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Stationary solutions for generalized Boussinesq models in exterior domains *

E.A. Notte-Cuello & M.A. Rojas-Medar

Abstract

We establish the existence of a stationary weak solution of a generalized Boussinesq model for thermally driven convection in exterior domains. We use the fact that the exterior domain can be approximated by interior domains.

1 Introduction

We study the stationary problem for equations governing a coupled mass and heat flow of a viscous incompressible fluid in generalized Boussinesq approximations. Assuming that the viscosity and the heat conductivity are temperature dependent in an exterior domain $\Omega \subset \mathbb{R}^3$, we study the equation

$$\begin{aligned} -\operatorname{div}(\nu(T)\nabla u) + u \cdot \nabla u - \alpha T g + \nabla p = 0 \\ \operatorname{div} u = 0 \\ -\operatorname{div}(\kappa(T)\nabla T) + u \cdot \nabla T = 0\,. \end{aligned} \tag{1}$$

Here $u(x) \in \mathbb{R}^3$ denotes the velocity of the fluid at a point $x \in \Omega$; $p(x) \in \mathbb{R}$ is the hydrostatic pressure; $T(x) \in \mathbb{R}$ is the temperature; $g(x)$ is the external force per unit of mass; $\nu(\cdot) > 0$ and $\kappa(\cdot) > 0$ are kinematic viscosity and thermal conductivity, respectively; and α is a positive constant associated to the coefficient of volume expansion. Without loss of generality, we have taken the reference temperature as zero. For a derivation of the above equations, see Drazin and Reid [1].

The expressions ∇, Δ, and div denote the gradient, Laplace, and divergence operators, respectively. The gradient is also denoted by grad. The i-th component of $u \cdot \nabla u$ is given by

$$(u \cdot \nabla u)_i = \sum_{j=1}^{3} u_j(\partial u_i/\partial x_j)\,; \quad u \cdot \nabla T = \sum_{j=1}^{3} u_j(\partial T/\partial x_j)\,.$$

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The boundary conditions and conditions at infinity are

$$u\,|_\Gamma = 0\,,\quad T\,|_\Gamma = T_0 > 0\,; \tag{2}$$

$$\lim_{|x|\to\infty} u(x) = 0\,,\quad \lim_{|x|\to\infty} T(x) = 0\,, \tag{3}$$

where Γ is the boundary of Ω.

Problem (1) was considered by Lorca and Boldrini [8] in a bounded domain with Dirichlet's conditions; while the reduced model, where ν and κ are positive constants, was studied by Morimoto [10] (in a bounded domain) and recently by Oeda [11] (in an exterior domain).

The evolution problem corresponding to (1) was analyzed by Lorca and Boldrini [9] in a bounded domain; when ν and κ are positive constants was discussed by many authors, see for instance, Korenev [6], Rojas-Medar and Lorca [14, 15] (in a bounded domain) and Hishida [5], Oeda [12], [13] (in an exterior domain). In another publication we will study the evolution problem corresponding to (1).

2 Preliminaries

Functions in this paper are either $\mathbb{R}$ or $\mathbb{R}^3$ valued, and we will not distinguish these two situations in our notation. To which case we refer to will be clear from the context.

Now, we give the precise definition of the exterior domain, Ω, where our boundary-value problem associated to the problem (1)-(3) has been formulated.

Let K be a compact subset of $\mathbb{R}^3$, whose boundary ∂K is of class C^2. The exterior domain is $\Omega = K^c$, and $\Gamma = \partial\Omega = \partial K$.

The extending domain method was introduced by Ladyzhenskaya [7] to study the Navier-Stokes equations in unbounded domains. As observed by Heywood [3] the method is useful in certain class of unbounded domains. Certainly, our domain is in this class. The basic idea is the following: The exterior domain Ω can be approximated by interior domains $\Omega_m = B_m \cap \Omega$, where B_m is a ball with radius m and center at 0, as $m \to \infty$.

In each interior domain Ω_m, we will prove the existence of a weak solution, by using the Galerkin method together with the Brouwer's fixed point theorem as in Heywood [3]. Next, by using the estimates given in Ladyzhenskaya's book [7] together with diagonal argument and Rellich's compactness theorem, we obtain the desirable weak solution to problem (1)-(3).

Let D denote Ω or Ω_m. Define function spaces as follows:

$$\begin{aligned}
W^{r,p}(D) &= \{u; D^\alpha u \in L^p(D), |\alpha| \le r\}\\
W_0^{r,p}(D) &= \text{ completion of } C_0^\infty(D) \text{ in } W^{r,p}(D)\\
C_{0,\sigma}^\infty(D) &= \{\varphi \in C_0^\infty(D); \operatorname{div}\varphi = 0\}\\
J(D) &= \text{ completion of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\nabla\phi\|\\
H(D) &= \text{ completion of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\phi\|\,.
\end{aligned}$$

Here $\|\cdot\|$ denotes the L^2-norm, $\|\cdot\|_p$ denotes the L^p-norm. We note that $J(D)$ can be characterized as

$$J(D) = \left\{\phi \in W^{1,2}(D); \phi|_\Gamma = 0, \operatorname{div}\phi = 0\right\},$$

as was proved by Heywood [3]. When $p = 2$, we write $W^{r,p}(D) \equiv H^r(D)$ and $W_0^{r,p}(D) \equiv H_0^r(D)$.

We make use of some inequalities with constants that depend only on the dimension and are independent of the domain (see [7] chapter I).

Lemma 1 *Suppose the space dimension is 3, with D bounded or unbounded. Then (a) For $u \in W_0^{1,2}(D)$ (or $J(D)$ or $H_0^1(D)$), we have*

$$\|u\|_{L^6(D)} \leq C_L \|\nabla u\|_{L^2(D)}$$

where $C_L = (48)^{1/6}$.
(b) (Hölder's inequality). If each integral makes sense. Then we have

$$|((u\cdot\nabla)v, w)| \leq 3^{\frac{1}{p}+\frac{1}{r}} \|u\|_{L^p(D)} \|\nabla v\|_{L^q(D)} \|w\|_{L^r(D)}$$

where $p, q, r > 0$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

The following assumptions will be needed throughout this paper.

(S1) $w_0 \subset K$ (w_0 is a neighborhood of the origin 0) and $K \subseteq B = B(0, d)$ which is a ball with radius d and center at 0.

(S2) $\partial\Omega = \Gamma = \partial K \in C^2$.

(S3) $g(x)$ is a bounded and continuous vector function in $\mathbb{R}^3 \backslash w_0$. Moreover $g \in L^p(\Omega)$ for $p \geq 6/5$.

We assume that the functions $\nu(\cdot)$ and $\kappa(\cdot)$ satisfy

$$0 < \nu_0(T_0) \leq \nu(\tau) \leq \nu_1(T_0)$$
$$0 < \kappa_0(T_0) \leq \kappa(\tau) \leq \kappa_1(T_0)$$

for all $\tau \in \mathbb{R}$, where

$$\nu_0(T_0) = \inf\{\nu(t); |t| \leq \sup_{\partial\Omega} |T_0|\}/2, \nu_1(T_0) = \sup\{\nu(t); |t| \leq \sup_{\partial\Omega} |T_0|\},$$

with analogous definitions for $\kappa_0(T_0)$ and $\kappa_1(T_0)$, and ν, κ, are continuous functions.

To transform the boundary condition on T to a homogeneous boundary condition, we introduce an auxiliary function S (see Gilbarg and Trudinger [2] p. 137).

Lemma 2 *There exists a function S which satisfies the following properties (i) $S(\Gamma) = T_0$. (ii) $S \in C_0^2(\mathbb{R}^3)$. (iii) for any $\epsilon > 0$ and $p \geq 1$, we can redefine S, if necessary, such that $\|S\|_{L^p} < \epsilon$.*

Now we make a change of variable: $\varphi = T - S$ to obtain

$$\begin{gathered}
-\operatorname{div}(\nu(\varphi+S)\nabla u) + u\cdot\nabla u - \alpha\varphi g - \alpha S g + \nabla p = 0\\
\operatorname{div} u = 0\\
-\operatorname{div}(\kappa(\varphi+S)\nabla\varphi) + u\cdot\nabla\varphi - \operatorname{div}(\kappa(\varphi+S)\nabla S) + u\cdot\nabla S = 0
\end{gathered}\tag{4}$$

in Ω, with boundary conditions

$$u = 0 \quad\text{and}\quad \varphi = 0 \quad \text{on } \partial\Omega \tag{5}$$

$$\lim_{|x|\to\infty} u(x) = 0\,;\quad \lim_{|x|\to\infty}\varphi(x) = 0\,. \tag{6}$$

Definition $(u,\varphi) \in J(\Omega)\times H_0^1(\Omega)$ is called a stationary weak solution of (4)-(6) if it satisfies

$$\begin{gathered}
(\nu(\varphi+S)\nabla u,\nabla v) + B(u,u,v) - \alpha(\varphi g, v) - \alpha(Sg, v) = 0 \qquad (7)\\
(\kappa(\varphi+S)\nabla\varphi,\nabla\psi) + b(u,\varphi,\psi) + (\kappa(\varphi+S)\nabla S,\nabla\psi) + b(u,S,\psi) = 0\,,
\end{gathered}$$

for all $v \in J(\Omega)$ and all $\psi \in H_0^1(\Omega)$. Where

$$\begin{gathered}
B(u,v,w) = (u\cdot\nabla v, w) = \int\int_\Omega \sum_{i,j=1}^3 u_j(x)(\partial v_i/\partial x_j)(x) w_i(x)\,dx\,,\\
b(u,\varphi,\psi) = (u\cdot\nabla\varphi,\psi) = \int\int_\Omega \sum_{i,j=1}^3 u_j(x)(\partial\varphi_i/\partial x_j)(x)\psi_i(x)\,dx\,.
\end{gathered}$$

Theorem 1 *(Existence) Under Assumptions (S1), (S2) and (S3), there exists a stationary weak solution of (7).*

3 Auxiliary problem.

Following the extending domain method, we first present a lemma which ensures the existence of weak solutions of interior problems in domains $\Omega_m = B_m \cap \Omega$. The interior problem is stated as follows:

$$\begin{gathered}
-\operatorname{div}(\nu(\varphi+S)\nabla u) + u\cdot\nabla u - \alpha\varphi g - \alpha S g + \nabla p = 0\\
\operatorname{div} u = 0\\
-\operatorname{div}(\kappa(\varphi+S)\nabla\varphi) + u\cdot\nabla\varphi - \operatorname{div}(\kappa(\varphi+S)\nabla S) + u\cdot\nabla S = 0\\
u = 0,\ \varphi = 0 \text{ on } \partial\Omega_m = \partial\Omega\cap\partial B_m
\end{gathered}\tag{P_m}$$

Definition $(u,\varphi) \in J(\Omega_m)\times H_0^1(\Omega_m)$ is called a stationary weak solution for (P_m) if it satisfies

$$\begin{gathered}
(\nu(\varphi+S)\nabla u,\nabla v) + B(u,u,v) - \alpha(\varphi g, v) - \alpha(Sg, v) = 0 \qquad (8)\\
(\kappa(\varphi+S)\nabla\varphi,\nabla\psi) + b(u,\varphi,\psi) + (\kappa(\varphi+S)\nabla S,\nabla\psi) + b(u,S,\psi) = 0\,,
\end{gathered}$$

for all $v \in J(\Omega_m)$, and for all $\psi \in H_0^1(\Omega_m)$.

Lemma 3 *Under Assumptions (S1), (S2), and (S3) we can construct a weak solution $(\overline{u}^m,\overline{\varphi}^m)$ of (P_m).*

Proof Let m be an arbitrary fixed number. Let $\{v_j\}_{j=1}^{\infty} \subset J(\Omega_m)$ and $\{\psi_j\}_{j=1}^{\infty} \subset H_0^1(\Omega_m)$ be a sequences of functions, linearly independent and such that the linear span of the v_j and ψ_j are dense in $J(\Omega_m)$ and $H_0^1(\Omega_m)$ respectively.

Since Ω_m is bounded, we can choose them such that

$$\begin{gathered}(\nabla v_j, \nabla v_k) = \delta_{ik}\,, \quad (\nabla\psi_j, \nabla\psi_k) = \delta_{jk} \\ u^n(x) = \textstyle\sum_{k=1}^n c_{n,k} v_k(x)\,, \quad \varphi^n(x) = \textstyle\sum_{k=1}^n d_{n,k}\psi_k(x)\,.\end{gathered}$$

Then we consider the system of equations

$$\begin{aligned}(\nu(\varphi^n + S)\nabla u^n, \nabla v_j) + B(u^n, u^n, v_j) - \alpha(\varphi^n g, v_j) - \alpha(Sg, v_j) &= 0 \\ (\kappa(\varphi^n + S)\nabla\varphi^n, \nabla\psi_j) + b(u^n, \varphi^n, \psi_j) & \\ +(\kappa(\varphi^n + S)\nabla S, \nabla\psi_j) + b(u^n, S, \psi_j) &= 0\,,\end{aligned} \tag{9}$$

where $1 \le j \le n$. Using the representations of u^n, φ^n, we have

$$\begin{aligned}\sum_{k=1}^n c_k(\nu(\varphi^n + S)\nabla v_k, \nabla v_j) + \sum_{k,l}^n c_k d_l B(v_k, v_l, v_j) & \\ -\sum_{k=1}^n \alpha d_k(g\psi_k, v_j) - \alpha(Sg, v_j) &= 0\,, \qquad (10) \\ \sum_{k=1}^n d_k(\kappa(\varphi^n + S)\nabla\psi_k, \nabla\psi_j) + \sum_{k,l}^n c_k d_l b(v_k, \psi_l, \psi_j) & \\ +(\kappa(\varphi^n + S)\nabla S, \nabla\psi_j) + \sum_{k=1}^n c_k b(v_k, S, \psi_j) &= 0\,,\end{aligned}$$

where $1 \le j \le n$. Put $(c;d) = (c_1, \dots, c_n, d_1, \dots, d_n)$, and
$P(c;d) = (P_1(c;d), \dots, P_{2n}(c;d))$. Then, from (10) we obtain

$$\begin{aligned}&\sum_{k=1}^n c_k \nu_0(T_0)(\nabla v_k, \nabla v_j) \\ &\quad\le \;|\sum_{k,l} c_k d_l B(v_k, v_j, v_l)| + |\sum_k \alpha d_k(g\psi_k, v_j)|\,, +|\alpha(Sg, v_j)| \\ &\sum_{k=1}^n d_k \kappa_0(T_0)(\nabla\psi_k, \nabla\psi_j) \qquad (11) \\ &\quad\le \;|\sum_{k,l} c_k d_l b(v_k, \psi_j, \psi_l)| + \kappa_1(T_0)|(\nabla S, \nabla\psi_j)| + |\sum_k c_k b(v_k, S, \psi_j)|\,;\end{aligned}$$

thus

$P_j(c;d)$

$$\leq \quad \frac{1}{\nu_0(T_0)}\left\{|\sum_{k,l} c_k d_l B(v_k,v_j,v_l)| + |\sum_k \alpha d_k(g\psi_k,v_j)| + |\alpha(Sg,v_j)|\right\},$$

$$P_{n+j}(c;d) \tag{12}$$

$$\leq \quad \frac{1}{\kappa_0(T_0)}\left\{|\sum_{k,l} c_k d_l b(v_k,\psi_j,\psi_l)| + \kappa_1(T_0)|(\nabla S,\nabla\psi_j)| + |\sum_k c_k b(v_k,S,\psi_j)|\right\}$$

where $1 \leq j \leq n$. Then our problem is reduced to obtaining a fixed point of $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Now we use Brouwer's fixed point theorem. Namely, if all possible solutions $(c;d)$ of the equation $(c;d) = \lambda P(c;d)$ for $\lambda \in [0,1]$ stay in a same ball $\|(c;d)\| \leq r$, then there exists a fixed point of P.

By multiplying $(11)_i$ (respectively. $(11)_{ii}$) by c_j (respectively. d_j), summing up with respect to j and noting $B(u^n,u^n,u^n)=0$, $b(u^n,\varphi^n,\varphi^n)=0$ we have

$$\begin{aligned}
\nu_0(T_0)\sum_{j=1}^n |c_j|^2 &= \nu_0(T_0)|\nabla u^n|^2 = \nu_0(T_0)\lambda\sum_{j=1}^n P_j(c;d)c_j \\
&\leq \lambda\alpha|(g\varphi^n,u^n)| + |(Sg,u^n)| \\
&\leq \lambda\alpha\left\{|g|_{3/2}|\varphi^n|_6|u^n|_6 + |g|_{3/2}|S|_6|u^n|_6\right\} \\
&\leq \lambda\alpha\left\{|g|_{3/2}\left(|\nabla\varphi^n| + |S|_6\right)|\nabla u^n|\right\}
\end{aligned}$$

then

$$|\nabla u^n|^2 \leq \frac{\lambda\alpha}{\nu_0(T_o)}|g|_{3/2}\left\{|\nabla\varphi^n| + |\nabla S|\right\}. \tag{13}$$

In the same manner, we find

$$|\nabla\varphi^n| \leq \frac{\lambda\kappa_1(T_0)}{\kappa_0(T_0)}|\nabla S| + \frac{\lambda}{\kappa_0(T_0)}|\nabla u^n||S|_3 \tag{14}$$

by substituting (14) into (13), we obtain

$$|\nabla u^n| \leq \frac{\lambda\alpha}{\nu_0(T_o)}|g|_{3/2}\left\{\frac{\lambda\kappa_1(T_0)}{\kappa_0(T_0)}|\nabla S| + \frac{\lambda}{\kappa_0(T_0)}|\nabla u^n||S|_3\right\} + \frac{\lambda\alpha}{\nu_0(T_o)}|g|_{3/2}|\nabla S|;$$

therefore,

$$\left(1 - \frac{\lambda^2\alpha}{\nu_0(T_o)\kappa_0(T_0)}|g|_{3/2}|S|_3\right)|\nabla u^n| \leq \frac{\lambda\alpha}{\nu_0(T_o)}|g|_{3/2}|\nabla S|\left(\frac{\kappa_1(T_0)}{\kappa_0(T_0)} + 1\right).$$

According to Lemma 2, with $p=3$, we can choose an extension S of T_0 such that

$$\gamma \equiv \frac{\alpha}{\nu_0(T_o)\kappa_0(T_0)}|g|_{3/2}|S|_3 < 1/2\,.$$

Then we have

$$|\nabla u^n| \leq \frac{\lambda\alpha}{(1-\lambda^2\gamma)\nu_0(T_o)}|g|_{3/2}|\nabla S|\left(\frac{\kappa_1(T_0)}{\kappa_0(T_0)} + 1\right). \tag{15}$$

By substituting the previous inequality in (14), we obtain

$$|\nabla\varphi^n| \le \frac{\lambda|\nabla S|}{\kappa_0(T_o)}\left(\kappa_1(T_0) + \frac{\lambda\alpha}{(1-\lambda^2\gamma)\nu_0(T_o)}|g|_{3/2}\left(\frac{\kappa_1(T_0)}{\kappa_0(T_0)}+1\right)|S|_3\right). \tag{16}$$

Since $0 \le \lambda \le 1$ and $\frac{1}{1-\lambda^2\gamma} \le \frac{1}{1-\gamma}$, from (15) and (16) we have

$$|\nabla u^n| \le \tfrac{\alpha}{(1-\gamma)\nu_0(T_o)}|g|_{3/2}|\nabla S|\left(\tfrac{\kappa_1(T_0)}{\kappa_0(T_0)}+1\right) \equiv r_1 \tag{17}$$

$$|\nabla\varphi^n| \le \tfrac{|\nabla S|}{\kappa_0(T_o)}\left(\kappa_1(T_0) + \tfrac{\lambda\alpha}{(1-\gamma)\nu_0(T_o)}|g|_{3/2}\left(\tfrac{\kappa_1(T_0)}{\kappa_0(T_0)}+1\right)|S|_3\right) \equiv r_2 \tag{18}$$

Therefore we have uniform estimates on u^n and φ^n. Indeed, r_1 and r_2 are both independent of λ, n, m. Hence solutions of $(c;d) = \lambda P(c;d)$ for $\lambda \in [0,1]$ lie in a $\mathbb{R}^{2n}-$ball $\left\{\sum_{j=1}^n \left(|c_j|^2 + |d_j|^2\right) \le r_1^2 + r_2^2\right\}$. Therefore, due to Brouwer's fixed point theorem, we have obtained a solution (u^n, φ^n) of the equations (8) with the property (after getting the fixed point, repeat the same calculation as $\lambda = 1$)

$$|\nabla u^n| \le r_1\,, \quad |\nabla\varphi^n| \le r_2\,. \tag{19}$$

Since $J(\Omega_m)$ (respectively. $H_0^1(\Omega_m)$) is compactly imbedded in $H(\Omega_m)$ (respectively. $L^2(\Omega_m)$) we can choose subsequences, which we again denote by (u^n, φ^n), and elements $\overline{u}^m \in J(\Omega_m)$, $\overline{\varphi}^m \in H_0^1(\Omega_m)$ such that $u^n \to \overline{u}^m$ weakly in $J(\Omega_m)$ and strongly in $H(\Omega_m)$ and also $\varphi^n \to \overline{\varphi}^m$ weakly in $H_0^1(\Omega_m)$, and strongly in $L^2(\Omega_m)$ and also everywhere in Ω_m.

Passing to the limit in (10) as $n \to \infty$, we find that $(\overline{u}^m, \overline{\varphi}^m)$ is a desired weak solution of (P_m).

Lemma 4 *Let us $(\overline{u}^m, \overline{\varphi}^m)$ be a weak solution for (P_m) obtained in the previous lemma. Put*

$$u^m(x) = \begin{cases} \overline{u}^m(x) & \text{if } x \in \Omega_m \\ 0 & \text{if } x \in \Omega\backslash\Omega_m \end{cases}$$

$$\varphi^m(x) = \begin{cases} \overline{\varphi}^m(x) & \text{if } x \in \Omega_m \\ 0 & \text{if } x \in \Omega\backslash\Omega_m. \end{cases}$$

Then it holds that $(u^m, \varphi^m) \in J(\Omega) \times H_0^1(\Omega)$ and furthermore

$$|\nabla u^m| \le r_1\,, \quad |\nabla\varphi^m| \le r_2 \tag{20}$$

where r_1 and r_2 be taken uniformly in m.

Proof It is easy to show $(u^m, \varphi^m) \in J(\Omega) \times H_0^1(\Omega)$. The estimates (20) are directly deduced from the (19) and the lower semi-continuity of the norm.

4 Proof of main theorem

Using the previous lemma, applying Rellich's compactness theorem, and the diagonal argument, we can choose subsequences which we again denote by (u^m, φ^m) and $u \in J(\Omega), \varphi \in H_0^1(\Omega)$ such that

$$u^m \to u \text{ weakly in } J(\Omega) \text{ and strongly in } L^2_{loc}(\Omega)$$
$$\varphi^m \to \varphi \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2_{loc}(\Omega)\,.$$

Once we get such subsequences and limits, we can show that (u, φ) becomes a stationary weak solution of (7). In fact, let us (ξ, ψ) be an arbitrary given test function. Then we find a bounded domain Ω' and a number m_0 such that suppξ, supp $\psi \subset \Omega'$ and $\Omega' \subset \Omega_{m_0} \subset \Omega_m$ for all $m \geq m_0$. Then

$$\begin{aligned}
&|\,(\nu(\varphi^m + S)\nabla\xi, \nabla u^m)_\Omega - (\nu(\varphi + S)\nabla\xi, \nabla u)_\Omega\,| \\
&\quad\leq\ |\,((\nu(\varphi^m + S) - \nu(\varphi + S))\nabla\xi, \nabla u^m)_{\Omega'}\,| + |\,(\nu(\varphi + S)\nabla\xi, \nabla(u^m - u))_{\Omega'}\,| \\
&\quad\leq\ |\nu(\varphi^m + S) - \nu(\varphi + S)|_\infty |\nabla\xi||\nabla u^m| + |\,(\nu(\varphi + S)\nabla\xi, \nabla(u^m - u))_{\Omega'}\,|
\end{aligned}$$

because the function ν is continuous and $\varphi^m \to \varphi$ strongly in $L^2_{loc}(\Omega)$, it is now immediate that $\nu(\varphi^m + S)$ converges strongly towards $\nu(\varphi + S)$. This, together with the weak convergence $u^m \to u$ in $J(\Omega)$, yields the convergence

$$|\,(\nu(\varphi^m + S)\nabla\xi, \nabla u^m)_\Omega - (\nu(\varphi + S)\nabla\xi, \nabla u)_\Omega\,| \to 0$$

as $m \to \infty$. The other convergences are analogously established. Thus, we see (u, φ) is a stationary weak solution for (7)

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E.A. Notte-Cuello
Dpto. de Matemática, U. de Antofagasta
Casilla 170, Antofagasta, Chile.
Email address: enotte@uantof.cl

M.A. Rojas-Medar
UNICAMP-IMECC, C.P.6065,13081-970, Campinas,
São Paulo, Brazil
Email address: marko@ime.inicamp.br