

RECOVERING A TIME- AND SPACE-DEPENDENT KERNEL IN A HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATION FROM A RESTRICTED DIRICHLET-TO-NEUMANN OPERATOR

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ABSTRACT. We prove that a space- and time-dependent kernel occurring in a hyperbolic integro-differential equation in three space dimensions can be uniquely reconstructed from the restriction of the Dirichlet-to-Neumann operator of the equation into a set of Dirichlet data of the form of products of a fixed time-dependent coefficient times arbitrary space-dependent functions.

1. INTRODUCTION

Motion of viscoelastic materials is governed by hyperbolic integro-differential equations involving time-dependent (and in the case of inhomogeneity also space-dependent) kernels [22, 23]. These kernels, which describe the relaxation of the material, are often unknown or scarcely known in practice. To determine these kernels, inverse problems are used.

Inverse problems for space-independent kernels in hyperbolic equations are well-studied (see e.g. [4, 5, 6, 7, 10, 11, 15, 16, 19, 27]). Some results are obtained in the case of space-dependent kernels, too. For instance, in [18, 19] the identification of kernels depending on partial space variables or satisfying certain spherical symmetry conditions, was studied. The papers [13, 16, 17] consider the determination of kernels representable as finite sums of products of known space-dependent and unknown time-dependent functions.

In this paper we consider a hyperbolic integro-differential equation in a three-dimensional domain, which contains a time- and space-dependent kernel. We do not assume any special form of this kernel and study an inverse problem to recover the kernel from the Dirichlet-to-Neumann operator (DNO) of the equation.

A global uniqueness result for an inverse problem involving DNO of an elliptic equation was first time proved in the paper [24]. Later on this result was extended to several identification problems of inhomogeneous media, in particular some problems to identify space- and time-dependent coefficients of evolutionary equations (see, e.g., [8, 9, 25]). In [3] a problem to identify a kernel contained in a lower order term of a hyperbolic equation was studied. In the mentioned papers the time-

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and space-dependent unknowns are recovered by means of the full DNO, i.e., DNO given in all space- and time-dependent Dirichlet data.

On the other hand, inverse problems using full DNO are highly over-determined. In this paper we will show how it is possible to reduce the amount of information related to DNO in a case of a special problem of such kind. Namely, we will prove that the identification of the kernel in the above-mentioned hyperbolic integro-differential equation does not require the full DNO. The kernel can be uniquely recovered from a restriction of DNO into a set of Dirichlet data which contain products of a fixed time-dependent coefficient times arbitrary space-dependent functions. A similar result for a parabolic problem was earlier proved by the author in [12].

2. PROBLEM FORMULATION AND RESULTS

Let us start by introducing some notation. Given a Banach space X , $\sigma \in \mathbb{R}$, $k \in \{0, 1, 2, \dots\}$ and $p \in [1, \infty]$ we define the following sets of abstract functions with values in X :

$$\begin{aligned} W_{\sigma}^{k,p}((0, \infty); X) &= \{z : (0, \infty) \rightarrow X : e^{-\sigma \cdot} z(\cdot) \in W^{k,p}((0, \infty); X)\}, \\ C_{\sigma}^k([0, \infty); X) &= \{z : [0, \infty) \rightarrow X : e^{-\sigma \cdot} z(\cdot) \in C^k([0, \infty); X)\}. \end{aligned}$$

Here $W^{k,p}((0, \infty); X)$ is the abstract Sobolev space and $C^k([0, \infty); X)$ consists of abstract functions bounded and continuous in $[0, \infty)$ together with their derivatives up to the order k . In case $k = 0$ we set $L_{\sigma}^p((0, \infty); X) := W_{\sigma}^{0,p}((0, \infty); X)$ and $C_{\sigma}([0, \infty); X) := C_{\sigma}^0([0, \infty); X)$. Moreover, in case $X = \mathbb{C}$, we use the simplified notation $W_{\sigma}^{k,p}(0, \infty) := W_{\sigma}^{k,p}((0, \infty); \mathbb{C})$ and $C_{\sigma}^k[0, \infty) := C_{\sigma}^k([0, \infty); \mathbb{C})$. Finally, given $k \in \{0, 1, 2, \dots\}$ and $\mu > 0$, we denote by $C^{k,\mu}(\Omega)$ the space of functions which are Hölder-continuous of degree λ in Ω together with their derivatives up to the order k .

Let Ω be a three-dimensional domain with a Lipschitz-boundary Γ . In case Ω is filled by the isotropic non-homogeneous viscoelastic material, the constitutive relations and the system of equations of motion of the material point of Ω contain time- and space-dependent relaxation functions $a(t, x)$ and $b(t, x)$, which correspond to the shear and bulk moduli, respectively (see [22, p. 122-127]). We suppose that these functions are unknown. Unfortunately, inverse problems to determine both $a(t, x)$ and $b(t, x)$ in the viscoelasticity system, are very complex. Therefore we essentially simplify the situation taking into consideration a partially theoretical scalar model of such kind. If the displacement field is solenoidal, then the system of equations of motion is reduced to three independent equations of divergence type, which can be summarized as

$$\rho(x) \partial_t^2 v(t, x) = \int_{-\infty}^t \operatorname{div} [a(t - \tau, x) \nabla \partial_{\tau} v(\tau, x)] d\tau, \quad x \in \Omega, t \in \mathbb{R}, \quad (2.1)$$

where v is an arbitrary linear combination of coordinates of the displacement and ρ is the density. We will pose and study an inverse problem that consists in determining the kernel $a(t, x)$ in (2.1) for $t \geq 0$ and $x \in \Omega$.

Let us assume $v(t, x) = 0$ for $t \leq 0$. Then the equation (2.1) for twice differentiable $v(t, \cdot)$ is equivalent to the integrated equation

$$\rho(x) \partial_t v(t, x) = \int_0^t (t - \tau) \operatorname{div} [a(t - \tau, x) \nabla \partial_{\tau} v(\tau, x)] d\tau, \quad x \in \Omega, t \in \mathbb{R}. \quad (2.2)$$

We interpret the latter equation as a generalized form of (2.1) and complement it with the initial and Dirichlet boundary conditions

$$v(0, x) = 0, \quad x \in \Omega, \quad v(t, x) = \psi(t, x), \quad x \in \Gamma, t > 0. \quad (2.3)$$

Let us associate with v the secondary variable $s(t, x) = \int_{-\infty}^t a(t - \tau, x) \nabla \partial_\tau v(\tau, x) d\tau$ and denote by $\nu(x)$ the outer normal of the boundary of Ω at the point $x \in \Gamma$. Given ρ, a and ψ , the solution v of the initial-boundary value problem (2.2), (2.3) determines the normal component of s at the boundary:

$$h(t, x) := -\nu(x) \cdot s(t, x) = \int_0^t a(t - \tau, x) \partial_\nu \partial_t v(\tau, x) d\tau, \quad x \in \Gamma, t > 0. \quad (2.4)$$

The operator D , which depends on ρ and a , and maps the Dirichlet data ψ via the solution v to the Neumann data h , is called the Dirichlet-to-Neumann operator of the problem (2.2), (2.3).

We begin by eliminating the derivative in the equation (2.2) by the change of variable

$$u(t, x) = \partial_t v(t, x) \Leftrightarrow v(t, x) = \int_0^t u(\tau, x) d\tau. \quad (2.5)$$

Defining $\varphi(t, x) := \partial_t \psi(t, x)$, the problem (2.2), (2.3) is transformed to

$$\rho(x)u(t, x) = \int_0^t (t - \tau) \operatorname{div} [a(t - \tau, x) \nabla u(\tau, x)] d\tau, \quad x \in \Omega, t > 0, \quad (2.6)$$

$$u(t, x) = \varphi(t, x), \quad x \in \Gamma, t > 0. \quad (2.7)$$

For the obtained problem have the following theorem which will be proved in section 3.

Theorem 2.1. *Let*

$$\rho \in L^\infty(\Omega), \quad \rho(x) \geq \rho_0 > 0, \quad x \in \Omega \quad (2.8)$$

and

$$a \in W_{\sigma_0}^{1,1}((0, \infty); W^{1,\infty}(\Omega)) \cap W_{\sigma_0}^{2,1}((0, \infty); L^\infty(\Omega)), \quad (2.9)$$

$$a(0, x) \geq a_0 > 0, \quad x \in \Omega$$

with some $\sigma_0 \in \mathbb{R}$. Moreover, let

$$\varphi \in W_{\sigma_1}^{5,1}((0, \infty); H^{3/2}(\Gamma)), \quad \varphi(0, x) = \partial_t \varphi(0, x) = \dots = \partial_t^4 \varphi(0, x) \equiv 0 \quad (2.10)$$

with some $\sigma_1 \in \mathbb{R}$.

Then there exists σ_a , which depends on a and satisfies the inequalities $\sigma_a > 0$, $\sigma_a \geq \sigma_0$, such that the problem (2.6), (2.7) has a solution u in the space $C_{\sigma_u}([0, \infty); H^2(\Omega))$, where $\sigma_u = \max\{\sigma_1, \sigma_a\}$. Moreover, $u(0, x) \equiv 0$. The solution is unique in the space $L_{\sigma_u}^1((0, \infty); H^2(\Omega))$.

Due to (2.4) and (2.5) the Neumann data h has the form

$$h(t, x) = \int_0^t a(t - \tau, x) \partial_\nu u(\tau, x) d\tau, \quad x \in \Gamma, t > 0. \quad (2.11)$$

Theorem 2.1 with the trace theorem implies the following result.

Corollary 2.2. *Under the assumptions of Theorem 2.1, $h \in C_{\sigma_u}^1([0, \infty); L^2(\Gamma))$.*

Let \tilde{D} be the operator which assigns to a function φ the function h via the solution u of the problem (2.6), (2.7). Observing Corollary 2.2 we see that \tilde{D} transforms the set of functions φ satisfying (2.10) into the space $C_{\sigma_u}^1([0, \infty); L^2(\Gamma))$, provided ρ and a meet the conditions (2.8) and (2.9). Moreover, $D\psi = \tilde{D}\partial_t\psi$. Therefore, the operator D transforms the space

$$\left\{ \psi \in W_{\sigma_1}^{6,1}((0, \infty); H^{3/2}(\Gamma)) \mid \psi(0, x) = \partial_t\psi(0, x) = \dots = \partial_t^5\psi(0, x) \equiv 0 \right\}$$

into the space $C_{\sigma_u}^1([0, \infty); L^2(\Gamma))$, provided ρ and a meet the conditions (2.8) and (2.9).

Let us choose a function f such that

$$f \in W_{\sigma_1}^{5,1}(0, \infty), \quad f \neq 0, \quad \text{Im } f = 0, \quad f(0) = f'(0) = \dots = f^{IV}(0) = 0, \quad (2.12)$$

where $\sigma_1 \in \mathbb{R}$. For given f , ρ and a satisfying (2.12), (2.8) and (2.9), respectively, let us define the operator $\lambda_a : H^{3/2}(\Gamma) \rightarrow C_{\sigma_u}^1([0, \infty); L^2(\Gamma))$ by the following relation

$$\lambda_a g := \tilde{D}(f(t)g(x)) = D\left(\int_0^t f(\tau)d\tau g(x)\right).$$

The main result of the paper is the following theorem, which asserts that a is uniquely recovered by λ_a , in other words, by the restriction of the Dirichlet-to-Neumann operator D in the set of Dirichlet data of the form $\psi(t, x) = \int_0^t f(\tau)d\tau g(x)$ with fixed f .

Theorem 2.3. *Let Γ_ϵ be some neighbourhood of Γ . Assume that (2.8), (2.12) are valid and $\rho \in C^{0,\mu}(\Gamma_\epsilon)$ with some $\mu > 0$. Furthermore, let a_1 and a_2 be two functions satisfying (2.9) and the relations*

$$a_j \in L_{\sigma_0}^1((0, \infty); W^{2,\infty}(\Omega) \cap C^{1,\mu}(\Gamma_\epsilon)), \quad \text{Im } a_j = 0, \quad j = 1, 2. \quad (2.13)$$

Then the equality $\lambda_{a_1} = \lambda_{a_2}$ implies $a_1 = a_2$.

The proof of this theorem will be given in section 5. It uses a preliminary result concerning the boundary identifiability proved in section 4.

3. AUXILIARY RESULTS

Let X be a complex Banach space and $z \in L_\sigma^1((0, \infty); X)$ with some $\sigma \in \mathbb{R}$. Then the Laplace transform of z , i.e.,

$$Z(p) = \mathcal{L}_{t \rightarrow p}(z) = \int_0^\infty e^{-pt} z(t) dt \quad (3.1)$$

exists in the half plane $\text{Re } p > \sigma$ and is holomorphic there (see [22]).

Lemma 3.1. *Let $z \in L_\sigma^1((0, \infty); X)$ with some $\sigma \in \mathbb{R}$. Then*

$$\|Z(p)\|_X \leq C(z, \text{Re } p) \quad \text{for } \text{Re } p > \sigma, \quad (3.2)$$

where

$$C(z, s) = \int_0^\infty e^{-(s-\sigma)t} \|e^{-\sigma t} z(t)\|_X dt \rightarrow 0 \quad \text{as } s \rightarrow \infty. \quad (3.3)$$

If, in addition, $z \in W_\sigma^{k,1}((0, \infty); X)$ with some $k \in \{1, 2, \dots\}$ then

$$\|p^k Z(p) - p^{k-1} z(0) - p^{k-2} z'(0) - \dots - z^{(k-1)}(0)\|_X \leq C(z^{(k)}, \text{Re } p) \quad \text{for } \text{Re } p > \sigma, \quad (3.4)$$

which in case $z(0) = z'(0) = \dots = z^{(k-1)}(0) = 0$ implies

$$|p|^k \|Z(p)\| \leq C(z^{(k)}, \operatorname{Re} p) \quad \text{for } \operatorname{Re} p > \sigma. \quad (3.5)$$

The proof of the lemma above can be found in [12].

Lemma 3.2. *Let $Z : \mathbb{C} \rightarrow X$ be holomorphic for $\operatorname{Re} p > \sigma$ with some $\sigma \in \mathbb{R}$ and*

$$|p|^2 \|Z(p)\|_X \leq c_0 \quad \text{for } \operatorname{Re} p > \sigma \quad (3.6)$$

with a constant c_0 . Then there exists a function $z \in C_\sigma([0, \infty); X)$ such that $Z(p) = \mathcal{L}_{t \rightarrow p}(z)$. Moreover, $z(0) = 0$.

The assertion of the lemma above follows from [22, Proposition 0.2].

Lemma 3.3. *Let a satisfy (2.9) with some $\sigma_0 \in \mathbb{R}$ and $A(p, x) = \mathcal{L}_{t \rightarrow p}(a(\cdot, x))$. Then*

$$|p| \|A(p, \cdot)\|_{W^{1, \infty}(\Omega)} \leq c_1 \quad \text{for } \operatorname{Re} p > \sigma_0 \quad (3.7)$$

with a constant c_1 and

$$\begin{aligned} pA(p, \cdot) &\rightarrow a(0, \cdot) \quad \text{as } \operatorname{Re} p \rightarrow \infty \\ &\text{in } W^{1, \infty}(\Omega) \text{ uniformly with respect to } \operatorname{Im} p. \end{aligned} \quad (3.8)$$

Moreover, there exists σ_a , satisfying the inequalities $\sigma_a > 0$, $\sigma_a \geq \sigma_0$, such that

$$|p| |A(p, x)| \geq \kappa > 0 \quad \text{for } \operatorname{Re} p > \sigma_a, \quad x \in \Omega, \quad (3.9)$$

$$\operatorname{Re} p a(0, x) - |p a(0, x) - p^2 A(p, x)| \geq \kappa > 0 \quad \text{for } \operatorname{Re} p > \sigma_a, \quad x \in \Omega. \quad (3.10)$$

Proof. Using the estimate (3.4) for a we obtain

$$\|pA(p, \cdot) - a(0, \cdot)\|_{W^{1, \infty}(\Omega)} \leq C(\partial_t a, \operatorname{Re} p) \quad \text{for } \operatorname{Re} p > \sigma_0, \quad (3.11)$$

$$\|p^2 A(p, \cdot) - pa(0, \cdot) - \partial_t a(0, \cdot)\|_{L^\infty(\Omega)} \leq C(\partial_t^2 a, \operatorname{Re} p) \quad \text{for } \operatorname{Re} p > \sigma_0. \quad (3.12)$$

In view of (3.3) and the assumption $a(0, x) \geq a_0 > 0$ the relation (3.11) implies (3.7) - (3.9) and (3.12) yields (3.10). \square

Lemma 3.4. *Let φ satisfy (2.10) with some $\sigma_1 \in \mathbb{R}$ and $\Phi(p, x) = \mathcal{L}_{t \rightarrow p}(\varphi(\cdot, x))$. Then*

$$|p|^5 \|\Phi(p, \cdot)\|_{H^{3/2}(\Gamma)} \leq c_2 \quad \text{for } \operatorname{Re} p > \sigma_1 \quad (3.13)$$

with a constant c_2 .

The assertion of this lemma follows from (3.5) and (3.3).

4. DIRECT PROBLEM

The direct problem (2.6), (2.7) is formally equivalent to the following elliptic boundary value problem derived by means of the Laplace transform:

$$(L_A U)(p, x) \equiv -\operatorname{div}(A(p, x) \nabla U(p, x)) + p\rho(x)U(p, x) = 0, \quad x \in \Omega, \quad (4.1)$$

$$U(p, x) = \Phi(p, x), \quad x \in \Gamma. \quad (4.2)$$

Here $p \in \mathbb{C}$, $A(p, x) = \mathcal{L}_{t \rightarrow p}(a(\cdot, x))$, $\Phi(p, x) = \mathcal{L}_{t \rightarrow p}(\varphi(\cdot, x))$, and $U(p, x) = \mathcal{L}_{t \rightarrow p}(u(\cdot, x))$.

Proposition 4.1. *Let ρ satisfy (2.8) and a satisfy (2.9) with some $\sigma_0 \in \mathbb{R}$. Moreover, let $\Phi(p, x)$ and $F(p, x)$ be given functions such that*

$$\Phi(p, \cdot) \in H^{3/2}(\Gamma), \quad F(p, \cdot) \in L^2(\Omega) \quad \text{for } \operatorname{Re} p > \sigma_1 \quad (4.3)$$

with some $\sigma_1 \in \mathbb{R}$. Then the problem

$$(L_A U)(p, x) = F(p, x), \quad x \in \Omega, \quad U(p, x) = \Phi(p, x), \quad x \in \Gamma \quad (4.4)$$

has a unique solution $U(p, \cdot) \in H^2(\Omega)$ for every $\operatorname{Re} p > \sigma_u$ with $\sigma_u = \max\{\sigma_a, \sigma_1\}$ and σ_a from Lemma 3.3. This solution satisfies the estimate

$$\|U(p, \cdot)\|_{H^2(\Omega)} \leq c_3 |p|^2 (\|F(p, \cdot)\|_{L^2(\Omega)} + |p| \|\Phi(p, \cdot)\|_{H^{3/2}(\Gamma)}), \quad \operatorname{Re} p > \sigma_u, \quad (4.5)$$

where the coefficient c_3 depends on ρ, a, Ω , but is independent of p, Φ and F .

Proof. Due to the assumption $\Phi(p, \cdot) \in H^{3/2}(\Gamma)$ for $\operatorname{Re} p > \sigma_1$, there exists a function $\tilde{\Phi}(p, x)$ such that $\tilde{\Phi}(p, \cdot) \in H^2(\Omega)$ and $\tilde{\Phi}(p, x)|_{x \in \Gamma} = \Phi(p, x)$ for $\operatorname{Re} p > \sigma_1$. Denoting $W = U - \tilde{\Phi}$, the problem (4.4) reduces to the following problem with homogeneous boundary condition for W :

$$(L_A W)(p, x) = F(p, x) - (L_A \tilde{\Phi})(p, x), \quad x \in \Omega, \quad W(p, x) = 0, \quad x \in \Gamma. \quad (4.6)$$

The sesquilinear form, associated with the operator L_A in the space $H_0^1(\Omega)$, reads

$$l_p(W_1, W_2) = \int_{\Omega} A(p, x) \nabla W_1(x) \cdot \nabla \overline{W_2}(x) dx + p \int_{\Omega} \rho(x) W_1(x) \overline{W_2}(x) dx. \quad (4.7)$$

By (2.8) and the assertion (3.7) of Lemma 3.3, l_p is bounded in $(H_0^1(\Omega))^2$ if $\operatorname{Re} p > \sigma_0$. We are going to prove the coercitivity of l_p . For any $W \in H_0^1(\Omega)$ we have

$$\begin{aligned} |l_p(W, W)| &\geq \left| \frac{1}{p} \int_{\Omega} a(0, x) |\nabla W(x)|^2 dx + p \int_{\Omega} \rho(x) |W(x)|^2 dx \right| \\ &\quad - \frac{1}{|p|^2} \int_{\Omega} |p a(0, x) - p^2 A(p, x)| |\nabla W(x)|^2 dx. \end{aligned} \quad (4.8)$$

In order to estimate the first term on the right-hand side of (4.8) from below we use the relation $|c_1 p^{-1} + c_2 p| \geq \operatorname{Re} p (c_1 |p|^{-2} + c_2)$, which, as easily can be verified, holds for any $c_1, c_2 \geq 0$ and $\operatorname{Re} p > 0$. Applying this relation with $c_1 = \int_{\Omega} a(0, x) |\nabla W(x)|^2 dx$ and $c_2 = \int_{\Omega} \rho(x) |W(x)|^2 dx$ in (4.8) we derive

$$\begin{aligned} |l_p(W, W)| &\geq \frac{1}{|p|^2} \int_{\Omega} [\operatorname{Re} p a(0, x) - |p a(0, x) - p^2 A(p, x)|] |\nabla W(x)|^2 dx \\ &\quad + \operatorname{Re} p \int_{\Omega} \rho(x) |W(x)|^2 dx \end{aligned}$$

for $\operatorname{Re} p > 0$. Applying the estimate (3.10) of Lemma 3.3 and the assumed inequality $\rho(x) \geq \rho_0 > 0$ we obtain

$$|l_p(W, W)| \geq c_4 \left(\frac{1}{|p|^2} \|\nabla W\|_{L^2(\Omega)}^2 + \|W\|_{L^2(\Omega)}^2 \right), \quad \operatorname{Re} p > \sigma_a \quad (4.9)$$

with a coefficient c_4 depending on a and ρ . Thus, l_p is coercive in $(H_0^1(\Omega))^2$ for $\operatorname{Re} p > \sigma_a$.

By (2.8), (4.3), (3.7) and the relation $\tilde{\Phi}(p, \cdot) \in H^2(\Omega)$ for $\operatorname{Re} p > \sigma_1$ the inclusion $F(p, \cdot) - (L_A \tilde{\Phi})(p, \cdot) \in L^2(\Omega) \subset H^{-1}(\Omega)$ holds for $\operatorname{Re} p > \sigma_u$. Consequently, due to the Lax-Milgram lemma the problem (4.6) has a unique solution $W(p, \cdot) \in H_0^1(\Omega)$

for $\operatorname{Re} p > \sigma_u$. This yields the existence and uniqueness of the solution $U(p, \cdot)$ of the problem (4.4) in the space $H^1(\Omega)$ for any $\operatorname{Re} p > \sigma_u$.

To prove the assertion $U(p, \cdot) \in H^2(\Omega)$ for $\operatorname{Re} p > \sigma_u$ we rewrite the problem (4.4) in the form

$$\Delta U(p, x) = F_U(p, x), \quad x \in \Omega, \quad U(p, x) = \Phi(p, x), \quad x \in \Gamma \quad (4.10)$$

for $\operatorname{Re} p > \sigma_u$, where Δ is the Laplacian and

$$F_U(p, x) = A(p, x)^{-1} [pp(x)U(p, x) - \nabla A(p, x) \cdot \nabla U(p, x) - F(p, x)]. \quad (4.11)$$

In view of the assumptions of Proposition 4.1 and the assertions of Lemma 3.3 we see that $F_U(p, \cdot) \in L^2(\Omega)$ for $\operatorname{Re} p > \sigma_u$. Taking this relation and the assumption $\Phi(p, \cdot) \in H^{3/2}(\Gamma)$ into account, the well-known theory of smoothness of the solution of the Dirichlet problem for the Poisson equation (see, e.g., [26, Theorem 27.2]) yields $U(p, \cdot) \in H^2(\Omega)$ for $\operatorname{Re} p > \sigma_u$ and the estimate

$$\|U(p, \cdot)\|_{H^2(\Omega)} \leq c_5 (\|F_U(p, \cdot)\|_{L^2(\Omega)} + \|\Phi(p, \cdot)\|_{H^{3/2}(\Gamma)}), \quad \operatorname{Re} p > \sigma_u, \quad (4.12)$$

where c_5 is a constant depending on Ω .

It remains to derive (4.5). We substitute $W(p, \cdot)$ for W_1 and W_2 in the formula (4.7), where W is the solution of (4.6), and apply the divergence theorem for the first integral in the right-hand side of this formula to get $l_p(W, W) = \int_{\Omega} [F(p, x) - (L_A \tilde{\Phi})(p, x)] \overline{W}(p, x) dx$. Thereupon we estimate this expression from above by means of the Cauchy-Schwartz inequality, use the definition of L_A and combine the obtained result with the estimate from below (4.9). This leads to the relation

$$\|W(p, \cdot)\|_{H^k(\Omega)} \leq c_6 |p|^k (\|F(p, \cdot)\|_{L^2(\Omega)} + |p| \|\tilde{\Phi}(p, \cdot)\|_{H^2(\Omega)}), \quad (4.13)$$

for $\operatorname{Re} p > \sigma_u$, where $k \in \{0; 1\}$ and c_6 depends on a, ρ . Note that we can choose $\tilde{\Phi}(p, \cdot) \in H^2(\Omega)$, satisfying the relation $\tilde{\Phi}(p, x)|_{x \in \Gamma} = \Phi(p, x)$, so that the inequality $\|\tilde{\Phi}(p, \cdot)\|_{H^2(\Omega)} \leq c_7 \|\Phi(p, \cdot)\|_{H^{3/2}(\Gamma)}$ holds for $\operatorname{Re} p > \sigma_u$ with a constant c_7 , which depends on Ω but is independent of Φ . Using the latter inequality, the estimate (4.13) and the obvious relation $1 < \sigma_u^{-1}|p|$ for $\operatorname{Re} p > \sigma_u$ in the formula $U = W + \tilde{\Phi}$ we obtain

$$\|U(p, \cdot)\|_{H^k(\Omega)} \leq c_8 |p|^k (\|F(p, \cdot)\|_{L^2(\Omega)} + |p| \|\Phi(p, \cdot)\|_{H^{3/2}(\Gamma)}), \quad (4.14)$$

for $\operatorname{Re} p > \sigma_u$, where $k \in \{0; 1\}$ and c_8 depends on a, ρ, Ω . Applying the assumptions imposed on ρ, F , the assertions (3.7), (3.9) of Lemma 3.3 and the estimate (4.14) in (4.11) we derive

$$\|F_U(p, \cdot)\|_{L^2(\Omega)} \leq c_9 |p|^2 (\|F(p, \cdot)\|_{L^2(\Omega)} + |p| \|\Phi(p, \cdot)\|_{H^{3/2}(\Gamma)}), \quad (4.15)$$

for $\operatorname{Re} p > \sigma_u$, where c_9 depends on a, ρ, Ω . Finally, (4.12) with (4.15) implies (4.5). \square

Proposition 4.2. *Let the assumptions of Proposition 4.1 hold for ρ, a and Φ . Also let $A(p, \cdot)$ and $\Phi(p, \cdot)$ be holomorphic in $\operatorname{Re} p > \sigma_u$ with values in $W^{1, \infty}(\Omega)$ and $H^{3/2}(\Gamma)$, respectively. Then the solution $U(p, \cdot)$ of (4.1), (4.2) is holomorphic in $\operatorname{Re} p > \sigma_u$ with values in $H^2(\Omega)$.*

Proof. Let p and q be arbitrary numbers such that $\operatorname{Re} p > \sigma_u$, $\operatorname{Re} q > \sigma_u$. Define $A_q(p, x) := A(q, x) - A(p, x)$, $\Phi_q(p, x) := \Phi(q, x) - \Phi(p, x)$, $U_q(p, x) := U(q, x) - U(p, x)$. From (4.1) and (4.2) we obtain the following problem for U_q :

$$\begin{aligned} (L_A U_q)(p, x) &= \operatorname{div} \{ A_q(p, x) \nabla [U_q(p, x) + U(p, x)] \} \\ &\quad - (q - p) \rho(x) [U_q(p, x) + U(p, x)], \\ U_q(p, x)|_{x \in \Gamma} &= \Phi_q(p, x). \end{aligned} \quad (4.16)$$

Applying estimate (4.5) of Proposition 4.1 to this problem we obtain

$$\begin{aligned} \|U_q(p, \cdot)\|_{H^2(\Omega)} &\leq c_3 |p|^2 \{ [\|A_q(p, \cdot)\|_{W^{1,\infty}(\Omega)} + |q - p| \|\rho\|_{L^\infty(\Omega)}] \\ &\quad \times [\|U_q(p, \cdot)\|_{H^2(\Omega)} + \|U(p, \cdot)\|_{H^2(\Omega)}] + |p| \|\Phi_q(p, \cdot)\|_{H^{3/2}(\Gamma)} \}. \end{aligned}$$

This estimate due the relations $\|A_q(p, \cdot)\|_{W^{1,\infty}(\Omega)} \rightarrow 0$ and $\|\Phi_q(p, \cdot)\|_{H^{3/2}(\Gamma)} \rightarrow 0$ as $q \rightarrow p$, following from the assumptions of the proposition, yields

$$\|U_q(p, \cdot)\|_{H^2(\Omega)} \rightarrow 0 \quad \text{as } q \rightarrow p. \quad (4.17)$$

Further, let $A'(p, x)$ and $\Phi'(p, x)$ be the derivatives of $A(p, x)$ and $\Phi(p, x)$ with respect to p , respectively, and let \widehat{U} be the solution of the problem

$$(L_A \widehat{U})(p, x) = \operatorname{div} (A'(p, x) \nabla U(p, x)), \quad \widehat{U}(p, x)|_{x \in \Gamma} = \Phi'(p, x). \quad (4.18)$$

Denoting $\widehat{U}_q(p, x) := \frac{U_q(p, x)}{q - p} - \widehat{U}(p, x) = \frac{U(q, x) - U(p, x)}{q - p} - \widehat{U}(p, x)$ and subtracting (4.18) from (4.16) divided by $q - p$ we obtain the problem

$$\begin{aligned} (L_A \widehat{U}_q)(p, x) &= \operatorname{div} \left\{ \left[\frac{A_q(p, x)}{q - p} - A'(p, x) \right] \nabla U(p, x) + \frac{A_q(p, x)}{q - p} \nabla U_q(p, x) \right\} \\ &\quad - \rho(x) U_q(p, x), \\ \widehat{U}_q(p, x)|_{x \in \Gamma} &= \frac{\Phi_q(p, x)}{q - p} - \Phi'(p, x). \end{aligned}$$

Using the estimate (4.5) for this problem we have

$$\begin{aligned} &\|\widehat{U}_q(p, \cdot)\|_{H^2(\Omega)} \\ &\leq c_3 |p|^2 \{ \|U(p, \cdot)\|_{H^2(\Omega)} \left\| \frac{A_q(p, \cdot)}{q - p} - A'(p, \cdot) \right\|_{W^{1,\infty}(\Omega)} \\ &\quad + \|U_q(p, \cdot)\|_{H^2(\Omega)} \\ &\quad \times \left[\left\| \frac{A_q(p, \cdot)}{q - p} \right\|_{W^{1,\infty}(\Omega)} + \|\rho\|_{L^\infty(\Omega)} \right] \\ &\quad + |p| \left\| \frac{\Phi_q(p, \cdot)}{q - p} - \Phi'(p, \cdot) \right\|_{H^{3/2}(\Gamma)} \}. \end{aligned} \quad (4.19)$$

Due to the assumptions of the proposition we have $\left\| \frac{A_q(p, \cdot)}{q - p} - A'(p, \cdot) \right\|_{W^{1,\infty}(\Omega)} \rightarrow 0$ and $\left\| \frac{\Phi_q(p, \cdot)}{q - p} - \Phi'(p, \cdot) \right\|_{H^{3/2}(\Gamma)} \rightarrow 0$ as $q \rightarrow p$. Using these relations as well as (4.17) in (4.19) we obtain $\|\widehat{U}_q(p, \cdot)\|_{H^2(\Omega)} \rightarrow 0$ as $q \rightarrow p$, or equivalently, $\frac{U(q, \cdot) - U(p, \cdot)}{q - p} \rightarrow \widehat{U}(p, \cdot)$ as $q \rightarrow p$ in $H^2(\Omega)$. This yields the differentiability, hence holomorphy of $U(p, \cdot)$ at p . \square

Proof of Theorem 2.1. In virtue of Proposition 4.1 the problem (4.1), (4.2) has a unique solution $U(p, \cdot) \in H^2(\Omega)$ for any $\operatorname{Re} p > \sigma_u$. Applying the estimate (4.5) to this solution and observing Lemma 3.4 we obtain

$$|p|^2 \|U(p, \cdot)\|_{H^2(\Omega)} \leq c_2 c_3 \quad \text{for } \operatorname{Re} p > \sigma_u.$$

Further, since A and Φ are the Laplace transforms of abstract functions a and φ with values in $W^{1,\infty}(\Omega)$ and $H^{3/2}(\Gamma)$, respectively, $A(p, \cdot) \in W^{1,\infty}(\Omega)$ and $\Phi(p, \cdot) \in H^{3/2}(\Gamma)$ are holomorphic in the half-plane $\operatorname{Re} p > \sigma_u$. Proposition 4.2 yields the holomorphy of $U(p, \cdot) \in H^2(\Omega)$ for $\operatorname{Re} p > \sigma_u$. Summing up, the assumptions of Lemma 3.2 are valid for U . Consequently, there exists a function $u \in C_{\sigma_u}([0, \infty); H^2(\Omega))$ such that $\mathcal{L}_{t \rightarrow p}(u(\cdot, x)) = U(p, x)$ and $u(0, x) \equiv 0$. Applying the inverse transform to (4.1), (4.2) we see that u satisfies the problem (2.6), (2.7). This proves the existence assertion of Theorem 2.1

To prove the uniqueness assertion let us suppose that (2.6), (2.7) has two solutions $u^j \in L_{\sigma_u}^1((0, \infty); H^2(\Omega))$, $j = 1, 2$. Denote $U^j(p, x) = \mathcal{L}_{t \rightarrow p}(u^j(\cdot, x))$, $j = 1, 2$. Then $U^j(p, \cdot) \in H^2(\Omega)$, $j = 1, 2$ solve (4.1), (4.2) for $\operatorname{Re} p > \sigma_u$. Proposition 4.1 implies $U^1(p, \cdot) = U^2(p, \cdot)$ for $\operatorname{Re} p > \sigma_u$. Finally, by the uniqueness of the inverse transform the relation $u^1(t, \cdot) = u^2(t, \cdot)$ for almost any $t \in (0, \infty)$ follows. \square

5. UNIQUENESS ON THE BOUNDARY

In this section we will prove that the assumption $\lambda_{a_1} = \lambda_{a_2}$ implies the equalities $a_1 = a_2$ and $\partial_\nu a_1 = \partial_\nu a_2$ on the boundary Γ .

We begin by introducing some additional notation. Let $u_{a,g}$ denote the solution of (2.6), (2.7) corresponding to the kernel a and the boundary condition $\varphi(t, x) = f(t)g(x)$. Define $U_{a,g}(p, x) = \mathcal{L}_{t \rightarrow p}(u_{a,g}(\cdot, x))$ and $F(p) = \mathcal{L}_{t \rightarrow p}(f)$. Then $U_{a,g}$ solves (4.1), (4.2) with the boundary condition $\Phi(p, x) = F(p)g(x)$. Further, let Λ_a stand for the operator that assigns to every function $g \in H^{3/2}(\Gamma)$ the Laplace transform of $\lambda_a g$, namely

$$\begin{aligned} (\Lambda_a g)(p, x) &= \mathcal{L}_{t \rightarrow p}((\lambda_a g)(\cdot, x)) \\ &= \mathcal{L}_{t \rightarrow p} \left(\int_0^\cdot a(\cdot - \tau, x) \partial_\nu u_{a,g}(\tau, x) dx \right) \\ &= A(p, x) \partial_\nu U_{a,g}(p, x), \quad x \in \Gamma, \operatorname{Re} p > \sigma_u. \end{aligned} \quad (5.1)$$

Finally, for any pair of functions a_1 and a_2 satisfying (2.9) we define $A_j(p, x) = \mathcal{L}_{t \rightarrow p}(a_j(\cdot, x))$, $j = 1, 2$ and $\sigma_{12} := \max\{\sigma_1, \sigma_{a_1}, \sigma_{a_2}\}$.

Let us prove some lemmas. First one is an analogue of the Alessandrini's equality for the inverse conductivity problem [1].

Lemma 5.1. *Let (2.8), (2.12) hold, a_1, a_2 satisfy (2.9) and $g_1, g_2 \in H^{3/2}(\Gamma)$. Then*

$$\begin{aligned} &\int_\Omega [A_1(p, x) - A_2(p, x)] \nabla U_{a_1, g_1}(p, x) \cdot \nabla U_{a_2, g_2}(p, x) dx \\ &= F(p) \int_\Gamma [(\Lambda_{a_1} - \Lambda_{a_2})g_1](p, x) g_2(x) d\Gamma_x, \quad \operatorname{Re} p > \sigma_{12}, \end{aligned} \quad (5.2)$$

where $d\Gamma_x$ is the Lebesgue surface measure of Γ .

Proof. Let a, b be arbitrary functions satisfying (2.9), $A = \mathcal{L}_{t \rightarrow p}(a)$, $B = \mathcal{L}_{t \rightarrow p}(b)$ and g, γ be arbitrary functions in $H^{3/2}(\Gamma)$. Multiplying the equation (4.1) for $U_{a,g}$

by $U_{b,\gamma}$, using the divergence formula and the definition of Λ_a we obtain the equality $E(A, g; B, \gamma) = 0$, where

$$\begin{aligned} E(A, g; B, \gamma) &= \int_{\Omega} [A(p, x) \nabla U_{a,g}(p, x) \cdot \nabla U_{b,\gamma}(p, x) + p\rho(x)U_{a,g}(p, x)U_{b,\gamma}(p, x)] dx \\ &\quad - F(p) \int_{\Gamma} (\Lambda_a g)(p, x) \gamma(x) d\Gamma_x. \end{aligned}$$

This equality implies the relation

$$E(A_1, g_1; A_2, g_2) - E(A_2, g_2; A_1, g_1) - E(A_2, g_1; A_2, g_2) + E(A_2, g_2; A_2, g_1) = 0,$$

which is identical to (5.2). \square

Lemma 5.2. *Let $z \in \Gamma$ and B be a neighbourhood of z . Let α and β be given functions such that $\alpha \in W^{1,\infty}(\Omega) \cap C^{1,\mu}(B)$, $\beta \in L^\infty(\Omega) \cap C^{0,\mu}(B)$ with some $\mu > 0$ and $\alpha(x) \geq \alpha_0 > 0$, $\beta(x) \geq \rho_0 > 0$ for $x \in \Omega \cup B$. Then, for any $y \in B \setminus \bar{\Omega}$, there exists a function $w_y : \Omega \cup B \setminus \{y\} \rightarrow \mathbb{R}$ satisfying the following conditions:*

(i) w_y solves the equation

$$-\operatorname{div}(\alpha(x)\nabla w_y(x)) + \beta(x)w_y(x) = 0, \quad x \in \Omega \cup B \setminus \{y\}; \quad (5.3)$$

(ii) $w_y(x)|_{x \in \Omega}$ belongs to $H^2(\Omega)$. Moreover, for any $x \in B \setminus \{y\}$ the relation

$$w_y(x) = \frac{\alpha(y)}{4\pi|x-y|} + \omega_y(x) \quad (5.4)$$

is valid, where

$$|\omega_y(x)| \leq c_{10}|x-y|^{\delta-1}, \quad |\partial_{x_i}\omega_y(x)| \leq c_{10}|x-y|^{\delta-2}, \quad i = 1, 2, 3 \quad (5.5)$$

with a coefficient c_{10} and an exponent $\delta \in (0, 1)$ independent of $x \in B \setminus \{y\}$ and $y \in B \setminus \bar{\Omega}$.

The proof of this lemma is given in [12]. Now we can prove the main result of the section.

Theorem 5.3. *Let the assumptions of Theorem 2.3 be satisfied for ρ , f and a_1 , a_2 . Then $\lambda_{a_1} = \lambda_{a_2}$ implies $a_1(t, z) = a_2(t, z)$ and $\partial_\nu a_1(t, z) = \partial_\nu a_2(t, z)$ for $z \in \Gamma$ and a.e. $t \in (0, \infty)$.*

Proof. The proof uses in an adapted form the method of singular solutions due to Alessandrini [2]. First, we note that the assumption $\lambda_{a_1} = \lambda_{a_2}$ implies $\Lambda_{a_1} = \Lambda_{a_2}$. This, by Lemma 5.1 yields

$$\int_{\Omega} [A_1 - A_2](p, x) \nabla U_{a_1, g_1}(p, x) \cdot \nabla U_{a_2, g_2}(p, x) dx = 0 \quad \text{for } \operatorname{Re} p > \sigma_{12}. \quad (5.6)$$

Let us choose some $p_0 \in \mathbb{R}$, $p_0 > \sigma_{12}$, such that $F(p_0) \neq 0$. In view of the assumptions $\operatorname{Im} a_j = 0$, $a_j(0, x) \geq a_0 > 0$ and the assertions (3.8), (3.9) of Lemma 3.3, the functions A_j satisfy the inequality $A_j(p_0, x) \geq \frac{\kappa}{p_0} > 0$ for $x \in \Omega$. Moreover, due to (2.13) we have $A_j(p_0, \cdot) \in W^{2,\infty}(\Omega) \cap C^{1,\mu}(\Gamma_\epsilon)$, $j = 1, 2$.

Let z be an arbitrary point on Γ . Observing the properties of $A_j(p_0, x)$ and the assumptions imposed on ρ , we can choose a neighbourhood $B_z \subset \Gamma_\epsilon$ of z such that the assumptions of Lemma 5.2 are fulfilled for $B = B_z$, $\beta = p_0\rho$ and $\alpha = A_j(p_0, \cdot)$

with $j \in \{1, 2\}$. Let $w_{j,y}$ with $j \in \{1, 2\}$ be a function which fulfills the assertions of Lemma 5.2 for $\alpha = \alpha_j = A_j(p_0, \cdot)$, $\beta = p_0\rho$. Thus,

$$-\operatorname{div}(A_j(p_0, x)\nabla w_{j,y}(x)) + p_0\rho(x)w_{j,y}(x) = 0, \quad x \in \Omega \cup B_z \setminus \{y\}, \quad j = 1, 2, \quad (5.7)$$

Observing the assertions of Lemma 5.2 and the inequality $\alpha_j(y) \geq \frac{\kappa}{p_0} > 0$, one can verify that

$$\nabla w_{1,y}(x) \cdot \nabla w_{2,y}(x) \sim \frac{\alpha_1(y)\alpha_2(y)}{|x - y|^4} \quad \text{as } y \rightarrow x. \quad (5.8)$$

Let us set $g_j(x) := w_{j,y}(x)|_{x \in \Gamma}$ for $j = 1, 2$. Then the function $U_{a_j, g_j}(p, x)$ solves (4.1), (4.2) with $A = A_j$ and $\Phi(p, x) = F(p)g_j(x) = F(p)w_{j,y}(x)|_{x \in \Gamma}$. Since (4.1) in case $p = p_0$ coincides with (5.7), we have $U_{a_j, g_j}(p_0, x) = F(p_0)w_{j,y}(x)$ for $x \in \Omega$. Using this relation in (5.6) and taking the inequality $F(p_0) \neq 0$ into account we obtain

$$I_y := \int_{\Omega} [A_1 - A_2](p_0, x)\nabla w_{1,y}(x) \cdot \nabla w_{2,y}(x)dx = 0. \quad (5.9)$$

On the other hand, by $A_j(p_0, \cdot) \in C^{1,\mu}(B_z)$ we have

$$(A_1 - A_2)(p_0, x) = (A_1 - A_2)(p_0, z) + \nabla(A_1 - A_2)(z) \cdot (x - z) + O(|x - z|^{1+\mu})$$

for $x \in B_z$. Thus, $I_y = I_y^0 + I_y^1 + I_y^2$, where

$$\begin{aligned} I_y^0 &= (A_1 - A_2)(p_0, z) \int_{\Omega \cap B_z} \nabla w_{1,y}(x) \cdot \nabla w_{2,y}(x)dx, \\ I_y^1 &= \nabla(A_1 - A_2)(p_0, z) \cdot \int_{\Omega \cap B_z} (x - z)\nabla w_{1,y}(x) \cdot \nabla w_{2,y}(x)dx, \\ I_y^2 &= \int_{\Omega \cap B_z} O(|x - z|^{1+\mu})\nabla w_{1,y}(x) \cdot \nabla w_{2,y}(x)dx \\ &\quad + \int_{\Omega \setminus B_z} [A_1 - A_2](p_0, x)\nabla w_{1,y}(x) \cdot \nabla w_{2,y}(x)dx. \end{aligned}$$

Observing that the singularity of $w_{j,y}(x)$ is located at $x = y$, where $w_{j,y}(x)$ satisfies the relation (5.8), and the distance between y and $\Omega \setminus B_z$ is bounded away from 0 as $y \rightarrow z$, we see that the term I_y^2 is bounded as $y \rightarrow z$. Moreover, supposing $(A_1 - A_2)(p_0, z) \neq 0$, we have $I_y \sim I_y^0$ and $|I_y^0| \rightarrow \infty$ as $y \rightarrow z$. But this contradicts (5.9). Thus, $(A_1 - A_2)(p_0, z) = 0$. Further, supposing $\nabla(A_1 - A_2)(p_0, z) \neq 0$, we have $I_y \sim I_y^1$ and $|I_y^1| \rightarrow \infty$ as $y \rightarrow z$. Again, this contradicts (5.9). Consequently, $\nabla(A_1 - A_2)(p_0, z) = 0$.

Since $z \in \Gamma$ was chosen arbitrarily, we have proven the equalities

$$A_1(p, \cdot) = A_2(p, \cdot), \quad \partial_\nu A_1(p, \cdot) = \partial_\nu A_2(p, \cdot) \quad (5.10)$$

in $C^{1,\lambda}(\Gamma)$ for any $p > \sigma_{12}$ such that $F(p) \neq 0$. The set $\{p : p > \sigma_{12}, F(p) \neq 0\}$ has an accumulation point because of the assumption $f \neq 0$. Therefore, by means of the analytic continuation we can extend the equalities (5.10) in $C^{1,\lambda}(\Gamma)$ to all $\operatorname{Re} p > \sigma_{12}$. Finally, by uniqueness of the inverse transform we obtain the assertion of the theorem. \square

6. UNIQUENESS IN THE WHOLE DOMAIN

In this section, we prove Theorem 2.3 For this end we need the following lemmas.

Lemma 6.1. *Let (2.8), (2.12) hold and a satisfy (2.9), (2.13). Moreover, let $g \in H^{3/2}(\Gamma)$. Define*

$$V_{a,g}(p, x) := A^{1/2}(p, x)U_{r,g}(p, x), \quad (6.1)$$

where $A^{1/2}(p, x)$ is the principal value of the square root of $A(p, x)$, namely

$$A^{1/2}(p, x) = |A(p, x)|^{1/2} \left(\cos \frac{\arg A(p, x)}{2} + i \sin \frac{\arg A(p, x)}{2} \right).$$

Then $V_{a,g}(p, \cdot)$ belongs to $H^2(\Omega)$ and solves the problem

$$-\Delta V_{a,g}(p, x) + K(p, x)V_{a,g}(p, x) = 0, \quad x \in \Omega, \quad (6.2)$$

$$V_{a,g}(p, x) = F(p)A^{1/2}(p, x)g(x), \quad x \in \Gamma \quad (6.3)$$

for $\operatorname{Re} p > \sigma_u$, where $K(p, \cdot)$ belongs to $L^\infty(\Omega)$ and is given by

$$K(p, x) = [\Delta A^{1/2}(p, x) + p\rho(x)A^{-1/2}(p, x)]A^{-1/2}(p, x) \quad (6.4)$$

for $\operatorname{Re} p > \sigma_a$.

Proof. In view of the assumption (2.13) we obtain $A(p, \cdot) \in W^{2,\infty}(\Omega)$ for $\operatorname{Re} p > \sigma_0$. This together with the assertion (3.9) of Lemma 3.3 yields

$$A^{1/2}(p, \cdot) \in W^{2,\infty}(\Omega), \quad |A^{1/2}(p, x)| \geq \frac{\kappa^{1/2}}{|p|^{1/2}} > 0, \quad x \in \Omega \quad (6.5)$$

for $\operatorname{Re} p > \sigma_a$. Applying these relations and (2.8) in (6.4) we deduce $K(p, \cdot) \in L^\infty(\Omega)$ for $\operatorname{Re} p > \sigma_a$. Further, due to Theorem 2.1 $U_{a,g}(p, \cdot) \in H^2(\Omega)$ for $\operatorname{Re} p > \sigma_u$. This in view of (6.1) and (6.5) yields $V_{a,g}(p, \cdot) \in H^2(\Omega)$ for $\operatorname{Re} p > \sigma_u$. Finally, observing that $U_{a,g}(p, x)$ solves the problem (4.1), (4.2) with $\Phi(p, x) = F(p)g(x)$, the change of the unknown by (6.1) in this problem results in (6.2) and (6.3) with K of the form (6.4). \square

Lemma 6.2. *Let the assumptions of Theorem 2.3 be valid for f, ρ, a_1 and a_2 . Assume that $g_1, g_2 \in H^{3/2}(\Gamma)$ and $K_j, j = 1, 2$, are defined by (6.4) with A replaced by A_j . Let $A_1(p, x) = A_2(p, x)$, $\partial_\nu A_1(p, x) = \partial_\nu A_2(p, x)$ for any $x \in \Gamma$ and $\operatorname{Re} p > \sigma_{12}$. Then*

$$\begin{aligned} & \int_{\Omega} [K_1(p, x) - K_2(p, x)] V_{a_1, g_1}(p, x) V_{a_2, g_2}(p, x) dx \\ &= F(p) \int_{\Gamma} [(\Lambda_{a_1} - \Lambda_{a_2})g_1](p, x)g_2(x)d\Gamma_x, \quad \operatorname{Re} p > \sigma_{12}. \end{aligned} \quad (6.6)$$

Proof. Let a, b be arbitrary functions satisfying (2.9), $A = \mathcal{L}_{t \rightarrow p}(a)$, $B = \mathcal{L}_{t \rightarrow p}(b)$, K be given by (6.4) and g, γ be arbitrary functions in $H^{3/2}(\Gamma)$. Multiplying the equation (6.2) for $V_{a,g}$ by $V_{b,\gamma}$ and using the divergence formula we obtain the equality $\hat{E}(A, g; B, \gamma) = 0$, where

$$\begin{aligned} \hat{E}(A, g; B, \gamma) &= \int_{\Omega} [\nabla V_{a,g}(p, x) \cdot \nabla V_{b,\gamma}(p, x) + K(p, x)V_{a,g}(p, x)V_{b,\gamma}(p, x)] dx \\ &\quad - \int_{\Gamma} \partial_\nu V_{a,g} V_{b,\gamma} d\Gamma_x. \end{aligned}$$

This equality implies the relation

$$\hat{E}(A_1, g_1; A_2, g_2) - \hat{E}(A_2, g_2; A_1, g_1) - \hat{E}(A_2, g_1; A_2, g_2) + \hat{E}(A_2, g_2; A_2, g_1) = 0,$$

which is identical to

$$\begin{aligned} & \int_{\Omega} [K_1 - K_2] V_{a,g} V_{b,\gamma} dx \\ &= \int_{\Gamma} [\partial_{\nu} V_{a_1, g_1} V_{a_2, g_2} - \partial_{\nu} V_{a_2, g_2} V_{a_1, g_1} + \partial_{\nu} V_{a_2, g_2} V_{a_2, g_1} - \partial_{\nu} V_{a_2, g_1} V_{a_2, g_2}] d\Gamma_x. \end{aligned} \quad (6.7)$$

By (6.3) and the proved equalities $A(p, x) = A_1(p, x) = A_2(p, x)$, $\partial_{\nu} A_1(p, x) = \partial_{\nu} A_2(p, x)$ for $x \in \Gamma$ we have

$$\begin{aligned} V_{a_2, g_1}(p, x) &= V_{a_1, g_1}(p, x) = F(p)A^{1/2}(p, x)g_1(x), \\ V_{a_2, g_2}(p, x) &= V_{a_1, g_2}(p, x) = F(p)A^{1/2}(p, x)g_2(x), \\ \partial_{\nu} V_{a_1, g_1}(p, x) - \partial_{\nu} V_{a_2, g_1}(p, x) \\ &= \partial_{\nu} [A_1(p, x)^{1/2} U_{a_1, g_1}(p, x) - \partial_{\nu} [A_2(p, x)^{1/2} U_{a_2, g_1}(p, x) \\ &\quad + A^{-1/2}(p, x) [A_1(p, x) \partial_{\nu} U_{a_1, g_1}(p, x) - A_2(p, x) \partial_{\nu} U_{a_2, g_1}(p, x)]] \\ &= A^{-1/2}(p, x) [(\Lambda_{a_1} - \Lambda_{a_2}) g_1](p, x) \end{aligned}$$

for $x \in \Gamma$. Using these relations in (6.7) we derive (6.6). \square

Lemma 6.3. *Let $z, \zeta, \eta \in \mathbb{R}^3$ and $\omega > 0$ satisfy the relations*

$$z \cdot \eta = z \cdot \zeta = \eta \cdot \zeta = 0, \quad |\eta| = 1, \quad |\zeta|^2 = \frac{|z|^2}{2} + \omega^2. \quad (6.8)$$

Define

$$\xi_1 = i\left(\frac{z}{2} + \omega\eta\right) + \zeta, \quad \xi_2 = i\left(\frac{z}{2} - \omega\eta\right) - \zeta. \quad (6.9)$$

Furthermore, let $\alpha_j \in L^{\infty}(\Omega)$, $j = 1, 2$. Then there exists $M \geq 0$ depending on α_1, α_2 such that the equations

$$-\Delta \hat{w}_j(x) + \alpha_j(x) \hat{w}_j(x) = 0, \quad j = 1, 2 \quad (6.10)$$

have solutions \hat{w}_j , $j = 1, 2$, which belong to $H^2(\Omega)$ and have the form

$$\hat{w}_j(x) = e^{x \cdot \xi_j} [1 + \psi_j(x)], \quad (6.11)$$

where ψ_j , $j = 1, 2$ satisfy the estimates

$$\|\psi_j\|_{L^2(\Omega)} \leq \frac{1}{|\xi_j|} \|\alpha_j\|_{L^{\infty}(\Omega)} \quad \text{for } |\xi_j| > M, \quad j = 1, 2. \quad (6.12)$$

The proof of the above lemma is given in [12].

Proof of Theorem 2.3. To prove this theorem, we will apply in an adapted form, the method due to Sylvester and Uhlmann [24]. Assume that $\lambda_{a_1} = \lambda_{a_2}$. Let us fix some $p_0 \in \mathbb{R}$, $p_0 > \sigma_{12}$, such that $F(p_0) \neq 0$. Further, let us choose $z \in \mathbb{R}^3$ and $\omega > 0$ and let $\zeta, \eta \in \mathbb{R}^3$ satisfy the relations (6.8). Define $\alpha_j(x) := K_j(p_0, x)$ for $j = 1, 2$, where K_j is given by (6.4) with A replaced by A_j . Finally, let \hat{w}_j , $j = 1, 2$, be the functions satisfying the assertions of Lemma 6.3 with these α_j . By virtue of

Theorem 5.3 we have $A_1(p, x) = A_2(p, x) =: A(p, x)$ for $x \in \Gamma$, $\operatorname{Re} p > \sigma_{12}$. Let us set

$$g_j(x) := \frac{\widehat{w}_j(x)}{F(p_0)A^{1/2}(p_0, x)} \quad \text{for } x \in \Gamma, \quad j = 1, 2. \tag{6.13}$$

The functions g_j , $j = 1, 2$ belong to $H^{3/2}(\Gamma)$ because of the relations $\widehat{w}_j \in H^2(\Omega)$, $F(p_0) \neq 0$ and (6.5). The problem (6.2), (6.3) for V_{a_j, g_j} reads

$$-\Delta V_{a_j, g_j}(p, x) + K_j(p, x)V_{a_j, g_j}(p, x) = 0, \quad x \in \Omega, \tag{6.14}$$

$$V_{a_j, g_j}(p, x) = F(p)A^{1/2}(p, x)g_j(x), \quad x \in \Gamma, \tag{6.15}$$

where $\operatorname{Re} p > \sigma_{12}$. Due to the relation $\alpha_j = K_j(p_0, \cdot)$ the equation (6.14) for V_{a_j, g_j} coincides with (6.10) for \widehat{w}_j in case $p = p_0$. Further, in view of (6.13) and (6.15) we have $V_{a_j, g_j}(p_0, x) = \widehat{w}_j(x)$ for $x \in \Gamma$ and $j = 1, 2$. Recall that \widehat{w}_j solves (6.10). Thus, by the uniqueness of the solution of the boundary value problem (6.14), (6.15), we obtain the equality $V_{a_j, g_j}(p_0, x) = \widehat{w}_j(x)$ for $x \in \Omega$ and $j = 1, 2$.

By the assumptions of Theorem 2.3, the supposed relation $\lambda_{a_1} = \lambda_{a_2}$ and Theorem 5.3 the assumptions of Lemma 6.2 are satisfied. Moreover, $\lambda_{a_1} = \lambda_{a_2}$ implies $\Lambda_{a_1} = \Lambda_{a_2}$. Hence, the equality (6.6) holds with 0 in the right-hand side. In case $p = p_0$ it reads

$$\int_{\Omega} [K_1(p_0, x) - K_2(p_0, x)] \widehat{w}_1(x) \widehat{w}_2(x) dx = 0.$$

This, in view of (6.11) and (6.9), implies

$$\int_{\Omega} [K_1(p_0, x) - K_2(p_0, x)] e^{ix \cdot z} [1 + \psi_1(x)][1 + \psi_2(x)] dx = 0. \tag{6.16}$$

From (6.8) and (6.9) we see that that $|\xi_j| \rightarrow \infty$ as $\omega \rightarrow \infty$ for fixed z . Hence, by and (6.12) $\|\psi_j\|_{L^2(\Omega)} \rightarrow 0$ as $\omega \rightarrow \infty$ for fixed z . Thus, passing to the limit $\omega \rightarrow \infty$ in (6.16) we deduce

$$\int_{\Omega} [K_1(p_0, x) - K_2(p_0, x)] e^{ix \cdot z} dx = 0.$$

Since this equality holds for arbitrary $z \in \mathbb{R}^3$, we obtain $K_1(p_0, x) = K_2(p_0, x)$ for $x \in \Omega$.

Due to this equality, (6.4), and Theorem 5.3, the difference $A_1^{1/2}(p_0, \cdot) - A_2^{1/2}(p_0, \cdot)$ solves the problem

$$-\Delta[A_1^{1/2}(p_0, x) - A_2^{1/2}(p_0, x)] + \tilde{\alpha}(p_0, x)[A_1^{1/2}(p_0, x) - A_2^{1/2}(p_0, x)] = 0, \tag{6.17}$$

$$A_1^{1/2}(p_0, x) - A_2^{1/2}(p_0, x) = 0, \quad x \in \Gamma, \tag{6.18}$$

where

$$\tilde{\alpha}(p_0, x)$$

$$= \frac{1}{A_2^{1/2}(p_0, x)} \left[\frac{p_0 \rho(x)}{A_1^{1/2}(p_0, x)} + \frac{1}{A_2^{1/2}(p_0, x)} (p_0 \rho(x) + \frac{\Delta A_2(p_0, x)}{2} - \frac{|\nabla A_2(p_0, x)|^2}{4A_2(p_0, x)}) \right].$$

Let us study the asymptotic behaviour of $\tilde{\alpha}$ as $p_0 \rightarrow \infty$. Observe that (2.13) and the assertions (3.2), (3.3) of Lemma 3.1 imply $\|\Delta A_2(p_0, \cdot)\|_{L^\infty(\Omega)} \rightarrow 0$ as $p_0 \rightarrow \infty$. Moreover, by the assertions (3.7) and (3.9) of Lemma 3.3

$$\frac{|\nabla A_2(p_0, x)|^2}{|A_2(p_0, x)|} = \frac{1}{p_0} \frac{|p_0 \nabla A_2(p_0, x)|^2}{p_0 |A_2(p_0, x)|} \leq \frac{c_1}{\kappa p_0} \rightarrow 0 \quad \text{as } p_0 \rightarrow \infty.$$

Consequently, in view of (2.8)

$$\tilde{\alpha}(p_0, x) \sim \frac{p_0 \rho(x)}{A_2^{1/2}(p_0, x)} \left[\frac{1}{A_1^{1/2}(p_0, x)} + \frac{1}{A_2^{1/2}(p_0, x)} \right] \quad \text{as } p_0 \rightarrow \infty$$

uniformly in $x \in \Omega$. The assumptions $\text{Im } a_j = 0$, $j = 1, 2$ and $a_j(0, x) \geq a_0 > 0$ imply that $A_j(p_0, x) > 0$ for sufficiently large p_0 . Hence, $\tilde{\alpha}(p_0, x) \geq 0$ for $p_0 > \tilde{\sigma}$ with some sufficiently large $\tilde{\sigma}$. This implies that the solution of the problem (6.17), (6.18) is unique if $p_0 > \tilde{\sigma}$. Consequently, $A_1(p_0, x) = A_2(p_0, x)$ for $x \in \Omega$ and $p_0 > \tilde{\sigma}$ such that $F(p_0) \neq 0$.

The set $\{p \mid p > \tilde{\sigma}, F(p) \neq 0\}$ has an accumulation point in view of the assumption $f \neq 0$. Hence, by means of analytic continuation we can extend the equality $A_1(p, \cdot) = A_2(p, \cdot)$ for all $\text{Re } p > \sigma_{12}$. Finally, by the uniqueness of the inverse transform we derive $a_1 = a_2$. \square

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