when the vector field on $\Re^2$

(1) is polynomial in the variables x and y (see [17], [8], [6], [5], [12], among other works); or
(2) has the Lojasiewicz property at infinity (see [19]).

The vector field $X(f)$ is not completely polynomial. However, it can be proven that $X(f)$ has the property of Lojasiewicz at infinity. Then, M and $X(f)$ can be compactified by the Poincaré compactification as a compact cylinder, and a vector field with two lines of singularities on the border of the cylinder.

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