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COEXISTENCE OF ALGEBRAIC AND NON-ALGEBRAIC LIMIT CYCLES FOR QUINTIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In the work by Giné and Grau [11], a planar differential system of degree nine admitting a nested configuration formed by an algebraic and a non-algebraic limit cycles explicitly given was presented. As an improvement, we obtain by a new method a similar result for a family of quintic polynomial differential systems.

1. Introduction

In the qualitative theory of autonomous and planar differential systems, the study of limit cycles is very attractive because of their relation with the applications to other areas of sciences; see for instance [9, 17]. Nevertheless, most of researchers on that domain focus their attention on the number, stability and location in the phase plane of the limit cycles for the system of degree $n = \max\{\deg P_n, \deg Q_n\}$,

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P_n(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q_n(x,y), \end{aligned} \tag{1.1}$$

where $P_n(x,y)$ and $Q_n(x,y)$ are coprime polynomials of $\mathbb{R}[x,y]$. We recall that a limit cycle of system (1.1) is an isolated periodic orbit in the set of its periodic orbits and it is said to be algebraic if it is contained in the zero set of an invariant algebraic curve of the system. We recall that an algebraic curve defined by $U(x,y) = 0$ is an invariant curve for (1.1) if there exists a polynomial $K(x,y)$ (called the cofactor) such that

$$P_n(x,y)\frac{\partial U}{\partial x} + Q_n(x,y)\frac{\partial U}{\partial x} = K(x,y)U(x,y). \tag{1.2}$$

Another interesting and also a natural problem is to express analytically the limit cycles. Until recently, the only limit cycles known in an explicit way were algebraic (see for instance [4, 5, 12, 14] and references therein). It is surprising that

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exact algebraic limit cycles where obtained by Abdelkadder [1] and Bendjeddou and Cheurfa [4] for a class of Liénard equation.

Limit cycles of planar polynomial differential systems are not in general algebraic. For instance, the limit cycle appearing in the van der Pol equation is non-algebraic as it is proved by Odani [15]. In the chronological order the first examples of systems were explicit non-algebraic limit cycles appeared are those of Gasull [10] and by Al-Dossary [2] for $n = 5$, Bendjeddou and al. [3] for $n = 7$ and by Benterki and Llibre [6] for $n = 3$. Another class of quintic systems with homogeneous nonlinearity has been studied via averaging theory by Benterki and Llibre [7] . The first result for the coexistence of algebraic and non-algebraic limit cycles goes back to Giné and Grau [11] for $n = 9$. These last authors transform their system into a Ricatti equation which is itself transformed into a variable coefficients second order linear differential equation using the classic linearization method. From the principal result of an earlier work (see details from page 5 of their paper) they obtain a first integral and by the way the explicit equations of the possible limit cycles.

In this work, we obtain by a more intuitive and understandable method a similar result for a class of systems of degree $n = 5$. We show that our system admits an invariant algebraic curve, corresponding of course to a particular solution of the Ricatti equation obtained when the suited transformations are performed on the system, so the first integral can be easily obtained. The limit cycles are also exactly given and form a nested configuration, the inner one is algebraic, while the outer is non-algebraic.

2. Main result

As a main result, we shall prove the following theorem.

Theorem 2.1. *The quintic two-parameters system*

$$\begin{aligned} \dot{x} &= P_5(x, y), \\ \dot{y} &= Q_5(x, y), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} P_5(x, y) &= x + x(x^2 + y^2 - 1)(ax^2 - 4bxy + ay^2) \\ &\quad + (x^2 + y^2)(-2x + 2y + x^3 + xy^2), \end{aligned}$$

and

$$\begin{aligned} Q_5(x, y) &= y + y(x^2 + y^2 - 1)(ax^2 - 4bxy + ay^2) \\ &\quad + (x^2 + y^2)(-2x - 2y + y^3 + x^2y), \end{aligned}$$

in which $a \in \mathbb{R}_+^*$ and $b \in \mathbb{R}^*$ possesses exactly two limit cycles: the circle (γ_1) : $x^2 + y^2 - 1 = 0$ surrounding a transcendental and stable limit cycle (γ_*) explicitly given in polar coordinates (r, θ), by the equation

$$r(\theta; r_*) = \sqrt{\frac{\exp(a\theta + b\cos 2\theta)(\frac{r_*^2}{(r_*^2-1)e^b} + f(\theta))}{-1 + \exp(a\theta + b\cos 2\theta)(\frac{r_*^2}{(r_*^2-1)e^b} + f(\theta))}}, \tag{2.2}$$

with

$$f(\theta) = \int_0^\theta \exp(-as - b\cos 2s)ds, \quad r_* = \sqrt{\frac{f(2\pi)e^{b+2\pi a}}{(f(2\pi) + 1)e^{2\pi a} - 1}},$$

if the following conditions are fulfilled:

$$4b^2 - a^2 < 0, \tag{2.3}$$

$$f(2\pi) \neq \frac{1+e^{-2\pi a}}{1-e^b}, \quad f(2\pi) \neq e^{-b-2\pi a}\Big(\frac{1+(e^{2\pi a}-1)r_*^2 \pm \sqrt{2}e^{\pi a}\sqrt{r_*}}{1-r_*^2}\Big). \tag{2.4}$$

Moreover, (γ_*) *defines an unstable limit cycle when* $b + \pi a = 0$.

Proof. Firstly, we have $yP_5(x,y) - xQ_5(x,y) = 2(x^2+y^2)^2$, thus the origin is the unique critical point at finite distance. Moreover it is not difficult to see that the circle $(\gamma_1) : x^2+y^2-1 = 0$ is an invariant curve, the associated cofactor being

$$K(x,y) = 2(x^2+y^2)P_2(x,y),$$

where $P_2(x,y) = (a+1)x^2 - 4bxy + (a+1)y^2 - 1$.

Of course (γ_1) defines a periodic solution of system (2.1), since it do not pass through the origin. To see whether or not (γ_1) is in fact a limit cycle, we can proceed as follow: Let T denotes be the period of (γ_1), we consider the integral $I(\gamma_1)$, where

$$I(\gamma_1) = \int_0^T Div(x(t),y(t))dt. \tag{2.5}$$

We know from [12] that can be computed via

$$I(\gamma_1) = \int_0^T K(x(t),y(t))dt. \tag{2.6}$$

From (2.3), we have $4b^2 - a^2 < 0$, so the curve $P_2(x,y) = 0$ do not cross (γ_1). But $P_2(0,0) < 0$, hence $K(x,y) < 0$ inside $(\gamma_1)/\{(0,0)\}$, so $I(\gamma_1) < 0$. Consequently (γ_1) defines a stable algebraic limit cycle for system (2.1). The search for the non-algebraic limit cycle, requires the integration of the system. In polar coordinates, this system becomes

$$\begin{aligned} \dot{r} &= (-2b\sin 2\theta + a + 1)r^5 + (2b\sin 2\theta - a - 2)r^3 + r, \\ \dot{\theta} &= -2r^2. \end{aligned} \tag{2.7}$$

Since $\dot{\theta}$ is negative for all t, the orbits $(r(t),\theta(t))$ of system (2.6) have the opposite orientation with respect to those $(x(t),y(t))$ of system (2.1). Taking θ as an independent variable, we obtain the equation

$$\frac{dr}{d\theta} = -\frac{1}{2}(-2b\sin 2\theta + a + 1)r^3 - \frac{1}{2}(2b\sin 2\theta - a - 2)r - \frac{1}{2r}. \tag{2.8}$$

Via the change of variables $\rho = r^2$, this equation is transformed into the Riccati equation

$$\frac{d\rho}{d\theta} = (2b\sin 2\theta - a - 1)\rho^2 + (-2b\sin 2\theta + a + 2)\rho - 1. \tag{2.9}$$

Fortunately, this equation is integrable, since it possesses the particular solution $\rho = 1$ corresponding of course to the limit cycle (γ_1). The general solution of this equation is

$$\rho(\theta) = \frac{\exp(a\theta + b\cos 2\theta)(k + f(\theta))}{-1 + \exp(a\theta + b\cos 2\theta)(k + f(\theta))},$$

with $f(\theta)=\int_0^\theta \exp(-as-b\cos 2s)ds$. Consequently, the general solution of (2.8) is

$$r(\theta;k)=\sqrt{\frac{\exp(a\theta+b\cos 2\theta)(k+f(\theta))}{-1+\exp(a\theta+b\cos 2\theta)(k+f(\theta))}}\,, \tag{2.10}$$

as given in the theorem.

By passing to Cartesian coordinates, we deduce the first integral is

$$\begin{aligned} F(x,y)=\Big(&\tfrac{x^2+y^2}{x^2+y^2-1}-\exp(a\arctan\frac{y}{x}+b\cos(2\arctan\frac{y}{x}))\\ &\times\int_0^{\arctan\frac{y}{x}}\exp(-b\cos 2s-as)ds\Big)\\ &\div\exp(a\arctan\frac{y}{x}+b\cos(2\arctan\frac{y}{x})). \end{aligned} \tag{2.11}$$

The trajectories of system (2.1) are the level curves $F(x,y)=k$, $k\in\mathbb{R}$ and since these curves are obviously all non-algebraic (if we exclude of course the curve (γ_1) corresponding to $k\to+\infty$), thus any other limit cycle, if exists, should also be non-algebraic.

To go a steep further, we remark that the solution such as $r(0;r_0)=r_0>0$, corresponds to the value $k=\frac{r_0^2}{(r_0^2-1)e^b}$ provided a rewriting of the general solution of (2.8) as

$$r(\theta;r_0)=\sqrt{g(\theta)}, \tag{2.12}$$

where

$$g(\theta)=\frac{\exp(a\theta+b\cos 2\theta)\big(\frac{r_0^2}{(r_0^2-1)e^b}+f(\theta)\big)}{-1+\exp(a\theta+b\cos 2\theta)\big(\frac{r_0^2}{(r_0^2-1)e^b}+f(\theta)\big)} \tag{2.13}$$

A periodic solution of system (2.1) must satisfy the condition

$$r(2\pi;r_0)=r_0, \tag{2.14}$$

provided two distinct values of r_0: $r_1=1$ and thanks to (2.4), the well defined second value

$$r_*=\sqrt{\frac{f(2\pi)e^{b+2\pi a}}{(f(2\pi)+1)e^{2\pi a}-1}}.$$

Obviously, the first value of r_0 corresponds to the algebraic limit cycle (γ_1).

By inserting the second value r_* of r_0 in (2.12), we obtain the second candidate solution given by the statement of the theorem through (2.2). In the sequel, the notation $r(\theta,r_*)$ or (γ_*) both refer to this curve solution.

To show that it is a periodic solution, we have to show that

- the function $\theta\to g(\theta)$ is 2π-periodic, where in this case

$$g(\theta)=\frac{\exp(a\theta+b\cos 2\theta)\big(\frac{e^{2\pi a}}{1-e^{2\pi a}}f(2\pi)+f(\theta)\big)}{-1+\exp(a\theta+b\cos 2\theta)\big(\frac{e^{2\pi a}}{1-e^{2\pi a}}f(2\pi)+f(\theta)\big)}\,. \tag{2.15}$$

- $g(\theta)>0$ for all $\theta\in[0,2\pi[$.

The last condition ensures that $r(\theta,r_*)$ is well defined for all $\theta\in[0,2\pi[$ and the periodic solution do not pass through the unique equilibrium point $(0,0)$ of system (2.1).

Periodicity. Let $\theta \in [0, 2\pi[$, then

$$g(\theta+2\pi) = \frac{\exp(a\theta + 2\pi a + b\cos 2\theta)\big(\frac{e^{2\pi a}}{1-e^{2\pi a}} f(2\pi) + f(\theta+2\pi)\big)}{-1 + \exp(a\theta + 2\pi a + b\cos 2\theta)\big(\frac{e^{2\pi a}}{1-e^{2\pi a}} f(2\pi) + f(\theta+2\pi)\big)}.$$

However,

$$\begin{aligned} f(\theta+2\pi) &= \int_0^{\theta+2\pi} \exp(-as - b\cos 2s)ds \\ &= f(2\pi) + \int_{2\pi}^{\theta+2\pi} \exp(-as - b\cos 2s)ds. \end{aligned}$$

In the integral case $\int_{2\pi}^{\theta+2\pi} \exp(-as - b\cos 2s)ds$, we make the change of variable $u = s - 2\pi$, we obtain

$$\begin{aligned} f(\theta+2\pi) &= f(2\pi) + \int_0^{\theta} \exp(-a(u+2\pi) - b\cos 2(u+2\pi)) \\ &= f(2\pi) + e^{-2\pi a} f(\theta). \end{aligned}$$

Taking into account (2.3), after some calculations we obtain that $g(\theta + 2\pi) = g(\theta)$, hence g is 2π-periodic.

Strict positivity of $g(\theta)$ for $\theta \in [0, 2\pi[$. Let $\phi(\theta) = \frac{e^{2\pi a}}{1-e^{2\pi a}} f(2\pi) + f(\theta)$. Since $\frac{d\phi}{d\theta}(\theta) = \exp(-a\theta - b\cos 2\theta) > 0$ for all $\theta \in [0, 2\pi[$, the function $\theta \to \phi(\theta)$ is strictly increasing with $\phi(0) = \frac{e^{2\pi a}}{1-e^{2\pi a}} f(2\pi)$ and $\phi(2\pi) = \frac{1}{1-e^{2\pi a}} f(2\pi)$. Since $a > 0$, then $\phi(2\pi) < 0 \implies \phi(\theta) < 0$, thus $\exp(a\theta + b\cos 2\theta)\phi(\theta) < 0$, hence $g(\theta) > 0$ for all $\theta \in [0, 2\pi[$.

To show that it is in fact a limit cycle, we consider (2.13), and introduce the Poincaré return map $r_0 \to \Pi(2\pi; r_0) = r(2\pi; r_0) = \sqrt{g(2\pi)}$, with the positive x-axis as section. We compute $\frac{d\Pi}{dr_0}(2\pi; r_0)$ at the value $r_0 = r_*$. We find that

$$\begin{aligned} &\frac{d\Pi}{dr_0}(2\pi; r_0)\big|_{r_0=r*} \\ &= e^{\pi a} r_* \frac{\sqrt{(e^{2\pi a} + Ae^{b+2\pi a} - 1)r_*^2 - Ae^{b+2\pi a}}}{\sqrt{((Ae^b+1)r_*^2 - Ae^b)((e^{2\pi a} + Ae^{b+2\pi a} - 1)r_*^2 - Ae^{b+2\pi a})^2}}. \end{aligned}$$

Taking into account (2.4), we deduce that $\frac{d\Pi}{dr_0}(2\pi; r_0)\big|_{r_0=r*} \neq 1$, and finally that (γ_*) is the expected non-algebraic limit cycle. Obviously (γ_*) lies inside (γ_1) when $r_* < 1$. Since the Poincaré return map do not possess other fixed points, the system (2.1) admits exactly two limit cycles. □

3. Example

As an example let $a = 4$, $b = 1$. then system (2.1) becomes

$$\begin{aligned} x' &= x + x(x^2+y^2-1)(4x^2 - 4xy + 4y^2) + (x^2+y^2)(-2x + 2y + x^3 + xy^2), \\ y' &= y + y(x^2+y^2-1)(4x^2 - 4xy + 4y^2) + (x^2+y^2)(-2x - 2y + y^3 + x^2y). \end{aligned} \tag{3.1}$$

Then we have $f(2\pi) = \int_0^{2\pi} \exp(-4s - \cos 2s)ds \simeq 0.121\,24$ and then

$$r_* = \sqrt{\frac{(0.121\,24)e^{1+8\pi}}{((0.121\,24)+1)e^{8\pi} - 1}} \simeq 0.542\,15. \tag{3.2}$$

It is easy to verify that all conditions of Theorem 2.1 are satisfied. We conclude that system (3.1) has two limit cycles. Since $r_* < 1$, the non-algebraic lies inside the algebraic one as shown on the Poincaré disc in Figure 3.1:

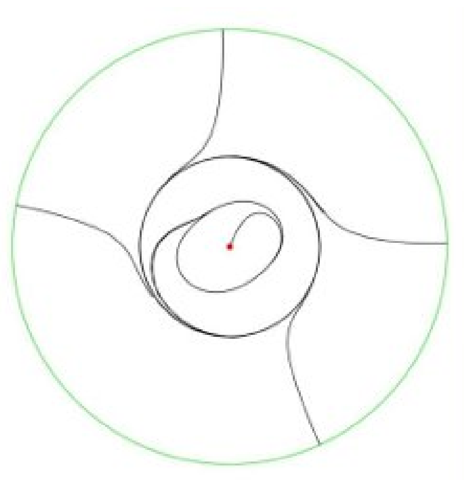

FIGURE 3.1. Limit Cycles of System (3.1)

Conclusion. In this work, we have extend the result obtained in [11] by reducing the degree of the differential system from $n = 9$ to $n = 5$. The method used is intuitive. Obtaining interesting results of this kind becomes more and more difficult for lower values of n. Nevertheless it is not forbidden to undertake the study of the following problems:

- coexistence of two explicit non-algebraic limit cycles for a quintic system;
- coexistence of explicit algebraic and non-algebraic limit cycles for $n = 3$;
- obtaining a quadratic system with exact non-algebraic limit cycle (this question is due to Benterki and Llibre [6]).

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