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CONTINUOUS SELECTIONS OF SET OF MILD SOLUTIONS OF EVOLUTION INCLUSIONS

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Abstract. We prove the existence of continuous selections of the set valued map $\xi \to \mathcal{S}(\xi)$ where $\mathcal{S}(\xi)$ is the set of all mild solutions of the evolution inclusions of the form

$$\begin{gathered}\dot{x}(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds \\ x(0) = \xi, \quad t \in I = [0,T],\end{gathered}$$

where F is a lower semi continuous set valued map Lipchitzean with respect to x in a separable Banach space X, A is the infinitesimal generator of a C_0-semi group of bounded linear operators from X to X, and $K(t,s)$ is a continuous real valued function defined on $I \times I$ with $t \geq s$ for all $t, s \in I$ and $\xi \in X$.

1. Introduction

Existence of solutions of differential inclusions and integrodifferential equations has been studied by many authors [1, 2, 3]. Existence of continuous selections of the solution sets of the Cauchy problem $\dot{x}(t) \in F(t,x(t)),\ x(0) = \xi$ was first proved by Cellina [5] for F Lipchitzean with respect to x defined on an open subset of $R \times R^n$ and taking compact uniformly bounded values. Cellina proved that the map that associates the set of solutions $\mathcal{S}(\xi)$ of the above Cauchy problem to the initial point ξ, admits a selection continuous from R^n to the space of absolutely continuous functions.

Extensions of Cellina's result to Lipchitzean maps with closed non empty values in a separable Banach space has been obtained in [4] and [6]. In [7] Staicu proved the existence of a continuous selection of the set valued map $\xi \to \mathcal{S}(\xi)$ where $\mathcal{S}(\xi)$ is the set of all mild solutions of the Cauchy problem

$$\dot{x}(t) \in Ax(t) + F(t,x(t)),\ x(0) = \xi$$

where A is the infinitesimal generator of a C_0 - semi group and F is Lipchitzean with respect to x. Staicu also proved the same result for the set of all weak solutions by considering that $-A$ is a maximal monotone map.

In this present work first we prove the existence of a continuous selection of the set valued map $\xi \to \mathcal{S}(\xi)$ where $\mathcal{S}(\xi)$ is the set of all mild solutions of the

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integrodifferential inclusions of the form

$$\dot{x}(t) \in Ax(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = \xi \ , \quad t \in I = [0,T] \tag{1.1}$$

where F is a set valued map Lipchitzean with respect to x in a separable Banach space X, A is the infinitesimal generator of a C_0-semi group of bounded linear operators from X to X and $K(t,s)$ is a continuous real valued function defined on $I \times I$ with $t \geq s$ for all $t,s \in I$ and $\xi \in X$. Then we extend our result for the evolution inclusions of the form

$$\dot{x}(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = \xi \ , \quad t \in I = [0,T] \,. \tag{1.2}$$

2. Preliminaries

Let $T > 0$, $I = [0,T]$ and denote by $\mathcal{L}$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with norm $\|\cdot\|$. Let 2^X be the family of all non empty subsets of X and $\mathcal{B}(X)$ be the family of Borel subsets of X.

If $x \in X$ and A is a subset of X, then we define

$$d(x,A) = \inf\{\|x-y\| : y \in A\}.$$

For any two closed and bounded non empty subsets A and B of X, we define *Housdorff distance* from A and B by

$$h(A,B) = \max\{\sup\{d(x,B) : x \in A\}, \sup\{d(y,A) : x \in B\}\}.$$

Let $C(I,X)$ denote the Banach space of all continuous functions $x : I \to X$ with norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in I\}.$$

Let $L^1(I,X)$ denote the Banach space of all Bochner integrable functions $x : I \to X$ with norm $\|x\|_1 = \int_0^T \|x(t)\|dt$. Let $\mathcal{D}$ be the family of all decomposable closed non empty subsets of $L^1(I,X)$.

A set valued map $\mathcal{G} : S \to 2^X$ is said to be *lower semi continuous (l.s.c)* if for every closed subset C of X the set $\{s \in S : \mathcal{G}(s) \subset C\}$ is closed in S.

A function $g : S \to X$ such that $g(s) \in \mathcal{G}(s)$ for all $s \in S$ is called a *selection* of $\mathcal{G}(\cdot)$. Let $\{G(t) : t \geq 0\}$ be a strongly continuous semi group of bounded linear operators from X to X. Here $G(t)$ is a mapping (operator) of X into itself for every $t \geq 0$ with

1. $G(0) = I$ (the identity mapping of X onto X)
2. $G(t+s) = G(t)G(s)$ for all $t,s \geq 0$.

Now we assume the following:

- (H1) $F : I \times X \to 2^X$ is a lower semi continuous set valued map taking non empty closed bounded values.
- (H2) F is $\mathcal{L} \otimes \mathcal{B}(X)$ measurable.
- (H3) There exists a $k \in L^1(I,R)$ such that the Hausdorff distance satisfies $h(F(t,x(t)), F(t,y(t))) \leq k(t)\|x(t) - y(t)\|$ for all $x,y \in X$ and a.e. $t \in I$
- (H4) There exists a $\beta \in L^1(I,R)$ such that $d(0, F(t,0)) \leq \beta(t)$ a.e. $t \in I$
- (H5) $K : D \to R$ is a real valued continuous function where $D = \{(t,s) \in I \times I : t \geq s\}$ such that $B = \sup\{\|K(t,s)\| : t \geq s\}$.

To prove our theorem we need the following two lemmas.

Lemma 2.1 ([5]). *Let $F : I \times S \to 2^X, S \subseteq X$, be measurable with non empty closed values, and let $F(t, \cdot)$ be lower semi continuous for each $t \in I$. Then the map $\xi \to G_F(\xi)$ given by*

$$G_F(\xi) = \{v \in L^1(I, X) : v(t) \in F(T, \xi) \quad \forall\ t \in I\}$$

is lower semi continuous from S into $\mathcal{D}$ if and only if there exists a continuous function $\beta : S \to L^1(I, R)$ such that for all $\xi \in S$, we have $d(0, F(t, \xi)) \leq \beta(\xi)(t)$ a.e. $t \in I$.

Lemma 2.2 ([5]). *Let $\zeta : S \to \mathcal{D}$ be a lower semi continuous set valued map and let $\varphi : S \to L^1(I, X)$ and $\psi : S \to L^1(I, X)$ be continuous maps. If for every $\xi \in S$ the set*

$$H(\xi) = cl\{v \in \zeta(\xi) : \|v(t) - \varphi(\xi)(t)\| < \psi(\xi)(t)\ a.e\ t \in I\}$$

is non empty, then the map $H : S \to \mathcal{D}$ defined above admits a continuous selection.

3. Integrodifferential inclusions

Definition. A function $x(\cdot, \xi) : I \to X$ is called *a mild solution* of (1.1) if there exists a function $f(\cdot, \xi) \in L^1(I, X)$ such that

(i) $f(t, \xi) \in F(t, x(t, \xi))$ for almost all $t \in I$
(ii) $x(t, \xi) = G(t)\xi + \int_0^t G(t - \tau) \int_0^\tau K(\tau, s) f(s, \xi) ds d\tau$ for each $t \in I$.

Theorem 3.1. *Let A be the infinitesimal generator of a C_0-semi group $\{G(t) : t \geq 0\}$ of bounded linear operators of X into X and the hypotheses (H1)–(H5) be satisfied. Then there exists a function $x(\cdot, \cdot) : I \times X \to X$ such that*

(i) *$x(\cdot, \xi) \in \mathcal{S}(\xi)$ for every $\xi \in X$ and*
(ii) *$\xi \to x(\cdot, \xi)$ is continuous from X into $C(I, X)$.*

Proof. Let $\epsilon > 0$ be given. For $n \in N$ let $\epsilon_n = \frac{1}{\epsilon^{n+1}}$. Let $M = \sup\{|G(t)| : t \in I\}$. For every $\xi \in X$ define $x_0(\cdot, \xi) : I \to X$ by

$$x_0(t, \xi) = G(t)\xi \tag{3.1}$$

Now

$$\|x_0(t, \xi_1) - x_0(t, \xi_2)\| = \|G(t)\xi_1 - G(t)\xi_2\| = |G(t)|\ \|\xi_1 - \xi_2\| \leq M\|\xi_1 - \xi_2\|$$

i.e. The map $\xi \to x_0(\cdot, \xi)$ is continuous from X to $C(I, X)$. For each $\xi \in X$ define $\alpha(\xi) : I \to R$ by

$$\alpha(\xi)(t) = \beta(t) + k(t)\|x_0(t, \xi)\|\,. \tag{3.2}$$

Now $|\alpha(\xi_1)(t) - \alpha(\xi_2)(t)| < k(t)\|\xi_1 - \xi_2\|$ i.e. $\alpha(\cdot)$ is continuous from X to $L^1(I, R)$. By (H4) and (3.2) we have

$$d(0, F(t, x_0(t, \xi)) < \beta(t) + k(t)\|x_0(t, \xi)\|$$

and so

$$d(0, F(t, x_0(t, \xi)) < \alpha(\xi)(t) \text{ for a.e. } t \in I \tag{3.3}$$

Define the set valued maps $G_0 : X \to 2^{L^1(I,X)}$ and $H_0 : X \to 2^{L^1(I,X)}$ by

$$G_0(\xi) = \{v \in L^1(I, X) : v(t) \in F(t, x_0(t)) \quad \text{a.e. } t \in I\},$$
$$H_0(\xi) = \text{cl}\{v \in G_0(\xi) : \|v(t)\| < \alpha(\xi)(t) + \epsilon_0\}\,.$$

By Lemma 2.2 and (3.3) there exists a continuous selection $h_0 : X \to L^1(I, X)$ of $H_0(\cdot)$. Define $m(t) = \int_0^t k(s)ds$ For $n \geq 1$ define $\beta_n(\xi)(t)$ by

$$\begin{aligned} \beta_n(\xi)(t) &= M^n B^n T^{n-1} \int_0^t \int_0^\tau \alpha(\xi)(s) \frac{[m(t) - m(s)]^{n-1}}{(n-1)} ds\, d\tau \\ &\quad + M^n B^n T^n \Big(\sum_{i=0}^n \epsilon_i \Big) \int_0^t \frac{[m(t) - m(\tau)]^{n-1}}{(n-1)} d \end{aligned} \tag{3.4}$$

Set $f_0(t, \xi) = h_0(\xi)(t)$. By the definition of h_0 we see that $f_0(t, \xi) \in F(t, x_0(t, \xi))$. Define

$$x_1(t, \xi) = G(t)\xi + \int_0^t G(t - \tau) \int_0^\tau K(\tau, s) f_0(s, \xi) ds d\tau \ \forall\ t \in I \backslash \{0\} \tag{3.5}$$

By the definition of $H_0(\cdot)$ we see that $\|f_0(t, \xi)\| < \alpha(\xi)(t) + \epsilon_0$ for all $t \in I\backslash\{0\}$. From (3.1) and (3.5) we have

$$\begin{aligned} \|x_1(t, \xi) - x_0(t, \xi)\| &\leq \int_0^t |G(t - \tau)| \int_0^\tau |K(\tau, s)| \|f_0(s, \xi)\| ds d\tau \\ &\leq MB \int_0^t \int_0^\tau \|f_0(s, \xi)\| ds d\tau \\ &\leq MB \int_0^t \int_0^\tau \{\alpha(\xi)(s) + \epsilon_0\} \\ &< MB \int_0^t \int_0^\tau \alpha(\xi)(s) ds d\tau + MBT \Big(\sum_{i=0}^1 \epsilon_i \Big) \int_0^t d\tau \\ &< \beta_1(\xi)(t). \end{aligned}$$

We claim that there are two sequences $\{f_n(\cdot, \xi)\}$ and $\{x_n(\cdot, \xi)\}$ such that for $n \geq 1$ the following properties are satisfied:

(a) the map $\xi \to f_n(\cdot, \xi)$ is continuous from X into $L^1(I, X)$
(b) $f_n(t, \xi) \in F(t, x_n(t, \xi))$ for each $\xi \in X$ a.e. $t \in I$
(c) $\|f_n(t, \xi) - f_{n-1}(t, \xi)\| \leq k(t)\beta_n(\xi)(t)$ for a.e. $t \in I$
(d) $x_{n+1}(t, \xi) = G(t)\xi + \int_0^t G(t - \tau) \int_0^\tau K(\tau, s) f_n(s, \xi) ds d\tau$, for all $t \in I$.

We shall claim the above by induction on n.We assume that already there exist functions $f_1 \dots f_n$ and $x_1 \dots x_n$ satisfying (a)–(d). Define $x_{n+1}(\cdot, \xi) : I \to X$ by

$$x_{n+1}(t, \xi) = G(t)\xi + \int_0^t G(t - \tau) \int_0^\tau K(\tau, s) f_n(s, \xi) ds d\tau, \quad \forall t \in I.$$

Then by (c) and (d), for $t \in I\backslash\{0\}$, we have

$$\begin{aligned}
&\|x_{n+1}(t,\xi) - x_n(t,\xi)\| \\
&= \Big\| \int_0^t G(t-\tau) \int_0^\tau K(\tau,s)\{f_n(s,\xi) - f_{n-1}(s,\xi)\} ds\, d\tau \Big\| \\
&\leq MB \int_0^t \int_0^\tau \|f_n(s,\xi) - f_{n-1}\| ds\, d\tau \qquad (3.6)\\
&\leq MB \int_0^t \int_0^\tau k(s)\beta_n(\xi)(s) ds d\tau \\
&\leq MB \int_0^t \int_0^\tau k(s)\Big\{ M^n B^n T^{n-1} \int_0^s \int_0^u \alpha(\xi)(v) \frac{[m(s)-m(v)]^{n-1}}{(n-1)} dv\, du \\
&\quad + M^n B^n T^n \Big(\sum_{i=0}^n \epsilon_i\Big) \int_0^s \frac{[m(s)-m(u)]^{n-1}}{(n-1)\;!} du \Big\} ds\, d\tau \qquad (3.7)\\
&< M^{n+1}B^{n+1}T^n \int_0^t \int_0^\tau \alpha(\xi)(s) \frac{[m(t)-m(s)]^n}{n} ds\, d\tau \\
&\quad + M^{n+1}B^{n+1}T^{n+1} \Big(\sum_{i=0}^{n+1} \epsilon_i\Big) \int_0^t \frac{[m(t)-m(\tau)]^n}{n} d\tau \qquad (3.8)\\
&< \beta_n(\xi)(t).
\end{aligned}$$

By (H3) we now have

$$d(f_n(t,\xi), F_n(t, x_{n+1}(t,\xi)) \leq k(t)\|x_{n+1}(t,\xi) - x_n(t,\xi)\| < k(t)\beta_{n+1}(\xi)(t) \qquad (3.9)$$

Define a set valued map $G_{n+1} : X \to 2^{L^1(I,X)}$ by

$$G_{n+1}(\xi) = \{v \in L^1(I,X) : v(t) \in F(t, x_{n+1}(t,\xi)) \quad \text{a.e. } t \in I\}$$

By Lemma 3.1 and (3.9), G_{n+1} is lower semi continuous from X into $\mathcal{D}$. Define a set valued map $H_{n+1} : X \to 2^{L^1(I,X)}$ by

$$H_{n+1} = \text{cl}\{v \in G_{n+1}(\xi) : \|v(t) - f_n(t,\xi)\| < k(t)\beta_{n-1}(t) \quad \text{a.e. } t \in I\} \qquad (3.10)$$

Therefore, $H_{n+1}(\xi)$ is non empty for each $\xi \in X$. By Lemma 2.2 and (3.10) we see that there exists a cotinuous selection

$$h_{n+1} : X \to L^1(I,X) \quad \text{of } H_{n+1}(\cdot).$$

Then $f_{n+1}(t,\xi) = h_{n+1}(\xi)(t)$ for each $\xi \in I$ and each $t \in I$ satisfies the properties (a)–(c) of our claim. By the property (c) and (3.6)–(3.9) we have

$$\begin{aligned}
\|x_{n+1}(\cdot,\xi) - x_n(\cdot,\xi)\|_\infty &\leq MB\|f_n(\cdot,\xi) - f_{n-1}(\cdot,\xi)\|_1 \\
&\leq \frac{(MBT\|k\|_1)^n}{n!}\big\{MB\|\alpha(\xi)\|_1 + MBT\epsilon\big\}
\end{aligned}$$

Therefore, the sequence $\{f_n(\cdot,\xi)\}$ is a Cauchy sequence in $L^1(I,X)$ and the sequence $\{x_n(\cdot),\xi)\}$ is a Cauchy sequence in $C(I,X)$. Let $f(\cdot,\xi) \in L^1(I,X)$ be the limit of the Cauchy sequence $\{f_n(\cdot,\xi)\}$ and $x(\cdot,\xi) \in C(I,X)$ be the limit of the Cauchy sequuence $\{x_n(\cdot),\xi)\}$.

Now we can easily show that the map $\xi \to f(\cdot,\xi)$ is continuous from X into $L^1(I,X)$ and the map $\xi \to x(\cdot,\xi)$ is continuous from X into $C(I,X)$ and for all

$\xi \in X$ and almost all $t \in I$, $f(t,\xi) \in F(t, x(t,\xi))$. Taking limit in (d) we obtain

$$x(t,\xi) = G(t)\xi + \int_0^t G(t-\tau) \int_0^\tau K(\tau,s) f(t,\xi) ds\, d\tau \quad \forall\, t \in I.$$

This completes the proof. □

4. Evolution Inclusions

Now we prove the existence of continuous selections of the set of mild solutions of evolution inclusions (1.2).

Definition A function $x(\cdot,\xi) : I \to X$ is called *a mild solution* of (1.2) if there exists a function $f(\cdot,\xi) \in L^1(I,X)$ such that

(i) $f(t,\xi) \in F(t, x(t,\xi))$ for almost all $t \in I$
(ii) $x(t,\xi) = G(t,0)\xi + \int_0^t G(t,\tau) \int_0^\tau K(\tau,s) f(s,\xi) ds d\tau$ for each $t \in I$.

We introduce the following norms, where ω is a constant,

$$\|x\|_2 = e^{-\omega t}\|x\|\,,$$
$$\|x\|_3 = \sup\{\|x(t)\|_2, t \in I\}\,,$$
$$\|x\|_4 = \int_0^t \|x\|_2 dt\,.$$

Theorem 4.1. *Let $A(t)$ be the infinitesimal generator of a C_0-semi group of a two parameter family $\{G(t,\tau) : t \geq 0, \tau \geq 0\}$ of bounded linear operators of X into X and the hypotheses (H1)—(H5) be satisfied. Then there exists a function $x(\cdot,\cdot) : I \times X \to X$ such that*

(i) *$x(\cdot,\xi) \in \mathcal{S}(\xi)$ for every $\xi \in X$ and*
(ii) *$\xi \to x(\cdot,\xi)$ is continuous from X into $C(I,X)$.*

Proof. Let $\|G(t,\tau)\| \leq Me^{\omega(t-\tau)}$ where ω is a constant. For every $\xi \in X$ define $x_0(\cdot,\xi) : I \to X$ by

$$x_0(t,\xi) = G(t,0)\xi$$

Now taking $\beta_n(\xi)(t)$ as in theorem 3.1, we can prove that there are two sequences $\{f_n(\cdot,\xi)\}$ and $\{x_n(\cdot,\xi)\}$ such that for $n \geq 1$ the following properties are satisfied:

(a) The map $\xi \to f_n(\cdot,\xi)$ is continuous from X into $L^1(I,X)$.
(b) $f_n(t,\xi) \in F(t, x_n(t,\xi))$ for each $\xi \in X$ a.e. $t \in I$
(c) $\|f_n(t,\xi) - f_{n-1}(t,\xi)\|_2 \leq k(t)\beta_n(\xi)(t)$ for a.e. $t \in I$
(d) $x_{n+1}(t,\xi) = G(t,0)\xi + \int_0^t G(t,\tau) \int_0^\tau K(\tau,s) f_n(s,\xi) ds d\tau$, for all $t \in I$.

Now we have

$$\begin{aligned} \|x_{n+1}(\cdot,\xi) - x_n(\cdot,\xi)\|_3 &\leq MB\|f_n(\cdot,\xi) - f_{n-1}(\cdot,\xi)\|_4 \\ &\leq \frac{(MBT\|k\|_4)^n}{n!}\big\{MB\|\alpha(\xi)\|_4 + MBT\epsilon\big\} \end{aligned}$$

Then the Cauchy sequence $\{x_n(.,\xi)\}$ converges to a limit $x(.,\xi) \in C(I,X)$. Now we can easily show that the map $\xi \to f(\cdot,\xi)$ is continuous from X into $L^1(I,X)$ and the map $\xi \to x(\cdot,\xi)$ is continuous from X into $C(I,X)$ and for all $\xi \in X$ and almost all $t \in I$, $f(t,\xi) \in F(t, x(t,\xi))$. Taking limit in (d) we obtain

$$x(t,\xi) = G(t,0)\xi + \int_0^t G(t,\tau) \int_0^\tau K(\tau,s) f(t,\xi) ds d\tau \quad \forall t \in I.$$

This completes the proof. □

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