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VARIATIONAL METHODS FOR A RESONANT PROBLEM WITH THE p-LAPLACIAN IN $\mathbb{R}^N$

BÉNÉDICTE ALZIARY, JACQUELINE FLECKINGER, PETER TAKÁČ

Abstract. The solvability of the resonant Cauchy problem

$$-\Delta_p u = \lambda_1 m(|x|)|u|^{p-2}u + f(x) \quad \text{in } \mathbb{R}^N; \quad u \in D^{1,p}(\mathbb{R}^N),$$

in the entire Euclidean space $\mathbb{R}^N$ ($N \geq 1$) is investigated as a part of the Fredholm alternative at the first (smallest) eigenvalue λ_1 of the positive p-Laplacian $-\Delta_p$ on $D^{1,p}(\mathbb{R}^N)$ relative to the weight $m(|x|)$. Here, Δ_p stands for the p-Laplacian, $m\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a weight function assumed to be radially symmetric, $m \not\equiv 0$ in $\mathbb{R}_+$, and $f\colon \mathbb{R}^N \to \mathbb{R}$ is a given function satisfying a suitable integrability condition. The weight $m(r)$ is assumed to be bounded and to decay fast enough as $r \to +\infty$. Let φ_1 denote the (positive) eigenfunction associated with the (simple) eigenvalue λ_1 of $-\Delta_p$. If $\int_{\mathbb{R}^N} f\varphi_1 \,\mathrm{d}x = 0$, we show that problem has at least one solution u in the completion $D^{1,p}(\mathbb{R}^N)$ of $C_c^1(\mathbb{R}^N)$ endowed with the norm $(\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x)^{1/p}$. To establish this existence result, we employ a saddle point method if $1 < p < 2$, and an improved Poincaré inequality if $2 \leq p < N$. We use weighted Lebesgue and Sobolev spaces with weights depending on φ_1. The asymptotic behavior of $\varphi_1(x) = \varphi_1(|x|)$ as $|x| \to \infty$ plays a crucial role.

1. Introduction

Spectral problems involving quasilinear degenerate or singular elliptic operators have been an interesting subject of investigation for quite some time; see e.g. Drábek [3] or Fučík et al. [10]. In our present work we focus our attention on the solvability of the Cauchy problem

$$-\Delta_p u = \lambda\, m(x)\, |u|^{p-2}u + f(x) \ \text{ in } \mathbb{R}^N; \qquad u \in D^{1,p}(\mathbb{R}^N), \tag{1.1}$$

in the entire Euclidean space $\mathbb{R}^N$ ($N \geq 1$). Here, Δ_p stands for the p-Laplacian defined by $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $\lambda \in \mathbb{R}$ is the spectral parameter, $m\colon \mathbb{R}^N \to \mathbb{R}_+$ is a weight function assumed to be radially symmetric, $m \not\equiv 0$ in $\mathbb{R}^N$, and $f\colon \mathbb{R}^N \to \mathbb{R}$ is a given function satisfying a suitable integrability condition. We look for a weak solution to problem (1.1) in the Sobolev space $D^{1,p}(\mathbb{R}^N)$ defined to

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be the completion of $C_{\mathrm{c}}^1(\mathbb{R}^N)$ under the Sobolev norm

$$\|u\|_{D^{1,p}(\mathbb{R}^N)} \stackrel{\text{def}}{=} \Big(\int_{\mathbb{R}^N} |\nabla u(x)|^p \,\mathrm{d}x \Big)^{1/p}.$$

If the weight $m(x)$ is measurable, bounded and decays at least as fast as $|x|^{-p-\delta}$ as $|x| \to \infty$, with some $\delta > 0$, the Sobolev imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; m)$ turns out to be compact, where $L^p(\mathbb{R}^N; m)$ denotes the weighted Lebesgue space of all measurable functions $u\colon \mathbb{R}^N \to \mathbb{R}$ with the norm

$$\|u\|_{L^p(\mathbb{R}^N;m)} \stackrel{\text{def}}{=} \Big(\int_{\mathbb{R}^N} |u(x)|^p \, m(x) \,\mathrm{d}x \Big)^{1/p} < \infty.$$

Hence, the Rayleigh quotient

$$\lambda_1 \stackrel{\text{def}}{=} \inf \Big\{ \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x \colon u \in D^{1,p}(\mathbb{R}^N) \ \text{ with } \ \int_{\mathbb{R}^N} |u|^p \, m \,\mathrm{d}x = 1 \Big\} \tag{1.2}$$

is positive and gives the first (smallest) eigenvalue λ_1 of $-\Delta_p$ relative to the weight m. Now take f from the dual space $D^{-1,p'}(\mathbb{R}^N)$ of $D^{1,p}(\mathbb{R}^N)$, $p' = p/(p-1)$, with respect to the standard duality $\langle \cdot\,, \cdot \rangle$ induced by the inner product on $L^2(\mathbb{R}^N)$. If $-\infty < \lambda < \lambda_1$ then the energy functional corresponding to equation (1.1),

$$\mathcal{J}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p} \int_{\mathbb{R}^N} |u|^p \, m(x) \,\mathrm{d}x - \int_{\mathbb{R}^N} f(x) u \,\mathrm{d}x \tag{1.3}$$

defined for $u \in D^{1,p}(\mathbb{R}^N)$, is weakly lower semicontinuous and coercive on $D^{1,p}(\mathbb{R}^N)$. Thus, $\mathcal{J}_\lambda$ possesses a global minimizer which provides a weak solution to equation (1.1).

The critical case $\lambda = \lambda_1$ is much more complicated when $p \neq 2$ because the linear Fredholm alternative cannot be applied. First, one has to have sufficient information on the first eigenvalue λ_1; we refer the reader to Fleckinger et al. [8, Sect. 2 and 3] or Stavrakakis and de Thélin [21]. One has

$$-\Delta_p \varphi_1 = \lambda_1 \, m(x) \, |\varphi_1|^{p-2} \varphi_1 \ \text{ in } \mathbb{R}^N; \qquad \varphi_1 \in D^{1,p}(\mathbb{R}^N) \setminus \{0\}, \tag{1.4}$$

and the eigenvalue λ_1 is simple, by a result due to Anane [1, Théorème 1, p. 727] and later generalized by Lindqvist [14, Theorem 1.3, p. 157]. Moreover, the corresponding eigenfunction φ_1 can be normalized by $\|\varphi_1\|_{L^p(\mathbb{R}^N;m)} = 1$ and $\varphi_1 > 0$ in $\mathbb{R}^N$, owing to the strong maximum principle [24, Prop. 3.2.1 and 3.2.2, p. 801] or [25, Theorem 5, p. 200]. We decompose the unknown function $u \in D^{1,p}(\mathbb{R}^N)$ as a direct sum

$$\begin{gathered} u = u^{\parallel} \cdot \varphi_1 + u^{\top} \quad \text{ where} \\ u^{\parallel} = \int_{\mathbb{R}^N} u \, \varphi_1 \, \mu(x) \,\mathrm{d}x \in \mathbb{R} \ \text{ and } \ \int_{\mathbb{R}^N} u^{\top} \varphi_1 \, \mu(x) \,\mathrm{d}x = 0, \end{gathered} \tag{1.5}$$

with the weight $\mu(x)$ given by $\mu \stackrel{\text{def}}{=} \varphi_1^{p-2} \, m$. It is quite natural that we treat the two components, $u^{\parallel}$ and $u^{\top}$, differently. The linearization of the equation

$$-\Delta_p u = \lambda_1 \, m(x) \, |u|^{p-2} u + f(x) \ \text{ in } \mathbb{R}^N; \qquad u \in D^{1,p}(\mathbb{R}^N), \tag{1.6}$$

about $u^{\parallel} \cdot \varphi_1$, and the corresponding "quadratization" of the functional $\mathcal{J}_{\lambda_1}$, play an important role in our approach. We will also see that the orthogonality condition

$$\int_{\mathbb{R}^N} f \, \varphi_1 \, \mu \,\mathrm{d}x \equiv \int_{\mathbb{R}^N} f \, \varphi_1^{p-1} \, m \,\mathrm{d}x = 0 \tag{1.7}$$

for f and φ_1 relative to the measure $\mu(x)\,\mathrm{d}x$ is sufficient, but not necessary for the solvability of problem (1.6).

Similarly as in Drábek and Holubová [5] for $1 < p < 2$, in Fleckinger and Takáč [9] for $2 \le p < \infty$, and in Takáč [22, 23] for any $1 < p < \infty$, where the domain $\Omega \subset \mathbb{R}^N$ is bounded, we apply the calculus of variations using the direct sum (1.5) in order to obtain a solution to equation (1.6). We use entirely different variational methods to treat the two cases $1 < p < 2$ and $2 \le p < N$: In the former case we apply a saddle point method from [5, 22, 23], whereas in the latter case we use a minimization method due to [9] which is based on an improved Poincaré inequality. Our variational methods are different from the standard ones because the functional $\mathcal{J}_{\lambda_1}$ needs not satisfy the Palais-Smale condition if f obeys the orthogonality condition (1.7); cf. del Pino, Drábek and Manásevich [17, Theorem 1.2(ii), p. 390].

This paper is organized as follows. In Section 2 we mention some elementary properties of the first eigenfunction φ_1 and introduce basic function spaces and notation. Section 3 contains our main results on the solvability of problem (1.6), Theorem 3.1 for $2 \le p < N$ and Theorem 3.3 for $1 < p < 2 \le N$, and some properties of the energy functional $\mathcal{J}_\lambda$ needed to establish the solvability, as well. Naturally, our approach requires the compactness of several Sobolev imbeddings in $\mathbb{R}^N$ with weights (Proposition 3.6) which we prove in Section 4. In Section 5 we establish a few auxiliary results for the quadratization of $\mathcal{J}_{\lambda_1}$. We use this quadratization to verify the improved Poincaré inequality (Lemma 3.7) for $2 \le p < N$ in Section 6. From this inequality we derive Theorem 3.1 in Section 7. For $1 < p < 2$ the quadratization of $\mathcal{J}_{\lambda_1}$ is employed in a saddle point method to prove Theorem 3.3 in Section 8. Finally, some asymptotic formulas for the eigenfunction φ_1 near infinity are established in the Appendix (Proposition 9.1).

The rate of decay of $\varphi_1(x)$ as $|x| \to \infty$ is, in fact, the main cause for our restriction $p < N$. The case $p \ge N$ seems to require a different technique.

2. Preliminaries

We now put our resonant problem (1.6) into a rigorous setting. Set $\mathbb{R}_+ = [0, \infty)$. For $x \in \mathbb{R}^N$ we denote by $r = |x| \ge 0$ the radial variable in $\mathbb{R}^N$.

2.1. **Hypotheses.** We assume $1 < p < N$ throughout this article unless indicated otherwise. Furthermore, the weight m is assumed to be radially symmetric, $m(x) \equiv m(|x|)$, $x \in \mathbb{R}^N$, where $m\colon \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue measurable function satisfying the following hypothesis:

(**H**) There exist constants $\delta > 0$ and $C > 0$ such that

$$0 < m(r) \le \frac{C}{(1+r)^{p+\delta}} \quad \text{for almost all } 0 \le r < \infty. \tag{2.1}$$

Remark 2.1. In fact, in hypothesis (H) above, instead of $m(r) > 0$ for almost all $0 \le r < \infty$, it suffices to assume only $m \ge 0$ a.e. in $\mathbb{R}^N$ and m does not vanish identically near zero, i.e., for every $r_0 > 0$ we have $m \not\equiv 0$ in $(0, r_0)$. However, if $m \equiv 0$ on a set $S \subset \mathbb{R}_+$ of positive Lebesgue measure, then the weighted spaces $\mathcal{H}_{\varphi_1} = L^2(\mathbb{R}^N; \varphi_1^{p-2} m)$, $L^p(\mathbb{R}^N; m)$, etc. defined below become linear spaces with a seminorm only. Moreover, all functions from their dual spaces $\mathcal{H}'_{\varphi_1} = L^2\big(\mathbb{R}^N; \varphi_1^{2-p} m^{-1}\big)$, $L^{p'}(\mathbb{R}^N; m^{-1/(p-1)})$, etc., respectively, must vanish identically (i.e., almost everywhere) in the "spherical shell" $\{x \in \mathbb{R}^N : |x| \in S\}$. This would make our presentation much less clear; therefore, we have decided to leave the necessary amendments in our arguments to an interested reader.

2.2. **The first eigenfunction φ_1.** Under hypothesis (H), the first eigenvalue λ_1 of $-\Delta_p$ on $\mathbb{R}^N$ relative to the weight $m(|x|)$ is simple and the eigenfunction φ_1 associated with λ_1 is commonly called a "ground state" for the Cauchy problem (1.4). The simplicity of λ_1 forces $\varphi_1(x) = \varphi_1(|x|)$ radially symmetric in $\mathbb{R}^N$. Hence, the eigenvalue problem (1.4) is equivalent to

$$-\left(|\varphi_1'|^{p-2}\varphi_1'\right)' - \frac{N-1}{r}\,|\varphi_1'|^{p-2}\varphi_1' = \lambda_1\, m(r)\, \varphi_1^{p-1} \quad \text{for } r > 0;$$
$$\text{subject to} \quad \int_0^\infty |\varphi_1'(r)|^p\, r^{N-1}\, \mathrm{d}r < \infty \quad \text{and} \quad \varphi_1(r) \to 0 \text{ as } r \to \infty.$$

It can be further rewritten as

$$\begin{gathered} -\left(r^{N-1}\,|\varphi_1'|^{p-2}\varphi_1'\right)' = \lambda_1\, m(r)\, r^{N-1}\, \varphi_1^{p-1} \quad \text{for } r > 0; \\ \varphi_1'(r) \to 0 \text{ as } r \to 0 \quad \text{and} \quad \varphi_1(r) \to 0 \text{ as } r \to \infty. \end{gathered} \tag{2.2}$$

Recalling hypothesis (H), from (2.2) we can deduce the following simple facts.

Lemma 2.2. *Let $1 < p < N$ and let hypothesis H be satisfied. Then the function $r \mapsto r^{\frac{N-1}{p-1}}\varphi_1'(r) \colon \mathbb{R}_+ \to \mathbb{R}$ is continuous and decreasing, and satisfies $\varphi_1'(r) < 0$ for all $r > 0$.*

To determine the asymptotic behavior of $\varphi_1(r)$ as $r \to \infty$, we will investigate the corresponding nonlinear eigenvalue problem (2.2) in Appendix 9. Higher smoothness of $\varphi_1 \colon \mathbb{R}_+ \to (0,\infty)$ can be obtained directly by integrating equation (2.2): $\varphi_1 \in C^{1,\beta}(\mathbb{R}_+)$ with $\beta = \min\{1, \frac{1}{p-1}\}$. We refer to Manásevich and Takáč [15, Eq. (33)] for details.

2.3. **Notation.** The closure and boundary of a set $S \subset \mathbb{R}^N$ are denoted by $\overline{S}$ and ∂S, respectively. We denote by $B_\varrho \overset{\text{def}}{=} \{x \in \mathbb{R}^N \colon |x| < \varrho\}$ the ball of radius $0 < \varrho < \infty$.

All Banach and Hilbert spaces used in this article are real. Given an integer $k \geq 0$ and $0 \leq \alpha \leq 1$, we denote by $C^{k,\alpha}(\mathbb{R}^N)$ the linear space of all k-times continuously differentiable functions $u \colon \mathbb{R}^N \to \mathbb{R}$ whose all (classical) partial derivatives of order $\leq k$ are locally α-Hölder continuous on $\mathbb{R}^N$. As usual, we abbreviate $C^k(\mathbb{R}^N) \equiv C^{k,0}(\mathbb{R}^N)$. The linear subspace of $C^k(\mathbb{R}^N)$ consisting of all C^k functions $u \colon \mathbb{R}^N \to \mathbb{R}$ with compact support is denoted by $C_{\mathrm{c}}^k(\mathbb{R}^N)$.

For $1 < p < 2$ we denote by $\mathcal{D}_{\varphi_1}$ the normed linear space of all functions $u \in D^{1,2}(\mathbb{R}^N)$ whose norm

$$\|u\|_{\mathcal{D}_{\varphi_1}} \overset{\text{def}}{=} \Big(\int_{\mathbb{R}^N} |\varphi_1'(|x|)|^{p-2} |\nabla u(x)|^2\, \mathrm{d}x\Big)^{1/2} \tag{2.3}$$

is finite. Hence, the imbedding $\mathcal{D}_{\varphi_1} \hookrightarrow D^{1,2}(\mathbb{R}^N)$ is continuous. For $p = 2$ we set $\mathcal{D}_{\varphi_1} = D^{1,2}(\mathbb{R}^N)$. Finally, for $2 < p < N$ we define $\mathcal{D}_{\varphi_1}$ to be the completion of $D^{1,p}(\mathbb{R}^N)$ in the norm (2.3). Thus, the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow \mathcal{D}_{\varphi_1}$ is continuous.

For $1 < p < N$ we denote by $\mathcal{H}_{\varphi_1}$ the weighted Lebesgue space of all measurable functions $u \colon \mathbb{R}^N \to \mathbb{R}$ with the norm

$$\|u\|_{\mathcal{H}_{\varphi_1}} \overset{\text{def}}{=} \Big(\int_{\mathbb{R}^N} |u|^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x\Big)^{1/2} < \infty$$

and with the inner product

$$(u, v)_{\mathcal{H}_{\varphi_1}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} u\, v\, \varphi_1^{p-2}\, m\, \mathrm{d}x \quad \text{for } u, v \in \mathcal{H}_{\varphi_1}.$$

The imbedding $\mathcal{H}_{\varphi_1} \hookrightarrow L^p(\mathbb{R}^N; m)$ is continuous for $1 < p \le 2$, and $L^p(\mathbb{R}^N; m) \hookrightarrow \mathcal{H}_{\varphi_1}$ is continuous for $2 \le p < N$, by Lemma 4.2. The Hilbert spaces $\mathcal{D}_{\varphi_1}$ and $\mathcal{H}_{\varphi_1}$ will play an important role throughout this article.

We use the standard inner product in $L^2(\mathbb{R}^N)$ defined by $\langle u, v\rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} uv\, \mathrm{d}x$ for $u, v \in L^2(\mathbb{R}^N)$. This inner product induces a duality between the Lebesgue spaces $L^p(\mathbb{R}^N; m)$ and $L^{p'}(\mathbb{R}^N; m^{-1/(p-1)})$, where $1 \le p < \infty$ and $1 < p' \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and between the Sobolev space $D^{1,p}(\mathbb{R}^N)$ and its dual $D^{-1,p'}(\mathbb{R}^N)$, as well. Similarly, $\mathcal{D}'_{\varphi_1}$ ($\mathcal{H}'_{\varphi_1}$, respectively) stands for the dual space of $\mathcal{D}_{\varphi_1}$ ($\mathcal{H}_{\varphi_1}$). We keep the same notation also for the duality between the Cartesian products of such spaces.

2.4. **Linearization and quadratic forms.** As usual, I is the identity matrix from $\mathbb{R}^{N\times N}$, the tensor product $\mathbf{a} \otimes \mathbf{b}$ stands for the $(N \times N)$-matrix $\mathbf{T} = (a_i b_j)_{i,j=1}^N$ whenever $\mathbf{a} = (a_i)_{i=1}^N$ and $\mathbf{b} = (b_i)_{i=1}^N$ are vectors from $\mathbb{R}^N$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^N}$ denotes the Euclidean inner product in $\mathbb{R}^N$. We introduce the abbreviation

$$\mathbf{A}(\mathbf{a}) \stackrel{\text{def}}{=} |\mathbf{a}|^{p-2}\big(I + (p-2)\,\frac{\mathbf{a} \otimes \mathbf{a}}{|\mathbf{a}|^2}\big) \quad \text{for } \mathbf{a} \in \mathbb{R}^N \setminus \{\mathbf{0}\}. \tag{2.4}$$

We set $\mathbf{A}(\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0} \in \mathbb{R}^{N\times N}$ for all $1 < p < \infty$. For $\mathbf{a} \ne \mathbf{0}$, $\mathbf{A}(\mathbf{a})$ is a positive definite, symmetric matrix. The spectrum of the matrix $|\mathbf{a}|^{2-p}\mathbf{A}(\mathbf{a})$ consists of the eigenvalues 1 and $p-1$. For all $\mathbf{a}, \mathbf{v} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ we thus obtain

$$0 < \min\{1,\, p-1\} \le \frac{\langle \mathbf{A}(\mathbf{a})\mathbf{v}, \mathbf{v}\rangle_{\mathbb{R}^N}}{|\mathbf{a}|^{p-2}|\mathbf{v}|^2} \le \max\{1,\, p-1\}. \tag{2.5}$$

The following auxiliary inequalities are Lemma A.2 ($p \ge 2$) and Remark A.3 ($p < 2$) from Takáč [22, p. 235], respectively; their proofs are straightforward. First, for any $2 \le p < \infty$, there exists a constant $c_p > 0$, such that for arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{v} \in \mathbb{R}^N$ we have

$$\begin{aligned} c_p \cdot \big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}|\big)^{p-2}|\mathbf{v}|^2 &\le \int_0^1 \langle \mathbf{A}(\mathbf{a} + s\mathbf{b})\mathbf{v}, \mathbf{v}\rangle (1-s)\, \mathrm{d}s \\ &\le \frac{p-1}{2}\big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}|\big)^{p-2}|\mathbf{v}|^2. \end{aligned} \tag{2.6}$$

On the other hand, given any $1 < p < 2$, there exists a constant $c_p > 0$, such that for arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{v} \in \mathbb{R}^N$, with $|\mathbf{a}| + |\mathbf{b}| > 0$, we have

$$\begin{aligned} \frac{p-1}{2}\big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}|\big)^{p-2}|\mathbf{v}|^2 &\le \int_0^1 \langle \mathbf{A}(\mathbf{a} + s\mathbf{b})\mathbf{v}, \mathbf{v}\rangle (1-s)\, \mathrm{d}s \\ &\le c_p \cdot \big(\max_{0\le s\le 1} |\mathbf{a} + s\mathbf{b}|\big)^{p-2}|\mathbf{v}|^2. \end{aligned} \tag{2.7}$$

These inequalities are needed to treat the linearization of $-\Delta_p$ at φ_1 below.

Next, as in [22, Sect. 1], we rewrite the first and second terms of the energy functional $\mathcal{J}_{\lambda_1}$ using the integral forms of the first- and second-order Taylor formulas; we set

$$\mathcal{F}(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, \mathrm{d}x - \frac{\lambda_1}{p}\int_{\mathbb{R}^N} |u|^p\, m\, \mathrm{d}x, \quad u \in D^{1,p}(\mathbb{R}^N). \tag{2.8}$$

We need to treat the Taylor formulas for $p \geq 2$ and $1 < p < 2$ separately.
Case $p \geq 2$. Let $\phi \in D^{1,p}(\mathbb{R}^N)$ be arbitrary. We take advantage of eq. (1.4) to obtain $\mathcal{J}(\varphi_1) = 0$ and consequently

$$\begin{aligned}\mathcal{F}(\varphi_1 + \phi) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\, \mathcal{F}(\varphi_1 + s\phi)\,\mathrm{d}s \\ &= \int_0^1 \int_{\mathbb{R}^N} |\nabla(\varphi_1 + s\phi)|^{p-2}\nabla(\varphi_1 + s\phi)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \lambda_1 \int_0^1 \int_{\mathbb{R}^N} |\varphi_1 + s\phi|^{p-2}(\varphi_1 + s\phi)\phi\, m\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{2.9}$$

Similarly, applying (1.4) once again, i.e., $\mathcal{F}'(\varphi_1) = 0$, we get

$$\mathcal{F}(\varphi_1 + \phi) = \mathcal{Q}_\phi(\phi, \phi) \tag{2.10}$$

where $\mathcal{Q}_\phi$ is the symmetric bilinear form on the Cartesian product $[D^{1,p}(\mathbb{R}^N)]^2$ defined as follows: Given any fixed $\phi \in D^{1,p}(\mathbb{R}^N)$, we set

$$\begin{aligned}\mathcal{Q}_\phi(v, w) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \Big\langle \big[\int_0^1 \mathbf{A}(\nabla(\varphi_1 + s\phi))(1-s)\,\mathrm{d}s\big]\nabla v,\, \nabla w \Big\rangle_{\mathbb{R}^N} \mathrm{d}x \\ &\quad - \lambda_1(p-1)\int_{\mathbb{R}^N} \big[\int_0^1 |\varphi_1 + s\phi|^{p-2}(1-s)\,\mathrm{d}s\big] v\, w\, m\,\mathrm{d}x\end{aligned} \tag{2.11}$$

for all $v, w \in D^{1,p}(\mathbb{R}^N)$. In particular, when $v \equiv w$ in $\mathbb{R}^N$, one obtains the quadratic form $\mathcal{Q}_\phi(v, v)$. If also $\phi \equiv 0$ then

$$\mathcal{Q}_0(v, v) = \frac{1}{2}\int_{\mathbb{R}^N} \langle \mathbf{A}(\nabla\varphi_1)\nabla v,\, \nabla v\rangle_{\mathbb{R}^N}\ \mathrm{d}x - \frac{1}{2}\lambda_1(p-1)\int_{\mathbb{R}^N} v^2\, \varphi_1^{p-2}\, m\,\mathrm{d}x. \tag{2.12}$$

The imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow \mathcal{D}_{\varphi_1}$ being dense, we extend the domain of the symmetric bilinear form $\mathcal{Q}_0$ defined by (2.12) to all of $\mathcal{D}_{\varphi_1} \times \mathcal{D}_{\varphi_1}$; see e.g. Kato [12, Chapt. VI, §1.3, p. 313].

Note that, due to the radial symmetry of φ_1, formula (2.4) yields

$$\mathbf{A}(\nabla\varphi_1) = |\varphi_1'(r)|^{p-2}\big(I + (p-2)\,\frac{x\otimes x}{r^2}\big) \quad \text{with} \quad \nabla\varphi_1 = \varphi_1'(r)\,\frac{x}{r} \tag{2.13}$$

for every $x \in \mathbb{R}^N$ with $r = |x| > 0$. Furthermore, our definition (1.2) of λ_1 and eq. (2.10) guarantee $\mathcal{Q}_{t\phi}(\phi, \phi) \geq 0$ for all $t \in \mathbb{R}\setminus\{0\}$. Letting $t \to 0$ we arrive at

$$\mathcal{Q}_0(\phi, \phi) \geq 0 \quad \text{for all } \phi \in D^{1,p}(\mathbb{R}^N). \tag{2.14}$$

Case $1 < p < 2$. Since $\mathcal{D}_{\varphi_1} \hookrightarrow D^{1,p}(\mathbb{R}^N)$ in this case, given any fixed $\phi \in D^{1,p}(\mathbb{R}^N)$, we define the symmetric bilinear form $\mathcal{Q}_\phi$ on the Cartesian product $\mathcal{D}_{\varphi_1} \times \mathcal{D}_{\varphi_1}$ by formula (2.11). Notice that the first integral in (2.11) converges absolutely by inequality (2.7). The absolute convergence of the second integral in (2.11) is obtained by similar arguments using also the continuity of the imbedding $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$, by Lemma 4.4.

3. Main results

Recall that $1 < p < N$ throughout this article unless indicated otherwise.

3.1. **Statements of Theorems.** The following two theorems are the main results of our present article.

Theorem 3.1. *Let $2 \le p < N$. If $f \in \mathcal{D}'_{\varphi_1}$ satisfies $\langle f, \varphi_1 \rangle = 0$, then problem* (1.6) *possesses a weak solution $u \in D^{1,p}(\mathbb{R}^N)$.*

This is a part of the Fredholm alternative for $-\Delta_p$ at λ_1. The proof is given in Section 7. In a bounded domain $\Omega \subset \mathbb{R}^N$, this theorem is due to Fleckinger and Takáč [9, Theorem 3.3, p. 958].

The orthogonality condition $\langle f, \varphi_1 \rangle = 0$ is sufficient, but *not* necessary to obtain existence for problem (1.6) provided $p \ne 2$, according to recent results obtained in Drábek, Girg and Manásevich [4, Theorem 1.3] for $N = 1$, in Drábek and Holubová [5, Theorem 1.1] for any $N \ge 1$ and $1 < p < 2$, and in Takáč [23, Theorems 3.1 and 3.5] for any $N \ge 1$.

Example 3.2. For $2 \le p < N$, the hypothesis $f \in \mathcal{D}'_{\varphi_1}$ is fulfilled, for example, if $f = f_1 + f_2$ where $f_1 \in L^2(B_\varepsilon; m^{-1})$ and $f_1 \equiv 0$ in $\mathbb{R}^N \setminus B_\varepsilon$, and $f_2 \equiv 0$ in B_ε and $f_2 \in L^2\big(\mathbb{R}^N \setminus B_\varepsilon; r^{-N+\frac{N-p}{p-1}}\big)$ for some $0 < \varepsilon \le 1$. This claim follows from the imbeddings in Lemma 4.4 combined with the asymptotic formulas in Proposition 9.1, where $\mathcal{H}'_{\varphi_1} = L^2\big(\mathbb{R}^N; \varphi_1^{2-p} m^{-1}\big)$ is the dual space of $\mathcal{H}_{\varphi_1}$, and $L^2\left(\mathbb{R}^N; |\varphi_1'|^{-p} \varphi_1^2\right)$ is the dual space of $L^2\left(\mathbb{R}^N; |\varphi_1'|^p \varphi_1^{-2}\right)$.

Theorem 3.3. *Let $N \ge 2$ and $1 < p < 2$. Assume that $f^\# \in D^{-1,p'}(\mathbb{R}^N)$ satisfies $\langle f^\#, \varphi_1 \rangle = 0$ and $f^\# \not\equiv 0$ in $\mathbb{R}^N$. Then there exist two numbers $\delta \equiv \delta(f^\#) > 0$ and $\varrho \equiv \varrho(f^\#) > 0$ such that problem* (1.1) *with $f = f^\# + \zeta\, m\varphi_1^{p-1}$ has at least one solution whenever $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$ and $\zeta \in (-\varrho, \varrho)$.*

The proof of this theorem is given in Section 8.

Remark 3.4. In the situation of Theorem 3.3, if $\lambda \in (\lambda_1 - \delta, \lambda_1)$ and $\zeta \in (-\varrho, \varrho)$, then problem (1.1) has *at least three* solutions $u_1, u_2, u_3 \in D^{1,p}(\mathbb{R}^N)$, such that

$$\int_{\mathbb{R}^N} u_2\, \varphi_1^{p-1}\, m \,\mathrm{d}x < \int_{\mathbb{R}^N} u_1\, \varphi_1^{p-1}\, m \,\mathrm{d}x < \int_{\mathbb{R}^N} u_3\, \varphi_1^{p-1}\, m \,\mathrm{d}x,$$

u_1 is a saddle point (which will be obtained in the proof of Theorem 3.3) and u_2, u_3 are local minimizers for the functional $\mathcal{J}_\lambda$ on $D^{1,p}(\mathbb{R}^N)$. The proof of this claim is given in Section 8, §8.3, after the proof of Theorem 3.3.

Example 3.5. For $1 < p \le 2$, the hypothesis $f \in \mathcal{D}'_{\varphi_1}$ is fulfilled if $|x|\, f(x) \in L^{p'}(\mathbb{R}^N)$ with $p' = p/(p-1)$, by the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; |x|^{-p})$ in Lemma 4.1.

The proofs of both theorems above hinge on the following imbeddings with weights.

Proposition 3.6. *Let $1 < p < N$ and let hypothesis* (H) *be satisfied. Then the following two imbeddings are compact:*

(a) $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; m)$;
(b) $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$.

The proof of this proposition is given in Section 4. The reader is referred to Berger and Schechter [2, Proof of Theorem 2.4, p. 277], Fleckinger, Gossez, and de Thélin [6, Lemma 2.3], or Schechter [19, 20] for related imbeddings and compactness results.

3.2. **Properties of the corresponding energy functional.** Weak solutions in $D^{1,p}(\mathbb{R}^N)$ to the Dirichlet boundary value problem (1.6) with $f \in D^{-1,p'}(\mathbb{R}^N)$ correspond to critical points of the energy functional $\mathcal{J}_{\lambda_1}\colon D^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined in (1.3) with $\lambda = \lambda_1$. Owing to the imbeddings in Proposition 3.6, all expressions in (1.3) are meaningful. For the cases $2 \le p < N$ and $1 < p < 2 \le N$, the geometry of the functional $\mathcal{J}_{\lambda_1}$ is completely different; cf. Fleckinger and Takáč [9, Theorem 3.1, p. 957] and Drábek and Holubová [5, Theorem 1.1, p. 184], respectively, in a bounded domain $\Omega \subset \mathbb{R}^N$.

In the former case, we have the following analogue of the *improved Poincaré inequality* from [9, Theorem 3.1, p. 957], which is of independent interest.

Lemma 3.7. *Let $2 \le p < N$ and let hypothesis* (H) *be satisfied. Then there exists a constant $c \equiv c(p,m) > 0$ such that the inequality*

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x - \lambda_1 \int_{\mathbb{R}^N} |u|^p \, m(x) \,\mathrm{d}x \\ &\ge c\Big(|u^{\parallel}|^{p-2} \int_{\mathbb{R}^N} |\nabla \varphi_1(x)|^{p-2} |\nabla u^{\top}|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u^{\top}|^p \,\mathrm{d}x \Big) \end{aligned} \tag{3.1}$$

holds for all $u \in D^{1,p}(\mathbb{R}^N)$.

Here, a function $u \in D^{1,p}(\mathbb{R}^N)$ is decomposed as the direct sum (1.5). If the constant c in (3.1) is replaced by zero, one obtains the classical Poincaré inequality; see e.g. Gilbarg and Trudinger [11, Ineq. (7.44), p. 164]. In analogy with the case $p = 2$, the *improved Poincaré inequality* (3.1) guarantees the solvability of the Cauchy boundary value problem (1.6) in the special case when $f \in \mathcal{D}'_{\varphi_1}$ satisfies $\langle f, \varphi_1 \rangle = 0$.

On the other hand, the "singular" case $1 < p < 2 \le N$ is much different and has to be treated by a minimax method introduced in Takáč [22, Sect. 7]. It uses the fact that the functional $\mathcal{J}_{\lambda_1}$ still remains coercive on

$$D^{1,p}(\mathbb{R}^N)^{\top} \stackrel{\text{def}}{=} \Big\{ u \in D^{1,p}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} u\, \varphi_1^{p-1}\, m \,\mathrm{d}x = 0 \Big\}, \tag{3.2}$$

the complement of $\mathrm{lin}\{\varphi_1\}$ in $D^{1,p}(\mathbb{R}^N)$ with respect to the direct sum (1.5), viz. $D^{1,p}(\mathbb{R}^N) = \mathrm{lin}\{\varphi_1\} \oplus D^{1,p}(\mathbb{R}^N)^{\top}$.

The following notion introduced in Drábek and Holubová [5, Def. 2.1, p. 185] is crucial.

Definition 3.8. We say that a continuous functional $\mathcal{E}\colon D^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ has a *simple saddle point geometry* if we can find $u, v \in D^{1,p}(\mathbb{R}^N)$ such that

$$\begin{gathered} \int_{\mathbb{R}^N} u\, \varphi_1^{p-1}\, m \,\mathrm{d}x < 0 < \int_{\mathbb{R}^N} v\, \varphi_1^{p-1}\, m \,\mathrm{d}x \quad \text{and} \\ \max\{\mathcal{E}(u),\, \mathcal{E}(v)\} < \inf\big\{ \mathcal{E}(w)\colon w \in D^{1,p}(\mathbb{R}^N)^{\top} \big\}. \end{gathered}$$

Note that on any continuous path $\theta\colon [-1,1] \to D^{1,p}(\mathbb{R}^N)$ with $\theta(-1) = u$ and $\theta(1) = v$ there is a point $w = \theta(t_0) \in D^{1,p}(\mathbb{R}^N)^{\top}$ for some $t_0 \in [-1,1]$. Hence, $\max\{\mathcal{E}(u),\, \mathcal{E}(v)\} < \mathcal{E}(w)$ shows that the function $\mathcal{E} \circ \theta\colon [-1,1] \to \mathbb{R}$ attains its maximum at some $t' \in (-1,1)$.

The following result is essential; in fact it replaces Lemma 3.7. For a bounded domain $\Omega \subset \mathbb{R}^N$, it was shown in Drábek and Holubová [5, Lemma 2.1, p. 185].

Lemma 3.9. *Let $1 < p < 2 \le N$. Assume $f \in D^{-1,p'}(\mathbb{R}^N)$ with $\langle f, \varphi_1 \rangle = 0$ and $f \not\equiv 0$ in $\mathbb{R}^N$. Then the functional $\mathcal{J}_{\lambda_1}$ has a simple saddle point geometry. Moreover, it is unbounded from below on $D^{1,p}(\mathbb{R}^N)$.*

Its proof will be given in Section 8, §8.1.

For $1 < p < 2$ we will obtain a weak solution to problem (1.1) by showing that the "minimax" (or rather "maximin") expression

$$\beta_\lambda \stackrel{\text{def}}{=} \sup_{a<\tau<b} \inf_{u^\top \in D^{1,p}(\mathbb{R}^N)^\top} \mathcal{J}_\lambda(\tau\varphi_1 + u^\top) \tag{3.3}$$

provides a critical value β_λ for the energy functional $\mathcal{J}_\lambda$ defined in (1.3). Here a, b ($-\infty < a < 0 < b < \infty$) are provided by the simple saddle point geometry of $\mathcal{J}_{\lambda_1}$ established in Lemma 3.9 above. Formula (3.3) is justified by Lemma 6.2 (§6.1) whenever $-\infty < \lambda < \Lambda_\infty$ and $f \in D^{-1,p'}(\mathbb{R}^N)$. We will provide a simple sufficient condition for the criticality of β_λ in Lemma 8.3 (§8.2). This condition is verified in the setting of our Theorem 3.3 as a consequence of Lemma 3.9.

4. Proof of Proposition 3.6

To prove this proposition, we need a few preliminary results.

4.1. **Some imbeddings with weights.** We begin with the classical Hardy inequality (Kufner [13, Theorem 5.2, p. 28]) which reads

$$\int_{\mathbb{R}^N} \Big(\frac{|u(x)|}{|x|}\Big)^p \,\mathrm{d}x \le \Big(\frac{p}{N-p}\Big)^p \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x, \quad u \in D^{1,p}(\mathbb{R}^N). \tag{4.1}$$

In particular, the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; |x|^{-p})$ is continuous.

Next, we show the continuity of some more imbeddings.

Lemma 4.1. *Let $1 < p < N$ and let hypothesis* (H) *be satisfied. Then the following imbeddings are continuous:*

$$D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; |x|^{-p}) \hookrightarrow L^p(\mathbb{R}^N; m); \tag{4.2}$$

$$D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; m), \tag{4.3}$$

where $p^* = Np/(N-p)$ denotes the critical Sobolev exponent.

Proof. The imbedding $L^p(\mathbb{R}^N; |x|^{-p}) \hookrightarrow L^p(\mathbb{R}^N; m)$ follows from inequality (2.1).

By a classical result (Gilbarg and Trudinger [11, Theorem 7.10, p. 166]), the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ is continuous. Notice that $(p/p^*) + (p/N) = 1$. Finally, given an arbitrary function $u \in C_c^0(\mathbb{R}^N)$, we combine the Hölder inequality with (2.1) to estimate

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^p\, m \,\mathrm{d}x &\le \Big(\int_{\mathbb{R}^N} |u|^{p^*} \,\mathrm{d}x\Big)^{p/p^*} \Big(\int_{\mathbb{R}^N} m^{N/p} \,\mathrm{d}x\Big)^{p/N} \\ &\le C\Big(\int_{\mathbb{R}^N} |u|^{p^*} \,\mathrm{d}x\Big)^{p/p^*} \Big(\int_{\mathbb{R}^N} (1+|x|)^{-N(1+\frac{\delta}{p})} \,\mathrm{d}x\Big)^{p/N}.\end{aligned}$$

The continuity of the imbedding $L^{p^*}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; m)$ follows because $C_c^0(\mathbb{R}^N)$ is dense in $L^{p^*}(\mathbb{R}^N)$. □

Lemma 4.2. *Let hypothesis* (H) *be satisfied. Then we have the following imbeddings:*

(i) $\mathcal{H}_{\varphi_1} \hookrightarrow L^p(\mathbb{R}^N; m)$ *if* $1 < p < 2$*;*

(ii) $\mathcal{H}_{\varphi_1} = L^2(\mathbb{R}^N; m)$ *if* $p = 2$;
(iii) $L^p(\mathbb{R}^N; m) \hookrightarrow \mathcal{H}_{\varphi_1}$ *if* $2 < p < N$.

Proof. We need to distinguish between the cases $1 < p < 2$ and $2 < p < N$.
Case $p < 2$. Let $u \in C_c^0(\mathbb{R}^N)$ be arbitrary. We apply Hölder's inequality again to estimate

$$\int_{\mathbb{R}^N} |u|^p \, m \, \mathrm{d}x \le \Big(\int_{\mathbb{R}^N} u^2 \, \varphi_1^{p-2} \, m \, \mathrm{d}x \Big)^{p/2} \Big(\int_{\mathbb{R}^N} \varphi_1^p \, m \, \mathrm{d}x \Big)^{(2-p)/2} = \|u\|_{\mathcal{H}_{\varphi_1}}^p ,$$

by $\int_{\mathbb{R}^N} \varphi_1^p \, m \, \mathrm{d}x = 1$. The space $C_c^0(\mathbb{R}^N)$ being dense in $\mathcal{H}_{\varphi_1}$, the imbedding in Part (i) follows.

Case $p > 2$. As above, for $u \in C_c^0(\mathbb{R}^N)$ we estimate

$$\int_{\mathbb{R}^N} u^2 \, \varphi_1^{p-2} \, m \, \mathrm{d}x \le \Big(\int_{\mathbb{R}^N} |u|^p \, m \, \mathrm{d}x \Big)^{2/p} \Big(\int_{\mathbb{R}^N} \varphi_1^p \, m \, \mathrm{d}x \Big)^{(p-2)/p} = \|u\|_{L^p(\mathbb{R}^N; m)}^2 .$$

The lemma is proved. □

Lemma 4.3. *Let hypothesis* (H) *be satisfied. The following imbeddings hold true:*
(i) $\mathcal{D}_{\varphi_1} \hookrightarrow D^{1,p}(\mathbb{R}^N)$ *if* $1 < p < 2$;
(ii) $\mathcal{D}_{\varphi_1} = D^{1,2}(\mathbb{R}^N)$ *if* $p = 2$;
(iii) $D^{1,p}(\mathbb{R}^N) \hookrightarrow \mathcal{D}_{\varphi_1}$ *if* $2 < p < N$.

Proof. Again, we distinguish between the cases $1 < p < 2$ and $2 < p < N$.
Case $p < 2$. Let $u \in C_c^1(\mathbb{R}^N)$ be arbitrary. Hölder's inequality yields

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u|^p \, |\varphi_1'(r)|^{(p-2)p/2} \, |\varphi_1'(r)|^{-(p-2)p/2} \, \mathrm{d}x \\ &\le \Big(\int_{\mathbb{R}^N} |\nabla u|^2 \, |\varphi_1'|^{p-2} \, \mathrm{d}x \Big)^{p/2} \Big(\int_{\mathbb{R}^N} |\varphi_1'|^p \, \mathrm{d}x \Big)^{(2-p)/2} \\ &= \lambda_1^{(2-p)/2} \, \|u\|_{\mathcal{D}_{\varphi_1}}^p , \end{aligned}$$

by $\int_{\mathbb{R}^N} |\varphi_1'|^p \, \mathrm{d}x = \lambda_1$. The desired imbedding in Part (i) now follows from the density of $C_c^1(\mathbb{R}^N)$ in $\mathcal{D}_{\varphi_1}$.
Case $p > 2$. Given $u \in C_c^0(\mathbb{R}^N)$, we estimate

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^2 \, |\varphi_1'(r)|^{p-2} \, \mathrm{d}x &\le \Big(\int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x \Big)^{2/p} \Big(\int_{\mathbb{R}^N} |\varphi_1'|^p \, \mathrm{d}x \Big)^{(p-2)/p} \\ &= \lambda_1^{(p-2)/p} \, \|u\|_{\mathcal{D}_{\varphi_1}}^2 . \end{aligned}$$

This proves the lemma. □

Lemma 4.4. *Let* $1 < p < N$ *and let hypothesis* (H) *be satisfied. Then both imbeddings* $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$ *and* $\mathcal{D}_{\varphi_1} \hookrightarrow L^2\left(\mathbb{R}^N; |\varphi_1'|^p \varphi_1^{-2}\right)$ *are continuous.*

Proof. We need to distinguish between the cases $1 < p < 2$ and $2 \le p < N$.

Case $p < 2$. Since $\mathcal{D}_{\varphi_1} \hookrightarrow D^{1,2}(\mathbb{R}^N)$ with $\varphi_1'(r) \, r^{\frac{N-1}{p-1}} \to -\frac{N-p}{p-1} c$ as $r \to \infty$, by (9.4), the linear subspace of $\mathcal{D}_{\varphi_1}$ consisting of all functions with compact support is dense in $\mathcal{D}_{\varphi_1}$. So take an arbitrary function $u \in \mathcal{D}_{\varphi_1}$ with compact support. Using $u \in D^{1,2}(\mathbb{R}^N)$ and the properties of φ_1, we deduce that both integrals below converge:

$$\int_{\mathbb{R}^N} u^2 \, |\varphi_1'|^p \, \varphi_1^{-2} \, \mathrm{d}x < \infty \quad \text{and} \quad \int_{\mathbb{R}^N} u^2 \, \varphi_1^{p-2} \, m \, \mathrm{d}x < \infty .$$

Consequently, we are allowed to apply the weak formulation of the eigenvalue problem (1.4) with the test function $u^2/\varphi_1 \in D^{1,1}(\mathbb{R}^N)$ to compute

$$\begin{aligned}
&\lambda_1 \int_{\mathbb{R}^N} u(x)^2\, \varphi_1(r)^{p-2}\, m(r)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} u(x)^2\, \varphi_1(r)^{-1}(-\Delta_p \varphi_1)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} |\varphi_1'(r)|^{p-2}\, \varphi_1'(r)\, \frac{x}{r} \cdot \nabla\big(\frac{u^2}{\varphi_1}\big)\, \mathrm{d}x \\
&= 2 \int_{\mathbb{R}^N} |\varphi_1'|^{p-2}\, \varphi_1'\, \frac{\partial u}{\partial r}\, \frac{u}{\varphi_1}\, \mathrm{d}x - \int_{\mathbb{R}^N} |\varphi_1'|^{p-2}\, \varphi_1'\, \frac{\partial \varphi_1}{\partial r}\, \big(\frac{u}{\varphi_1}\big)^2\, \mathrm{d}x.
\end{aligned}$$

Adding the last integral and estimating the second last one by the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}
&\lambda_1 \int_{\mathbb{R}^N} u^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x + \int_{\mathbb{R}^N} u^2\, |\varphi_1'|^p\, \varphi_1^{-2}\, \mathrm{d}x \\
&\le 2 \Big(\int_{\mathbb{R}^N} |\varphi_1'|^{p-2} \big(\frac{\partial u}{\partial r}\big)^2\, \mathrm{d}x \Big)^{1/2} \Big(\int_{\mathbb{R}^N} u^2\, |\varphi_1'|^p\, \varphi_1^{-2}\, \mathrm{d}x \Big)^{1/2} \\
&\le 2 \int_{\mathbb{R}^N} |\varphi_1'|^{p-2} \big(\frac{\partial u}{\partial r}\big)^2\, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} u^2\, |\varphi_1'|^p\, \varphi_1^{-2}\, \mathrm{d}x,
\end{aligned} \tag{4.4}$$

and therefore,

$$\begin{aligned}
&\lambda_1 \int_{\mathbb{R}^N} u^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} u^2\, |\varphi_1'|^p\, \varphi_1^{-2}\, \mathrm{d}x \\
&\le 2 \int_{\mathbb{R}^N} |\varphi_1'|^{p-2}\, \big(\frac{\partial u}{\partial r}\big)^2\, \mathrm{d}x \le 2\, \|u\|^2_{\mathcal{D}_{\varphi_1}}.
\end{aligned} \tag{4.5}$$

It follows that both imbeddings $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$ and $\mathcal{D}_{\varphi_1} \hookrightarrow L^2\left(\mathbb{R}^N; |\varphi_1'|^p \varphi_1^{-2}\right)$ are continuous.

Case $p \ge 2$. The linear space $C^1_{\rm c}(\mathbb{R}^N)$ is dense in both $D^{1,p}(\mathbb{R}^N)$ and $\mathcal{D}_{\varphi_1}$, by definition. So take an arbitrary function $u \in C^1_{\rm c}(\mathbb{R}^N)$. The same procedure as for $p < 2$ above leads us to the inequalities in (4.5). Again, both imbeddings $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$ and $\mathcal{D}_{\varphi_1} \hookrightarrow L^2\left(\mathbb{R}^N; |\varphi_1'|^p \varphi_1^{-2}\right)$ are continuous. The lemma is proved. $\square$

Next we will show that our hypothesis (H) guarantees also the compactness of both imbeddings $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; m)$ and $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$ for $1 < p < N$. In order to prove this compactness, given any $\varrho \in (0, \infty)$, we introduce a cut-off function $\psi_\varrho \colon \mathbb{R}_+ \to [0,1]$ as follows: Take any C^1 function $\psi_1 \colon \mathbb{R}_+ \to [0,1]$ such that $\psi_1(r) = 1$ for $0 \le r \le 1$, $\psi_1(r) = 0$ for $2 \le r < \infty$, and $\psi_1'(r) \le 0$ for $1 \le r \le 2$. We define $\psi_\varrho(x) \equiv \psi_\varrho(r) \overset{\text{def}}{=} \psi_1(r/\varrho)$ for all $x \in \mathbb{R}^N$ and $r = |x|$. Notice that its radial derivative $\psi_\varrho'(r) = (1/\varrho)\, \psi_1'(r/\varrho)$ satisfies

$$|\psi_\varrho'(r)| \le C_1\, r^{-1} \quad \text{for all } r \ge 0, \tag{4.6}$$

where $C_1 = 2 \cdot \sup_{\mathbb{R}_+} |\psi_1'| < \infty$ is a constant. Obviously, $\psi_\varrho(r) = 1$ for $0 \le r \le \varrho$, $\psi_\varrho(r) = 0$ for $2\varrho \le r < \infty$, and $\psi_\varrho'(r) \le 0$ for $\varrho \le r \le 2\varrho$. Now we define the corresponding cut-off operator $T_\varrho \colon L^1_{\rm loc}(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ by $T_\varrho u \overset{\text{def}}{=} \psi_\varrho u$ for all $u \in L^1_{\rm loc}(\mathbb{R}^N)$. These linear operators are uniformly bounded from $D^{1,p}(\mathbb{R}^N)$ ($\mathcal{D}_{\varphi_1}$, respectively) into itself for all $\varrho > 0$ sufficiently large as is shown in the following lemma.

Lemma 4.5. *Let $1 < p < N$ and let hypothesis* (H) *be satisfied. Then there exist constants $C_2 > 0$, $C_3 > 0$ and $R_1 > 0$, such that for all $\varrho \geq R_1$ we have*

$$\|\psi_\varrho u\|_{D^{1,p}(\mathbb{R}^N)} \leq C_2 \|u\|_{D^{1,p}(\mathbb{R}^N)} \quad \textit{for all } u \in D^{1,p}(\mathbb{R}^N); \tag{4.7}$$

$$\|\psi_\varrho u\|_{\mathcal{D}_{\varphi_1}} \leq C_3 \|u\|_{\mathcal{D}_{\varphi_1}} \quad \textit{for all } u \in \mathcal{D}_{\varphi_1}. \tag{4.8}$$

Proof. We give the proof for the case $1 < p < 2$ only and leave minor changes for $2 \leq p < N$ to the reader. Let $\varrho > 0$. For an arbitrary function $u \in D^{1,p}(\mathbb{R}^N)$ we have

$$\nabla(\psi_\varrho u) = \psi_\varrho(r)\,\nabla u(x) + u(x)\,\psi_\varrho'(r)\,r^{-1}x \quad \text{for } x \in \mathbb{R}^N \text{ and } r = |x|.$$

Therefore, by the Minkowski inequality followed by (4.6) and the Hardy inequality (4.1), we have

$$\begin{aligned}
\|\psi_\varrho u\|_{D^{1,p}(\mathbb{R}^N)} &= \Big(\int_{\mathbb{R}^N} |\nabla(\psi_\varrho u)|^p \,\mathrm{d}x\Big)^{1/p} \\
&\leq \Big(\int_{\mathbb{R}^N} |\psi_\varrho|^p |\nabla u|^p \,\mathrm{d}x\Big)^{1/p} + \Big(\int_{\mathbb{R}^N} |\psi_\varrho'|^p |u|^p \,\mathrm{d}x\Big)^{1/p} \\
&\leq \|u\|_{D^{1,p}(\mathbb{R}^N)} + C_1 \Big(\int_{\mathbb{R}^N} |u(x)|^p\,|x|^{-p} \,\mathrm{d}x\Big)^{1/p} \\
&\leq C_2\,\|u\|_{D^{1,p}(\mathbb{R}^N)},
\end{aligned} \tag{4.9}$$

where $C_2 = 1 + pC_1/(N-p)$. This proves (4.7).

Similarly, for every $u \in \mathcal{D}_{\varphi_1}$ we have

$$\begin{aligned}
\|\psi_\varrho u\|_{\mathcal{D}_{\varphi_1}} &= \Big(\int_{\mathbb{R}^N} |\varphi_1'|^{p-2} |\nabla(\psi_\varrho u)|^2 \,\mathrm{d}x\Big)^{1/2} \\
&\leq \Big(\int_{\mathbb{R}^N} |\varphi_1'|^{p-2} |\psi_\varrho|^2 |\nabla u|^2 \,\mathrm{d}x\Big)^{1/2} + \Big(\int_{\mathbb{R}^N} |\varphi_1'|^{p-2} |\psi_\varrho'|^2 u^2 \,\mathrm{d}x\Big)^{1/2} \\
&\leq \|u\|_{\mathcal{D}_{\varphi_1}} + \Big(\int_{\mathbb{R}^N} |\varphi_1'|^{p-2} |\psi_\varrho'|^2 u^2 \,\mathrm{d}x\Big)^{1/2}.
\end{aligned} \tag{4.10}$$

The last integral is estimated as follows. Using the limit formula (9.21) we have

$$\varphi_1^{-1}|\varphi_1'| \geq \frac{N-p}{2(p-1)r} \quad \text{for all } r \geq R_1, \tag{4.11}$$

where $R_1 > 0$ is a sufficiently large constant. We combine this inequality with (4.6) to conclude that

$$|\psi_\varrho'(r)| \leq C_4\,\varphi_1^{-1}|\varphi_1'| \quad \text{for all } r \geq R_1, \tag{4.12}$$

where $C_4 = 2(p-1)C_1/(N-p)$. Applying this estimate to the last integral in (4.10), and recalling $\psi_\varrho'(r) = 0$ whenever $0 \leq r \leq \varrho$, for every $\varrho \geq R_1$ we get

$$\|\psi_\varrho u\|_{\mathcal{D}_{\varphi_1}} \leq \|u\|_{\mathcal{D}_{\varphi_1}} + C_4 \Big(\int_{\mathbb{R}^N} |\varphi_1'|^p |\varphi_1|^{-2} u^2 \,\mathrm{d}x\Big)^{1/2}.$$

Finally, we invoke inequality (4.5) to estimate the last integral. The desired estimate (4.8) follows with the constant $C_3 > 0$ given by $C_3 = 1 + 2C_4$. □

Denoting by J (J_{φ_1}, respectively) the continuous imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; m)$ ($\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$), we now show that the operators

$$JT_\varrho \colon D^{1,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N; m) \qquad (J_{\varphi_1}T_\varrho \colon \mathcal{D}_{\varphi_1} \to \mathcal{H}_{\varphi_1})$$

converge to J (J_{φ_1}) in the uniform operator topology as $\varrho \to \infty$.

Lemma 4.6. *Let $1 < p < N$ and let hypothesis* (H) *be satisfied. Then, as $\varrho \to \infty$, we have*

$$\|(1-\psi_\varrho)u\|_{L^p(\mathbb{R}^N;m)} \to 0 \quad \textit{uniformly for} \quad \|u\|_{D^{1,p}(\mathbb{R}^N)} \le 1; \tag{4.13}$$

$$\|(1-\psi_\varrho)u\|_{\mathcal{H}_{\varphi_1}} \to 0 \quad \textit{uniformly for} \quad \|u\|_{\mathcal{D}_{\varphi_1}} \le 1. \tag{4.14}$$

Proof. From hypothesis (H) we get

$$m(r)\, r^p \le \frac{C\, r^p}{(1+r)^{p+\delta}} < \frac{C}{(1+r)^\delta} \quad \text{for all } r > 0.$$

Hence, for any $\varrho > 0$,

$$\begin{aligned}\int_{|x|\ge\varrho} |u|^p\, m \,\mathrm{d}x &\le \frac{C}{(1+\varrho)^\delta} \int_{|x|\ge\varrho} |u|^p\, |x|^{-p} \,\mathrm{d}x \\ &\le \frac{C}{(1+\varrho)^\delta} \big(\frac{p}{N-p}\big)^p \|u\|^p_{D^{1,p}(\mathbb{R}^N)},\end{aligned}$$

by the Hardy inequality (4.1). Letting $\varrho \to \infty$ we obtain the convergence (4.13).

Similarly as above, we combine hypothesis (H) and inequality (4.11) to compare the weights

$$\frac{\varphi_1(r)^{p-2}\, m(r)}{|\varphi_1'(r)|^p\, \varphi_1(r)^{-2}} \le \frac{C_5\, r^p}{(1+r)^{p+\delta}} < \frac{C_5}{(1+r)^\delta} \quad \text{for all } r \ge R_1,$$

where

$$C_5 = \big(\frac{2(p-1)}{N-p}\big)^p C.$$

We use this inequality to estimate the second integral on the left-hand side in (4.5), thus arriving at

$$\lambda_1 \int_{\mathbb{R}^N} u^2\, \varphi_1^{p-2}\, m \,\mathrm{d}x + \frac{(1+\varrho)^\delta}{2C_5} \int_{|x|\ge\varrho} u^2\, \varphi_1^{p-2}\, m \,\mathrm{d}x \le 2\, \|u\|^2_{\mathcal{D}_{\varphi_1}}$$

for every $\varrho \ge R_1$. Letting $\varrho \to \infty$ we obtain the conclusion (4.14) immediately. $\square$

4.2. **Rest of the proof of Proposition 3.6.** According to Lemmas 4.1 and 4.4, it remains to show that the imbeddings $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; m)$ and $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$ are compact. We take advantage of the well-known approximation theorem (see Kato [12, Chapt. III, §4.2, p. 158]) which states that the set of all compact linear operators $S\colon X \to Y$, where X and Y are Banach spaces, is a Banach space. In our setting this means that, by Lemma 4.6, it suffices to show that the operators

$$JT_\varrho\colon D^{1,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N; m) \quad \text{and} \quad J_{\varphi_1} T_\varrho\colon \mathcal{D}_{\varphi_1} \to \mathcal{H}_{\varphi_1},$$

respectively, are compact for each $\varrho > 0$ large enough.

Recall $B_r = \{x \in \mathbb{R}^N\colon |x| < r\}$ for $0 < r < \infty$. A function $u \in L^2(B_r)$ or $u \in L^2(\mathbb{R}^N \setminus B_r; m)$, respectively, is naturally extended to all of $\mathbb{R}^N$ by setting $u(x) = 0$ for all $x \in \mathbb{R}^N \setminus B_r$ or for all $x \in B_r$. We observe that

$$W_0^{1,p}(B_r) = \{u \in D^{1,p}(\mathbb{R}^N)\colon u = 0 \text{ almost everywhere in } \mathbb{R}^N \setminus B_r\}$$

and set

$$\mathcal{D}_{\varphi_1}(B_r) \stackrel{\text{def}}{=} \{u \in \mathcal{D}_{\varphi_1}\colon u = 0 \text{ almost everywhere in } \mathbb{R}^N \setminus B_r\}.$$

Clearly, $W_0^{1,p}(B_r)$ is a closed linear subspace of $D^{1,p}(\mathbb{R}^N)$ and the same is true of $\mathcal{D}_{\varphi_1}(B_r)$ in $\mathcal{D}_{\varphi_1}$.

Proof of Part (a). By Lemma 4.5, the cut-off operators

$$T_\varrho\colon D^{1,p}(\mathbb{R}^N) \to W_0^{1,p}(B_{2\varrho}) \subset D^{1,p}(\mathbb{R}^N)$$

are uniformly bounded for all $\varrho \geq R_1$. Furthermore, the imbedding $W_0^{1,p}(B_{2\varrho}) \hookrightarrow L^p(B_{2\varrho})$ being compact by Rellich's theorem, and $L^p(B_{2\varrho}) \hookrightarrow L^p(\mathbb{R}^N; m)$ being continuous by (2.1), we conclude that $JT_\varrho\colon D^{1,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N; m)$ is compact as well, whenever $\varrho \geq R_1$.

Proof of Part (b). We need to treat the two cases $1 < p < 2$ and $2 \leq p < N$ separately.

Case $p < 2$. By Lemma 4.5, the operators $T_\varrho\colon \mathcal{D}_{\varphi_1} \to \mathcal{D}_{\varphi_1}(B_{2\varrho}) \subset \mathcal{D}_{\varphi_1}$ are uniformly bounded for all $\varrho \geq R_1$. Furthermore, the imbedding $\mathcal{D}_{\varphi_1}(B_{2\varrho}) \hookrightarrow W_0^{1,2}(B_{2\varrho})$ is continuous by $\gamma_1 \stackrel{\text{def}}{=} \inf_{(0,\infty)} |\varphi_1'|^{p-2} > 0$. Finally, the imbedding $W_0^{1,2}(B_{2\varrho}) \hookrightarrow L^2(B_{2\varrho})$ being compact by Rellich's theorem, and $L^2(B_{2\varrho}) \hookrightarrow \mathcal{H}_{\varphi_1}$ being continuous by (2.1), we conclude that $J_{\varphi_1} T_\varrho\colon \mathcal{D}_{\varphi_1} \to \mathcal{H}_{\varphi_1}$ is compact as well, whenever $\varrho \geq R_1$.

Case $p \geq 2$. First, taking an arbitrary function $u \in C^1(\mathbb{R}^N)$ with compact support, we derive inequalities (4.4) and (4.5). In particular, inequalities in (4.4) entail

$$\begin{aligned} \lambda_1 \int_{\mathbb{R}^N} u^2\, \varphi_1^{p-2}\, m \,\mathrm{d}x &\leq 2\Big(\int_{\mathbb{R}^N} |\varphi_1'|^{p-2} \big(\frac{\partial u}{\partial r}\big)^2 \,\mathrm{d}x \Big)^{1/2} \Big(\int_{\mathbb{R}^N} u^2\, |\varphi_1'|^p\, \varphi_1^{-2} \,\mathrm{d}x \Big)^{1/2} \\ &\leq 2\, \|u\|_{\mathcal{D}_{\varphi_1}} \Big(\int_{\mathbb{R}^N} u^2\, |\varphi_1'|^p\, \varphi_1^{-2} \,\mathrm{d}x \Big)^{1/2}. \end{aligned} \tag{4.15}$$

We need to show that, besides inequalities (4.5), we have also

$$\int_{B_R} |\varphi_1'|^p\, \varphi_1^{-2}\, u^2 \,\mathrm{d}x \leq 9 \cdot \log \left(\frac{\varphi_1(0)}{\varphi_1(R)} \right) \cdot \|u\|^2_{\mathcal{D}_{\varphi_1}} \quad \text{for every } R > 0. \tag{4.16}$$

To this end, fix any $x' \in \mathbb{R}^N$ with $|x'| = 1$, and take $x = rx'$ with $0 \leq r \leq R$. We use eq. (2.2) to compute

$$\begin{aligned} &r^{N-1}\, |\varphi_1'(r)|^{p-1}\, \varphi_1(r)^{-1}\, u(rx')^2 = -\big(r^{N-1}\, |\varphi_1'|^{p-2} \varphi_1' \big)\, \varphi_1^{-1}\, u^2 \\ &= -\int_0^r \frac{\partial}{\partial s} \left[s^{N-1}\, |\varphi_1'(s)|^{p-2} \varphi_1(s)'\, \varphi_1(s)^{-1}\, u(sx')^2 \right] \mathrm{d}s \\ &= \lambda_1 \int_0^r m(s)\, s^{N-1}\, \varphi_1(s)^{p-2}\, u(sx')^2 \,\mathrm{d}s \\ &\quad + \int_0^r s^{N-1}\, |\varphi_1'(s)|^p\, \varphi_1(s)^{-2}\, u(sx')^2 \,\mathrm{d}s \\ &\quad + 2 \int_0^r s^{N-1}\, |\varphi_1'(s)|^{p-1}\, \varphi_1(s)^{-1}\, u(sx')\, \frac{\partial u}{\partial s}(sx') \,\mathrm{d}s. \end{aligned}$$

Estimating the last integral by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} r^{N-1}\,|\varphi_1'(r)|^{p-1}\,\varphi_1(r)^{-1}\,u(rx')^2 \le \lambda_1 \int_0^r m(s)\,\varphi_1(s)^{p-2}\,u(sx')^2\,s^{N-1}\,\mathrm{d}s \\ + 2\int_0^r |\varphi_1'(s)|^p\,\varphi_1(s)^{-2}\,u(sx')^2\,s^{N-1}\,\mathrm{d}s \\ + \int_0^r |\varphi_1'(s)|^{p-2}\,\Big(\frac{\partial u}{\partial s}(sx')\Big)^2\,s^{N-1}\,\mathrm{d}s. \end{aligned}$$

Next, setting $y = sx'$, we integrate this inequality with respect to x' over the unit sphere $S_1 = \partial B_1 \subset \mathbb{R}^N$ endowed with the surface measure σ to get

$$\begin{aligned} & r^{N-1}\,|\varphi_1'(r)|^{p-1}\,\varphi_1(r)^{-1}\int_{S_1} u(rx')^2\,\mathrm{d}\sigma(x') \\ & \le \lambda_1\int_{B_r} u^2\,\varphi_1^{p-2}\,m\,\mathrm{d}y + 2\int_{B_r} u^2\,|\varphi_1'|^p\,\varphi_1^{-2}\,\mathrm{d}y \\ & \quad + \int_{B_r} |\varphi_1'|^{p-2}\,\big(\frac{\partial u}{\partial s}\big)^2\,\mathrm{d}y \le 8\,\|u\|_{\mathcal{D}_{\varphi_1}}^2 + \|u\|_{\mathcal{D}_{\varphi_1}}^2 = 9\,\|u\|_{\mathcal{D}_{\varphi_1}}^2, \end{aligned} \tag{4.17}$$

by ineq. (4.5). Finally, upon multiplication by $-\varphi_1'/\varphi_1$ followed by integration over $0 \le r \le R$, we arrive at the desired inequality (4.16).

Again, by Lemma 4.5, the operators $T_\varrho\colon \mathcal{D}_{\varphi_1} \to \mathcal{D}_{\varphi_1}(B_{2\varrho}) \subset \mathcal{D}_{\varphi_1}$ are uniformly bounded for all $\varrho \ge R_1$. In order to show that $JT_\varrho\colon \mathcal{D}_{\varphi_1} \to \mathcal{H}_{\varphi_1}$ is compact, it suffices to verify that the imbedding $\mathcal{D}_{\varphi_1}(B_{2\varrho}) \hookrightarrow \mathcal{H}_{\varphi_1}$ is compact. So let $\varrho \ge R_1$ be fixed.

Consider an arbitrary bounded sequence $\{u_n\}_{n=1}^\infty$ in the Hilbert space $\mathcal{D}_{\varphi_1}(B_{2\varrho})$. Hence, there exists a weakly convergent subsequence denoted again by $\{u_n\}_{n=1}^\infty$, i.e., $u_n \rightharpoonup u$ in $\mathcal{D}_{\varphi_1}(B_{2\varrho})$ as $n \to \infty$. Replacing $u_n - u$ by u_n, we may assume $u_n \rightharpoonup 0$ weakly in $\mathcal{D}_{\varphi_1}(B_{2\varrho})$. In addition, we may assume $\|u_n\|_{\mathcal{D}_{\varphi_1}} \le 1$ for all $n = 1, 2, \dots$. Next, we show that $u_n \to 0$ strongly in $L^2\big(B_{2\varrho}; |\varphi_1'|^p\varphi_1^{-2}\big)$. Choose $\varepsilon > 0$. Fix $R_0 > 0$ small enough, such that

$$9\cdot\log\big(\frac{\varphi_1(0)}{\varphi_1(R_0)}\big) \le \frac{\varepsilon}{2},$$

by $\lim_{r\to 0}\varphi_1(r) = \varphi_1(0) > 0$. Hence, inequality (4.16) entails

$$\int_{B_{R_0}} |\varphi_1'|^p\,\varphi_1^{-2}\,u_n^2\,\mathrm{d}x \le \frac{\varepsilon}{2} \quad \text{for } n = 1, 2, \dots. \tag{4.18}$$

Since $\gamma_2 \overset{\text{def}}{=} \inf_{[R_0,2\varrho]} |\varphi_1'|^{p-2} > 0$, by Lemma 2.2, the sequence $\{u_n\}_{n=1}^\infty$ is bounded in the Sobolev space $W^{1,2}(B_{2\varrho} \setminus B_{R_0})$, by inequalities (4.5). The imbedding $W^{1,2}(B_{2\varrho} \setminus B_{R_0}) \hookrightarrow L^2(B_{2\varrho} \setminus B_{R_0})$ being compact by Rellich's theorem, we conclude that $u_n \to 0$ strongly in $L^2(B_{2\varrho} \setminus B_{R_0})$. Consequently, there is an integer $n_0 \ge 1$ large enough, such that

$$\int_{B_{2\varrho}\setminus B_{R_0}} |\varphi_1'|^p\,\varphi_1^{-2}\,u_n^2\,\mathrm{d}x \le \frac{\varepsilon}{2} \quad \text{for every } n \ge n_0. \tag{4.19}$$

We combine estimates (4.18) and (4.19) to obtain

$$\int_{B_{2\varrho}} |\varphi_1'|^p\,\varphi_1^{-2}\,u_n^2\,\mathrm{d}x \le \varepsilon \quad \text{for every } n \ge n_0.$$

This means that $u_n \to 0$ strongly in $L^2\left(B_{2\varrho}; |\varphi_1'|^p \varphi_1^{-2}\right)$. Finally, from inequality (4.15) we deduce $u_n \to 0$ strongly also in $\mathcal{H}_{\varphi_1}$. Hence, the imbedding $\mathcal{D}_{\varphi_1}(B_{2\varrho}) \hookrightarrow \mathcal{H}_{\varphi_1}$ is compact as claimed.

We have completed the proof of Proposition 3.6.

5. Properties of the quadratization at φ_1

In this section we state a few analog results to those in Takáč [22, Sect. 4] that are employed later in the proofs of Theorem 3.1 and Lemma 3.7.

Note that inequality (2.5) entails

$$\min\{1,\, p-1\}\|v\|^2_{\mathcal{D}_{\varphi_1}} \le \int_{\mathbb{R}^N} \langle \mathbf{A}(\nabla\varphi_1)\nabla v, \nabla v\rangle_{\mathbb{R}^N}\, \mathrm{d}x \le \max\{1,\, p-1\}\|v\|^2_{\mathcal{D}_{\varphi_1}} \quad (5.1)$$

for $v \in \mathcal{D}_{\varphi_1}$. Several important properties of $\mathcal{D}_{\varphi_1}$ are established below. The following result is obvious.

Lemma 5.1. *We have $\mathcal{Q}_0(\varphi_1, \varphi_1) = 0$ and $0 \le \mathcal{Q}_0(v, v) < \infty$ for all $v \in \mathcal{D}_{\varphi_1}$.*

We denote by $\mathcal{A}_{\varphi_1}$ the Lax-Milgram representation of the symmetric bilinear form $2 \cdot \mathcal{Q}_0$ on $\mathcal{D}_{\varphi_1} \times \mathcal{D}_{\varphi_1}$ (see [12, Chapt. VI, Eq. (2.3), p. 323]). In our setting this means that $\mathcal{A}_{\varphi_1} : \mathcal{D}_{\varphi_1} \to \mathcal{D}'_{\varphi_1}$ is a bounded linear operator such that

$$\langle \mathcal{A}_{\varphi_1} v, w\rangle = 2 \cdot \mathcal{Q}_0(v, w) \quad \text{for all } v, w \in \mathcal{D}_{\varphi_1}. \quad (5.2)$$

Identifying the dual space of $\mathcal{D}'_{\varphi_1}$ with $\mathcal{D}_{\varphi_1}$ (see Yosida [26, Theorem IV.8.2, p. 113]), we find that $\mathcal{A}_{\varphi_1}$ is selfadjoint in the following sense:

$$\langle \mathcal{A}_{\varphi_1} v, w\rangle = \langle v, \mathcal{A}_{\varphi_1} w\rangle \quad \text{for all } v, w \in \mathcal{D}_{\varphi_1}.$$

Note that our definition of $\mathcal{Q}_0$ yields $\mathcal{A}_{\varphi_1}\varphi_1 = 0$. Since the imbedding $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$ is compact, the null space of $\mathcal{A}_{\varphi_1}$ denoted by

$$\ker(\mathcal{A}_{\varphi_1}) = \{v \in \mathrm{dom}(\mathcal{A}_{\varphi_1}) \colon \mathcal{A}_{\varphi_1} v = 0\}$$

is finite-dimensional, by the Riesz-Schauder theorem [12, Theorem III.6.29, p. 187].

Lemma 5.1 provides another variational formula for λ_1, namely,

$$\lambda_1 = \inf\left\{ \frac{\int_{\mathbb{R}^N} \langle \mathbf{A}(\nabla\varphi_1)\nabla u, \nabla u\rangle_{\mathbb{R}^N}\, \mathrm{d}x}{(p-1)\int_{\mathbb{R}^N} |u|^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x} : 0 \not\equiv u \in \mathcal{D}_{\varphi_1} \right\}, \quad (5.3)$$

cf. eq. (1.2). This is a generalized Rayleigh quotient formula for the first (smallest) eigenvalue of the selfadjoint operator $(p-1)^{-1}\mathcal{A}_{\varphi_1} + \lambda_1 \varphi_1^{p-2} m \colon \mathcal{D}_{\varphi_1} \to \mathcal{D}'_{\varphi_1}$, where $\mathcal{A}_{\varphi_1}$ has been defined in (5.2). The following result determines all minimizers for (5.3):

Proposition 5.2. *Let $1 < p < N$ and let hypothesis* (H) *be satisfied. Then a function $u \in \mathcal{D}_{\varphi_1}$ satisfies $\mathcal{Q}_0(u, u) = 0$ if and only if $u = \kappa\varphi_1$ for some constant $\kappa \in \mathbb{R}$.*

The analogue of this proposition for a bounded domain $\Omega \subset \mathbb{R}^N$ with a sufficiently regular boundary $\partial\Omega$ is due to Takáč [22, Prop. 4.4, p. 202]. Our proof of Proposition 5.2 below is a simplification of that given in [22].

Proof. Proof of Proposition 5.2 Recall that the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$ is compact, by Proposition 3.6(b). Let u be any (nontrivial) minimizer for λ_1 in (5.3). If u

changes sign in $\mathbb{R}^N$, denote $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Then we have, using Gilbarg and Trudinger [11, Theorem 7.8, p. 153],

$$\begin{aligned}\lambda_1 &= \frac{\int_{\mathbb{R}^N} (u^+)^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x}{\int_{\mathbb{R}^N} u^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x} \cdot \frac{\int_{\mathbb{R}^N} \langle \mathbf{A}(\nabla\varphi_1)\nabla u^+, \nabla u^+\rangle_{\mathbb{R}^N}\, \mathrm{d}x}{(p-1)\int_{\mathbb{R}^N} (u^+)^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x} \\ &\quad + \frac{\int_{\mathbb{R}^N} (u^-)^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x}{\int_{\mathbb{R}^N} u^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x} \cdot \frac{\int_{\mathbb{R}^N} \langle \mathbf{A}(\nabla\varphi_1)\nabla u^-, \nabla u^-\rangle_{\mathbb{R}^N}\, \mathrm{d}x}{(p-1)\int_{\mathbb{R}^N} (u^-)^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x} \\ &\geq \Big(\frac{\int_{\mathbb{R}^N} (u^+)^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x}{\int_{\mathbb{R}^N} u^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x} + \frac{\int_{\mathbb{R}^N} (u^-)^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x}{\int_{\mathbb{R}^N} u^2\, \varphi_1^{p-2}\, m\, \mathrm{d}x} \Big) \lambda_1 = \lambda_1 .\end{aligned}$$

Consequently, both u^+ and u^- are (nontrivial) minimizers for λ_1.

Next, we show that if $u \in \ker(\mathcal{A}_{\varphi_1})$ then u is a constant multiple of φ_1. Since φ_1 satisfies (1.4), it is of class C^∞ in $\mathbb{R}^N \setminus \{0\}$, by classical regularity theory [11, Theorem 8.10, p. 186]. Now, for each $\gamma \in \mathbb{R}$ fixed, consider the function $v_\gamma \overset{\text{def}}{=} u - \gamma\varphi_1$ in $\mathbb{R}^N$. Then both v_γ^+ and v_γ^- belong to $\ker(\mathcal{A}_{\varphi_1})$ and thus satisfy the equation

$$-\nabla\cdot\big(\mathbf{A}(\nabla\varphi_1)\nabla v_\gamma^\pm\big) = \lambda_1(p-1)\varphi_1^{p-2} m v_\gamma^\pm \geq 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{5.4}$$

Again, we have $v_\gamma^\pm \in C^\infty(\mathbb{R}^N \setminus \{0\})$. So we may apply the strong maximum principle [11, Theorem 3.5, p. 35] to eq. (5.4) to conclude that either $v_\gamma^+ \equiv 0$ in $\mathbb{R}^N \setminus \{0\}$, or else $v_\gamma^+ > 0$ throughout $\mathbb{R}^N \setminus \{0\}$, and similarly for v_γ^-. This means that $\operatorname{sign}(u-\gamma\varphi_1) \equiv \text{const}$ in $\mathbb{R}^N \setminus \{0\}$. Moving γ from $-\infty$ to $+\infty$, we get $u \equiv \kappa\varphi_1$ in $\mathbb{R}^N \setminus \{0\}$ for some constant $\kappa \in \mathbb{R}$. This means $u = \kappa\varphi_1$ in $\mathcal{D}_{\varphi_1}$, as claimed. □

6. An improved Poincaré inequality ($2 \leq p < N$)

We need a few more technical tools from Fleckinger and Takáč [9, Sect. 5] to prove Lemma 3.7. Although our present situation requires only a few changes in the space setting in [9], we provide complete proofs of all results for the convenience of the reader.

Remark 6.1. Except when $u^\parallel = 0$, we may replace $u \in D^{1,p}(\mathbb{R}^N)$ by $v = u/u^\parallel$ in inequality (3.1) and thus restate it equivalently as follows, for all $v^\top \in D^{1,p}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} v^\top\, \varphi_1^{p-1}\, m\, \mathrm{d}x = 0$:

$$\mathcal{Q}_{v^\top}(v^\top, v^\top) = \mathcal{F}(\varphi_1 + v^\top) \geq \frac{c}{p}\left(\|v^\top\|^2_{\mathcal{D}_{\varphi_1}} + \|v^\top\|^p_{D^{1,p}(\mathbb{R}^N)} \right). \tag{6.1}$$

This remark indicates that our proof of inequality (3.1) should distinguish between the cases when the ratio $\|u^\top\|_{D^{1,p}(\mathbb{R}^N)}/|u^\parallel|$ is bounded away from zero by a constant $\gamma > 0$, say,

$$\|u^\top\|_{D^{1,p}(\mathbb{R}^N)}/|u^\parallel| \geq \gamma,$$

and when it is sufficiently small, say,

$$\|u^\top\|_{D^{1,p}(\mathbb{R}^N)}/|u^\parallel| \leq \gamma$$

where $\gamma > 0$ is small enough. The former case is treated in a standard way analogous to (1.2), whereas the latter case requires a more sophisticated approach based on the second-order Taylor formula (2.11) applied to the expression $\mathcal{Q}_{v^\top}(v^\top, v^\top)$ on the left-hand side in (6.1) where $v = u/u^\parallel$. For either of these cases we need a separate auxiliary result: We derive two formulas for Rayleigh quotients outside and inside an arbitrarily small cone around the axis spanned by φ_1, respectively.

6.1. **Minimization outside a cone around φ_1.** We allow $1 < p < N$ throughout this paragraph. Given any number $0 < \gamma < \infty$, we set

$$\mathcal{C}_\gamma \stackrel{\text{def}}{=} \Big\{ u \in D^{1,p}(\mathbb{R}^N) \colon \|u^\top\|_{D^{1,p}(\mathbb{R}^N)} \le \gamma |u^\parallel| \Big\},$$
$$\mathcal{C}'_\gamma \stackrel{\text{def}}{=} \Big\{ u \in D^{1,p}(\mathbb{R}^N) \colon \|u^\top\|_{D^{1,p}(\mathbb{R}^N)} \ge \gamma |u^\parallel| \Big\}.$$

Note that $\mathcal{C}_\gamma$ is a closed cone in $D^{1,p}(\mathbb{R}^N)$ and $\mathcal{C}'_\gamma$ is the closure of $\mathcal{C}^c_\gamma$, the complement of $\mathcal{C}_\gamma$ in $D^{1,p}(\mathbb{R}^N)$. We consider also the hyperplane

$$\mathcal{C}'_\infty \stackrel{\text{def}}{=} \Big\{ u \in D^{1,p}(\mathbb{R}^N) \colon u^\parallel = 0 \Big\} = \bigcap_{0<\gamma<\infty} \mathcal{C}'_\gamma .$$

For $0 < \gamma \le \infty$ we define

$$\Lambda_\gamma \stackrel{\text{def}}{=} \inf \Big\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x}{\int_{\mathbb{R}^N} |u|^p \, m \,\mathrm{d}x} \colon u \in \mathcal{C}'_\gamma \setminus \{0\} \Big\}. \tag{6.2}$$

The next result is an analogue of [9, Lemma 5.1, p. 963] proved for a bounded domain $\Omega \subset \mathbb{R}^N$.

Lemma 6.2. *Let $1 < p < N$ and $0 < \gamma \le \infty$. Then we have $\Lambda_\gamma > \lambda_1$.*

Proof. Assume the contrary, that is, $\Lambda_\gamma = \lambda_1$ for some $0 < \gamma < \infty$. Pick a minimizing sequence $\{u_n\}_{n=1}^\infty$ in $\mathcal{C}'_\gamma$ such that

$$\int_{\mathbb{R}^N} |u_n|^p \, m \,\mathrm{d}x = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla u_n|^p \,\mathrm{d}x \to \lambda_1 \quad \text{as } n \to \infty.$$

Since $D^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space, the minimizing sequence contains a weakly convergent subsequence in $D^{1,p}(\mathbb{R}^N)$ which we denote by $\{u_n\}_{n=1}^\infty$ again. Consequently, $u_n \to u$ strongly in $L^p(\mathbb{R}^N; m)$, by Proposition 3.6(a), and $\nabla u_n \rightharpoonup \nabla u$ weakly in $[L^p(\mathbb{R}^N)]^N$ as $n \to \infty$. We deduce that $\int_{\mathbb{R}^N} |u|^p \, m \,\mathrm{d}x = 1$ and

$$\lambda_1^{1/p} \le \|\nabla u\|_{L^p(\mathbb{R}^N)} \le \liminf_{n\to\infty} \|\nabla u_n\|_{L^p(\mathbb{R}^N)} = \lambda_1^{1/p}.$$

As the standard norm on the space $D^{1,p}(\mathbb{R}^N)$ is uniformly convex, by Clarkson's inequalities, we must have $u_n \to u$ strongly in $D^{1,p}(\mathbb{R}^N)$, by the proof of Milman's theorem (see Yosida [26, Theorem V.2.2, p. 127]). This means that

$$u_n^\parallel = \int_{\mathbb{R}^N} u_n \, \varphi_1^{p-1} \, m \,\mathrm{d}x \to u^\parallel = \int_{\mathbb{R}^N} u \, \varphi_1^{p-1} \, m \,\mathrm{d}x,$$
$$u_n^\top = u_n - u_n^\parallel \varphi_1 \to u^\top = u - u^\parallel \varphi_1 \text{ strongly in } D^{1,p}(\mathbb{R}^N),$$

as $n \to \infty$. The set $\mathcal{C}'_\gamma$ being closed in $D^{1,p}(\mathbb{R}^N)$, we thus have $u \in \mathcal{C}'_\gamma$.

On the other hand, from $\|u\|_{L^p(\mathbb{R}^N;m)} = 1$ and $\|\nabla u\|_{L^p(\mathbb{R}^N)} = \lambda_1^{1/p}$, combined with the simplicity of the first eigenvalue λ_1, one deduces that $u = \pm\varphi_1$, a contradiction to $u \in \mathcal{C}'_\gamma$. The lemma is proved. $\square$

6.2. **Minimization inside a cone around φ_1.** For $\phi \in D^{1,p}(\mathbb{R}^N)$, $\phi \not\equiv 0$ in $\mathbb{R}^N$, let us define

$$\tilde{\Lambda} \stackrel{\text{def}}{=} \liminf_{\substack{\|\phi\|_{D^{1,p}(\mathbb{R}^N)} \to 0 \\ \langle \phi, \varphi_1^{p-1} m \rangle = 0}} \frac{\int_{\mathbb{R}^N} \left\langle \left[\int_0^1 \mathbf{A}\left(\nabla(\varphi_1 + s\phi) \right) (1-s)\, \mathrm{d}s \right] \nabla\phi,\, \nabla\phi \right\rangle_{\mathbb{R}^N} \mathrm{d}x}{\int_{\mathbb{R}^N} \left[\int_0^1 |\varphi_1 + s\phi|^{p-2} (1-s)\, \mathrm{d}s \right] |\phi|^2\, m\, \mathrm{d}x} \tag{6.3}$$

with the abbreviation (2.4). Using the quadratic form $\mathcal{Q}_\phi$ defined in (2.11), we notice that

$$\tilde{\Lambda} - \lambda_1(p-1) = \liminf_{\substack{\|\phi\|_{D^{1,p}(\mathbb{R}^N)} \to 0 \\ \langle \phi, \varphi_1^{p-1} m \rangle = 0}} \frac{\mathcal{Q}_\phi(\phi, \phi)}{\int_{\mathbb{R}^N} \left[\int_0^1 |\varphi_1 + s\phi|^{p-2} (1-s)\, \mathrm{d}s \right] |\phi|^2\, m\, \mathrm{d}x} \geq 0.$$

The next result parallels [9, Lemma 5.2, p. 964] shown for a bounded domain $\Omega \subset \mathbb{R}^N$.

Lemma 6.3. *Let $2 \leq p < N$. We have $\tilde{\Lambda} > \lambda_1(p-1)$.*

Before giving the proof of this inequality, we first recall that the kernels of the quadratic forms $\mathcal{Q}_\phi(v,v)$ and $\mathcal{Q}_0(v,v)$ defined in (2.11) and (2.12), respectively, can be compared by inequalities (2.6) for $p \geq 2$, and (2.7) for $p < 2$, so that we can use the Hilbert space $\mathcal{D}_{\varphi_1}$ not only for $\mathcal{Q}_0$ but also for $\mathcal{Q}_\phi$.

Next, we introduce the following notations where $t \in \mathbb{R}$ and $\phi \in D^{1,p}(\mathbb{R}^N)$:

$$\mathcal{P}_0(t,\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \Big[\int_0^1 |\varphi_1 + st\phi|^{p-2} (1-s)\, \mathrm{d}s \Big] \phi^2\, m\, \mathrm{d}x,$$

$$\mathcal{P}_1(t,\phi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \Big\langle \Big[\int_0^1 \mathbf{A}(\nabla(\varphi_1 + st\phi))(1-s)\, \mathrm{d}s \Big] \nabla\phi,\, \nabla\phi \Big\rangle_{\mathbb{R}^N} \mathrm{d}x.$$

Hence, equation (6.3) takes the form

$$\tilde{\Lambda} = \liminf_{\substack{\|\phi\|_{D^{1,p}(\mathbb{R}^N)} \to 0 \\ \langle \phi, \varphi_1^{p-1} m \rangle = 0}} \frac{\mathcal{P}_1(t,\phi)}{\mathcal{P}_0(t,\phi)} \quad \text{with any fixed } t \in \mathbb{R} \setminus \{0\}.$$

Furthermore, due to inequalities (2.6), the expressions $\mathcal{P}_0(t,\phi)$ and $\mathcal{P}_1(t,\phi)$, respectively, are equivalent to

$$\begin{aligned} \mathcal{N}_0(t,\phi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \Big(\varphi_1^{p-2} + |t|^{p-2} |\phi|^{p-2} \Big) \phi^2\, m\, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \varphi_1^{p-2} \phi^2\, m\, \mathrm{d}x + |t|^{p-2} \|\phi\|^p_{L^p(\mathbb{R}^N;m)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_1(t,\phi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \big(|\nabla\varphi_1|^{p-2} + |t|^{p-2} |\nabla\phi|^{p-2} \big) |\nabla\phi|^2\, \mathrm{d}x \\ &= \|\phi\|^2_{\mathcal{D}_{\varphi_1}} + |t|^{p-2} \|\phi\|^p_{D^{1,p}(\mathbb{R}^N)}, \end{aligned}$$

that is, there are two constants $c_1, c_2 > 0$ independent from t and ϕ such that

$$c_1 \mathcal{N}_i(t,\phi) \leq \mathcal{P}_i(t,\phi) \leq c_2 \mathcal{N}_i(t,\phi); \quad i = 0, 1. \tag{6.4}$$

Proof of Lemma 6.3. On the contrary, assume that $\tilde{\Lambda} \le \lambda_1(p-1)$. Pick a minimizing sequence $\{\phi_n\}_{n=1}^{\infty}$ in $D^{1,p}(\mathbb{R}^N)$ such that $\phi_n \not\equiv 0$ in $\mathbb{R}^N$, $\langle \phi_n, \varphi_1^{p-1} m\rangle = 0$, $\|\phi_n\|_{D^{1,p}(\mathbb{R}^N)} \to 0$, and

$$\frac{\mathcal{P}_1(1,\phi_n)}{\mathcal{P}_0(1,\phi_n)} \longrightarrow \tilde{\Lambda} \le \lambda_1(p-1) \quad \text{as } n \to \infty.$$

Next, set $t_n = \mathcal{P}_0(1,\phi_n)^{1/2}$ and $V_n = \phi_n / t_n$ for $n = 1, 2, \dots$. Hence, we have $t_n \to 0$, $\mathcal{P}_0(t_n, V_n) = 1$, and $\mathcal{P}_1(t_n, V_n) \to \tilde{\Lambda}$ as $n \to \infty$. Inequalities (6.4) guarantee that both sequences $\|V_n\|_{\mathcal{D}_{\varphi_1}}$ and $t_n^{1-(2/p)} \|V_n\|_{D^{1,p}(\mathbb{R}^N)}$ are bounded, and so we may extract a subsequence denoted again by $\{V_n\}_{n=1}^{\infty}$ such that $V_n \rightharpoonup V$ weakly in $\mathcal{D}_{\varphi_1}$ and $t_n^{1-(2/p)} V_n \rightharpoonup z$ weakly in $D^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. Using the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow \mathcal{D}_{\varphi_1}$, we get $z \equiv 0$ in $\mathbb{R}^N$. Furthermore, both imbeddings $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; m)$ and $\mathcal{D}_{\varphi_1} \hookrightarrow \mathcal{H}_{\varphi_1}$ being compact by Proposition 3.6, we have also $V_n \to V$ strongly in $\mathcal{H}_{\varphi_1}$ and $t_n^{1-(2/p)} V_n \to 0$ strongly in $L^p(\mathbb{R}^N; m)$. It follows that $\langle V, \varphi_1^{p-1} m\rangle = 0$ and

$$\begin{aligned}
\mathcal{P}_0(0,V) &= \frac{1}{2} \int_{\mathbb{R}^N} \varphi_1^{p-2} V^2 \,\mathrm{d}x = 1, \\
\mathcal{P}_1(0,V) &= \frac{1}{2} \langle \mathbf{A}(\nabla\varphi_1)\nabla V,\, \nabla V\rangle \le \tilde{\Lambda} \le \lambda_1(p-1).
\end{aligned}$$

Consequently, Proposition 5.2 forces $V = \kappa\varphi_1$ in $\mathbb{R}^N$, where $\kappa \in \mathbb{R}$ is a constant, $\kappa \neq 0$ by $\mathcal{P}_0(0,V) = 1$. But this is a contradiction to $\langle V, \varphi_1^{p-1} m\rangle = 0$. We conclude that $\tilde{\Lambda} > \lambda_1(p-1)$ as claimed. $\square$

6.3. **Proof of Lemma 3.7.** If $u \in D^{1,p}(\mathbb{R}^N)$ satisfies $\langle u, \varphi_1\rangle = 0$, then equation (6.2) implies

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x - \lambda_1 \int_{\mathbb{R}^N} |u|^p \, m \,\mathrm{d}x &\ge \big(1 - \frac{\lambda_1}{\Lambda_\infty}\big) \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x \\
&= \big(1 - \frac{\lambda_1}{\Lambda_\infty}\big) \int_{\mathbb{R}^N} |\nabla u^\top|^p \,\mathrm{d}x
\end{aligned} \tag{6.5}$$

where $\lambda_1/\Lambda_\infty < 1$ by Lemma 6.2. Thus, we may assume $\langle u, \varphi_1\rangle \neq 0$ and so we need to prove only inequality (6.1). We will apply Lemmas 6.2 and 6.3 to the following two cases, respectively.

Case $\|v^\top\|_{D^{1,p}(\mathbb{R}^N)} \ge \gamma$: Here, $\gamma > 0$ is an arbitrary, but fixed number. In analogy with inequality (6.5) above, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\varphi_1 + \nabla v^\top|^p \,\mathrm{d}x - \lambda_1 \int_{\mathbb{R}^N} |\varphi_1 + v^\top|^p \, m \,\mathrm{d}x \\
&\ge \big(1 - \frac{\lambda_1}{\Lambda_\gamma}\big) \int_{\mathbb{R}^N} |\nabla\varphi_1 + \nabla v^\top|^p \,\mathrm{d}x \ge c_\gamma \int_{\mathbb{R}^N} |\nabla v^\top|^p \,\mathrm{d}x
\end{aligned} \tag{6.6}$$

for all $v^\top \in D^{1,p}(\mathbb{R}^N)$ such that $\langle v^\top, \varphi_1^{p-1} m\rangle = 0$ and $\|v^\top\|_{D^{1,p}(\mathbb{R}^N)} \ge \gamma$, where $c_\gamma > 0$ is a constant independent from $v^\top$. The last inequality follows from the boundedness of the orthogonal projections $u \mapsto u^\parallel \cdot \varphi_1$ and $u \mapsto u^\top$ in $D^{1,p}(\mathbb{R}^N)$. Recalling the imbedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow \mathcal{D}_{\varphi_1}$, we deduce from (6.6) that inequality (6.1) is valid provided $\|v^\top\|_{D^{1,p}(\mathbb{R}^N)} \ge \gamma$.

Case $\|v^\top\|_{D^{1,p}(\mathbb{R}^N)} \le \gamma$: Here, $\gamma > 0$ is sufficiently small. According to equation (6.3) and Lemma 6.3 we have

$$\begin{aligned} \mathcal{Q}_{v^\top}(v^\top, v^\top) &= \mathcal{P}_1(1, v^\top) - \lambda_1(p-1)\,\mathcal{P}_0(1, v^\top) \\ &\ge \Big(1 - \frac{\lambda_1(p-1)}{\tilde{\Lambda}}\Big)\mathcal{P}_1(1, v^\top) \\ &\ge \tilde{c}\cdot\mathcal{N}_1(1, v^\top) \end{aligned} \tag{6.7}$$

for all $v^\top \in D^{1,p}(\mathbb{R}^N)$ such that $\langle v^\top, \varphi_1^{p-1} m\rangle = 0$ and $\|v^\top\|_{D^{1,p}(\mathbb{R}^N)} \le \gamma$, where $\gamma > 0$ is sufficiently small and $\tilde{c} > 0$ is a constant independent from $v^\top$. Recall that the expressions $\mathcal{P}_i(1, v^\top)$ and $\mathcal{N}_i(1, v^\top)$ $(i = 0, 1)$ have been defined after Lemma 6.3. From (6.7) we deduce that inequality (6.1) is valid also when $\|v^\top\|_{D^{1,p}(\mathbb{R}^N)} \le \gamma$.

Remark 6.4. Assume $2 < p < N$ and let $f \in \mathcal{D}'_{\varphi_1}$ satisfy $\langle f, \varphi_1\rangle = 0$. Recall that $D^{1,p}(\mathbb{R}^N) \hookrightarrow \mathcal{D}_{\varphi_1}$. Although the functional $\mathcal{J}_{\lambda_1}$, defined in (1.3) with $\lambda = \lambda_1$, is no longer coercive on $D^{1,p}(\mathbb{R}^N)$, it is still not only bounded from below, but also "very close" to being coercive on the weighted Sobolev space $\mathcal{D}_{\varphi_1}$, as a direct consequence of improved Poincaré's inequality (3.1). This property of $\mathcal{J}_{\lambda_1}$ will be used in the next section to prove the existence theorem (Theorem 3.1) for problem (1.6).

7. Proof of Theorem 3.1

Our proof of Theorem 3.1 combines the improved Poincaré inequality (3.1) with a generalized Rayleigh quotient formula. To this end, we may assume that $f \in \mathcal{D}'_{\varphi_1}$ satisfies $f \not\equiv 0$ in $\mathbb{R}^N$ and $\langle f, \varphi_1\rangle = 0$. Define the number M_f, for $0 \le M_f \le \infty$, by

$$M_f \stackrel{\text{def}}{=} \sup_{\substack{v\in D^{1,p}(\mathbb{R}^N)\\ v\notin\{\kappa\varphi_1\,:\,\kappa\in\mathbb{R}\}}} \frac{|\langle f, v\rangle|^p}{\int_{\mathbb{R}^N} |\nabla v|^p\,\mathrm{d}x - \lambda_1\int_{\mathbb{R}^N} |v|^p\, m\,\mathrm{d}x}\,. \tag{7.1}$$

Clearly, $M_f > 0$. Moreover, inequality (3.1) entails

$$|\langle f, v\rangle|^p \le \|f\|^p_{D^{-1,p'}(\mathbb{R}^N)}\,\|v^\top\|^p_{D^{1,p}(\mathbb{R}^N)} \le C_f\Big(\int_{\mathbb{R}^N} |\nabla v|^p\,\mathrm{d}x - \lambda_1\int_{\mathbb{R}^N} |v|^p\, m\,\mathrm{d}x\Big)$$

for all $v \in D^{1,p}(\mathbb{R}^N)$, where $C_f = c^{-1}\,\|f\|^p_{D^{-1,p'}(\mathbb{R}^N)}$ is a constant. This shows that $M_f \le C_f < \infty$. In a similar way we arrive at

$$\begin{aligned} |v^\parallel|^{p-2}\,|\langle f, v\rangle|^2 &\le |v^\parallel|^{p-2}\left(\|f\|_{\mathcal{D}'_{\varphi_1}}\right)^2\,\|v^\top\|^2_{\mathcal{D}_{\varphi_1}} \\ &\le C'_f\Big(\int_{\mathbb{R}^N} |\nabla v|^p\,\mathrm{d}x - \lambda_1\int_{\mathbb{R}^N} |v|^p\, m\,\mathrm{d}x\Big) \quad \text{for all } v \in D^{1,p}(\mathbb{R}^N), \end{aligned} \tag{7.2}$$

where $C'_f = c^{-1}(\|f\|_{\mathcal{D}'_{\varphi_1}})^2$ is a constant, and $\|\cdot\|_{\mathcal{D}'_{\varphi_1}}$ stands for the dual norm on $\mathcal{D}'_{\varphi_1}$. ¿From (7.1) and inequality (7.2) we can draw the following conclusion: If $v \in D^{1,p}(\mathbb{R}^N)$ is such that $v^\top \not\equiv 0$ in $\mathbb{R}^N$ and

$$\frac{|\langle f, v\rangle|^p}{\int_{\mathbb{R}^N} |\nabla v|^p\,\mathrm{d}x - \lambda_1\int_{\mathbb{R}^N} |v|^p\, m\,\mathrm{d}x} \ge \frac{1}{2}M_f,$$

then $\langle f, v\rangle \ne 0$ and

$$|v^\parallel|^{p-2} \le 2(C'_f/M_f)\,|\langle f, v\rangle|^{p-2} \le (C''_f)^{p-2}\,\|v^\top\|^{p-2}_{D^{1,p}(\mathbb{R}^N)},$$

where $C''_f = [2(C'_f/M_f)]^{1/(p-2)} \|f\|_{D^{-1,p'}(\mathbb{R}^N)}$ is a constant, i.e.,

$$|v^{\|}| \le C''_f \, \|v^{\top}\|_{D^{1,p}(\mathbb{R}^N)}. \tag{7.3}$$

Next, take any maximizing sequence $\{v_n\}_{n=1}^{\infty}$ in $D^{1,p}(\mathbb{R}^N)$ for the generalized Rayleigh quotient (7.1), that is, $v_n^{\top} \not\equiv 0$ in $\mathbb{R}^N$ and

$$\frac{|\langle f, v_n\rangle|^p}{\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x - \lambda_1 \int_{\mathbb{R}^N} |v_n|^p \, m \,\mathrm{d}x} \longrightarrow M_f \quad \text{as } n \to \infty. \tag{7.4}$$

Since both, the numerator and the denominator are p-homogeneous, we may assume $\|v_n\|_{D^{1,p}(\mathbb{R}^N)} = 1$ for all $n \ge 1$. The Sobolev space $D^{1,p}(\mathbb{R}^N)$ being reflexive, we may pass to a convergent subsequence $v_n \rightharpoonup w$ weakly in $D^{1,p}(\mathbb{R}^N)$; hence, also $v_n \to w$ strongly in $L^p(\mathbb{R}^N; m)$, by Proposition 3.6(a), and $\langle f, v_n\rangle \to \langle f, w\rangle$ as $n \to \infty$. We insert these limits into (7.4) to obtain

$$\int_{\mathbb{R}^N} |\nabla w|^p \,\mathrm{d}x - \lambda_1 \int_{\mathbb{R}^N} |w|^p \, m \,\mathrm{d}x \le 1 - \lambda_1 \int_{\mathbb{R}^N} |w|^p \, m \,\mathrm{d}x = M_f^{-1} |\langle f, w\rangle|^p. \tag{7.5}$$

In particular, we have $w \not\equiv 0$ in $\mathbb{R}^N$, therefore also $w^{\top} \not\equiv 0$ by (7.3), and consequently $|\langle f, w\rangle| \ne 0$ by (7.5). We combine (7.1) with (7.5) to get $\int_{\mathbb{R}^N} |\nabla w|^p \,\mathrm{d}x = 1$. Hence, the supremum M_f in (7.1) is attained at w in place of v.

Finally, we can apply the calculus of variations to the inequality

$$\int_{\mathbb{R}^N} |\nabla v|^p \,\mathrm{d}x - \lambda_1 \int_{\mathbb{R}^N} |v|^p \, m \,\mathrm{d}x - M_f^{-1} |\langle f, v\rangle|^p \ge 0 \quad \text{for } v \in D^{1,p}(\mathbb{R}^N)$$

to derive

$$-\Delta_p w - \lambda_1 \, m \, |w|^{p-2} w = M_f^{-1} |\langle f, w\rangle|^{p-2} \langle f, w\rangle \cdot f(x) \quad \text{in } \mathbb{R}^N.$$

It follows that $u \stackrel{\text{def}}{=} M_f^{1/(p-1)} \langle f, w\rangle^{-1} \cdot w$ is a weak solution of problem (1.6). Theorem 3.1 is proved.

8. Proof of Theorem 3.3

In contrast to the case $2 \le p < N$ in Section 6, Remark 6.4, for $1 < p < 2$ the functional $\mathcal{J}_{\lambda_1}$ will turn out to be unbounded from below on $D^{1,p}(\mathbb{R}^N)$ along curves "close" to $\pm\tau\varphi_1$ as $\tau \to +\infty$, even though it still remains coercive on the complement $D^{1,p}(\mathbb{R}^N)^{\top}$ of $\mathrm{lin}\{\varphi_1\}$ in $D^{1,p}(\mathbb{R}^N)$ defined in (3.2). Again, we take advantage of the direct sum $D^{1,p}(\mathbb{R}^N) = \mathrm{lin}\{\varphi_1\} \oplus D^{1,p}(\mathbb{R}^N)^{\top}$ defined in (1.5). These facts show that the functional $\mathcal{J}_{\lambda_1}$ has a simple saddle point geometry. Such a scenario is typically suitable for a saddle point theorem which guarantees the existence of a critical point for $\mathcal{J}_{\lambda_1}$ by means of a minimax formula for a critical value of $\mathcal{J}_{\lambda_1}$. Here we make use of a "very direct" minimax method introduced in Takáč [22, Sect. 7] which does not require any Palais-Smale condition. In this section we adapt this method to our setting. In a closely related work Drábek and Holubová [5, Theorem 1.1] applied the saddle point theorem from Rabinowitz [18, Theorem 4.6, p. 24] to establish an existence result for problem (1.6) when $\Omega \subset \mathbb{R}^N$ is a bounded domain and $1 < p < 2$.

As we allow the function $f \in D^{-1,p'}(\mathbb{R}^N)$ in the energy functional (1.3) to vary, we write $\mathcal{J}_\lambda(u) \equiv \mathcal{J}_\lambda(u; f)$.

8.1. **Simple saddle point geometry.** For $\lambda = \lambda_1$ it will be convenient to use the notation

$$\mathcal{E}_f(u) \stackrel{\text{def}}{=} \mathcal{J}_{\lambda_1}(u; f) \quad \text{for } u \in D^{1,p}(\mathbb{R}^N).$$

Proof of Lemma 3.9. We infer from Lemma 6.2 that $\Lambda_\infty > \lambda_1$ in formula (6.2). This shows that the functional $\mathcal{E}_f$ is coercive on $\mathcal{C}'_\infty = D^{1,p}(\mathbb{R}^N)^\top$. Hence, being also weakly lower semicontinuous, $\mathcal{E}_f$ possesses a global minimizer $u_0^\top$ over $D^{1,p}(\mathbb{R}^N)^\top$,

$$\mathcal{E}_f(u_0^\top) = \inf_{w \in D^{1,p}(\mathbb{R}^N)^\top} \mathcal{E}_f(w) > -\infty.$$

Now let us look for the functions u and v, respectively, in Definition 3.8 in the forms of

$$u_\pm = \pm\tau\varphi_1 + \tau^{1-(p/2)}\phi \quad \text{with } \tau \in (0,\infty) \text{ sufficiently large}, \tag{8.1}$$

where $\phi \in C_c^1(\mathbb{R}^N)$ is a function chosen as follows:

($\boldsymbol{\Phi}$) $\langle f, \phi\rangle = 1$ and $0 \notin K$ where

$$K = \operatorname{supp}(\phi) \stackrel{\text{def}}{=} \overline{\{x \in \mathbb{R}^N : \phi(x) \neq 0\}} \quad (\subset \mathbb{R}^N)$$

denotes the support of ϕ.

The existence of ϕ is verified as follows. Since $f \in D^{-1,p'}(\mathbb{R}^N)$ satisfies $f \not\equiv 0$ in $\mathbb{R}^N$, there is a function $\phi_0 \in C_c^1(\mathbb{R}^N)$ such that $\langle f, \phi_0\rangle = 1$. On the contrary to ($\boldsymbol{\Phi}$), suppose that the support $K_0 = \operatorname{supp}(\phi_0)$ of ϕ_0 always contains $0 \in \mathbb{R}^N$. This is equivalent to saying that $\langle f, \phi\rangle = 0$ whenever $\phi \in C_c^1(\mathbb{R}^N)$ is such that $0 \notin \operatorname{supp}(\phi)$. Now choose a C^1 function $\psi \colon \mathbb{R}_+ \to [0,1]$ such that $\psi(r) = 1$ if $0 \le r \le 1$, $0 \le \psi(r) \le 1$ if $1 \le r \le 2$, and $\psi(r) = 0$ if $2 \le r < \infty$. Define $\psi_n(x) \stackrel{\text{def}}{=} \psi(n|x|)$ for all $x \in \mathbb{R}^N$; $n = 1, 2, \dots$. Then $0 \notin \operatorname{supp}((1-\psi_n)\phi_0)$ which yields $\langle f, (1-\psi_n)\phi_0\rangle = 0$. Hence $\langle f, \psi_n\phi_0\rangle = \langle f, \phi_0\rangle = 1$. However, this is contradicted by $\|\psi_n\phi_0\|_{D^{1,p}(\mathbb{R}^N)} \to 0$ as $n \to \infty$, which follows easily from

$$\|\nabla(\psi_n\phi_0)\|_{L^p(\mathbb{R}^N)} \le \|\phi_0\|_{L^\infty(\mathbb{R}^N)}\|\nabla\psi_n\|_{L^p(\mathbb{R}^N)} + \|\nabla\phi_0\|_{L^\infty(\mathbb{R}^N)}\|\psi_n\|_{L^p(\mathbb{R}^N)}$$

with both

$$\|\nabla\psi_n\|_{L^p(\mathbb{R}^N)} = n^{1-(N/p)}\|\nabla\psi\|_{L^p(\mathbb{R}^N)} \to 0,$$
$$\|\psi_n\|_{L^p(\mathbb{R}^N)} = n^{-(N/p)}\|\psi\|_{L^p(\mathbb{R}^N)} \to 0$$

as $n \to \infty$, by $1 < p < 2 \le N$.

So let $\phi \in C_c^1(\mathbb{R}^N)$ satisfy condition ($\boldsymbol{\Phi}$). For $\tau \in (0,\infty)$ we compute

$$\int_{\mathbb{R}^N} u_\pm\, \varphi_1^{p-1}\, m \,\mathrm{d}x = \pm\tau + \tau^{1-(p/2)} \int_{\mathbb{R}^N} \phi\, \varphi_1^{p-1}\, m \,\mathrm{d}x, \tag{8.2}$$

by $\int_{\mathbb{R}^N} \varphi_1^p\, m \,\mathrm{d}x = 1$. It follows that

$$\int_{\mathbb{R}^N} u_-\, \varphi_1^{p-1}\, m \,\mathrm{d}x < 0 < \int_{\mathbb{R}^N} u_+\, \varphi_1^{p-1}\, m \,\mathrm{d}x \quad \text{for all } \tau > 0 \text{ large enough.}$$

Next we use eqs. (2.10) and (2.11) together with $\langle f, \varphi_1\rangle = 0$ to obtain

$$\begin{aligned} &\mathcal{E}_f(u_\pm) = \mathcal{J}_{\lambda_1}(\pm\tau\varphi_1 + \tau^{1-(p/2)}\phi) = \\ &\mathcal{Q}_{\pm\tau^{-p/2}\phi}(\phi,\phi) - \tau^{1-(p/2)}\langle f, \phi\rangle = \mathcal{Q}_{\pm\tau^{-p/2}\phi}(\phi,\phi) - \tau^{1-(p/2)}. \end{aligned} \tag{8.3}$$

We recall that the quadratic forms $\mathcal{Q}_{\pm\tau^{-p/2}\phi}$ are given by formula (2.11). Since $\inf_K |\nabla\varphi_1| > 0$, $\inf_K \varphi_1 > 0$, and ϕ is supported in $K \subset \mathbb{R}^N \setminus \{0\}$, we conclude that both summands in $\mathcal{Q}_{\pm\tau^{-p/2}\phi}(\phi,\phi)$ are bounded independently from $\tau \geq \tau_0$, provided $\tau_0 \in (0,\infty)$ is large enough. Finally, from (8.3) we deduce that $\mathcal{E}_f(u_\pm) \to -\infty$ as $\tau \to +\infty$. The conclusion of the lemma follows. $\square$

8.2. **A minimax method.** We allow $1 < p < N$ throughout the entire paragraph even though we apply the results to the minimax expression in (3.3) only for $p < 2$.

We assume that $0 \leq \lambda \leq \Lambda_\infty - \eta$ and $f \in D^{-1,p'}(\mathbb{R}^N)$. Here, η is an arbitrary, but fixed number with $0 < \eta < \Lambda_\infty - \lambda_1$. Furthermore, in view of Lemma 6.2 with $\gamma = \gamma_\eta$, we find a constant $0 < \gamma_\eta < \infty$ large enough, so that $\Lambda_{\gamma_\eta} \geq \Lambda_\infty - \frac{1}{2}\eta$, and set

$$c = \frac{1}{2}\Big(1 - (\Lambda_\infty - \eta)\Lambda_{\gamma_\eta}^{-1}\Big) > 0.$$

Note that for any fixed $\tau \in \mathbb{R}$ the functional $u^\top \mapsto \mathcal{J}_\lambda(\tau\varphi_1 + u^\top)$ is coercive on the (closed linear) subspace $D^{1,p}(\mathbb{R}^N)^\top$ of $D^{1,p}(\mathbb{R}^N)$. This claim follows from the following inequalities which are valid whenever $|\tau| \leq T \leq \gamma_\eta^{-1}\|u^\top\|_{D^{1,p}(\mathbb{R}^N)}$, for any fixed $T \in (0,\infty)$:

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(\tau\varphi_1 + u^\top)|^p \,\mathrm{d}x - (\Lambda_\infty - \eta)\int_{\mathbb{R}^N} |(\tau\varphi_1 + u^\top)|^p \, m \,\mathrm{d}x \\
&\geq \Big(1 - \frac{\Lambda_\infty - \eta}{\Lambda_{\gamma_\eta}}\Big)\int_{\mathbb{R}^N} |\nabla(\tau\varphi_1 + u^\top)|^p \,\mathrm{d}x \\
&\geq \Big(1 - \frac{\Lambda_\infty - \eta}{\Lambda_{\gamma_\eta}}\Big)\big|\|\nabla u^\top\|_{L^p(\mathbb{R}^N)} - |\tau|\cdot\|\nabla\varphi_1\|_{L^p(\mathbb{R}^N)}\big|^p \\
&\geq c\,\|\nabla u^\top\|^p_{L^p(\mathbb{R}^N)} - c_T,
\end{aligned} \tag{8.4}$$

with another constant $0 < c_T < \infty$ depending solely on T. The first inequality in (8.4) is easily derived from formula (6.2). Consequently, any global minimizer $u_\tau^\top$ for the functional $u^\top \mapsto \mathcal{J}_\lambda(\tau\varphi_1 + u^\top)$ on $D^{1,p}(\mathbb{R}^N)^\top$ satisfies the estimate $\|u_\tau^\top\|_{D^{1,p}(\mathbb{R}^N)} \leq C_T < \infty$, where C_T is a constant independent from $\lambda \in [0,\Lambda_\infty - \eta]$ and $\tau \in [-T,T]$. Such a global minimizer always exists and verifies the Euler-Lagrange equation

$$\begin{aligned}
&-\Delta_p(\tau\varphi_1 + u_\tau^\top) - \lambda\, m(x)\,|\tau\varphi_1 + u_\tau^\top|^{p-2}(\tau\varphi_1 + u_\tau^\top) \\
&= f^\top(x) + \zeta_\tau\cdot m(x)\,\varphi_1(x)^{p-1} \quad \text{in } \mathbb{R}^N,
\end{aligned} \tag{8.5}$$

with a Lagrange multiplier $\zeta_\tau \in \mathbb{R}$. Thus, we may define

$$j_\lambda(\tau) \stackrel{\text{def}}{=} \min_{u^\top \in D^{1,p}(\mathbb{R}^N)^\top} \mathcal{J}_\lambda(\tau\varphi_1 + u^\top). \tag{8.6}$$

In the rest of our proof of Theorem 3.3 in §8.3 we will show that for $1 < p < 2$ the function $j_\lambda\colon \mathbb{R} \to \mathbb{R}$ attains a local maximum under the conditions of Theorem 3.3.

In analogy with the notation $\mathcal{J}_\lambda(u) \equiv \mathcal{J}_\lambda(u;f)$, we write also $j_\lambda(\tau) \equiv j_\lambda(\tau;f)$ if $f \in D^{-1,p'}(\mathbb{R}^N)$ varies, to avoid possible confusion.

Lemma 8.1. *Let $1 < p < N$. The mapping*

$$(\tau,\lambda,f) \mapsto j_\lambda(\tau;f)\colon \mathbb{R}\times[0,\Lambda_\infty - \eta]\times D^{-1,p'}(\mathbb{R}^N) \to \mathbb{R} \tag{8.7}$$

is continuous. In particular, if $0 < T < \infty$ and K is a compact set in $D^{-1,p'}(\mathbb{R}^N)$, then

$$\big\{j_\lambda(\,\cdot\,;f)\colon [-T,T] \to \mathbb{R}\colon\ (\lambda,f) \in [0,\Lambda_\infty - \eta]\times K\big\} \tag{8.8}$$

is a family of (uniformly) equicontinuous functions.

Proof. Let $\tau_n \to \tau_0$ in $\mathbb{R}$, $\mu_n \to \mu_0$ in $[0, \Lambda_\infty - \eta]$, and $f_n \to f_0$ in $D^{-1,p'}(\mathbb{R}^N)$ as $n \to \infty$. Suppose that $j_{\mu_n}(\tau_n; f_n)$ does *not* converge to $j_{\mu_0}(\tau_0; f_0)$ as $n \to \infty$. Passing to a subsequence if necessary, we may assume

$$\liminf_{n\to\infty} |j_{\mu_n}(\tau_n; f_n) - j_{\mu_0}(\tau_0; f_0)| > 0. \tag{8.9}$$

Consider any global minimizer $u_n^\top$ for the functional $u^\top \mapsto \mathcal{J}_{\mu_n}(\tau_n\varphi_1 + u^\top; f_n)$ on $D^{1,p}(\mathbb{R}^N)^\top$; $n = 1, 2, \dots$. The sequence $\{u_n^\top\}_{n=1}^\infty$ is bounded in $D^{1,p}(\mathbb{R}^N)$, by ineq. (8.4), and hence, it contains a weakly convergent subsequence (indexed by n again) $u_n^\top \rightharpoonup w^\top$ in $D^{1,p}(\mathbb{R}^N)^\top$ as $n \to \infty$. ¿From the weak lower semicontinuity of $\mathcal{J}_\lambda$ on $D^{1,p}(\mathbb{R}^N)$ we obtain

$$\begin{aligned} \liminf_{n\to\infty} j_{\mu_n}(\tau_n; f_n) &= \liminf_{n\to\infty} \mathcal{J}_{\mu_n}(\tau_n\varphi_1 + u_n^\top; f_n) \\ &\ge \mathcal{J}_{\mu_0}(\tau_0\varphi_1 + w^\top; f_0) \ge j_{\mu_0}(\tau_0; f_0). \end{aligned} \tag{8.10}$$

On the other hand, if $u_0^\top$ is any global minimizer for the functional $u^\top \mapsto \mathcal{J}_{\mu_0}(\tau_0\varphi_1 + u^\top; f_0)$ on $D^{1,p}(\mathbb{R}^N)^\top$, then one has

$$\begin{aligned} \limsup_{n\to\infty} j_{\mu_n}(\tau_n; f_n) &\le \lim_{n\to\infty} \mathcal{J}_{\mu_n}(\tau_n\varphi_1 + u_0^\top; f_n) \\ &= \mathcal{J}_{\mu_0}(\tau_0\varphi_1 + u_0^\top; f_0) = j_{\mu_0}(\tau_0; f_0). \end{aligned} \tag{8.11}$$

We combine inequalities (8.10) and (8.11) to get

$$\lim_{n\to\infty} j_{\mu_n}(\tau_n; f_n) = j_{\mu_0}(\tau_0; f_0)$$

which contradicts (8.9). The continuity of $(\tau, \lambda, f) \mapsto j_\lambda(\tau; f)$ is proved.

Finally, the equicontinuity of the family (8.8) is a consequence of the uniform continuity of the mapping (8.7) on the compact set $[-T, T] \times [0, \Lambda_\infty - \eta] \times K$. $\square$

Remark 8.2. We claim that in the proof of Lemma 8.1, $w^\top$ is a global minimizer for the functional $u^\top \mapsto \mathcal{J}_{\mu_0}(\tau_0\varphi_1 + u^\top; f_0)$ on $D^{1,p}(\mathbb{R}^N)^\top$ and we have also $u_n^\top \to w^\top$ strongly in $D^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. First of all, (8.10) and (8.11) imply

$$j_{\mu_n}(\tau_n; f_n) = \mathcal{J}_{\mu_n}(\tau_n\varphi_1 + u_n^\top; f_n) \to \mathcal{J}_{\mu_0}(\tau_0\varphi_1 + w^\top; f_0) = j_{\mu_0}(\tau_0; f_0).$$

Combining this result with $\tau_n \to \tau_0$, $\mu_n \to \mu_0$, $f_n \to f_0$ in $D^{-1,p'}(\mathbb{R}^N)$, $u_n^\top \rightharpoonup w^\top$ weakly in $D^{1,p}(\mathbb{R}^N)$, and $u_n^\top \to w^\top$ strongly in $L^p(\mathbb{R}^N; m)$, we arrive at

$$\|\tau_n\varphi_1 + u_n^\top\|_{D^{1,p}(\mathbb{R}^N)} \to \|\tau_0\varphi_1 + w^\top\|_{D^{1,p}(\mathbb{R}^N)} \quad \text{as } n \to \infty.$$

Thus, the uniform convexity of the standard norm on $D^{1,p}(\mathbb{R}^N)$ forces $\tau_n\varphi_1 + u_n^\top \to \tau_0\varphi_1 + w^\top$ strongly in $D^{1,p}(\mathbb{R}^N)$. Our claim now follows as $\tau_n \to \tau_0$.

Obviously, if the function $j_\lambda\colon \mathbb{R} \to \mathbb{R}$ has a local minimum at some point $\tau_0 \in \mathbb{R}$, and $u_0^\top$ is a global minimizer for the functional $u^\top \mapsto \mathcal{J}_\lambda(\tau_0\varphi_1 + u^\top)$ on $D^{1,p}(\mathbb{R}^N)^\top$, then $u_0 = \tau_0\varphi_1 + u_0^\top$ is a local minimizer for $\mathcal{J}_\lambda$ on $D^{1,p}(\mathbb{R}^N)$ and thus a weak solution to problem (1.1). Our next lemma displays a similar result if j_λ has a *local maximum* at $\tau_0 \in \mathbb{R}$; it claims that β_λ in (3.3) is a critical value of $\mathcal{J}_\lambda$.

Lemma 8.3. *Let $0 \le \lambda \le \Lambda_\infty - \eta$ and $f \in D^{-1,p'}(\mathbb{R}^N)$. Assume that the function $j_\lambda\colon \mathbb{R} \to \mathbb{R}$ attains a local maximum β_λ at some point $\tau_0 \in \mathbb{R}$. Then there exists $u_0^\top \in D^{1,p}(\mathbb{R}^N)^\top$ such that $u_0^\top$ is a global minimizer for the functional $u^\top \mapsto \mathcal{J}_\lambda(\tau_0\varphi_1 + u^\top)$ on $D^{1,p}(\mathbb{R}^N)^\top$, $u_0 = \tau_0\varphi_1 + u_0^\top$ is a critical point for $\mathcal{J}_\lambda$, and $\mathcal{J}_\lambda(u_0) = \beta_\lambda$.*

Proof. Given an arbitrary numerical sequence $\{\tau_n\}_{n=1}^{\infty}$ with $\tau_n \to \tau_0$ in $\mathbb{R}$ as $n \to \infty$ and $\tau_n \neq \tau_0$ for all $n \geq 1$, we can deduce from Remark 8.2 that this sequence contains a subsequence denoted again by $\{\tau_n\}_{n=1}^{\infty}$, such that for each $n = 0, 1, 2, \dots$, $u_n^{\top}$ is a global minimizer for the functional $u^{\top} \mapsto \mathcal{J}_\lambda(\tau_n\varphi_1 + u^{\top})$ and $u_n^{\top} \to u_0^{\top}$ strongly in $D^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. It follows that

$$\begin{aligned}\mathcal{J}_\lambda(\tau_n\varphi_1 + u_n^{\top}) - \mathcal{J}_\lambda(\tau_0\varphi_1 + u_n^{\top}) &\leq \mathcal{J}_\lambda(\tau_n\varphi_1 + u_n^{\top}) - \mathcal{J}_\lambda(\tau_0\varphi_1 + u_0^{\top}) \\ &= j_\lambda(\tau_n) - j_\lambda(\tau_0) \leq 0\end{aligned} \tag{8.12}$$

for all integers $n \geq 1$ sufficiently large; again, we may assume it for all $n \geq 1$.

On the other hand, denoting

$$\phi_n(s) \stackrel{\text{def}}{=} \tau_0\varphi_1 + u_n^{\top} + s(\tau_n - \tau_0)\varphi_1 \quad \text{for} \quad 0 \leq s \leq 1;\ n \geq 1,$$

we have

$$\mathcal{J}_\lambda(\tau_n\varphi_1 + u_n^{\top}) - \mathcal{J}_\lambda(\tau_0\varphi_1 + u_n^{\top}) = (\tau_n - \tau_0)\int_0^1 \langle \mathcal{J}_\lambda'(\phi_n(s)),\, \varphi_1\rangle \,\mathrm{d}s$$

where

$$\begin{aligned}\langle \mathcal{J}_\lambda'(\phi_n(s)),\, \varphi_1\rangle &= \int_{\mathbb{R}^N} |\nabla\phi_n(s)|^{p-2}\,\nabla\phi_n(s)\cdot\nabla\varphi_1\,\mathrm{d}x \\ &\quad - \lambda\int_{\mathbb{R}^N} |\phi_n(s)|^{p-2}\,\phi_n(s)\,\varphi_1\,m\,\mathrm{d}x - \int_{\mathbb{R}^N} f\varphi_1\,\mathrm{d}x.\end{aligned}$$

Since $\phi_n(s) \to u_0 = \tau_0\varphi_1 + u_0^{\top}$ strongly in $D^{1,p}(\mathbb{R}^N)$ and uniformly for $0 \leq s \leq 1$, we arrive at

$$\begin{aligned}&(\tau_n - \tau_0)^{-1}\left[\mathcal{J}_\lambda(\tau_n\varphi_1 + u_n^{\top}) - \mathcal{J}_\lambda(\tau_0\varphi_1 + u_n^{\top})\right] \\ &\longrightarrow\ \langle \mathcal{J}_\lambda'(u_0),\, \varphi_1\rangle = \zeta_0\,\|\varphi_1\|_{L^p(\mathbb{R}^N;m)} = \zeta_0 \quad \text{as } n \to \infty,\end{aligned} \tag{8.13}$$

where

$$\begin{aligned}\langle \mathcal{J}_\lambda'(u_0),\, \varphi_1\rangle &= \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\,\nabla u_0\cdot\nabla\varphi_1\,\mathrm{d}x \\ &\quad - \lambda\int_{\mathbb{R}^N} |u_0|^{p-2}u_0\varphi_1\,m\,\mathrm{d}x - \int_{\mathbb{R}^N} f\varphi_1\,\mathrm{d}x\end{aligned}$$

and $\zeta_0 \in \mathbb{R}$ is the Lagrange multiplier given by $\mathcal{J}_\lambda'(u_0) = \zeta_0\,m\,\varphi_1^{p-1}$.

Finally, if we choose τ_n such that the sign of $(\tau_n - \tau_0)$ does not change for all $n = 1, 2, \dots$, then (8.12) and (8.13) yield $\zeta_0 \leq 0$ if $\operatorname{sgn}(\tau_n - \tau_0) = 1$, and $\zeta_0 \geq 0$ if $\operatorname{sgn}(\tau_n - \tau_0) = -1$. Since both alternatives are possible, we conclude that $\zeta_0 = 0$ which shows $\mathcal{J}_\lambda'(u_0) = 0$, i.e., u_0 is a weak solution to problem (1.6) as desired. In particular, $\mathcal{J}_\lambda(u_0)$ is a critical value of $\mathcal{J}_\lambda$. □

Remark 8.4. As an easy consequence of (8.12), (8.13) in the proof of Lemma 8.3, we conclude that the function $j_\lambda\colon \mathbb{R} \to \mathbb{R}$ is differentiable at τ_0 with $j_\lambda'(\tau_0) = 0$.

8.3. **Rest of the proof of Theorem 3.3.** We deduce from Lemma 3.9 that there exist $a, b \in \mathbb{R}$ such that $a < 0 < b$ and

$$\max\{j_{\lambda_1}(a; f^{\#}),\, j_{\lambda_1}(b; f^{\#})\} < j_{\lambda_1}(0; f^{\#}).$$

The "continuity" Lemma 8.1 shows that there exist numbers $\delta \equiv \delta(f^{\#}) > 0$ and $\varrho \equiv \varrho(f^{\#}) > 0$ such that also with $f = f^{\#} + \zeta\,m\varphi_1^{p-1}$ we have

$$\max\{j_\lambda(a; f),\, j_\lambda(b; f)\} < j_\lambda(0; f)$$

for all $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$ and all $\zeta \in (-\varrho, \varrho)$. Now we can apply Lemma 8.3 to conclude that the functional $\mathcal{J}_\lambda(\,\cdot\,; f)$ possesses a critical point $u_1 = \tau_1\varphi_1 + u_1^\top$, with some $\tau_1 \in (a, b)$ and $u_1^\top \in D^{1,p}(\mathbb{R}^N)^\top$. This proves Theorem 3.3.

Proof of Remark 3.4. If $\lambda < \lambda_1$ then we have $j_\lambda(\tau; f) \to +\infty$ as $|\tau| \to \infty$. Consequently, for $\lambda \in (\lambda_1 - \delta, \lambda_1)$ and $\zeta \in (-\varrho, \varrho)$, the continuous function $j_\lambda(\,\cdot\,; f) \colon \mathbb{R} \to \mathbb{R}$ possesses also a local minimizer in each of the intervals $(-\infty, \tau_1)$ and (τ_1, ∞), say, τ_2 and τ_3, respectively. Our definition of $j_\lambda(\,\cdot\,; f)$ now shows that $u_2 = \tau_2\varphi_1 + u_2^\top$ and $u_3 = \tau_3\varphi_1 + u_3^\top$ are local minimizers for $\mathcal{J}_\lambda(\,\cdot\,; f)$, with some $u_2^\top, u_3^\top \in D^{1,p}(\mathbb{R}^N)^\top$, as claimed. □

9. Appendix: Asymptotics of the eigenfunction φ_1

To determine the asymptotic behavior of the first eigenfunction φ_1 of the p-Laplacian Δ_p on $\mathbb{R}^N$ subject to a weight $m(|x|)$, for $1 < p < N$, we consider a strictly positive, radially symmetric function $u \colon \mathbb{R}^N \to (0, \infty)$ of class C^1, $u(x) \equiv u(r)$ with $r \equiv |x|$, $x \in \mathbb{R}^N$, which satisfies the following partial differential equation (in the sense of distributions on $\mathbb{R}^N$):

$$-\Delta_p u = m(|x|)\, u^{p-1} \quad \text{for } x \in \mathbb{R}^N; \quad u(|x|) \to 0 \ \text{ as } |x| \to \infty. \tag{9.1}$$

We weaken the strict positivity in hypothesis (H) on the weight m as follows:

(**H'**) There exist constants $\delta > 0$ and $C > 0$ such that

$$0 \le m(r) \le \frac{C}{(1 + r)^{p+\delta}} \quad \text{for almost all } \ 0 \le r < \infty, \tag{9.2}$$

and $m \not\equiv 0$ in $\mathbb{R}_+$.

Under this hypothesis, we are able to establish the following asymptotic behavior of $u(r)$ and $u'(r)$ as $r \to \infty$.

Proposition 9.1. *There exists a constant $c > 0$ such that*

$$\lim_{r\to\infty} \Big(u(r)\, r^{\frac{N-p}{p-1}} \Big) = c, \tag{9.3}$$

$$\lim_{r\to\infty} \Big(u'(r)\, r^{\frac{N-1}{p-1}} \Big) = -\tfrac{N-p}{p-1}\, c. \tag{9.4}$$

For the related Cauchy problem,

$$-\Delta_p u(|x|) = f(u(|x|)) \ \text{ for } x \in \mathbb{R}^N; \qquad u(|x|) \to 0 \ \text{ as } |x| \to \infty, \tag{9.5}$$

with $f(u) \ge 0$ for $u > 0$ sufficiently small, the inequalities

$$u(r)\, r^{\frac{N-p}{p-1}} \ge c_1 > 0 \quad \text{and} \quad -\, u'(r)\, r^{\frac{N-1}{p-1}} \ge c_2 > 0$$

for all sufficiently large $r > 0$ (with some constants c_1 and c_2) have been established in the work of Ni and Serrin [16, Theorem 6.1]. Their method of proof applies also to our case. For the inequality

$$-\Delta_p u \le m(|x|)\, u^{p-1} \quad \text{for } x \in \mathbb{R}^N; \quad u(x) \to 0 \ \text{ as } |x| \to \infty, \tag{9.6}$$

with $u(x)$ not necessarily radially symmetric, $u(x) > 0$, but with the weight $m(r)$ decaying at infinity faster than ours, an upper estimate on the decay of u at infinity can be found in Fleckinger, Harrell and de Thélin [7, Theorem IV.2].

In the proof of Proposition 9.1 we need a few auxiliary results.

The Cauchy problem (9.1) is equivalent to

$$\begin{aligned} &-(|u'|^{p-2}u')' - \frac{N-1}{r}\,|u'|^{p-2}u' = m(r)\,u^{p-1} \quad \text{for } r>0;\\ &u'(r)\to 0 \text{ as } r\to 0 \quad\text{and}\quad u(r)\to 0 \text{ as } r\to\infty. \end{aligned} \tag{9.7}$$

This problem can be rewritten as

$$\begin{aligned} &-(r^{N-1}\,|u'|^{p-2}u')' = m(r)\,r^{N-1}\,u^{p-1} \quad \text{for } r>0;\\ &u'(r)\to 0 \text{ as } r\to 0 \quad\text{and}\quad u(r)\to 0 \text{ as } r\to\infty. \end{aligned} \tag{9.8}$$

We reduce this second-order differential equation to a first-order equation by introducing the Riccati-type transformation

$$U(r) \overset{\text{def}}{=} -r^{p-1}\Big|\frac{u'(r)}{u(r)}\Big|^{p-2}\frac{u'(r)}{u(r)} \quad \text{for } r>0, \quad U(0)\overset{\text{def}}{=} 0. \tag{9.9}$$

By (9.8), the function $r\mapsto r^{\frac{N-1}{p-1}}\,u'(r)$ is nonincreasing for $0<r<\infty$ which implies $u'(r)\le 0$ for all $r>0$, and therefore also $U(r)\ge 0$. Hence, for $r>0$,

$$\begin{aligned} U'(r) &= -(p-1)r^{p-2}\Big|\frac{u'}{u}\Big|^{p-2}\frac{u'}{u} - (p-1)r^{p-1}\Big|\frac{u'}{u}\Big|^{p-2}\Big[\frac{u''}{u} - \Big(\frac{u'}{u}\Big)^2\Big]\\ &= \frac{p-1}{r}\,U(r) - r^{p-1}\,\frac{(|u'|^{p-2}u')'}{|u|^{p-2}u} + \frac{p-1}{r}\,U(r)^{\frac{p}{p-1}}. \end{aligned}$$

Inserting the second derivative expression from equation (9.7), we arrive at

$$U'(r) = -\frac{N-p}{r}\,U(r) + \frac{p-1}{r}\,U(r)^{\frac{p}{p-1}} + m(r)\,r^{p-1}.$$

This is a differential equation for the unknown function U which we rewrite as

$$U'(r) = \frac{p-1}{r}\,U(r)\Big(U(r)^{\frac{1}{p-1}} - \frac{N-p}{p-1}\Big) + m(r)\,r^{p-1} \quad \text{for } r>0. \tag{9.10}$$

An upper bound for $U(r)$ is obtained first:

Lemma 9.2. *We have*

$$U(r) \le c_{N,p} \overset{\text{def}}{=} \Big(\frac{N-p}{p-1}\Big)^{p-1} \quad \textit{for all } r\ge 0. \tag{9.11}$$

Proof. Clearly, by (9.9), the function $U\colon \mathbb{R}_+\to\mathbb{R}$ is continuous and, by (9.10), it is differentiable almost everywhere with the derivative U' being locally bounded. Now, in contradiction with (9.11), suppose that there exists a number $r_0\ge 0$ such that $U(r_0)>c_{N,p}$. Let

$$r_1 \overset{\text{def}}{=} \sup\{r' : r'\ge r_0 \text{ and } U(r)>c_{N,p} \text{ for all } r_0\le r\le r'\}.$$

Next we show that $r_1=\infty$. Indeed, equation (9.10) with $U(r)>c_{N,p}$ and $m(r)\ge 0$ for $r_0\le r<r_1$ implies $U'(r)>0$. This shows that the function $U(r)$ is strictly increasing for $r_0\le r<r_1$. Consequently, $r_1<\infty$ would yield $U(r_1)=c_{N,p}<U(r_0)<U(r_1)$ which is impossible.

Hence, there is a constant $\gamma>0$ such that the expression inside the parenthesis in eq. (9.10) satisfies

$$U(r)^{\frac{1}{p-1}} - \frac{N-p}{p-1} \ge \gamma\,U(r)^{\frac{1}{p-1}} \quad \text{for all } r\ge r_0.$$

Applying this inequality to equation (9.10) we obtain

$$U'(r) \geq \frac{p-1}{r}\,\gamma\,U(r)^{\frac{p}{p-1}} \quad \text{for all } r \geq r_0.$$

We integrate this inequality over the interval $[r_0, r]$ to get

$$U(r_0)^{-\frac{1}{p-1}} - U(r)^{-\frac{1}{p-1}} \geq \gamma\,\log(r/r_0) \quad \text{for all } r \geq r_0.$$

Recalling $U(r) > 0$ and letting $r \to \infty$, we arrive at $U(r_0)^{-\frac{1}{p-1}} \geq +\infty$, which is a contradiction. Inequality (9.11) is proved. $\square$

Define the function

$$a(r) \stackrel{\text{def}}{=} \frac{p-1}{r}\Big(\frac{N-p}{p-1} - U(r)^{\frac{1}{p-1}}\Big) \quad \text{for } r > 0. \tag{9.12}$$

Note that $a(r) \geq 0$ by Lemma 9.2, and

$$a(r) = \frac{N-p}{r} + (p-1)\,\frac{u'(r)}{u(r)} = (p-1)\,\frac{\mathrm{d}}{\mathrm{d}r}\log\Big(u(r)\,r^{\frac{N-p}{p-1}}\Big).$$

We substitute this function into eq. (9.10) and use integrating factor to integrate it over any interval $[r_0, r]$ with $r_0 > 0$ fixed and $r \geq r_0$. We thus obtain

$$U(r) - U(r_0)\,e^{-\int_{r_0}^{r} a(s)\,\mathrm{d}s} = \int_{r_0}^{r} m(s)\,s^{p-1}\,e^{-\int_{s}^{r} a(t)\,\mathrm{d}t}\,\mathrm{d}s. \tag{9.13}$$

Furthermore, we introduce the abbreviation

$$A(r) \stackrel{\text{def}}{=} \int_{r_0}^{r} a(s)\,\mathrm{d}s = (p-1)\,\log\frac{u(r)\,r^{\frac{N-p}{p-1}}}{u(r_0)\,r_0^{\frac{N-p}{p-1}}} \quad \text{for } r \geq r_0. \tag{9.14}$$

Lemma 9.3. *We have $a(r) \geq 0$ for all $r > 0$ and*

$$\int_{r_0}^{\infty} a(r)\,\mathrm{d}r < \infty \quad \textit{for every } r_0 > 0. \tag{9.15}$$

Proof. The function $A(r)$ is nondecreasing for $r_0 \leq r < \infty$. Now, suppose that $\lim_{r\to\infty} A(r) = +\infty$. From equation (9.13) we deduce

$$0 \leq U(r) - U(r_0)\,e^{-A(r)} \leq \int_{r_0}^{\infty} m(s)\,s^{p-1}\,e^{-(A(r)-A(s))}\,\mathrm{d}s \quad \text{for } r \geq r_0. \tag{9.16}$$

Due to our hypothesis (H'), we are allowed to apply Lebesgue's dominated convergence theorem to the last integral to obtain, as $r \to \infty$, $0 \leq \lim_{r\to\infty} U(r) \leq 0$, i.e., $\lim_{r\to\infty} U(r) = 0$.

This shows that, given any number η such that $0 < \eta < N - p$, there exists a number $r_\eta \geq r_0$ such that

$$a(r) = \frac{p-1}{r}\Big(\frac{N-p}{p-1} - U(r)^{\frac{1}{p-1}}\Big) \geq \frac{N-p-\eta}{r} \quad \text{for all } r \geq r_\eta.$$

Since r_0 is arbitrary, $r_0 > 0$, we may take $r_0 = r_\eta$. Upon integration, we get

$$A(r) \geq (N-p-\eta)\int_{r_0}^{r} \frac{\mathrm{d}s}{s} = \log\big((r/r_0)^{N-p-\eta}\big) \quad \text{for all } r \geq r_0. \tag{9.17}$$

We apply inequalities (9.2) and (9.17) to equation (9.13) to obtain for all $r \geq 0$,

$$\begin{aligned} U(r) &\leq U(r_0)\big(\frac{r}{r_0}\big)^{-(N-p-\eta)} + C\int_{r_0}^{r} \frac{s^{p-1}}{(1+s)^{p+\delta}}\big(\frac{r}{s}\big)^{-(N-p-\eta)}\,\mathrm{d}s \\ &\leq U(r_0)\big(\frac{r}{r_0}\big)^{-(N-p-\eta)} + \frac{C\,r^{-(N-p-\eta)}}{N-p-\eta-\delta}\Big(r^{N-p-\eta-\delta} - r_0^{N-p-\eta-\delta}\Big). \end{aligned} \tag{9.18}$$

Note that in inequality (9.2), the constant $\delta > 0$ may be chosen arbitrarily small; we choose it such that $0 < \delta < N-p-\eta$. Hence, (9.18) yields

$$U(r) \leq C_0\,r^{-\delta} \quad \text{for all } r \geq r_0,$$

where $C_0 > 0$ is a constant. With our definition of U we have equivalently

$$-\frac{u'(r)}{u(r)} \leq C_0^{\frac{1}{p-1}}\,r^{-1-\frac{\delta}{p-1}} \quad \text{for all } r \geq r_0.$$

Upon integration we get

$$-\log\frac{u(r)}{u(r_0)} \leq C_0'\big(r_0^{-\frac{\delta}{p-1}} - r^{-\frac{\delta}{p-1}}\big) \quad \text{for all } r \geq r_0,$$

where $C_0' > 0$ is a constant. Recalling $u(r) \to 0$ as $r \to \infty$, we arrive at $+\infty \leq C_0'\,r_0^{-\frac{\delta}{p-1}}$ which is absurd. The proof of the lemma is complete. $\square$

Finally, we determine the limit of the function U at infinity.

Lemma 9.4. *We have*

$$\lim_{r\to\infty} U(r) = c_{N,p} = \big(\frac{N-p}{p-1}\big)^{p-1}. \tag{9.19}$$

Proof. The limit

$$A(\infty) \stackrel{\text{def}}{=} \lim_{r\to\infty} A(r) = \int_{r_0}^{\infty} a(r)\,\mathrm{d}r$$

exists and satisfies $0 \leq A(\infty) < \infty$, by (9.14) and (9.15). We apply this fact and hypothesis (H') to equation (9.13) to obtain the existence of the limit

$$U(\infty) \stackrel{\text{def}}{=} \lim_{r\to\infty} U(r) = U(r_0)\,e^{-A(\infty)} + \int_{r_0}^{\infty} m(s)\,s^{p-1}\,e^{-(A(\infty)-A(s))}\,\mathrm{d}s, \tag{9.20}$$

using Lebesgue's dominated convergence theorem. We have $U(\infty) \leq c_{N,p}$ by (9.11). However, if $U(\infty) < c_{N,p}$ then there exist constants $\gamma > 0$ and $r_1 \geq r_0$ such that

$$a(r) = \frac{p-1}{r}\Big(\frac{N-p}{p-1} - U(r)^{\frac{1}{p-1}}\Big) \geq \frac{\gamma}{r} \quad \text{for all } r \geq r_1.$$

But this inequality contradicts (9.15). We have proved (9.19). $\square$

Finally, we are ready to derive formulas (9.3) and (9.4).

Proof of Proposition 9.1. We combine (9.14) and (9.15) to conclude that the limit

$$c_0 \stackrel{\text{def}}{=} \lim_{r\to\infty} \log\Big(u(r)\,r^{\frac{N-p}{p-1}} \Big/ u(r_0)\,r_0^{\frac{N-p}{p-1}}\Big)$$

exists and satisfies $0 \leq c_0 < \infty$. The desired formula (9.3) follows immediately with $c \stackrel{\text{def}}{=} e^{c_0}\,u(r_0)\,r_0^{\frac{N-p}{p-1}} > 0$. The convergence formula (9.19) reads

$$-\,r\,\frac{u'(r)}{u(r)} \longrightarrow \frac{N-p}{p-1} \quad \text{as } r \to \infty. \tag{9.21}$$

We combine this result with (9.3) to get (9.4). The proposition is proved. □

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Bénédicte Alziary
CEREMATH & UMR MIP, Université Toulouse 1 – Sciences Sociales, 21 allées de Brienne, F–31042 Toulouse Cedex, France
E-mail address: alziary@univ-tlse1.fr

Jacqueline Fleckinger
CEREMATH & UMR MIP, Université Toulouse 1 – Sciences Sociales, 21 allées de Brienne, F–31042 Toulouse Cedex, France
E-mail address: jfleck@univ-tlse1.fr

Peter Takáč
Fachbereich Mathematik, Universität Rostock, Universitätsplatz 1, D–18055 Rostock, Germany
E-mail address: peter.takac@mathematik.uni-rostock.de