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EXISTENCE OF BOUNDED SOLUTIONS FOR NONLINEAR DEGENERATE ELLIPTIC EQUATIONS IN ORLICZ SPACES

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Abstract. We prove the existence of bounded solutions for the nonlinear elliptic problem
$$-\operatorname{div} a(x,u,\nabla u) = f \quad \text{in } \Omega,$$
with $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, where
$$a(x,s,\xi)\cdot\xi \geq \overline{M}^{-1} M(h(|s|))M(|\xi|),$$
and $h : \mathbb{R}^+ \to]0,1]$ is a continuous monotone decreasing function with unbounded primitive. As regards the N-function M, no Δ_2-condition is needed.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$. We consider the equation
$$\begin{gathered} -\operatorname{div}(a(x,u)\overline{M}^{-1}(M(|\nabla u|))\frac{\nabla u}{|\nabla u|}) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$
where
$$\overline{M}^{-1}(M(\frac{1}{(1+|s|)^\theta})) \leq a(x,s) \leq \beta, \tag{1.2}$$
with $0 \leq \theta \leq 1$, and β is a positive constant.

For $M(t) = t^2$, existence of bounded solutions of (1.1) was proved under (1.2) in [4] and in [5] when $f \in L^m(\Omega)$ with $m > \frac{N}{2}$. This result was then extended in [3], to the study of the problem
$$\begin{gathered} -\operatorname{div} a(x,u,\nabla u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$
in the Sobolev space $W_0^{1,p}(\Omega)$, under the condition
$$a(x,s,\xi)\cdot\xi \geq \frac{\alpha}{(1+|s|)^{\theta(p-1)}}|\xi|^p, \tag{1.4}$$
when $f \in L^m(\Omega)$ with $m > \frac{N}{p}$.

In this paper, we prove the existence of bounded solutions of (1.3) in the setting of Orlicz spaces under a more general condition than (1.4) adapted to this situation.

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The main tools used to get a priori estimates in our proof are symmetrization techniques. such techniques are widely used in the literature for linear and nonlinear equations (see [3] and the references quoted therein). We remark that our result is in some sense complementary to one contained in [13].

As examples of equations to which our result can be applied, we list:

$$
\begin{gathered}
-\operatorname{div}\Big(\frac{\alpha}{(e+|u|)^{\gamma}\log(e+|u|)}\frac{e^{|\nabla u|^p}-1}{|\nabla u|^2}\nabla u\Big)=f \quad \text{in } \Omega,\\
u=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $\alpha>0$, $\gamma<1$ and $M(t)=e^{t^p}-1$ with $1<p<N$; and

$$
\begin{gathered}
-\operatorname{div}\Big(\frac{\alpha}{(1+|u|)^{\gamma}}|\nabla u|^{p-2}\nabla u \log^{q}(e+|\nabla u|)=f \quad \text{in } \Omega,\\
u=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $\alpha>0$ and $0\le\gamma\le1$, here $M(t)=t^p\log^q(e+t)$ with $1<p<N$ and $q\in\mathbb{R}$.

2. Preliminaries

Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be an N-function; i.e., M is continuous, convex, with $M(t)>0$ for $t>0$, $\frac{M(t)}{t}\to0$ as $t\to0$ and $\frac{M(t)}{t}\to\infty$ as $t\to\infty$. The N-function conjugate to M is defined as $\overline{M}(t)=\sup\{st-M(t), s\ge0\}$. We will extend these N-functions into even functions on all $\mathbb{R}$. We recall that (see [1])

$$
M(t)\le t\overline{M}^{-1}(M(t))\le 2M(t) \quad \text{for all } t\ge0 \tag{2.1}
$$

and the Young's inequality: for all $s,t\ge0$, $st\le\overline{M}(s)+M(t)$. If for some $k>0$,

$$
M(2t)\le kM(t) \quad \text{for all } t\ge0, \tag{2.2}
$$

we said that M satisfies the Δ_2-condition, and if (2.2) holds only for t greater than or equal to t_0, then M is said to satisfy the Δ_2-condition near infinity.

Let P and Q be two N-functions. the notation $P\ll Q$ means that P grows essentially less rapidly than Q, i.e.

$$
\forall\epsilon>0,\quad \frac{P(t)}{Q(\epsilon t)}\to0 \quad \text{as } t\to\infty,
$$

that is the case if and only if

$$
\frac{Q^{-1}(t)}{P^{-1}(t)}\to0 \quad \text{as } t\to\infty.
$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence class of) real-valued measurable functions u on Ω such that:

$$
\int_\Omega M(u(x))dx<\infty \quad (\text{resp.}\int_\Omega M(\frac{u(x)}{\lambda})dx<\infty \text{ for some } \lambda>0).
$$

Endowed with the norm

$$
\|u\|_M=\inf\{\lambda>0:\int_\Omega M(\frac{u(x)}{\lambda})dx<\infty\},
$$

$L_M(\Omega)$ is a Banach space and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. the closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The Orlicz-Sobolev space $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$).

This is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the norm closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

We say that a sequence $\{u_n\}$ converges to u for the modular convergence in $W^1L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M(\frac{D^\alpha u_n - D^\alpha u}{\lambda})dx \to 0 \quad \text{for all } |\alpha| \leq 1;$$

this implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If M satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then the modular convergence coincides with norm convergence. Recall that the norm $\|Du\|_M$ defined on $W_0^1L_M(\Omega)$ is equivalent to $\|u\|_{1,M}$ (see [10]).

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open Ω has the segment property then the space $D(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (see [10]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined. For an exhaustive treatments one can see for example [1] or [12].

We will use the following lemma, (see[6]), which concerns operators of Nemytskii Type in Orlicz spaces. It is slightly different from the analogous one given in [12].

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$,*

$$|f(x,s)| \leq c(x) + k_1 P^{-1}M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f, defined by $N_f(u)(x) = f(x, u(x))$, is strongly continuous from $P(E_M, \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

We recall the definition of decreasing rearrangement of a measurable function $w : \Omega \to \mathbb{R}$. If one denotes by $|E|$ the Lebesgue measure of a set E, one can define the distribution function $\mu_w(t)$ of w as:

$$\mu_w(t) = |\{x \in \Omega : |w(x)| > t\}|, \quad t \geq 0.$$

The decreasing rearrangement w^* of w is defined as the generalized inverse function of μ_w:

$$w^*(\sigma) = \inf\{t \in \mathbb{R} : \mu_w(t) \leq \sigma\}, \quad \sigma \in (0, \Omega).$$

It is shown in [15] that w^* is everywhere continuous and

$$w^*(\mu_w(t)) = t \tag{2.3}$$

for every t between 0 and $\operatorname{ess\,sup}|w|$. More details can be found for example in [2, 13, 14].

3. Assumptions and main result

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property and M is an N-function twice continuously differentiable and strictly increasing, and P is an N-function such that $P \ll M$.

Let $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying, for a.e. $x \in \Omega$, and for all $s \in \mathbb{R}$ and all ξ, $\eta \in \mathbb{R}^N$, $\xi \neq \eta$,

$$a(x, s, \xi) \cdot \xi \geq \overline{M}^{-1} M(h(|s|))M(|\xi|) \tag{3.1}$$

where $h : \mathbb{R}^+ \to \mathbb{R}^*_+$ is a continuous monotone decreasing function such that $h(0) \leq 1$ and its primitive $H(s) = \int_0^s h(t)dt$ is unbounded,

$$|a(x, s, \xi)| \leq a_0(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\xi|) \tag{3.2}$$

where $a_0(x)$ belongs to $E_{\overline{M}}(\Omega)$ and k_1, k_2, k_3, k_4 to $\mathbb{R}^*_+$,

$$(a(x, s, \xi) - a(x, s, \eta)) \cdot (\xi - \eta) > 0. \tag{3.3}$$

Let A: $D(A) \subset W^1_0 L_M(\Omega) \to W^{-1} L_{\overline{M}}(\Omega)$ be a mapping (non-everywhere defined) given by

$$A(u) := -\operatorname{div} a(x, u, \nabla u),$$

We are interested, in proving the existence of bounded solutions to the nonlinear problem

$$\begin{gathered} A(u) := -\operatorname{div}(a(x, u, \nabla u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.4}$$

As regards the data f, we assume one of the following two conditions: Either

$$f \in L^N(\Omega), \tag{3.5}$$

or

$$\begin{gathered} f \in L^m(\Omega) \quad \text{with } m = rN/(r+1) \text{ for some } r > 0, \\ \text{and } \int_{\cdot}^{+\infty} \Big(\frac{t}{M(t)}\Big)^r dt < +\infty. \end{gathered} \tag{3.6}$$

We will use the following concept of solutions:

Definition 3.1. Let $f \in L^1(\Omega)$, a function $u \in W^1_0 L_M(\Omega)$ is said to be a weak solution of (3.4), if $a(\cdot, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N$ and

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx = \int_\Omega f v \, dx$$

holds for all $v \in D(\Omega)$.

Our main result is the following.

Theorem 3.2. *Under the assumptions* (3.1), (3.2), (3.3) *and either* (3.5) *or* (3.6), *there exists at least one weak solution of* (3.4) *in* $W^1_0 L_M(\Omega) \cap L^\infty(\Omega)$.

Remark 3.3. In the case where $M(t) = t^p$, with $p > 1$, assumptions (3.5) and (3.6) imply that $m > \frac{N}{p}$. Our result extends those in [5] and [4] where $M(t) = t^2$ and [3] where $M(t) = t^p$, with $p > 1$.

Remark 3.4. Note that the result of theorem (3.1) is independent of the function h which eliminates the coercivity of the operator A. The result is not surprising, since if we look for bounded solutions then the operator A becomes coercive.

Remark 3.5. The principal difficulty in dealing with the problem (3.4) is the non coerciveness of the operator A, this is due to the hypothesis (3.1), so the classical methods used to prove the existence of a solution for (3.4) can not be applied (see [11] and also [9]). To get rid of this difficulty, we will consider an approximation method in which we introduce a truncation. The main tool of the proof will be L^∞ a priori estimates, obtained by mean of a comparison result, which then imply the $W_0^1L_M(\Omega)$ estimate, since if u is bounded the operator A becomes uniformly coercive.

4. Proof of theorem 3.2

For $s \in \mathbb{R}$ and $k > 0$ set: $T_k(s) = \max(-k, \min(k, s))$ and $G_k(s) = s - T_k(s)$. Let $\{f_n\} \subset W^{-1}E_{\overline{M}}(\Omega)$ be a sequence of smooth functions such that

$$f_n \to f \quad \text{strongly in } L^{m^*}(\Omega)$$

and

$$\|f_n\|_{m^*} \leq \|f\|_{m^*},$$

where m^* denotes either N or m, according as we assume (3.5) or (3.6), and consider the operator:

$$A_n(u) = -\operatorname{div} a(x, T_n(u), \nabla u).$$

By assumption (3.1), we have

$$\begin{aligned}\langle A_n(u), u\rangle &= \int_\Omega a(x, T_n(u), \nabla u) \cdot \nabla u \, dx \\ &\geq \overline{M}^{-1}(M(h(n))) \int_\Omega M(|\nabla u|) dx.\end{aligned}$$

Thus, A_n satisfies the classical conditions from which derives, thanks to the fact that $f_n \in W^{-1}E_{\overline{M}}(\Omega)$, the existence of a solution $u_n \in W_0^1L_M(\Omega)$, (see [11] and also [9]), such that

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla v dx = \int_\Omega f_n v dx \tag{4.1}$$

holds for all $v \in W_0^1L_M(\Omega)$. To prove the L^∞ a priori estimates, we will need the following comparison lemma, whose proof will be given in the appendix.

Lemma 4.1. *Let $B(t) = \frac{M(t)}{t}$ and $\mu_n(t) = |\{x \in \Omega : |u_n(x)| > t\}|$, for all $t > 0$. We have for almost every $t > 0$:*

$$h(t) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}} B^{-1}\Big(\frac{\int_{\{|u_n|>t\}} |f_n| dx}{\overline{M}^{-1}(M(1))NC_N^{1/N}\mu_n(t)^{1-\frac{1}{N}}}\Big) \tag{4.2}$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$.

step 1: L^∞-bound. If we assume (3.5), using the inequality $\int_{\{|u_n|>t\}}|f_n|dx \leq \|f\|_N \mu_n(t)^{1-\frac{1}{N}}$, (4.2) becomes

$$h(t) \leq \frac{2M(1)(-\mu_n'(t))}{\overline{M}^{-1}(M(1))NC_N^{1/N}\mu_n(t)^{1-\frac{1}{N}}} B^{-1}\Big(\frac{\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{1/N}}\Big).$$

Then we integrate between 0 and s, we get

$$H(s) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} B^{-1}\Big(\frac{\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{1/N}}\Big) \int_0^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}}dt;$$

hence, a change of variables yields

$$H(s) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} B^{-1}\Big(\frac{\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{1/N}}\Big) \int_{\mu_n(s)}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

By (2.3) we get

$$H(u_n^*(\sigma)) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} B^{-1}\Big(\frac{\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{1/N}}\Big) \int_{\sigma}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

So that

$$H(u_n^*(0)) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} B^{-1}\Big(\frac{\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{1/N}}\Big) N|\Omega|^{1/N}.$$

Since $u_n^*(0) = \|u_n\|_\infty$, the assumption made on H (i.e., $\lim_{s\to+\infty} H(s) = +\infty$) shows that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Moreover if we denote by H^{-1} the inverse function of H, one has:

$$\|u_n\|_\infty \leq H^{-1}\Big(\frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} B^{-1}\Big(\frac{\|f\|_N}{\overline{M}^{-1}(M(1))NC_N^{1/N}}\Big) N|\Omega|^{1/N}\Big). \tag{4.3}$$

Now, we assume that (3.6) is filled. Then, using the inequality

$$\int_{\{|u_n|>t\}} |f_n|dx \leq \|f\|_m \mu_n(t)^{1-\frac{1}{m}}$$

in (4.2), we obtain

$$H(s) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} \int_0^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}} B^{-1}\Big(\frac{\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{1/N}\mu_n(t)^{\frac{1}{m}-\frac{1}{N}}}\Big)dt.$$

A change of variables gives

$$H(s) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} \int_{\mu_n(s)}^{|\Omega|} B^{-1}\Big(\frac{\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{1/N}\sigma^{\frac{1}{m}-\frac{1}{N}}}\Big)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

As above, (2.3) gives

$$H(u_n^*(\tau)) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} \int_{\tau}^{|\Omega|} B^{-1}\Big(\frac{\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{1/N}\sigma^{\frac{1}{m}-\frac{1}{N}}}\Big)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

Then, we have

$$H(\|u_n\|_\infty) \leq \frac{2M(1)}{\overline{M}^{-1}(M(1))NC_N^{1/N}} \int_0^{|\Omega|} B^{-1}\Big(\frac{\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{1/N}\sigma^{\frac{1}{m}-\frac{1}{N}}}\Big)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

A change of variables gives

$$H(\|u_n\|_\infty) \le \frac{2M(1)\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^r C_N^{\frac{r+1}{N}}} \int_{c_0}^{+\infty} rt^{-r-1}B^{-1}(t)dt,$$

where $c_0 = \frac{\|f\|_m}{\overline{M}^{-1}(M(1))NC_N^{1/N}|\Omega|^{\frac{1}{rN}}}$. Then, an integration by parts yields

$$H(\|u_n\|_\infty) \le \frac{2M(1)\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^r C_N^{\frac{r+1}{N}}}\Big(\frac{B^{-1}(c_0)}{c_0^r} + \int_{B^{-1}(c_0)}^{+\infty}\big(\frac{s}{M(s)}\big)^r ds\Big).$$

The assumption made on H guarantees that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Indeed, denoting by H^{-1} the inverse function of H, one has

$$\|u_n\|_\infty \le H^{-1}\Big(\frac{2M(1)\|f\|_m^r}{(\overline{M}^{-1}(M(1)))^{r+1}N^r C_N^{\frac{r+1}{N}}}\Big(\frac{B^{-1}(c_0)}{c_0^r} + \int_{B^{-1}(c_0)}^{+\infty}\big(\frac{s}{M(s)}\big)^r ds\Big)\Big). \tag{4.4}$$

Consequently, in both cases the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$, so that in the sequel, we will denote by c the constant appearing either in (4.3) or in (4.4), that is

$$\|u_n\|_\infty \le c. \tag{4.5}$$

Step 2: Estimation in $W_0^1L_M(\Omega)$. It is now easy to obtain an estimate in $W_0^1L_M(\Omega)$ under either (3.5) or (3.6). Let m^* denotes either N or m according as we assume (3.5) or (3.6). Taking u_n as test function in (4.1), one has

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot \nabla u_n dx = \int_\Omega f_n u_n dx.$$

Then by (3.1) and (4.5), we obtain

$$\int_\Omega M(|\nabla u_n|)dx \le \frac{c\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}}}{\overline{M}^{-1}(M(h(c)))}. \tag{4.6}$$

Hence, the sequence $\{u_n\}$ is bounded in $W_0^1L_M(\Omega)$. Therefore, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function u in $W_0^1L_M(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}) \tag{4.7}$$

and

$$u_n \to u \quad \text{in } E_M(\Omega) \text{ strongly and a.e. in } \Omega. \tag{4.8}$$

Step 3: Almost everywhere convergence of the gradients. Let us begin with the following lemma which we will use later.

Lemma 4.2. *The sequence $\{a(x, T_n(u_n), \nabla u_n)\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$.*

Proof. We will use the dual norm of $(L_{\overline{M}}(\Omega))^N$. Let $\varphi \in (E_M(\Omega))^N$ such that $\|\varphi\|_M = 1$. By (3.3) we have

$$\Big(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \frac{\varphi}{k_4})\Big)\cdot\Big(\nabla u_n - \frac{\varphi}{k_4}\Big) \ge 0.$$

Let $\lambda = 1 + k_1 + k_3$, by using (3.2), (4.5), (4.6) and Young's inequality we get

$$
\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n)\varphi dx \\
&\leq k_4 \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx - k_4 \int_\Omega a(x, T_n(u_n), \frac{\varphi}{k_4}) \cdot \nabla u_n \, dx \\
&\quad + \int_\Omega a(x, T_n(u_n), \frac{\varphi}{k_4}) \cdot \varphi dx \\
&\leq k_4 c \|f\|_{m^*} |\Omega|^{1-\frac{1}{m^*}} + k_4 \lambda \frac{c\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}}}{\overline{M}^{-1} M(h(c))} \\
&\quad + (1 + k_4)\Big(\int_\Omega \overline{M}(a_0(x))dx + k_1 \overline{MP}^{-1} M(k_2 c)|\Omega|\Big) + k_3(1 + k_4) + \lambda,
\end{aligned}
$$

which completes the proof. □

From (4.5) and (4.8) we obtain that $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, so that by [8, Theorem 4] there exists a sequence $\{v_j\}$ in $D(\Omega)$ such that $v_j \to u$ in $W_0^1 L_M(\Omega)$ as $j \to \infty$ for the modular convergence and almost everywhere in Ω, moreover $\|v_j\|_\infty \leq (N + 1)\|u\|_\infty$.

For $s > 0$, we denote by χ_j^s the characteristic function of the set

$$\Omega_j^s = \{x \in \Omega : |\nabla v_j(x)| \leq s\}$$

and by χ^s the characteristic function of the set $\Omega^s = \{x \in \Omega : |\nabla u(x)| \leq s\}$. Testing by $u_n - v_j$ in (4.1), we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n - \nabla v_j)dx = \int_\Omega f_n(u_n - v_j)dx \tag{4.9}$$

Denote by $\epsilon_i(n, j)$, $(i = 0, 1, \dots)$, various sequences of real numbers which tend to 0 when n and $j \to \infty$, i.e.

$$\lim_{j\to\infty} \lim_{n\to\infty} \epsilon_i(n, j) = 0.$$

For the right-hand side of (4.9), we have

$$\int_\Omega f_n(u_n - v_j)dx = \epsilon_0(n, j). \tag{4.10}$$

The left-hand side of (4.9) is written as

$$
\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n - \nabla v_j)\, dx \\
&= \int_\Omega \big(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla v_j \chi_j^s)\big) \cdot \big(\nabla u_n - \nabla v_j \chi_j^s\big)\, dx \\
&\quad + \int_\Omega a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s)dx \\
&\quad - \int_{\Omega\setminus\Omega_j^s} a(x, T_n(u_n), \nabla u_n) \cdot \nabla v_j \, dx
\end{aligned}
\tag{4.11}
$$

We will pass to the limit over n and j, for s fixed, in the second and the third terms of the right-hand side of (4.11). By Lemma 4.2, we deduce that there exists

$l \in (L_{\overline{M}}(\Omega))^N$ and up to a subsequence $a(x, T_n(u_n), \nabla u_n) \rightharpoonup l$ weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\prod L_{\overline{M}}, \prod E_M)$. Since $\nabla v_j \chi_{\Omega\setminus\Omega_j^s} \in (E_M(\Omega))^N$, we have by letting $n \to \infty$,

$$-\int_{\Omega\setminus\Omega_j^s} a(x, T_n(u_n), \nabla u_n) \cdot \nabla v_j dx \to -\int_{\Omega\setminus\Omega_j^s} l \cdot \nabla v_j dx.$$

Using the modular convergence of v_j, we get as $j \to \infty$

$$-\int_{\Omega\setminus\Omega_j^s} l \cdot \nabla v_j \, dx \to -\int_{\Omega\setminus\Omega^s} l \cdot \nabla u \, dx.$$

Hence, we have proved that the third term

$$-\int_{\Omega\setminus\Omega_j^s} a(x, T_n(u_n), \nabla u_n) \cdot \nabla v_j dx = -\int_{\Omega\setminus\Omega^s} l \cdot \nabla u dx + \epsilon_1(n, j). \tag{4.12}$$

For the second term, as $n \to \infty$, we have

$$\int_\Omega a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot \big(\nabla u_n - \nabla v_j \chi_j^s\big) \, dx \to \int_\Omega a(x, u, \nabla v_j \chi_j^s) \cdot \big(\nabla u - \nabla v_j \chi_j^s\big) \, dx,$$

since $a(x, T_n(u_n), \nabla v_j \chi_j^s) \to a(x, u, \nabla v_j \chi_j^s)$ strongly in $(E_{\overline{M}}(\Omega))^N$ as $n \to \infty$ by lemma 2.1 and (4.8), while $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_M(\Omega))^N$ by (4.7). And since $\nabla v_j \chi_j^s \to \nabla u \chi^s$ strongly in $(E_M(\Omega))^N$ as $j \to \infty$, we obtain

$$\int_\Omega a(x, u, \nabla v_j \chi_j^s) \cdot \big(\nabla u - \nabla v_j \chi_j^s\big) \, dx \to 0$$

as $j \to \infty$. So that

$$\int_\Omega a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot \big(\nabla u_n - \nabla v_j \chi_j^s\big) \, dx = \epsilon_2(n, j). \tag{4.13}$$

Consequently, combining (4.10), (4.12) and (4.13), we obtain

$$\begin{aligned} &\int_\Omega \big(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla v_j \chi_j^s)\big) \cdot \big(\nabla u_n - \nabla v_j \chi_j^s\big) \, dx \\ &= \int_{\Omega\setminus\Omega^s} l \cdot \nabla u \, dx + \epsilon_3(n, j). \end{aligned} \tag{4.14}$$

On the other hand

$$\begin{aligned} &\int_\Omega (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u \chi^s)) \cdot (\nabla u_n - \nabla u \chi^s) \, dx \\ &= \int_\Omega \big(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla v_j \chi_j^s)\big) \cdot \big(\nabla u_n - \nabla v_j \chi_j^s\big) \, dx \\ &\quad + \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \big(\nabla v_j \chi_j^s - \nabla u \chi^s\big) \, dx \\ &\quad - \int_\Omega a(x, T_n(u_n), \nabla u \chi^s) \cdot (\nabla u_n - \nabla u \chi^s) \, dx \\ &\quad + \int_\Omega a(x, T_n(u_n), \nabla v_j \chi_j^s) \cdot \big(\nabla u_n - \nabla v_j \chi_j^s\big) \, dx. \end{aligned}$$

We can argue as above in order to obtain

$$\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\big(\nabla v_j\chi_j^s-\nabla u\chi^s\big)\,dx=\epsilon_4(n,j),\\
&\int_\Omega a(x,T_n(u_n),\nabla u\chi^s)\cdot(\nabla u_n-\nabla u\chi^s)\,dx=\epsilon_5(n,j),\\
&\int_\Omega a(x,T_n(u_n),\nabla v_j\chi_j^s)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,dx=\epsilon_6(n,j).
\end{aligned}$$

Then, by (4.14) we have

$$\begin{aligned}
&\int_\Omega (a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u\chi^s))\cdot(\nabla u_n-\nabla u\chi^s)\,dx\\
&=\epsilon_7(n,j)+\int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx.
\end{aligned}$$

For $r\le s$, we write

$$\begin{aligned}
0&\le\int_{\Omega^r}(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n-\nabla u)\,dx\\
&\le\int_{\Omega^s}(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n-\nabla u)\,dx\\
&=\int_{\Omega^s}(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u\chi^s))\cdot(\nabla u_n-\nabla u\chi^s)\,dx\\
&\le\int_{\Omega}(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u\chi^s))\cdot(\nabla u_n-\nabla u\chi^s)\,dx\\
&\le\epsilon_7(n,j)+\int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx.
\end{aligned}$$

Which implies by passing at first to the limit superior over n and then over j,

$$\begin{aligned}
0&\le\limsup_{n\to\infty}\int_{\Omega^r}(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n-\nabla u)\,dx\\
&\le\int_{\Omega\setminus\Omega^s} l\cdot\nabla u dx.
\end{aligned}$$

Letting $s\to+\infty$ in the previous inequality, we conclude that as $n\to\infty$,

$$\int_{\Omega^r}(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n-\nabla u)\,dx\to 0. \tag{4.15}$$

Let B_n be defined by

$$B_n=(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n-\nabla u)\,.$$

As a consequence of (4.15), one has $B_n\to 0$ strongly in $L^1(\Omega^r)$, extracting a subsequence, still denoted by $\{u_n\}$, we get $B_n\to 0$ a.e in Ω^r. Then, there exists a subset Z of Ω^r, of zero measure, such that: $B_n(x)\to 0$ for all $x\in\Omega^r\setminus Z$. Using (3.2), we obtain for all $x\in\Omega^r\setminus Z$,

$$B_n(x)\ge\overline{M}^{-1}M(h(c))M(|\nabla u_n(x)|)-c_1(x)\left(1+\overline{M}^{-1}M(k_4|\nabla u_n(x)|)+|\nabla u_n(x)|\right),$$

where c is the constant appearing in (4.5) and $c_1(x)$ is a constant which does not depend on n. Thus, the sequence $\{\nabla u_n(x)\}$ is bounded in $\mathbb{R}^N$, then for a subsequence $\{u_{n'}(x)\}$, we have

$$\nabla u_{n'}(x) \to \xi \quad \text{in } \mathbb{R}^N,$$
$$(a(x,u(x),\xi) - a(x,u(x),\nabla u(x))) \cdot (\xi - \nabla u(x)) = 0.$$

Since $a(x,s,\xi)$ is strictly monotone, we have $\xi = \nabla u(x)$, and so $\nabla u_n(x) \to \nabla u(x)$ for the whole sequence. It follows that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega^r.$$

Consequently, as r is arbitrary, one can deduce that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{4.16}$$

Step 4: Passage to the limit. Let v be a function in $D(\Omega)$. Taking v as test function in (4.1), one has

$$\int_\Omega a(x,T_n(u_n),\nabla u_n) \cdot \nabla v dx = \int_\Omega f_n v\, dx.$$

Lemma 4.2, (4.8) and (4.16) imply that

$$a(x,T_n(u_n),\nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M),$$

so that one can pass to the limit in the previous equality to obtain

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla v dx = \int_\Omega f v\, dx.$$

Moreover, from (4.5) and (4.8) we have $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$. This completes the proof of theorem 3.2.

Remark 4.3. Note that the L^∞-bound in step 1 can be proven under the weaker assumption

$$\|f\|_{m,\infty} = \sup_{s>0} s^{\frac{1}{m}-1} \int_0^s f^*(t) dt < \infty,$$

which is equivalent to say that f belongs to the Lorentz space $L(m,\infty)$. Indeed, one can use the inequality

$$\int_{\{|u_n|>t\}} |f_n| dx \le \int_0^{\mu_n(t)} f^*(t) dt$$

(see [13, 14]) in (4.1) to obtain: If f belongs to $L(N,\infty)$, then

$$h(t) \le \frac{2M(1)(-\mu_n'(t))}{\overline{M}^{-1}(M(1)) N C_N^{1/N} \mu_n(t)^{1-\frac{1}{N}}} B^{-1}\Big(\frac{\|f\|_{N,\infty}}{\overline{M}^{-1}(M(1)) N C_N^{1/N}}\Big),$$

and if we assume that f belongs to $L(m,\infty)$ with $m < N$ and

$$\int_{\cdot}^{+\infty} \Big(\frac{t}{M(t)}\Big)^{\frac{m}{N-m}} dt < +\infty,$$

we obtain

$$h(t) \le \frac{2M(1)(-\mu_n'(t))}{\overline{M}^{-1}(M(1)) N C_N^{1/N} \mu_n(t)^{1-\frac{1}{N}}} B^{-1}\Big(\frac{\|f\|_{m,\infty}}{\overline{M}^{-1}(M(1)) N C_N^{1/N} \mu_n(t)^{\frac{1}{m}-\frac{1}{N}}}\Big).$$

As above, starting with those inequalities we obtain the desired result. Observe that when h is a constant function, this L^∞-bound was proved in [13].

5. Appendix

In this section, we prove lemma 4.1 based on techniques inspired from those in [13].

Proof of Lemma 4.1. Testing by $v = T_k(G_t(u_n))$, which lies in $W_0^1 L_M(\Omega)$ thanks to [7, Lemma 2], in (4.1) one has

$$\int_{\{t<|u_n|\le t+k\}} a(x, T_n(u_n), \nabla u_n)\cdot \nabla u_n dx \le k\int_{\{|u_n|>t\}} |f_n| dx.$$

Then (3.1) yields

$$\frac{1}{k}\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1} M(h(|u_n|))M(|\nabla u_n|)dx \le \int_{\{|u_n|>t\}} |f_n| dx.$$

Letting $k \to 0^+$ we obtain

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} \overline{M}^{-1} M(h(|u_n|))M(|\nabla u_n|)dx \le \int_{\{|u_n|>t\}} |f_n| dx, \tag{5.1}$$

for almost every $t > 0$. The hypotheses made on the N-function M, which are not a restriction, allow to affirm that the function $C(t) = \frac{1}{B^{-1}(t)}$ is decreasing and convex (see [13]). Hence, Jensen's inequality yields

$$\begin{aligned}
&C\Big(\frac{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx}{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n| dx}\Big)\\
&= C\Big(\frac{\int_{\{t<|u_n|\le t+k\}} B(|\nabla u_n|)\overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n| dx}{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n| dx}\Big)\\
&\le \frac{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))dx}{\int_{\{t<|u_n|\le t+k\}} \overline{M}^{-1}(M(h(|u_n|)))|\nabla u_n| dx}\\
&\le \frac{\overline{M}^{-1}(M(h(t)))(-\mu_n(t+k)+\mu_n(t))}{\overline{M}^{-1}(M(h(t+k)))\int_{\{t<|u_n|\le t+k\}} |\nabla u_n| dx}.
\end{aligned}$$

Taking into account that $\overline{M}^{-1}(M(h(t))) \le \overline{M}^{-1}(M(1))$, using the convexity of C and then letting $k \to 0^+$, we obtain for almost every $t > 0$,

$$\begin{aligned}
&\frac{\overline{M}^{-1}(M(1))}{\overline{M}^{-1}(M(h(t)))}\, C\Big(\frac{-\frac{d}{dt}\int_{\{|u_n|>t\}} \overline{M}^{-1}(M(h(|u_n|)))M(|\nabla u_n|)dx}{\overline{M}^{-1}(M(1))(-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n| dx)}\Big)\\
&\le \frac{-\mu_n'(t)}{-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n| dx}.
\end{aligned}$$

Now we recall the following inequality from [13]:

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n| dx \ge N C_N^{1/N} \mu_n(t)^{1-\frac{1}{N}} \quad \text{for almost every } t > 0. \tag{5.2}$$

The monotonicity of the function C, (5.1) and (5.2) yield

$$\begin{aligned} &\frac{1}{\overline{M}^{-1}(M(h(t)))} \\ &\leq \frac{-\mu_n'(t)}{\overline{M}^{-1}(M(1))NC_N^{1/N}\mu_n(t)^{1-\frac{1}{N}}}B^{-1}\Big(\frac{\int_{\{|u_n|>t\}}|f_n|dx}{\overline{M}^{-1}(M(1))NC_N^{1/N}\mu_n(t)^{1-\frac{1}{N}}}\Big). \end{aligned}$$

Using (2.1) and the fact that $0 < h(t) \leq 1$, we obtain (4.2). □

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