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SPECTRAL ANALYSIS OF SINGULAR HAMILTONIAN SYSTEMS WITH AN EIGENPARAMETER IN THE BOUNDARY CONDITION

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Abstract. In this article we study a non-self-adjoint eigenparameter dependent singular differential 1D Hamiltonian system with the singular end points a and b in the Hilbert space $L^2_P((a,b);\mathbb{C}^2)$ and we consider that this 1D Hamiltonian system is in the limit-circle cases at a and b. For this purpose we use the maximal dissipative operator associated with the considered problem whose spectral analysis is sufficient for boundary value problem. Self-adjoint dilation theory of Sz.-Nagy-Foiaş developed for the dissipative operators is used. Moreover we construct incoming and outgoing spectral representations of the self-adjoint dilation. This representations allows us to determine the scattering matrix. Therefore a functional model of the dissipative operator is constructed. Moreover, a functional model of the dissipative operator is constructed and its characteristic function in terms of solutions of the corresponding Hamiltonian system is described. Therefore using the obtained results for the characteristic function theory, theorems on completeness of the system of eigenvectors and associated vectors of the dissipative operator and Hamiltonian boundary value problem have been proved.

1. Introduction

One of the important problems in the spectral theory of operators include a spectral parameter both in the equation and boundary conditions. Eigenparameter dependent boundary value problems occurs in various problems of physics and engineering. In particular such problems occurring in physical processed can be found in [6, 7, 17]. Moreover one can find numerous studies devoted to eigenparameter dependent boundary value problems in [1, 2, 3, 5, 7, 8, 11, 17, 19, 20, 22, 24, 26, 27, 29]

This article mainly considers a non-self-adjoint eigenparameter dependent one dimensional (1D) singular differential Hamiltonian boundary value problem given by (2.8)-(2.10). It is known that contour integration method of resolvent is one of the main methods used in the spectral analysis of boundary value problem (2.8)-(2.10). However this method needs a well estimation of the resolvent on expanding contours separating the spectrum. It is better to note that the applicability of this

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method is restricted to weak perturbations of self-adjoint operators and operators with sparse discrete spectrum. The resolvent $\mathcal{R}_\lambda$ corresponding to the boundary value problem (2.8)-(2.10) can not be investigated directly, because there is no asymptotics of solutions associated with the system (2.8) concerning the spectral parameter λ. Therefore contour integration method is not useful for the boundary value problem (2.8)-(2.10).

For studying the spectral properties of the boundary value problem (2.8)-(2.10), the characteristic function theory of the model operator is suitable. Using the fundamental results given in [18, 21], characteristic function is constructed with studying the self-adjoint dilations. This self-adjoint dilations allow us to study the scattering problem and the characteristic function is realized as a scattering matrix (see [15]). For readers we should noted that, for the papers including the non-self-adjoint dissipative 1D singular differential Hamiltonian (or Dirac-type) systems with λ-independent boundary conditions, see, for example [4].

This article is organized as follows. In Section 2, the maximal dissipative operator $\mathcal{A}_\beta$ associated with the boundary value problem (2.8)-(2.10) is constructed and we establish the self-adjoint dilation $\mathbb{S}_\beta$ of the operator $\mathcal{A}_\beta$. In the section 3 we show that the scattering theory of Lax-Phillips [15] is applicable for the operator $\mathbb{S}_\beta$ and we can reveal the scattering matrix $\overline{\Theta}_\beta$ through the solution of system (2.8). In the incoming spectral presentation of the dilation, the operator $\mathcal{A}_\beta$ is converted to the model dissipative operator with the characteristic function Θ_β, which is, in its turn, unitary equivalent to $\mathcal{A}_\beta$. Finally, we derive the theorems on factorization of the characteristic function and completeness of the system of eigenvectors and associated vectors of the operator $\mathcal{M}_\beta$, and boundary value problem (2.8)-(2.10).

2. Construction of the maximal dissipative operator and its self-adjoint dilation

We consider the 1D Hamiltonian system

$$\begin{gathered} L_1(x) := J\frac{dx(t)}{dt} + Q(t)x(t) = \lambda P(t)x(t), \\ t \in \Omega := (a,b),\ -\infty \le a < b \le +\infty, \end{gathered} \tag{2.1}$$

where λ is a complex parameter, endpoints a and b are singular for L_1,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} x^{(1)}(t) \\ x^{(2)}(t) \end{pmatrix},$$
$$P(t) = \begin{pmatrix} p(t) & b(t) \\ b(t) & c(t) \end{pmatrix}, \quad Q(t) = \begin{pmatrix} q(t) & k(t) \\ k(t) & r(t) \end{pmatrix},$$

$P(t) > 0$ for almost all $t \in \Omega$ and the entries of the (2×2) matrices $P(t)$ and $Q(t)$ are real-valued, Lebesgue measurable and locally integrable functions on Ω.

Let us consider the differential expression $L(x) := P^{-1}(t)L_1(x)$ and the Hilbert space $\mathcal{L}_P^2(\Omega, E)$ $(E := \mathbb{C}^2)$ including all vector-valued functions x such that

$$\int_a^b (P(t)x(t), x(t))_E dt < +\infty$$

and with the inner product $(x,y) := \int_a^b (P(t)x(t), y(t))_E dt$.

We denote by $\mathcal{D}_{\max}$ the linear set consisting of all vectors $x \in \mathcal{L}_P^2(\Omega; E)$ such that $x^{(1)}$ and $x^{(2)}$ are locally absolutely continuous functions on Ω, and $L(x) \in \mathcal{L}_P^2(\Omega; E)$. We define the *maximal operator* $\mathcal{M}_{\max}$ on $\mathcal{D}_{\max}$ by the equality $\mathcal{M}_{\max}x = L(x)$.

For two arbitrary vectors $x, y \in \mathcal{D}_{\max}$, we obtain the Green's formula as

$$(\mathcal{M}_{\max}x, y) - (x, \mathcal{M}_{\max}y) = \mathcal{W}_b[x, \overline{y}] - \mathcal{W}_a[x, \overline{y}], \tag{2.2}$$

where

$$\mathcal{W}_t[x, \overline{y}] := x^{(1)}(t)\overline{y}^{(2)}(t) - x^{(2)}(t)\overline{y}^{(1)}(t) \quad (t \in \Omega),$$
$$\mathcal{W}_a[x, \overline{y}] := \lim_{t \to a^+}[x, \overline{y}]_t, \quad \mathcal{W}_b[x, \overline{y}] := \lim_{t \to b^-}\mathcal{W}_t[x, \overline{y}].$$

Let $\mathcal{D}_{\min}$ be the set of all vectors $x \in \mathcal{D}_{\max}$ satisfying

$$\mathcal{W}_b[x, \overline{y}] - \mathcal{W}_a[x, \overline{y}] = 0, \ \forall y \in \mathcal{D}_{\max}. \tag{2.3}$$

We denote by $\mathcal{M}_{\min}$ the restriction of the operator $\mathcal{M}_{\max}$ to $\mathcal{D}_{\min}$. It is known that the operator $\mathcal{M}_{\min}$ is a *minimal symmetric operator* with deficiency indices $(0, 0)$, $(1, 1)$ or $(2, 2)$, and $\mathcal{M}_{\max} = \mathcal{M}_{\min}^*$ (see [4, 6, 9, 10, 12, 13, 14, 16, 25, 28]). Note that for defect index $(0, 0)$, the operator $\mathcal{M}_{\min}$ is self-adjoint, i.e., $\mathcal{M}_{\min}^* = \mathcal{M}_{\min} = \mathcal{M}_{\max}$.

In this article, we assume that $\mathcal{M}_{\min}$ has deficiency index (2,2), i.e., the limit-circle case holds for the differential expression L at a and b (see [3, 4, 6, 9, 10, 12, 13, 14, 16, 25, 28]). There are several sufficient conditions that ensure the limit-circle case (see [3, 9, 10, 12, 13, 14, 16, 23, 28]).

We denote by θ and ϕ the solutions of the system

$$L_1(x) = 0, \ t \in \Omega \tag{2.4}$$

satisfying the initial conditions

$$\theta^{(1)}(c) = 1, \quad \theta^{(2)}(c) = 0, \quad \phi^{(1)}(c) = 0, \quad \phi^{(2)}(c) = 1, \quad c \in \Omega. \tag{2.5}$$

From conditions (2.5) we have

$$\mathcal{W}_t[\theta, \phi] = \mathcal{W}_c[\theta, \phi] = 1 \quad (a \le t \le b), \tag{2.6}$$

since the Wronskian of two solutions of (2.4) is independent of t. Moreover these solutions are linearly independent if and only if their Wronskian in non-zero. Therefore, θ and ϕ form a fundamental system of solutions for the system (2.4). Since $\mathcal{M}_{\min}$ has deficiency indices $(2, 2)$, $\theta, \phi \in \mathcal{L}_P^2(\Omega; E)$ and furthermore $\theta, \phi \in \mathcal{D}_{\max}$. Therefore, the domain $\mathcal{D}_{\min}$ of the operator $\mathcal{M}_{\min}$ includes definitely the vectors $x \in \mathcal{D}_{\max}$ satisfying the boundary conditions (see [4])

$$\mathcal{W}_a[x, \theta] = \mathcal{W}_a[x, \phi] = \mathcal{W}_b[x, \theta] = \mathcal{W}_b[x, \phi] = 0. \tag{2.7}$$

This article mainly considers the boundary value problem

$$J\frac{dx(t)}{dt} + Q(t)x(t) = \lambda P(t)x(t), \quad x \in \mathcal{D}_{\max}, \ t \in \Omega, \tag{2.8}$$

$$\delta_1 A_1^-(x) - \delta_2 A_2^-(x) = \lambda(\delta_1' A_1^-(x) - \delta_2' A_2^-(x)), \tag{2.9}$$

$$A_1^+(x) - \beta A_2^+(x) = 0, \ \operatorname{Im}\beta > 0, \tag{2.10}$$

where $\lambda \in \mathbb{C}$, $\delta_1, \delta_2, \delta_1', \delta_2' \in \mathbb{R} := (-\infty, \infty)$,

$$\delta := \begin{vmatrix} \delta_1' & \delta_1 \\ \delta_2' & \delta_2 \end{vmatrix} > 0,$$

and $A_1^-(x) := \mathcal{W}_a[x, \theta]$, $A_2^-(x) := \mathcal{W}_a[x, \phi]$, $A_1^+(x) := \mathcal{W}_b[x, \theta]$, $A_2^+(x) := \mathcal{W}_b[x, \phi]$.

We adopt the following notation:

$$S_-(x) := \delta_1 A_1^-(x) - \delta_2 A_2^-(x),$$
$$S'_-(x) := \delta'_1 A_1^-(x) - \delta'_2 A_2^-(x),$$
$$S_+(x) = A_1^+(x) - \beta A_2^+(x).$$

For arbitrary $x, y \in \mathcal{D}_{\max}$, the following equalities are obtained by direct calculations,

$$\mathcal{W}_a[x,y] = \frac{1}{\delta}[S_-(x)S'_-(y) - S'_-(x)S_-(y)], \tag{2.11}$$
$$\mathcal{W}_t[x,y] = \mathcal{W}_t[x,\theta]\mathcal{W}_t[y,\phi] - \mathcal{W}_t[x,\phi]\mathcal{W}_t[y,\theta]_t \quad (a \le t \le b), \tag{2.12}$$
$$S_-(\overline{y}) = \overline{S_-(y)}, \quad A_1^+(\overline{y}) = \overline{A_1^+(y)}, \quad A_2^+(\overline{y}) = \overline{A_2^+(y)}.$$

We denote by ρ_λ and ω_λ the solutions of (2.8) satisfying

$$A_1^-(\rho_\lambda) = \delta_2 - \delta'_2\lambda, \quad A_2^-(\rho_\lambda) = \delta_1 - \delta'_1\lambda, \quad A_1^+(\omega_\lambda) = \beta, \quad A_2^+(\omega_\lambda) = 1.$$

Using (2.11) we obtain that

$$\begin{aligned}\Delta(\lambda) :=& \mathcal{W}_t[\omega_\lambda,\rho_\lambda] = -\mathcal{W}_t[\rho_\lambda,\omega_\lambda] = -\mathcal{W}_a[\rho_\lambda,\omega_\lambda]\\ =& -\frac{1}{\delta}[S_-(\rho_\lambda)S'_-(\omega_\lambda) - S'_-(\rho_\lambda)S_-(\omega_\lambda)] = -\lambda S'_-(\omega_\lambda) + S_-(\omega_\lambda).\end{aligned} \tag{2.13}$$

Moreover, equality (2.7) gives us

$$\begin{aligned}\Delta(\lambda) &= -\mathcal{W}_t[\rho_\lambda,\omega_\lambda] = -\mathcal{W}_b[\rho_\lambda,\omega_\lambda]\\ &= -A_1^+(\rho_\lambda)A_2^+(\omega_\lambda) + A_2^+(\rho_\lambda)A_1^+(\omega_\lambda)\\ &= -A_1^+(\rho_\lambda) + \beta A_2^+(\rho_\lambda) = -S_+(\rho_\lambda).\end{aligned} \tag{2.14}$$

The spectrum of the boundary value problem (2.8)-(2.10) coincide with the zeros of the function Δ. Since Δ is analytic and not identically zero (ρ_λ and ω_λ are linearly independent), it follows that the function Δ has at most a countable number of isolated zeros with finite multiplicity and possible limit points at infinity.

Consider the Hilbert space $\mathcal{H} := \mathcal{L}_P^2(\Omega; E) \oplus \mathbb{C}$ consisting of vector-valued functions with values in $\mathbb{C}^3$ equipped with the inner product

$$(\hat{x},\hat{y})_{\mathcal{H}} = \int_a^b (P(t)x_1(t), y_1(t))_E dt + \frac{1}{\delta}x_2\overline{y}_2,$$

where

$$\hat{x}(t) = \begin{pmatrix} x_1(t)\\ x_2\end{pmatrix}, \quad \hat{y}(t) = \begin{pmatrix} y_1(t)\\ y_2\end{pmatrix}.$$

Let $\mathcal{D}(\mathcal{A}_\beta)$ be the linear set of all vectors $\hat{x} = \begin{pmatrix} x_1\\ x_2\end{pmatrix} \in \mathcal{H}$ with $x_1 \in \mathcal{D}_{\max}$, $S_+(x_1) = 0$ and $x_2 = S'_-(x_1)$. We construct the operator $\mathcal{A}_\beta$ on $\mathcal{D}(\mathcal{A}_\beta)$ by the equality

$$\mathcal{A}_\beta\hat{x} = \widetilde{L}(\hat{x}) := \begin{pmatrix} L(x_1)\\ S_-(x_1)\end{pmatrix}.$$

Since a linear operator $\mathfrak{T}$ (with dense domain $\mathcal{D}(\mathfrak{T})$) acting on some Hilbert space $\mathfrak{H}$ is called *dissipative* (*accumulative*) if $\operatorname{Im}(\mathfrak{T}f,f) \ge 0$) for all $f \in \mathcal{D}(\mathfrak{T})$ and *maximal dissipative* (*maximal accumulative*) if it does not have a proper dissipative extension, we can state the following result.

Theorem 2.1. *The operator $\mathcal{A}_\beta$ is maximal dissipative in the space $\mathcal{H}$.*

Proof. Note that $\mathcal{D}(\mathcal{A}_\beta)$ is dense set in $\mathcal{H}$. For $\hat{x} \in \mathcal{D}(\mathcal{A}_\beta)$. Using (2.11) we obtain

$$\begin{aligned}
&(\mathcal{A}_\beta\hat{x}, \hat{x}) - (\hat{x}, \mathcal{A}_\beta\hat{x}) \\
&= \mathcal{W}_b[x_1, \overline{x}_1] - \mathcal{W}_a[x_1, \overline{x}_1] + \frac{1}{\delta}[S_-(x_1)\overline{S'_-(x_1)} - S'_-(x_1)\overline{S_-(x_1)}] \\
&= \mathcal{W}_b[x_1, \overline{x}_1] = A_1^+(x_1)A_2^+(\overline{x}_1) - A_2^+(x_1)A_1^+(\overline{x}_1) \\
&= \beta A_2^+(x_1)A_2^+(\overline{x}_1) - \overline{\beta}A_2^+(x_1)A_2^+(\overline{x}_1) \\
&= (\beta - \overline{\beta})|A_2^+(x_1)|^2.
\end{aligned}$$

Therefore $\operatorname{Im}(\mathcal{A}_\beta\hat{x}, \hat{x}) = \operatorname{Im}\beta|A_2^+(x_1)|^2 \geq 0$. Hence we obtain that $\mathcal{A}_\beta$ is a dissipative in $\mathcal{H}$. One can show that $(\mathcal{A}_\beta - \lambda I)\mathcal{D}(\mathcal{A}_\beta) = \mathcal{H}$ for $\operatorname{Im}\lambda < 0$. Consequently $\mathcal{A}_\beta$ is a maximal dissipative operator in the space $\mathcal{H}$ and this completes the proof. □

Let $\mathfrak{T}$ denote the linear operator with the domain $\mathcal{D}(\mathfrak{T})$ acting in the Hilbert space $\mathfrak{H}$. A complex number λ_0 is called an *eigenvalue* of an operator $\mathfrak{T}$ if there exists a non-zero element $z_0 \in \mathcal{D}(\mathfrak{T})$ such that $\mathfrak{T}z_0 = \lambda_0 z_0$. Then z_0 is called an *eigenvector* of $\mathfrak{T}$ for λ_0. The eigenvector corresponding to λ_0 spans a subspace of $\mathcal{D}(\mathfrak{T})$. This subspace is called the *eigenspace* of λ_0 and the *geometric multiplicity* of λ_0 is the dimension of its eigenspace. The vectors $z_1, z_2, \dots, z_k$ are called the *associated vectors* of the eigenvector z_0 if they belong to $\mathcal{D}(\mathfrak{T})$ and $\mathfrak{T}z_j = \lambda_0 z_j + z_{j-1}$, $j = 1, 2, \dots, k$. The non-zero vector $z \in \mathcal{D}(\mathfrak{T})$ is called a *root vector* of the operator $\mathfrak{T}$ corresponding to the eigenvalue λ_0, if all powers of $\mathfrak{T}$ are defined on this element and $(\mathfrak{T} - \lambda_0 I)^m z = 0$ for some integer m. The set of all root vectors of $\mathfrak{T}$ that corresponds to the same eigenvalue λ_0 with the vector $z = 0$ forms a linear set $\mathfrak{N}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathfrak{N}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathfrak{N}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathfrak{T}$ that corresponds to the eigenvalue λ_0. As a consequence, we conclude that the completeness of the system of all eigenvectors and associated vectors of $\mathfrak{T}$ is equivalent to the completeness of the system of all root vectors of this operator.

Definition 2.2. If the following conditions are satisfied

$$L(x_0) = \lambda_0 x_0,\ S_-(x_0) - \lambda_0 S'_-(x_0) = 0, \quad S_+(x_0) = 0, \tag{2.15}$$

$$\begin{gathered}
L(x_s) - \lambda_0 x_s - x_{s-1} = 0, \quad S_-(x_s) - \lambda_0 S'_-(x_s) - S'_-(x_{s-1}) = 0, \\
S_+(x_s) = 0, \quad s = 1, 2, \dots, m,
\end{gathered} \tag{2.16}$$

then the system of vectors $x_0, x_1, \dots, x_m$ is called a chain of eigenvectors and associated vectors corresponding to the eigenvalue λ_0 of the boundary value problem (2.8)-(2.10).

Lemma 2.3. *The eigenvalues of the boundary value problem* (2.8)-(2.10) *including their multiplicity and the eigenvalues of the maximal dissipative operator $\mathcal{A}_\beta$ coincide with each other. Each chain of eigenvectors and associated vectors of the boundary value problem* (2.8)-(2.10)*, meeting the requirements of the eigenvalue λ_0, corresponds to the chain of eigenvectors and associated vectors $\hat{x}_0, \hat{x}_1, \dots, \hat{x}_m$ of the operator $\mathcal{A}_\beta$ corresponding to the same eigenvalue λ_0. In this case, we have*

$$\hat{x}_k = \begin{pmatrix} x_k \\ S'_-(x_k) \end{pmatrix}, \quad k = 0, 1, \dots, m\,. \tag{2.17}$$

Proof. Let $\hat{x}_0 \in \mathcal{D}(\mathcal{A}_\beta)$ and consider the equality $\mathcal{A}_\beta \hat{x}_0 = \lambda_0 \hat{x}_0$. Then one obtains that $L(x_0) = \lambda_0 x_0$, $S_-(x_0) - \lambda_0 S'_-(x_0) = 0$, $S_+(x_0) = 0$, that is, x_0 is an eigenvector of the boundary value problem (2.8)-(2.10). Conversely, consider that (2.15) is satisfied. Then we obtain

$$\begin{pmatrix} x_0 \\ S'_-(x_0) \end{pmatrix} = \hat{x}_0 \in \mathcal{D}(\mathcal{A}_\beta)$$

and $\mathcal{A}_\beta \hat{x}_0 = \lambda_0 \hat{x}_0$. This implies that $\hat{x}_0$ is an eigenvector of the operator $\mathcal{A}_\beta$.

Moreover let $\hat{x}_0, \hat{x}_1, \dots, \hat{x}_m$ be a chain of the eigenvectors and associated vectors of the operator $\mathcal{A}_\beta$ corresponding to the eigenvalue λ_0. Therefore taking in mind that $\hat{x}_k \in \mathcal{D}(\mathcal{A}_\beta)$ $(k = 0, 1, \dots, m)$ and the equality $\mathcal{A}_\beta \hat{x}_0 = \lambda_0 \hat{x}_0$, $\mathcal{A}_\beta \hat{x}_s = \lambda_0 \hat{x}_s + \hat{x}_{s-1}$, $s = 1, 2, \dots, m$, we arrive at the equality (2.15) holds, where $x_0, x_1, \dots, x_m$ are taken to be the first components of the vectors $\hat{x}_0, \hat{x}_1, \dots, \hat{x}_m$. Conversely, on the basis of the elements $x_0, x_1, \dots, x_m$ corresponding to (2.8)-(2.10), we can construct the vectors $\hat{x}_k = \begin{pmatrix} x_k \\ S'_-(x_k) \end{pmatrix}$ for which $\hat{x}_k \in \mathcal{D}(\mathcal{A}_\beta)$ $(k = 0, 1, \dots, m)$ and $\mathcal{A}_\beta \hat{x}_0 = \lambda_0 \hat{x}_0$, $\mathcal{A}_\beta \hat{x}_s = \lambda_0 \hat{x}_s + \hat{x}_{s-1}$, $s = 1, 2, \cdots$. This completes the proof. □

Let us consider the direct sum space $\mathbb{H} := \mathcal{L}^2(-\infty, 0) \oplus \mathcal{H} \oplus \mathcal{L}^2(0, \infty)$, where $\mathcal{L}^2(-\infty, 0)$ is called the 'incoming' channel and $\mathcal{L}^2(0, \infty)$ is called the 'outgoing' channel. This direct sum space is called it the *main Hilbert space of the dilation.* Consider the operator $\mathbb{S}_\beta$ in the main Hilbert space generated by the expression

$$\mathbb{S}\langle u_-, \hat{x}, u_+ \rangle = \langle i \frac{du_-}{d\xi}, \widetilde{L}(\hat{x}), i \frac{du_+}{d\zeta} \rangle, \quad \langle u_-, \hat{x}, u_+ \rangle \in \mathcal{D}(\mathbb{S}_\beta), \tag{2.18}$$

where $\mathcal{D}(\mathbb{S}_\beta)$ is the set consisting of vectors $\langle u_-, \hat{x}, u_+ \rangle$ satisfying the conditions: $u_- \in \mathcal{W}_2^1(-\infty, 0)$, $u_+ \in \mathcal{W}_2^1(0, \infty)$, $\hat{x} \in \mathcal{H}$, $\hat{x}(t) = \begin{pmatrix} x_1(t) \\ x_2 \end{pmatrix}$, $x_1 \in \mathcal{D}_{\max}$, $x_2 = S'_-(x_1)$, $\mathcal{W}_b[x_1, \theta] - \beta \mathcal{W}_b[x_1, \phi] = \gamma u_-(0)$, $\mathcal{W}_b[x_1, \theta] - \overline{\beta} \mathcal{W}_b[x_1, \phi] = \gamma u_+(0)$ $(\gamma^2 := 2 \operatorname{Im} \beta$, $\gamma > 0)$, where $\mathcal{W}_2^1$ is the Sobolev space.

Theorem 2.4. *The operator $\mathbb{S}_\beta$ is self-adjoint in $\mathbb{H}$ and it is a self-adjoint dilation of the dissipative operator $\mathcal{A}_\beta$*

Proof. Taking $F, G \in \mathcal{D}(\mathbb{S}_\beta)$, where $F = \langle u_-, \hat{x}, u_+ \rangle$ and $G = \langle v_-, \hat{y}, v_+ \rangle$, then we obtain that

$$\begin{aligned}
&(\mathbb{S}_\beta F, G)_{\mathbb{H}} - (F, \mathbb{S}_\beta G)_{\mathbb{H}} \\
&= \mathcal{W}_b[x_1, \bar{y}_1] - \mathcal{W}_a[x_1, \bar{y}_1] + \frac{1}{\gamma}(S_-(x_1)\overline{S'_-(y_1)} - S'_-(x_1)\overline{S_-(y_1)}) \\
&\quad + iu_-(0)\overline{v_-}(0) - iu_+(0)\overline{v_+}(0) \\
&= \mathcal{W}_b[x_1, \overline{y}_1] + iu_-(0)\overline{v_-}(0) - iu_+(0)\overline{v_+}(0) \\
&= \mathcal{W}_b[x_1, \overline{y}_1] - \frac{1}{i\gamma^2}(\mathcal{W}_b[x_1, \theta] - \beta \mathcal{W}_b[x_1, \phi])(\overline{\mathcal{W}_b[y_1, \theta]} \\
&\quad - \overline{\beta}\overline{\mathcal{W}_b[y_1, \phi]}) + \frac{1}{i\gamma^2}(\mathcal{W}_b[x_1, \theta] - \overline{\beta}\mathcal{W}_b[x_1, \phi])(\overline{\mathcal{W}_b[y_1, \theta]} - \beta\overline{\mathcal{W}_b[y_1, \phi]}) \\
&= \mathcal{W}_b[x_1, \overline{y}_1] - \frac{1}{i\gamma^2}\{\mathcal{W}_b[x_1, \theta]\overline{\mathcal{W}_b[y_1, \theta]} - \overline{\beta}\mathcal{W}_b[x_1, \theta]\overline{\mathcal{W}_b[y_1, \phi]} \\
&\quad - \beta \mathcal{W}_b[x_1, \phi]\overline{\mathcal{W}_b[y_1, \theta]} + |\beta|^2 \mathcal{W}_b[x_1, \phi]\overline{\mathcal{W}_b[y_1, \phi]}\} + \frac{1}{i\gamma^2}\{\mathcal{W}_b[x_1, \theta]\overline{\mathcal{W}_b[y_1, \theta]}
\end{aligned}$$

$$
\begin{aligned}
&- \beta\mathcal{W}_b[x_1,\theta]\overline{\mathcal{W}_b[y_1,\phi]} - \overline{\beta}\mathcal{W}_b[x_1,\phi]\overline{\mathcal{W}_b[y_1,\theta]} + |\beta|^2\mathcal{W}_b[x_1,\phi]\overline{\mathcal{W}_b[y_1,\phi]}\}\\
&= \mathcal{W}_b[x_1,\overline{y}_1] - \frac{1}{i\gamma^2}\{(-\overline{\beta}+\beta)\mathcal{W}_b[x_1,\theta]\overline{\mathcal{W}_b[y_1,\phi]} + (-\beta+\overline{\beta})\mathcal{W}_b[x_1,\phi]\overline{\mathcal{W}_b[y_1,\theta]}\}\\
&= \mathcal{W}_b[x_1,\overline{y}_1] - \mathcal{W}_b[x_1,\theta]\overline{\mathcal{W}_b[y_1,\phi]} + \mathcal{W}_b[x_1,\phi]\overline{\mathcal{W}_b[y_1,\theta]}. \qquad (2.19)
\end{aligned}
$$

From this equality and (2.12) one obtains that $(\mathbb{S}_\beta F, G)_{\mathbb{H}} - (F, \mathbb{S}_\beta G)_{\mathbb{H}} = 0$, that is, $\mathbb{S}_\beta$ is a symmetric operator in $\mathbb{H}$. Hence $\mathcal{D}(\mathbb{S}_\beta) \subseteq \mathcal{D}(\mathbb{S}_\beta^*)$.

Now we shall prove that $\mathbb{S}_\beta^* \subseteq \mathbb{S}_\beta$. Consider the bilinear form $(\mathbb{S}_\beta F, G)_{\mathbb{H}}$ on elements $G = \langle v_-, \hat{y}, v_+\rangle \in \mathcal{D}(\mathbb{S}_\beta^*)$, where $F = \langle u_-, 0, u_+\rangle$ such that $u_\mp \in W_2^1(\mathbb{R}_\mp)$, $u_\mp(0) = 0$ ($\mathbb{R}_- := (-\infty, 0], \mathbb{R}_+ := [0,\infty)$). Integration by parts gives that $\mathbb{S}_\beta^* G = \langle i\frac{dv_-}{d\xi}, \hat{y}^*, i\frac{dv_+}{d\zeta}\rangle$, where $v_\mp \in W_2^1(\mathbb{R}_\mp)$, $\hat{y}^* \in \mathcal{H}$. Moreover taking $F = \langle 0, \hat{x}, 0\rangle \in \mathcal{D}(\mathbb{S}_\beta)$, we obtain

$$
\mathbb{S}_\beta^* G = \mathbb{S}_\beta^*\langle v_-, \hat{y}, v_+\rangle = \langle i\frac{dv_-}{d\xi}, \widetilde{L}(\hat{y}), i\frac{dv_+}{d\zeta}\rangle, \quad y_1 \in \mathcal{D}_{\max},\ y_2 = S_-'(y_1). \qquad (2.20)
$$

This equality implies that $(\mathbb{S}F, G)_{\mathbb{H}} = (F, \mathbb{S}G)_{\mathbb{H}}$, for all $F \in \mathcal{D}(\mathbb{S}_\beta)$, where the operator $\mathbb{S}$ is given by (2.18). Hence the sum of the integrated terms in the bilinear form $(\mathbb{S}F, G)_{\mathbb{H}}$ must be zero:

$$
\begin{aligned}
&\mathcal{W}_b[x_1,\overline{y}_1] - \mathcal{W}_a[x_1,\overline{y}_1] + \frac{1}{\delta}[S_-(x_1)\overline{S_-'(y_1)} - S_-'(x_1)\overline{S_-(y_1)}]\\
&+ iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} = 0.
\end{aligned} \qquad (2.21)
$$

On the other hand, from (2.11) we obtain

$$
\mathcal{W}_b[x_1,y_1] + iu_-(0)\overline{v_-(0)} - iu_+(0)\overline{v_+(0)} = 0. \qquad (2.22)
$$

Moreover the boundary conditions for $\mathbb{S}_\beta$ imply that

$$
\mathcal{W}_b[x_1,\theta] = \gamma u_-(0) + \frac{i\beta}{\gamma}(u_-(0) - u_+(0)),\ \mathcal{W}_b[x_1,\phi] = \frac{i}{\gamma}(u_-(0) - u_+(0)).
$$

Therefore (2.12) and (2.22) give us

$$
\begin{aligned}
&[\gamma u_-(0) + \frac{i\beta}{\gamma}(u_-(0) - u_+(0))]\overline{\mathcal{W}_b[y_1,\phi]} - \frac{i}{\gamma}(u_-(0) - u_+(0))\overline{\mathcal{W}_b[y_1,\theta]}\\
&= iu_+(0)\overline{v_+(0)} - iu_-(0)\overline{v_-(0)}.
\end{aligned} \qquad (2.23)
$$

If we compare the coefficients of $u_-(0)$ in (2.23), we derive that

$$
\frac{i\gamma^2 - \beta}{\gamma}\overline{\mathcal{W}_b[y_1,\phi]} + \frac{1}{\gamma}\overline{\mathcal{W}_b[y_1,\theta]} = \overline{v_-(0)}
$$

or

$$
\mathcal{W}_b[y_1,\theta] - \beta\mathcal{W}_b[y_1,\phi] = \gamma v_-(0). \qquad (2.24)
$$

Analogously, comparing the coefficients of $u_+(0)$ in (2.23), we find that

$$
\mathcal{W}_b[y_1,\theta] - \overline{\beta}\mathcal{W}_b[y_1,\phi] = \gamma v_+(0). \qquad (2.25)
$$

In conclusion, conditions (2.24) and (2.25) prove that $\mathcal{D}(\mathbb{S}_\beta^*) \subseteq \mathcal{D}(\mathbb{S}_\beta)$, which implies in turn that $\mathbb{S}_\beta = \mathbb{S}_\beta^*$.

It is known that the self-adjoint operator $\mathbb{S}_\beta$ generates the unitary group $\mathbb{Y}(s) = \exp(i\mathbb{S}_\beta s)$ ($s \in \mathbb{R}$) on $\mathbb{H}$. Let $\mathcal{P} : \mathbb{H} \to \mathcal{H}$ and $\mathcal{P}_1 : \mathcal{H} \to \mathbb{H}$ denote the mappings acting according to the formulae $\mathcal{P} : \langle u_-, \hat{x}, u_+\rangle \to \hat{x}$ and $\mathcal{P}_1 : \hat{x} \to \langle 0, \hat{x}, 0\rangle$. We can construct a family $\{\mathcal{Y}(s)\}$ which is a strongly continuous semi-group of completely non-unitary contractions on $\mathcal{H}$ as $\mathcal{Y}(s) := \mathcal{P}\mathbb{Y}(s)\mathcal{P}_1$, $s \geq 0$. Now consider the operator $T_\beta \hat{x} = \lim_{s\to+0}(is)^{-1}(\mathcal{Y}_s\hat{x} - \hat{x})$. T_β is called the generator of the semi-group $\mathcal{Y}(s)$. The domain of T_β consists of all the vectors for which the limit exists. The operator T_β is maximal dissipative. The operator $\mathbb{S}_\beta$ is called the *self-adjoint dilation* of T_β. If we verify the equality $T_\beta = \mathcal{A}_\beta$, then we will have shown that $\mathbb{S}_\beta$ is a self-adjoint dilation of $\mathcal{A}_\beta$. For this purpose we shall get the equality [4, 21]

$$\mathcal{P}(\mathbb{S}_\beta - \lambda I)^{-1}\mathcal{P}_1\hat{x} = (\mathcal{A}_\beta - \lambda I)^{-1}\hat{x}, \quad \hat{x} \in \mathcal{H},\ \operatorname{Im}\lambda < 0. \tag{2.26}$$

Let $(\mathbb{S}_\beta - \lambda I)^{-1}\mathcal{P}_1\hat{x} = G = \langle v_-, \hat{y}, v_+\rangle$. Then $(\mathbb{S}_\beta - \lambda I)G = \mathcal{P}_1\hat{x}$, and hence, $\widetilde{L}(\hat{y}) - \lambda\hat{y} = \hat{x}$, $v_-(\xi) = v_-(0)e^{-i\lambda\xi}$ and $v_+(\zeta) = v_+(0)e^{-i\lambda\zeta}$. Since $G \in \mathcal{D}(\mathbb{S}_\beta)$, we have $v_- \in \mathcal{L}^2(\mathbb{R}_-)$, which implies that $v_-(0) = 0$, and thus $\hat{y}$ satisfies the boundary condition $\mathcal{W}_b[y_1,\theta] - \beta\mathcal{W}_b[y_1,\phi] = 0$. Therefore, $\hat{y} \in \mathcal{D}(\mathcal{A}_\beta)$. Moreover we have $v_+(0) = \gamma^{-1}\{\mathcal{W}_b[y_1,\theta] - \overline{\beta}\mathcal{W}_b[y_1,\phi]\}$, since point λ with $\operatorname{Im}\lambda < 0$ can not be an eigenvalue of a dissipative operator. Thus we get that

$$(\mathbb{S}_\beta - \lambda I)^{-1}\mathcal{P}_1\hat{x} = \langle 0, (\mathcal{A}_\beta - \lambda I)^{-1}\hat{x}, \gamma^{-1}(\mathcal{W}_b[y_1,\theta] - \overline{\beta}\mathcal{W}_b[y_1,\phi])\rangle$$

for $\hat{x} \in \mathcal{H}$ and $\operatorname{Im}\lambda < 0$. Applying the mapping $\mathcal{P}$, we get that (2.26) is satisfied and

$$\begin{aligned}(\mathcal{A}_\beta - \lambda I)^{-1} &= \mathcal{P}(\mathbb{S}_\beta - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty \mathbb{Y}(s)e^{-i\lambda s}ds\mathcal{P}_1\\ &= -i\int_0^\infty \mathcal{Y}_s e^{-i\lambda s}ds = (\mathcal{A}_\beta - \lambda I)^{-1}, \quad \operatorname{Im}\lambda < 0.\end{aligned}$$

Consequently $\mathcal{A}_\beta = T_\beta$ and this completes the proof. □

3. Scattering theory of dilation, functional model of dissipative operator and completeness theorems of the dissipative operator and the boundary value problem (2.8)-(2.10)

Lax-Phillips scattering theory [15] may be applied with the help of the unitary group $\{\mathbb{Y}(s)\}$. To be more precise, following properties are satisfied:

(i) $\mathbb{Y}(s)\mathbb{D}_- \subset \mathbb{D}_-$, $s \leq 0$ and $\mathbb{Y}(s)\mathbb{D}_+ \subset \mathbb{D}_+$, $s \geq 0$;
(ii) $\cap_{s\leq 0}\mathbb{Y}(s)\mathbb{D}_- = \cap_{s\geq 0}\mathbb{Y}(s)\mathbb{D}_+ = \{0\}$;
(iii) $\overline{\cap_{s\geq 0}\mathbb{Y}(s)\mathbb{D}_-} = \overline{\cap_{s\leq 0}\mathbb{Y}(s)\mathbb{D}_+} = \mathbb{H}$;
(iv) $\mathbb{D}_- \perp \mathbb{D}_+$, where $\mathbb{D}_- = \langle\mathcal{L}^2(\mathbb{R}_-), 0, 0\rangle$ and $\mathbb{D}_+ = \langle 0, 0, \mathcal{L}^2(\mathbb{R}_+)\rangle$ are called incoming and outgoing subspaces.

Property (iv) is obvious. Let us prove property (i) for $\mathbb{D}_+$ (the proof for $\mathbb{D}_-$ is similar). Consider the equality

$$\mathcal{R}_\lambda F := (\mathbb{S}_\beta - \lambda I)^{-1}\langle 0, 0, u_+\rangle = \langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}u_+(s)ds\rangle,$$

where $\operatorname{Im}\lambda < 0$, $F \in \mathbb{D}_+$. Therefore $\mathcal{R}_\lambda F \in \mathbb{D}_+$. Taking $G \perp \mathbb{D}_+$ one obtain

$$0 = (\mathcal{R}_\lambda F, G)_{\mathbb{H}} = -i\int_0^\infty e^{-i\lambda s}(\mathbb{Y}(s)F, G)_{\mathbb{H}}ds, \quad \operatorname{Im}\lambda < 0.$$

This equation implies that $(\mathbb{Y}(s)F, G)_{\mathbb{H}} = 0$ for all $s \geq 0$ and therefore $\mathbb{Y}(s)\mathbb{D}_+ \subset \mathbb{D}_+$ for $s \geq 0$. This proves the property (i).

To verify property (ii), we consider the mappings $\mathcal{P}^+ : \mathbb{H} \to \mathcal{L}^2(\mathbb{R}_+)$ and $\mathcal{P}_1^+ : \mathcal{L}^2(\mathbb{R}_+) \to \mathbb{D}_+$ acting according to the formulas $\mathcal{P}^+ : \langle u_-, \hat{x}, u_+\rangle \to u_+$ and $\mathcal{P}_1^+ : u \to \langle 0, 0, u\rangle$, respectively. We notice that the semi-group of isometries $\mathbb{Y}^+(s) :=$

$\mathcal{P}^+\mathbb{Y}(s)\mathcal{P}_1^+$ $(s \geq 0)$ is a one-sided shift in $\mathcal{L}^2(\mathbb{R}_+)$. In fact, the generator of the semi-group of the one-sided shift $\mathcal{V}(s)$ in $\mathcal{L}^2(\mathbb{R}_+)$ is the differential operator $i(d/d\zeta)$ satisfying the boundary condition $u(0) = 0$. On the other side, the generator $\mathcal{B}$ of the semi-group of isometries $\mathbb{Y}^+(s)$ $(s \geq 0)$ is the operator $\mathcal{B}u = \mathcal{P}^+\mathbb{S}_\beta\mathcal{P}_1^+u = \mathcal{P}^+\mathbb{S}_\beta\langle 0, 0, u\rangle = \mathcal{P}^+\langle 0, 0, i(d/d\zeta)u\rangle = i(d/d\zeta)u$, where $u \in W_2^1(\mathbb{R}_+)$ and $u(0) = 0$. Since a semi-group is uniquely determined by its generator, we get $\mathbb{Y}^+(s) = \mathcal{V}(s)$, and so,

$$\cap_{s\geq 0}\mathbb{Y}^+(s)\mathbb{D}_+ = \langle 0, 0, \cap_{s\geq 0}\mathcal{V}(s)\mathcal{L}^2(\mathbb{R}_+)\rangle = \{0\}, \quad s \geq 0,$$

i.e., property (ii) is proved.

According to the Lax-Phillips scattering theory, the scattering matrix is defined with the help of the theory of spectral representations. During this process, we will have also proved property (iii) of the incoming and outgoing subspaces.

We shall remind that the linear operator $\mathfrak{B}$ (with domain $\mathcal{D}(\mathfrak{B})$) acting in the Hilbert space $\mathfrak{H}$ is called *completely non-self-adjoint* (or *pure*) if there is no invariant subspace $\mathfrak{M} \subseteq \mathcal{D}(\mathfrak{B})$ $(\mathfrak{M} \neq \{0\})$ of the operator $\mathfrak{B}$ on which the restriction $\mathfrak{B}$ to $\mathfrak{M}$ is self-adjoint.

Lemma 3.1. *The operator $\mathcal{A}_\beta$ is completely non-self-adjoint (*pure*).*

Proof. Let $\mathcal{A}'_\beta$ be a self-adjoint part of $\mathcal{A}_\beta$ with domain $\mathcal{D}(\mathcal{A}'_\beta) = \mathcal{H}' \cap \mathcal{D}(\mathcal{A}_\beta)$ in the non-trivial subspace $\mathcal{H}' \subset \mathcal{H}$. If $\hat{x} \in \mathcal{D}(\mathcal{A}'_\beta)$. Let $\hat{x} \in \mathcal{D}(\mathcal{A}'^*_\beta)$ and $A_1^+(x_1) - \beta A_2^+(x_1) = 0$, $A_1^+(x_1) - \overline{\beta}A_2^+(x_1) = 0$, $x_2 = S'_-(x_1)$. Therefore we have $\mathcal{W}_b[x_1, \theta] = \mathcal{W}_b[x_1, \phi] = 0$, $x_2 = S'_-(x_1)$. This implies that $\hat{x}(t, \lambda) \equiv 0$ for the eigenvectors $\hat{x}(t, \lambda)$ of the operator $\mathcal{A}'_\beta$ that lie in $\mathcal{H}'$ and are eigenvectors of $\mathcal{A}_\beta$. Therefore by the theorem on expansion in eigenvectors of the self-adjoint operator $\mathcal{A}'_\beta$ we find that $\mathcal{H}' = \{0\}$. Consequently the operator $\mathcal{A}_\beta$ is pure and the lemma is proved. □

Let us set

$$\mathbb{H}_- = \overline{\cup_{s\geq 0}\mathbb{Y}(s)\mathbb{D}_-}, \quad \mathbb{H}_+ = \overline{\cup_{s\geq 0}\mathbb{Y}(s)\mathbb{D}_+}.$$

Lemma 3.2. *The equality $\mathbb{H}_- + \mathbb{H}_+ = \mathbb{H}$ holds.*

Proof. Consider the subspace $\mathbb{H}' = \mathbb{H} \ominus (\mathbb{H}_- + \mathbb{H}_+)$. Using property (i) of the subspace $\mathbb{D}_+$, we obtain that the subspace $\mathbb{H}'$ is invariant relative to the group $\{\mathbb{Y}(s)\}$. Moreover $\mathbb{H}'$ can be considered as $\mathbb{H}' = \langle 0, \mathcal{H}', 0\rangle$, where $\mathcal{H}'$ is a subspace in $\mathcal{H}$. Therefore, if the subspace $\mathbb{H}'$ (and hence also $\mathcal{H}'$) were non-trivial, then the unitary group $\{\mathbb{Y}'(s)\}$ restricted to this subspace would be a unitary part of the group $\{\mathbb{Y}(s)\}$. Therefore the restriction $\mathcal{A}'_\beta$ of $\mathcal{A}_\beta$ to $\mathcal{H}'$ would be a self-adjoint operator in $\mathcal{H}'$. Consequently the purity of the operator $\mathcal{A}_\beta$ implies that $\mathcal{H}' = \{0\}$. Thus, the proof is complete. □

Consider the solutions $\chi_\lambda(t)$ and $\psi_\lambda(t)$ of the system (2.8) satisfying the conditions

$$A_1^-(\chi_\lambda) = \frac{\delta'_2}{\delta}, \quad A_2^-(\chi_\lambda) = \frac{\delta'_1}{\delta}, \quad A_1^-(\psi_\lambda) = \delta_2 - \delta'_2\lambda, \quad A_2^-(\psi_\lambda) = \delta_1 - \delta'_1\lambda.$$

For convenience, we adopt the following notation:

$$\kappa(\lambda) := \frac{\mathcal{W}_b[\chi_\lambda, \phi]}{\mathcal{W}_b[\psi_\lambda, \phi]}, \quad \sigma(\lambda) := -\frac{\mathcal{W}_b[\psi_\lambda, \theta]}{\mathcal{W}_b[\psi_\lambda, \phi]}, \quad \hat{\psi}_\lambda(t) := \begin{pmatrix} \psi_\lambda(t) \\ \delta \end{pmatrix}, \tag{3.1}$$

$$\Theta_\beta(\lambda) := \frac{\sigma(\lambda) + \beta}{\sigma(\lambda) + \overline{\beta}}. \tag{3.2}$$

The fact that σ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis follows from (3.1). Further it is possible to show that the function σ satisfies $\operatorname{Im}\lambda \operatorname{Im}\sigma(\lambda) < 0$ for all $\operatorname{Im}\lambda \neq 0$, and $\overline{\sigma(\lambda)} = \sigma(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles of σ.

Consider the vector

$$\Psi_\lambda^-(t,\xi,\zeta) = \langle e^{-i\lambda\xi}, \gamma\kappa(\lambda)\left\{(\sigma(\lambda)+\beta)\mathcal{W}_b[\chi_\lambda,\phi]\right\}_\lambda^{-1}\hat{\psi}(t), \overline{\Theta}_\beta(\lambda)e^{-i\lambda\zeta}\rangle. \tag{3.3}$$

which, for real values of λ, do not belong to the space $\mathbb{H}$ and satisfies the equation $\mathbb{S}\Psi = \lambda\Psi$ with the corresponding boundary conditions for the operator $\mathbb{S}_\beta$.

Define the map $\Phi_- : F \to \widetilde{F}_-(\lambda)$ by $(\Phi_- F)(\lambda) := \widetilde{F}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(F, \Psi_\lambda^-)_{\mathbb{H}}$ on the vectors $F = \langle u_-, \hat{x}, u_+\rangle$. Here, the functions u_-, u_+, and x_1 are smooth, compactly supported functions.

Lemma 3.3. *$\mathbb{H}_-$ is isometrically mapped by the transformation Φ_- onto $\mathcal{L}^2(\mathbb{R})$. Parseval equality and the inversion formula hold for all vectors $F, G \in \mathbb{H}_-$ as follows*

$$(F,G)_{\mathbb{H}} = (\widetilde{F}_-, \widetilde{G}_-)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \widetilde{F}_-(\lambda)\overline{\widetilde{G}_-(\lambda)}d\lambda, \quad F = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{F}_-(\lambda)\Psi_\lambda^- d\lambda,$$

where $\widetilde{F}_-(\lambda) = (\Phi_- F)(\lambda)$ and $\widetilde{G}_-(\lambda) = (\Phi_- G)(\lambda)$.

Proof. For $F, G \in \mathbb{D}_-$, $F = \langle u_-, 0, 0\rangle$, and $G = \langle v_-, 0, 0\rangle$, we have

$$\widetilde{F}_-(\lambda) = \frac{1}{\sqrt{2\pi}}(F, \Psi_\lambda^-)_{\mathbb{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} u_-(\xi)e^{i\lambda\xi}d\xi \in \mathcal{H}_-^2,$$

where $\mathcal{H}_\pm^2$ describe the Hardy classes in $\mathcal{L}^2(\mathbb{R})$ consisting of the functions that are analytically extendable to the upper and lower half-planes, respectively, and with the help of the Parseval equality for Fourier integrals

$$(F,G)_{\mathbb{H}} = \int_{-\infty}^{\infty} u_-(\xi)\overline{v_-(\xi)}d\xi = \int_{-\infty}^{\infty} \widetilde{F}_-(\lambda)\overline{\widetilde{G}_-(\lambda)}d\lambda = (\Phi_- F, \Phi_- G)_{\mathcal{L}^2}.$$

To extend the Parseval equality to the whole of $\mathbb{H}_-$, consider the dense set of $\mathbb{H}'_-$ in $\mathbb{H}_-$ consisting of the vectors obtained as follows from the smooth, compactly supported functions in $\mathbb{D}_- : F \in \mathbb{H}'_-$ if $F = \mathbb{Y}(s)F_0$, $F_0 = \langle u_-, 0, 0\rangle$, $u_- \in C_0^\infty(-\infty, 0)$, where $s = s_F$ is a non-negative number depending on F. In this case, if $F, G \in \mathbb{H}'_-$, then we have $\mathbb{Y}(-s)F, \mathbb{Y}(-s)G \in \mathbb{D}_-$ for $s > s_F$ and $s > s_G$, and, moreover, the first components of these vectors lie in $C_0^\infty(-\infty, 0)$. Consequently being unitary of the operators $\mathbb{Y}(s)$ $(s \in \mathbb{R})$ we obtain from the equality $\Phi_-\mathbb{Y}(s)F = (\mathbb{Y}(s)F, U_\lambda^-)_{\mathbb{H}}$ $= e^{i\lambda s}(F, U_\lambda^-)_{\mathbb{H}} = e^{i\lambda s}\Phi_- F$ that

$$\begin{aligned}(F,G)_{\mathbb{H}} &= (\mathbb{Y}(-s)F, \mathbb{Y}(-s)G)_{\mathbb{H}} = (\Phi_-\mathbb{Y}(-s)F, \Phi_-\mathbb{Y}(-s)G)_{\mathcal{L}^2} \\ &= (e^{-i\lambda s}\Phi_- F, e^{-i\lambda s}\Phi_- G)_{\mathcal{L}^2} = (\widetilde{F}, \widetilde{G})_{\mathcal{L}^2}.\end{aligned} \tag{3.4}$$

Passing to the closure in (3.4), we get the Parseval equality for the whole space $\mathbb{H}_-$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of integrals over finite intervals. Hence the fact that Φ_- maps $\mathbb{H}_-$ onto the whole of $\mathcal{L}^2(\mathbb{R})$ follows from the following

$$\Phi_-\mathbb{H}_- = \overline{\cup_{s\geq 0}\Phi_-\mathbb{Y}(s)\mathbb{D}_-} = \overline{\cup_{s\geq 0}e^{-i\lambda s}\mathcal{H}_-^2} = \mathcal{L}^2(\mathbb{R}).$$

Therefore the lemma is proved. □

Consider the vector

$$\Psi_\lambda^+(t,\xi,\zeta) = \langle \Theta_\beta(\lambda) e^{-i\lambda\xi}, \gamma\kappa(\lambda)\left\{(\sigma(\lambda)+\overline{\beta})\mathcal{W}_b[\chi_\lambda,\phi]\right\}\hat{\psi}_\lambda(t), e^{-i\lambda\zeta}\rangle, \tag{3.5}$$

which, for real λ do not belong to the space $\mathbb{H}_-$, satisfies the equation $\mathbb{S}\Psi = \lambda\Psi$ and the corresponding boundary conditions for the operator $\mathbb{S}_\beta$. Define the transformation $\Phi_+ : F \to \widetilde{F}_+(\lambda)$ on vectors $F = \langle u_-, \hat{x}, u_+\rangle$, in which the functions u_-, u_+, and x are smooth, compactly supported functions, by setting $(\Phi_+ F)(\lambda) := \widetilde{F}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(F, \Psi_\lambda^+)_{\mathbb{H}}$. Next results can be verified in a similar manner used in the proof of Lemma 3.3.

Lemma 3.4. *$\mathbb{H}_+$ is isometrically maps by the transformation Φ_+ onto $\mathcal{L}^2(\mathbb{R})$, and for all vectors $F, G \in \mathbb{H}$, the Parseval equality and the inversion formula hold:*

$$(F,G)_{\mathbb{H}} = (\widetilde{F}_+, \widetilde{G}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty} \widetilde{F}_+(\lambda)\overline{\widetilde{G}_+(\lambda)}d\lambda, \quad F = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{F}_+(\lambda)\Psi_\lambda^+ d\lambda,$$

where $\widetilde{F}_+(\lambda) = (\Phi_+ F)(\lambda)$ and $\widetilde{G}_+(\lambda) = (\Phi_+ G)(\lambda)$.

Using (3.2), we get that $|\Theta_\beta(\lambda)| = 1$ for $\lambda \in \mathbb{R}$. Consequently we obtain from the definitions of the vectors Ψ_λ^- and Ψ_λ^+ that

$$\Psi_\lambda^- = \Psi_\lambda^+ \overline{\Theta_\beta(\lambda)} \quad (\lambda \in \mathbb{R}). \tag{3.6}$$

Lemmas 3.3 and 3.4 imply that $\mathbb{H}_- = \mathbb{H}_+$ and Lemma 3.2 implies that $\mathbb{H} = \mathbb{H}_- = \mathbb{H}_+$. Thus, property (iii) has been validated for the incoming and outgoing subspaces.

The transformation Φ_- is the incoming spectral representation for the group $\{\mathbb{Y}(s)\}$. In fact, the transformation Φ_- isometrically maps $\mathbb{H}$ onto $\mathcal{L}^2(\mathbb{R})$ with the subspace $\mathbb{D}_-$ mapped onto $\mathcal{H}_-^2$ and the operators $\mathbb{Y}(s)$ are transformed into the operators of multiplication by $e^{i\lambda s}$. Similarly, the transformation Φ_+ is the outgoing spectral representation for $\{\mathbb{Y}(s)\}$. Equality given by (3.6) implies that the passage from the Φ_+-representation of the vector $F \in \mathbb{H}$ to its Φ_--representation is realized by multiplication of the function $\Theta_\beta(\lambda) : \widetilde{F}_-(\lambda) = \Theta_\beta(\lambda)\widetilde{F}_+(\lambda)$. According to [15], we see that the *scattering function* (*matrix*) of the group $\{\mathbb{Y}(s)\}$ with respect to the subspaces $\mathbb{D}_-$ and $\mathbb{D}_+$, is the coefficient by which the Φ_--representation of a vector $F \in \mathbb{H}$ must be multiplied to get the corresponding Φ_+-representation: $\widetilde{F}_+(\lambda) = \overline{\Theta_\beta(\lambda)}\widetilde{F}_-(\lambda)$. Therefore we can state the following theorem.

Theorem 3.5. *The function $\overline{\Theta}_\beta$ is the scattering function (matrix) of the unitary group $\{\mathbb{Y}(s)\}$ (of the self-adjoint operator $\mathbb{S}_\beta$).*

We shall remind that the analytic function Θ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|\Theta(\lambda)| \le 1$ for $\lambda \in \mathbb{C}_+$ and $|\Theta(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$. Let Θ be an arbitrary non-constant inner function (see [18]) on the upper half-plane. The subspace $\mathcal{N} = \mathcal{H}_+^2 \ominus \Theta\mathcal{H}_+^2$ is not the trivial space and is a subspace of the Hilbert space $\mathcal{H}_+^2$. We consider the semi-group of operators $\mathcal{X}(s)$ $(s \ge 0)$ acting in $\mathcal{N}$ according to the formula $\mathcal{X}(s)u = \mathcal{P}[e^{i\lambda s}u]$, $u = u(\lambda) \in \mathcal{N}$, where $\mathcal{P}$ is the orthogonal projection from $\mathcal{H}_+^2$ onto $\mathcal{N}$. The generator of the semi-group $\{\mathcal{X}(s)\}$ is defined by $\mathcal{T}u = \lim_{s\to+0}[(is)^{-1}(\mathcal{X}(s)u - u)]$, where $\mathcal{T}$ is a maximal dissipative operator acting in $\mathcal{N}$ and with the domain $\mathcal{D}(\mathcal{T})$ consisting of all functions $u \in \mathcal{N}$ for which the limit exists. The operator $\mathcal{T}$ is called a *model dissipative operator*. It is better to note that this model dissipative operator, which is associated with the names of Lax and Phillips [15], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [18]. The basic assertion is that Θ is the *characteristic function* of the operator $\mathfrak{T}$.

Consider the space $\mathbb{H} = \mathbb{D}_- \oplus \mathcal{N} \oplus \mathbb{D}_+$, where $\mathcal{N} = \langle 0, \mathcal{H}, 0\rangle$. Under the unitary transformation Φ_- we have

$$\begin{gathered} \mathbb{H} \to \mathcal{L}^2(\mathbb{R}), \quad F \to \widetilde{F}_-(\lambda) = (\Phi_- F)(\lambda),\ \mathbb{D}_- \to \mathcal{H}^2_-, \quad \mathbb{D}_+ \to \Theta_\beta \mathcal{H}^2_+, \\ \mathcal{N} \to \mathcal{H}^2_+ \ominus \Theta_\beta \mathcal{H}^2_+, \quad \mathbb{Y}(s)F \to (\Phi_- \mathbb{Y}(s)\Phi_-^{-1}\widetilde{F}_-)(\lambda) = e^{i\lambda s}\widetilde{F}_-(\lambda). \end{gathered} \tag{3.7}$$

Therefore, according to the model operator theory, (3.7) implies that our operator $\mathcal{A}_\beta$ is unitarily equivalent to the model dissipative operator with the characteristic function Θ_β. Since the characteristic functions of unitarily equivalent dissipative operators coincide [4, 18, 21], we have proved the following theorem.

Theorem 3.6. *The characteristic function of the dissipative operator $\mathcal{A}_\beta$ coincides with the function Θ_β defined by* (3.2).

It is known that one can take the complete information about the spectral properties of the maximal dissipative operator $\mathcal{A}_\beta$. For example, the absence of a singular factor $s(\lambda)$ of the characteristic function Θ_β in the factorization $\Theta_\beta(\lambda) = s(\lambda)B(\lambda)$ ($B(\lambda)$ is a Blaschke product) guarantees the completeness of the system of eigenvectors and associated vectors of the operator $\mathcal{A}_\beta$ (see [1, 5, 18, 21]).

Theorem 3.7. *For all values of β with $\operatorname{Im}\beta > 0$, with the possible exception of a single value $\beta = \beta_0$, the characteristic function Θ_β of the dissipative operator $\mathcal{A}_\beta$ is a Blaschke product. The spectrum of $\mathcal{A}_\beta$ is purely discrete and belongs to the open upper half-plane. The operator $\mathcal{A}_\beta$ ($\beta \neq \beta_0$) has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity. The system of all eigenvectors and associated vectors (or root vectors) of the operator $\mathcal{A}_\beta$ ($\beta \neq \beta_0$) is complete in the space $\mathcal{H}$.*

Proof. Using that $\operatorname{Im}\lambda \operatorname{Im}\sigma(\lambda) < 0$ for all $\operatorname{Im}\lambda \neq 0$, and $\overline{\sigma(\lambda)} = \sigma(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles of $\sigma(\lambda)$ and (3.2), one can obtain that $|\Theta_\beta(\lambda)| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $|\Theta_\beta(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$. This implies that $\Theta_\beta(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex λ-plane. Therefore, it can be factored as follows

$$\Theta_\beta(\lambda) = e^{i\lambda b}B_\beta(\lambda), \quad b = b(\beta) \geq 0, \tag{3.8}$$

where $B_\beta(\lambda)$ is a Blaschke product. Therefore

$$|\Theta_\beta(\lambda)| = |e^{i\lambda b}||B_\beta(\lambda)| \leq e^{-b(\beta)\operatorname{Im}\lambda}, \quad \operatorname{Im}\lambda \geq 0. \tag{3.9}$$

Moreover (3.2) yields

$$\sigma(\lambda) = \frac{\beta - \overline{\beta}\Theta_\beta(\lambda)}{\Theta_\beta(\lambda) - 1}. \tag{3.10}$$

If $b(\beta) > 0$ for a given value β ($\operatorname{Im}\beta > 0$), then by (3.9) we have $\lim_{t\to+\infty}\Theta_\beta(it) = 0$, which together with (3.10) implies that $\lim_{t\to+\infty}\sigma(it) = -\beta$. Since $\sigma(\lambda)$ is independent of β, $b(\beta)$ can be non-zero at not more than a single point $\beta = \beta_0$ (and, further $\beta_0 = -\lim_{t\to+\infty}\sigma(it)$). The theorem is proved. $\square$

Since, by Lemma 2.3, the eigenvalues of the boundary value problem (2.8)-(2.10) and the eigenvalues of the operator $\mathcal{A}_\beta$ coincide, including their multiplicity and, furthermore, for the eigenvectors and associated vectors of the boundary problems

(2.8)-(2.10), the formula (2.17) is fulfilled. Then Theorem 3.7 can be stated as follows.

Theorem 3.8. *The spectrum of the boundary value problem* (2.8)-(2.10) *is purely discrete and belongs to the open upper half-plane. For all values of β with $\operatorname{Im}\beta > 0$, with the possible exception of a single value $\beta = \beta_0$, the boundary value problem* (2.8)-(2.10) $(\beta \neq \beta_0)$ *has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity. The system of eigenvectors and associated vectors of this problem $(\beta \neq \beta_0)$ is complete in the space $\mathcal{L}_P^2(\Omega; E)$.*

Since a linear operator $\mathfrak{T}$ acting in the Hilbert space $\mathfrak{H}$ is maximal accumulative if and only if $-\mathfrak{T}$ is maximal dissipative, all results concerning maximal dissipative operators can be immediately transferred to maximal accumulative operators. Then the Theorem 3.7 yields the following result.

Corollary 3.9. *For $\operatorname{Im}\beta < 0$ the spectrum of the boundary value problem* (2.8)-(2.10) *is purely discrete and belongs to the open lower half-plane. For all values of β with $\operatorname{Im}\beta < 0$, with the possible exception of a single value $\beta = \beta_1$, the boundary value problem* (2.8)-(2.10) $(\beta \neq \beta_1)$ *has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity. The system of eigenvectors and associated vectors of this problem $(\beta \neq \beta_1)$ is complete in the space $\mathcal{L}_P^2(\Omega; E)$.*

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