Electronic Journal of Differential Equations, Vol. 2005(2005), No. 28, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A NEW GREEN FUNCTION CONCEPT FOR FOURTH-ORDER DIFFERENTIAL EQUATIONS

KAMIL ORUCOGLU

Abstract. A linear completely nonhomogeneous generally nonlocal multi-point problem is investigated for a fourth-order differential equation with generally nonsmooth coefficients satisfying some general conditions such as p-integrability and boundedness. A system of five integro-algebraic equations called an adjoint system is introduced for this problem. A concept of a Green functional is introduced as a special solution of the adjoint system. This new type of Green function concept, which is more natural than the classical Green-type function concept, and an integral form of the nonhomogeneous problems can be found more naturally. Some applications are given for elastic bending problems.

1. Introduction

The Green functions of linear boundary-value problems for ordinary differential equations with sufficiently smooth coefficients have been investigated in detail in several studies [14, 17, 18, 19, 20]. In this work, a linear, generally nonlocal multi-point problem is investigated for a differential equation of fourth-order. The coefficients of the equation are assumed to be generally nonsmooth functions satisfying some general conditions such as p-integrability and boundedness. The operator of this equation, in general, does not have a formal adjoint operator or any extension of the traditional type on a space of distributions [11, 18]. In addition, the considered problem does not have a meaningful traditional type adjoint problem, even for simple cases of a differential equation and nonlocal conditions. Due to these facts, some serious difficulties arise in application of the classical methods for such a problem. As it follows from [14, p. 87], similar difficulties arise even for classical type boundary-value problems if the coefficients of the differential equation are, for example, continuous nonsmooth functions. For this reason, a new approach is introduced for the investigation of the considered problem and other similar problems. This approach is based on [1, 2, 3] and on methods of functional analysis. The main idea of this approach is related to the use of a new concept of the adjoint problem named "adjoint system". Such an adjoint system, in fact, includes five "integro-algebraic" equations with an unknown elements $(f_4(\zeta), f_3, f_2, f_1, f_0)$

2000 *Mathematics Subject Classification.* 34A30, 34B05, 34B10, 34B27, 45A05, 45E35, 45J05.
Key words and phrases. Green function; linear operator; multipoint; nonlocal problem; nonsmooth coefficient; differential equation.

Submitted November 19, 2004. Published March 6, 2005.

in which $f_4(\zeta)$ is a function, and f_j, $j = 0, 1, 2, 3$ are real numbers. One of these equations is an integral equation with respect to $f_4(\zeta)$ and generally includes f_j as parameters. The other four can be considered a system of four algebraic equations with respect to $(f_0, f_1 f_2, f_3)$, and they may include some integral functionals defined on $f_4(\zeta)$. The form of our adjoint system depends on the operators of the equation and the conditions. The role of our adjoint system is similar to that of the adjoint operator equation in the general theory of the linear operator equations in Banach spaces [7, 14, 13]. The integral representation of the solution is obtained by a concept of the "Green functional" which is introduced as a special solution $f(x) = (f_4(\zeta, x), f_3(x), f_2(x), f_1(x), f_0(x))$ of the corresponding adjoint system having a special free term depending on x as a parameter. The superposition principle for the equation is given by the first element $f_4(\zeta, x)$ of the Green functional $f(x)$; the other four elements $f_j(x)$, $(j = 0, 1, 2, 3)$ correspond to the unit effects of the conditions. If the homogeneous problem has a nontrivial solution, then the Green functional does not exist. The present approach for the Green functionals is constructive. In principle, this approach is different from the classical methods for constructing Green type functions [19].

2. Statement of the problem

Let $\mathbb{R}$ be the set of the real numbers. Let $G = (x_0, x_1)$ be a bounded open interval in $\mathbb{R}$, Let $L_p(G)$, with $1 \le p < \infty$, be the space of p-integrable functions on G. Let $L_\infty(G)$ be the space of measurable and essentially bounded functions on G, and $W_p^{(4)}(G)$, $1 \le p \le \infty$, be the space of all functions $u = u(x) \in L_p(G)$ having derivatives $d^k u/dx^k \in L_p(G)$, where $k = 1, \dots, 4$. The norm in the space $W_p^{(4)}(G)$ is defined as

$$\|u\|_{W_p^{(4)}(G)} = \sum_{k=0}^{4} \|\frac{d^k u}{dx^k}\|_{L_p(G)} .$$

We consider the differential equation

$$(V_4 u)(x) \equiv u^{(iv)}(x) + A_0(x)u(x) + A_1(x)u'(x) + A_2(x)u''(x) + A_3(x)u'''(x) = z_4(x), \tag{2.1}$$

$x \in G$, subject to the following generally nonlocal multipoint-boundary conditions

$$\begin{gathered} V_0 u \equiv u(x_0) = z_0; \\ V_1 u \equiv u'(x_0) = z_1; \\ V_2 u \equiv \alpha_1 u(\beta) + \alpha_2 u''(x_1) + \alpha_3 u'(x_1) = z_2; \\ V_3 u \equiv u(x_1) = z_3. \end{gathered} \tag{2.2}$$

Problem (2.1)-(2.2) is considered in the space $W_p = W_p^{(4)}(G)$. Furthermore, it is assumed that the following conditions are satisfied: $A_j \in L_p(G)$ are given functions, where $j = 0, 1, 2, 3$; α_j are given numbers; $\beta \in \bar{G}$ is given point with $x_0 < \beta < x_1$; $z_4 \in L_p(G)$ is given function, and z_j are given numbers.

Problem (2.1)-(2.2) is a linear completely nonhomogeneous problem which can be considered an operator equation:

$$Vu = z, \tag{2.3}$$

with the linear operator $V = (V_4, V_3, V_2, V_1, V_0)$ and $z = (z_4(x), z_3, z_2, z_1, z_0)$.

The conditions given above show that V is bounded from W_p to the Banach space $E_p = L_p(G) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ consisting of element $z = (z_4(x), z_3, z_2, z_1, z_0)$ with

$$\|z\|_{E_p} = \|z_4\|_{L_p(G)} + |z_3| + |z_2| + |z_1| + |z_0|, \quad 1 \le p \le \infty.$$

If, for a given $z \in E_p$, problem (2.1)-(2.2) has a unique solution $u \in W_p$ with $\|u\|_{W_p} \le c_0 \|z\|_{E_p}$, then this problem is called a well-posed problem, where c_0 is a constant independent of z. Problem (2.1)-(2.2) is well-posed if and only if V is a (linear) homeomorphism between W_p and E_p.

3. Adjoint space of the solution space

Problem (2.1)-(2.2) is investigated by means of a new concept of the adjoint problem. This concept is introduced following [2, 3] by the adjoint operator $V^\star$ of V. Furthermore, some isomorphic decompositions of the space W_p of the solutions and its adjoint space $W_p^\star$ will be employed.

It is well known that any function $u \in W_p$ can be represented as

$$\begin{aligned} u(x) =& u(\alpha) + u'(\alpha)(x-\alpha) + u''(\alpha)\frac{(x-\alpha)^2}{2} \\ &+ u'''(\alpha)\frac{(x-\alpha)^3}{6} + \int_\alpha^x \frac{(x-\zeta)^3}{6} u^{(iv)}(\zeta) d\zeta, \end{aligned} \tag{3.1}$$

where $\alpha \in \bar{G}$ is a given point. Furthermore, the trace or the value operators $D_0 u = u(\gamma)$, $D_1 u = u'(\gamma)$, $D_2 u = u''(\gamma)$ and $D_3 u = u'''(\gamma)$ are bounded and surjective from W_p onto $\mathbb{R}$ for a given $\gamma \in \bar{G}$. In addition, the values $u(\alpha)$, $u'(\alpha)$, $u''(\alpha)$, $u'''(\alpha)$ and $u^{(iv)}(x)$ are unrelated elements of the function $u \in W_p$ in the following sense: For arbitrary numbers ν_j and an arbitrary function $\nu_4 \in L_p(G)$, there exists one and only one $u \in W_p$ such that $u(\alpha) = \nu_0$, $u'(\alpha) = \nu_1$, $u''(\alpha) = \nu_2$, $u'''(\alpha) = \nu_3$, and $u^{(iv)}(x) = \nu_4(x)$. These assertions show that there exists a linear homeomorphism between W_p and E_p. That is, the space W_p has the isomorphic decomposition $W_p = L_p(G) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Theorem 3.1. *If $1 \le p < \infty$, then any linear bounded functional $F \in W_p^\star$ can be represented as*

$$F(u) = \int_{x_0}^{x_1} u^{(iv)}(x)\varphi_4(x)dx + u'''(x_0)\varphi_3 + u''(x_0)\varphi_2 + u'(x_0)\varphi_1 + u(x_0)\varphi_0 \tag{3.2}$$

with a unique element $\varphi = (\varphi_4(x), \varphi_3, \varphi_2, \varphi_1, \varphi_0) \in E_q$, where $p + q = pq$. Any linear bounded functional $F \in W_\infty^\star$ can be represented as

$$F(u) = \int_{x_0}^{x_1} u^{(iv)}(x)d\varphi_4 + u'''(x_0)\varphi_3 + u''(x_0)\varphi_2 + u'(x_0)\varphi_1 + u(x_0)\varphi_0 \tag{3.3}$$

with a unique element $\varphi = (\varphi_4(e), \varphi_3, \varphi_2, \varphi_1, \varphi_0) \in \hat{E}_1 = (BA(\Sigma, \mu)) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, where μ is the Lebesque measure on $\mathbb{R}$, Σ is σ-algebra of the μ-measurable subsets $e \subset G$ and $BA(\Sigma, \mu)$ is the space of bounded additive functions $\varphi_4(e)$ defined on Σ with $\varphi_4(e) = 0$ when $\mu(e) = 0$ [13, p. 192]. *The inverse is also valid, that is, if $\varphi \in E_q$, then* (3.2) *is bounded on W_p, $1 \le p < \infty$, and if $\varphi \in \hat{E}_1$, then* (3.3) *is bounded on W_∞.*

Proof. The operator N given by $Nu = (u^{(iv)}(x), u'''(x_0), u''(x_0), u'(x_0), u(x_0))$ is bounded from W_p onto E_p, and has a bounded inverse N^{-1} defined as

$$\begin{aligned} u(x) =&(N^{-1}g)(x) \\ \equiv& \int_{x_0}^{x} \frac{(x-\zeta)^3}{6} g_4(\zeta)d\zeta + g_3\frac{(x-x_0)^3}{6} + g_2\frac{(x-x_0)^2}{2} + g_1(x-x_0) + g_0, \\ & g = (g_4(x), g_3, g_2, g_1, g_0) \in E_p. \end{aligned}$$

Clearly, the kernel of N is trivial and the image of N is equal to E_p. Therefore, there exists a bounded adjoint operator $N^\star : E_p^\star \to W_p^\star$ with $\ker N^\star = \{0\}$ and $\operatorname{Im} N^\star = W_p^\star$. That is, for a given $F \in W_p^\star$ there exists a unique $\psi \in E_p^\star$ such that

$$F = N^\star\psi \quad \text{or} \quad F(u) = \psi(Nu), \quad u \in W_p. \tag{3.4}$$

If $1 \le p < \infty$, then $E_p^\star = E_q$ (in the sense of a isomorphism (see [13, p. 191]). Therefore, the functional ψ can be represented as

$$\psi(g) = \int_{x_0}^{x_1} \varphi_4(x)g_4(x)dx + \varphi_3 g_1 + \varphi_2 g_2 + \varphi_1 g_1 + \varphi_0 g_0, \quad g \in E_p, \tag{3.5}$$

with a unique element $\varphi = (\varphi_4(x), \varphi_3, \varphi_2, \varphi_1, \varphi_0) \in E_q$. Part 2 of (3.4) and (3.5) show that any $F \in W_p^\star$ is uniquely represented as (3.2). Clearly, for a given $\varphi \in E_q$, the functional F given by (3.2) is bounded on W_p. Thus, (3.2) is a general form of the functionals $F \in W_p^\star$. The case $p = \infty$ can be proven in a similar way. □

Theorem 3.1 shows that $W_p^\star = E_q$ for all $1 \le p < \infty$, and $W_\infty^\star = E_\infty^\star = \hat{E}_1$. Furthermore, we can also consider the space E_1 as a subspace of the space $\hat{E}_1$.

4. Adjoint operator and adjoint system of the integro-algebraic equations

The question of finding an explicit form of the adjoint operator $V^\star$ is considered in this section. For this reason, any element $f = (f_4(x), f_3, f_2, f_1, f_0) \in E_q$ is considered as a linear bounded functional on E_p. Furthermore, it is also assumed that

$$f(Vu) = \int_{x_0}^{x_1} f_4(x)(V_4u)(x)dx + f_3(V_3u) + f_2(V_2u) + f_1(V_1u) + f_0(V_0u), \tag{4.1}$$

$u \in W_p$. By substituting the expressions (2.1) and (2.2) of V_4 and V_i, $i = 0, 1, 2, 3$, and also the expression (3.1) (with $\alpha = x_0$) of $u \in W_p$ into (4.1), we obtain

$$\begin{aligned} f(Vu) = \int_{x_0}^{x_1} & f_4(x)\{u^{(iv)}(x) + A_0(x)\{u(x_0) + u'(x_0)(x-x_0) \\ & + u''(x_0)\frac{(x-x_0)^2}{2} + u'''(x_0)\frac{(x-x_0)^3}{6} + \int_{x_0}^{x} \frac{(x-\zeta)^3}{6} u^{(iv)}(\zeta)d\zeta\} \\ & + A_1(x)\{u'(x_0) + u''(x_0)(x-x_0) + u'''(x_0)\frac{(x-x_0)^2}{2} \\ & + \int_{x_0}^{x} \frac{(x-\zeta)^2}{2})u^{(iv)}(\zeta)d\zeta\} + A_2(x)\{u''(x_0) + u'''(x_0)(x-x_0) \\ & + \int_{x_0}^{x} (x-\zeta)u^{(iv)}(\zeta)d\zeta\} + A_3(x)\{u'''(x_0) + \int_{x_0}^{x} u^{(iv)}(\zeta)d\zeta\}\}dx \end{aligned}$$

$$
\begin{aligned}
&+ f_3\{u(x_0) + u'(x_0)(x_1 - x_0) + u''(x_0)\frac{(x_1 - x_0)^2}{2} + u'''(x_0)\frac{(x_1 - x_0)^3}{6} \\
&+ \int_{x_0}^{x_1} \frac{(x_1 - \zeta)^3}{6} u^{(iv)}(\zeta)d\zeta\} + f_2\{\alpha_1[u(x_0) + u'(x_0)(\beta - x_0) \\
&+ u''(x_0)\frac{(\beta - x_0)^2}{2} + u'''(x_0)\frac{(\beta - x_0)^3}{6} + \int_{x_0}^{\beta} \frac{(\beta - \zeta)^3}{6} u^{(iv)}(\zeta)d\zeta] \\
&+ \alpha_2[u''(x_0) + u'''(x_0)(x_1 - x_0) + \int_{x_0}^{x_1} (x_1 - \zeta)u^{(iv)}(\zeta)d\zeta] \\
&+ \alpha_3[u'(x_0) + u''(x_0)(x_1 - x_0) + u'''(x_0)\frac{(x_1 - x_0)^2}{2} \\
&+ \int_{x_0}^{x_1} \frac{(x_1 - \zeta)^2}{2} u^{(iv)}(\zeta)d\zeta] + f_1 u'(x_0) + f_0 u(x_0).
\end{aligned}
$$

After some calculations, the following identity is obtained

$$
\begin{aligned}
f(Vu) \equiv \int_{x_0}^{x_1} f_4(x)(V_4u)(x)dx + \sum_{i=0}^{3} f_i(V_iu) = \int_{x_0}^{x_1} (\omega_4 f)(\zeta)u^{(iv)}(\zeta)d\zeta \\
+ (\omega_3 f)u'''(x_0) + (\omega_2 f)u''(x_0) + (\omega_1 f)u'(x_0) + (\omega_0 f)u(x_0) \\
\equiv (\omega f)(u), \quad \forall f \in E_q, \quad \forall u \in W_p,\ 1 \le p \le \infty,
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
(\omega_4 f)(\zeta) =& f_4(\zeta) + \int_{\zeta}^{x_1} f_4(s)[A_0(s)\frac{(s - \zeta)^3}{6} + A_1(s)\frac{(s - \zeta)^2}{2} \\
&+ A_2(s)(s - \zeta) + A_3(s)]ds + f_3\frac{(x_1 - \zeta)^3}{6} \\
&+ f_2[\alpha_1\frac{(\beta - \zeta)^3}{6}H(\beta - \zeta) + \alpha_2(x_1 - \zeta) + \alpha_3\frac{(x_1 - \zeta)^2}{2}]; \\
\omega_3 f =& \int_{x_0}^{x_1} f_4(s)[A_0(s)\frac{(s - x_0)^3}{6} + A_1(s)\frac{(s - x_0)^2}{2} \\
&+ A_2(s)(s - x_0) + A_3(s)]ds + f_3\frac{(x_1 - x_0)^3}{6} \\
&+ f_2[\alpha_1\frac{(\beta - x_0)^3}{6} + \alpha_2(x_1 - x_0) + \alpha_3\frac{(x_1 - x_0)^2}{2}]; \\
\omega_2 f =& \int_{x_0}^{x_1} f_4(s)[A_0(s)\frac{(s - x_0)^2}{2} + A_1(s)(s - x_0) + A_2(s)]ds \\
&+ f_3\frac{(x_1 - x_0)^2}{2} + f_2[\alpha_1\frac{(\beta - x_0)^2}{2} + \alpha_2 + \alpha_3(x_1 - x_0)]; \\
\omega_1 f =& \int_{x_0}^{x_1} f_4(s)[A_0(s)(s - x_0) + A_1(s)]ds + f_3(x_1 - x_0) \\
&+ f_2[\alpha_1(\beta - x_0) + \alpha_3] + f_1; \\
\omega_0 f =& \int_{x_0}^{x_1} f_4(s)A_0(s)ds + f_3 + f_2\alpha_1 + f_0
\end{aligned}
\tag{4.3}
$$

and $H(x)$ is the Heaviside function on $\mathbb{R}$.

The operators ω_4, ω_3, ω_2, ω_1, and ω_0 are linear and bounded from the space E_q consisting of the element $(f_4(x), f_3, f_2, f_1, f_0)$ into the spaces $L_q(G)$, $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}$, respectively. Therefore, the operator $\omega = (\omega_4, \omega_3, \omega_2, \omega_1, \omega_0)$ given by $\omega f = (\omega_4 f, \omega_3 f, \omega_2 f, \omega_1, \omega_0)$ becomes linear and bounded from E_q into itself. The identity (4.2) and Theorem 3.1 shows that when $1 \le p < \infty$, the operator ω is an adjoint operator for the operator V, that is, $V^\star = \omega$. When $p = \infty$ the operator ω is bounded from E_1 into E_1; in this case, ω becomes the restriction of the adjoint operator $V^\star : E^\star_\infty \to W^\star_\infty$ of V onto $E_1 \subset E^\star_\infty$.

Equation (2.3) can be reduced to the following equivalent equation:

$$VSg = z \tag{4.4}$$

with an unknown $g = (g_4, g_3, g_2, g_1, g_0) \in E_p$ by the transformation $u = Sg$, where $S = N^{-1}$. If $u = Sg$, then $u^{(iv)}(x) = g_4(x)$, $u'''(x_0) = g_3$, $u''(x_0) = g_2$, $u'(x_0) = g_1$ and $u(x_0) = g_0$. Therefore, (4.2) can be rewritten as

$$\begin{aligned} f(VSg) \equiv \int_{x_0}^{x_1} f_4(x)(V_4 Sg)(x)dx + \sum_{i=0}^{3} f_i(V_i Sg) = \int_{x_0}^{x_1} (\omega_4 f)(\zeta) g_4(\zeta) d\zeta \\ + (\omega_3 f)g_3 + (\omega_2 f)g_2 + (\omega_1 f)g_1 + (\omega_0 f)g_0 \equiv (\omega f)(g), \\ \forall f \in E_q, \quad \forall g \in E_p,\ 1 \le p \le \infty. \end{aligned} \tag{4.5}$$

This shows that $V^\star = (VS)^\star = \omega$ if $1 \le p < \infty$, and $\omega^\star = VS$ if $1 < p \le \infty$. That is, at least one of the operators VS and ω becomes an adjoint operator for the other one of them. Therefore, the equation

$$\omega f = \varphi \tag{4.6}$$

with an unknown function $f = (f_4(x), f_3, f_2, f_1, f_0) \in E_q$ and a given function $\varphi = (\varphi_4(x), \varphi_3, \varphi_2, \varphi_1, \varphi_0)$ in E_q can be considered as an adjoint equation of (4.4) (or of (2.3)) for all $1 \le p \le \infty$. Equation (4.6) can be written in explicit form as the system of equations

$$\begin{aligned} (\omega_2 f)(\zeta) = \varphi_2(\zeta), \quad \zeta \in X; \\ \omega_3 f = \varphi_3, \\ \omega_2 f = \varphi_2, \\ \omega_1 f = \varphi_1, \\ \omega_0 f = \varphi_0. \end{aligned} \tag{4.7}$$

The expressions (4.3) show that the first equation in (4.7) is an integral equation with respect to $f_4(\zeta)$ and it includes f_3 and f_2 as parameters; furthermore, equations 2 and 3 in (4.7) and equations 4 and 5 in (4.7) become a system of four algebraic equations with respect to (f_3, f_2, f_1, f_0) and these equations include some integral functionals defined on $f_4(\zeta)$. That is, (4.7) is a system of five integro-algebraic equations. This system is introduced by the identity (4.3) which, in fact, is an integration by parts formula in a nonclassical form. The traditional type adjoint problem is defined by the classical Green's formula of the integration by parts [19], and, therefore, has a sense only for some restricted classes of the problems.

5. Solvability conditions of completely nonhomogeneous problems

The operator is taken as $Q = \omega - I_q$, where I_q is the identity operator on E_q, i.e. $I_q f = f$ for all $f \in E_q$. This operator can also be defined as $Q =$

$(Q_4, Q_3, Q_2, Q_1, Q_0)$ with

$$\begin{aligned}(Q_4 f)(\zeta) &= (\omega_4 f)(\zeta) - f_4(\zeta), \quad \zeta \in G;\\ Q_i f &= \omega_i f - f_i, \quad i = 0, \dots 3.\end{aligned} \tag{5.1}$$

The expressions (4.3) and the conditions imposed on A_i show that Q_4 is a compact operator from E_q into $L_q(G)$, and also Q_i are compact operators from E_q into $\mathbb{R}$, where $1 < p < \infty$. That is, $Q : E_q \to E_q$ is a compact operator, and, therefore, has a compact adjoint operator $Q^\star : E_p \to E_p$. Since $\omega = Q + I_q$ and $VS = Q^\star + I_p$, where $I_p = I_q^\star$, we have that the equations (4.4) and (4.6) are canonical Fredholm type equations; furthermore, S becomes a right regularizer of (2.3) (see [14, p. 52]). Consequently, the following theorem is proven.

Theorem 5.1. *Assume that $1 < p < \infty$. Then $Vu = 0$ has either only the trivial solution or a finite number linearly independent solutions in W_p:*
(i) If $Vu = 0$ has only the trivial solution in W_p, then $\omega f = 0$ also has only the trivial solution in E_q. Then, the operators $V : W_p \to E_p$ and $\omega : E_q \to E_q$ become linear homeomorphisms.
(ii) If $Vu = 0$ has m linearly independent solutions $u_1, \dots, u_m$ in W_p, then $\omega f = 0$ has also m linearly independent solutions

$$f^{(1)} = (f_4^{(1)}(x), f_3^{(1)}, f_2^{(1)} f_1^{(1)}, f_0^{(1)}), \dots, f^{(m)} = (f_4^{(m)}(x), f_3^{(1)}, f_2^{(1)}, f_1^{(m)}, f_0^{(m)})$$

in E_q. In this case, the equations (2.3) and (4.6) have the solutions $u \in W_p$ and $f \in E_q$, for given $z \in E_p$ and $\varphi \in E_q$, if and only if the conditions

$$\int_{x_0}^{x_1} f_4^{(i)}(\zeta) z_4(\zeta) d\zeta + f_3^{(i)} z_3 + f_2^{(i)} z_2 + f_1^{(i)} z_1 + f_0^{(i)} z_0 = 0, \quad i = 1, \dots, m, \tag{5.2}$$

and

$$\int_{x_0}^{x_1} \varphi_4(\zeta) u_i^{(iv)}(\zeta) d\zeta + \varphi_3 u_i'''(x_0) + \varphi_2 u_i''(x_0) + \varphi_1 u_i'(x_0) + \varphi_0 u_i(x_0) = 0, \tag{5.3}$$

$i = 1, \dots, m$, are satisfied, respectively.

6. Green Functional

The following equation given in the form of the functional identity is considered

$$(\omega f)(u) = u(x), \quad \forall u \in W_p \tag{6.1}$$

in which $f = (f_4(\zeta), f_3, f_2, f_1, f_0) \in E_q$ is an unknown element and $x \in \bar{G}$ is a parameter.

Definition Assume that $f(x) = (f_4(\zeta, x), f_3(x), f_2(x), f_1(x), f_0(x)) \in E_q$ is an element with the parameter $x \in \bar{G}$. If $f = f(x)$ is the solution of (6.1) for a given $x \in \bar{G}$, then $f(x)$ is called as a Green functional of V (or of (2.3)).

The operator $I_{W_p,C}$ of the imbedding of W_p into the space $C(\bar{G})$ of the continuous functions on $\bar{G}$ is bounded. Then, the linear functional $\theta(x)$ given by $\theta(x)(u) = u(x)$ is bounded on W_p for a given $x \in \bar{G}$. This and $(\omega f)(u) = (V^\star f)(u)$ show that the equation (6.1) can also be written as (see [3, 4])

$$V^\star f = \theta(x).$$

That is, the equation (6.1) can be considered as a special case of the adjoint equation $V^\star f = \psi$ when $\psi = \theta(x)$.

Now, by employing (3.1) with $x = x_0$ and (4.3), the equation (6.1) is written as

$$\begin{aligned} &\int_{x_0}^{x_1} (\omega_4 f)(\zeta) u^{(iv)}(\zeta) d\zeta + (\omega_3 f) u'''(x_0) + (\omega_2 f) u''(x_0) \\ &+ (\omega_1 f) u'(x_0) + (\omega_0 f) u(x_0) \\ &= \int_{x_0}^{x} \frac{(x-\zeta)^3}{6} u^{(iv)}(\zeta) d\zeta + u'''(x_0) \frac{(x-x_0)^3}{6} \\ &\quad + u''(x_0) \frac{(x-x_0)^2}{2} + u'(x_0)(x-x_0) + u(x_0), \quad \forall u \in W_p. \end{aligned} \tag{6.2}$$

The elements $u^{(iv)}(\zeta) \in L_p(G)$, $u'''(x_0), u''(x_0), u'(x_0) \in \mathbb{R}$ and $u(x_0) \in \mathbb{R}$ of the functions $u \in W_p$ are unrelated. Then,

$$\begin{gathered} (\omega_4 f)(\zeta) = \frac{(x-\zeta)^3}{6} H(x-\zeta), \quad \zeta \in G; \\ \omega_3 f = \frac{(x-x_0)^3}{6}; \\ \omega_2 f = \frac{(x-x_0)^2}{2}; \\ \omega_1 f = (x-x_0); \\ \omega_0 f = 1. \end{gathered} \tag{6.3}$$

This shows that the equation (6.1) is equivalent to the system (6.3) which is a special case of the adjoint system (4.7) when

$$\begin{gathered} \varphi_4(\zeta) = \frac{(x-\zeta)^3}{6} H(x-\zeta), \quad \varphi_3 = \frac{(x-x_0)^3}{6}, \\ \varphi_2 = \frac{(x-x_0)^2}{2}, \quad \varphi_1 = (x-x_0), \quad \varphi_0 = 1\,. \end{gathered}$$

Therefore, $f(x)$ is the Green functional if and only if it is a solution of the integro-algebraic equations (6.3) for an arbitrary $x \in \bar{G}$. For a solution u of (2.3) and a Green functional $f(x)$, the identity (4.2) can be written as

$$\begin{aligned} &\int_{x_0}^{x_1} f_4(\zeta, x) z_4(\zeta) d\zeta + f_3(x) z_3 + f_2(x) z_2 + f_1(x) z_1 + f_0(x) z_0 \\ &= \int_{x_0}^{x_1} \frac{(x-\zeta)^3}{6} H(x-\zeta) u^{(iv)}(\zeta) d\zeta + u'''(x_0) \frac{(x-x_0)^3}{6} \\ &\quad + u''(x_0) \frac{(x-x_0)^2}{2} + u'(x_0)(x-x_0) + u(x_0). \end{aligned} \tag{6.4}$$

The right-hand side of (6.4) is equal to $u(x)$. Therefore, the following theorem holds.

Theorem 6.1. *If (2.3) has at least one Green functional $f(x)$, then an arbitrary solution $u \in W_p$ of (2.3) can be represented as*

$$u(x) = \int_{x_0}^{x_1} f_4(\zeta, x) z_4(\zeta) d\zeta + f_3(x) z_3 + f_2 z_2 + f_1(x) z_1 + f_0(x) z_0. \tag{6.5}$$

Furthermore, $Vu = 0$ has only one trivial solution.

If at least one of the operators $V : W_p \to E_p$ or $\omega : E_q \to E_q$ is a homeomorphism, then the other one is also a homeomorphism; furthermore, there exists a unique Green functional, where $1 \le p \le \infty$. The Green functional exists and is unique. The necessary and sufficient conditions for the existence of a Green functional are given by the following theorem for the case $1 < p < \infty$.

Theorem 6.2. *If there exists a Green functional, then it is unique. There exists a Green functional if and only if $Vu = 0$ has only the trivial solution.*

Proof. If there exists a Green functional, then $Vu = 0$ has the unique solution $u = 0$ (Theorem 6.1). In this case $\omega : E_q \to E_q$ becomes a homeomorphism (Theorem 5.1). Therefore, the Green functional, as a solution of (6.3), is unique. The second part of the theorem follows from Theorem 5.1. □

Remark. Assume that $Vu = 0$ has a nontrivial solution. Then (2.3) does not have a Green functional (Theorem 6.1). In this case, $Vu = z$ usually has no solution unless the right-hand side z is a particular type. For example, $Vu = z$ has no solution if

$$\int_{x_0}^{x_1} f_4(z) z_4(x) dx + f_3 z_3 + f_2 z_2 + f_1 z_1 + f_0 z_0 = 0 \tag{6.6}$$

is not true at least for one solution $f = (f_4(\zeta), f_3, f_2, f_1, f_0)$ of the homogenous adjoint equation $\omega f = 0$. In this case, the representation of the existing solution of $Vu = z$ is obtained by a concept of the generalized Green functional [3].

7. Comparison with the classical Green type function

Consider the following problem which is a special case of (2.3):

$$\begin{gathered}
(V_4 u)(x) \equiv u^{(iv)}(x) + A(x)u = z_4(x), \quad x \in G; \\
V_0 u \equiv u(x_0) = z_0, \\
V_1 u \equiv u'(x_0) = z_1, \\
V_1 u \equiv u''(x_0) = z_2, \\
V_0 u \equiv u(x_1) = z_3.
\end{gathered} \tag{7.1}$$

In this case, system (6.3) can be written as

$$\begin{aligned}
(\omega_4 f)(\zeta) &\equiv f_4(\zeta) + \int_{\zeta}^{x_1} f_4(s) A(s) \frac{(s-\zeta)^3}{6} ds + f_3 \frac{(x_1-\zeta)^3}{6} + f_2 (x_1 - \zeta) \\
&= \frac{(x-\zeta)^3}{6} H(x-\zeta), \quad \zeta \in G; \\
\omega_3 f &\equiv \int_{x_0}^{x_1} f_4(s) A(s) \frac{(s-x_0)^3}{6} ds + f_3 \frac{(x_1-x_0)^3}{6} + f_2 (x_1 - x_0) = \frac{(x-x_0)^3}{6}; \\
\omega_2 f &\equiv \int_{x_0}^{x_1} f_4(s) A(s) \frac{(s-x_0)^2}{2} ds + f_3 \frac{(x_1-x_0)^2}{2} + f_2 = \frac{(x-x_0)^2}{2}; \\
\omega_1 f &\equiv \int_{x_0}^{x_1} f_4(s) A(s)(s-x_0) ds + f_3 (x_1 - x_0) + f_1 = (x - x_0); \\
\omega_0 f &\equiv \int_{x_0}^{x_1} f_4(s) A(s) ds + f_3 (x_1 - x_0) + f_0 = 1.
\end{aligned} \tag{7.2}$$

From parts 2 and 3 of (7.2) and parts 4 and 5 of (7.2) it is obtained that

$$\begin{aligned}
f_3 &= \frac{3\Delta(x-x_0)^2-(x-x_0)^3}{2\Delta^3}+\int_{x_0}^{x_1} f_4(s)A(s)\{\frac{(s-x_0)^3)-3\Delta(s-x_0)^2}{2\Delta^3}\}ds;\\
f_2 &= \frac{(x-x_0)^3-\Delta(x-x_0)^2}{4\Delta}+\int_{x_0}^{x_1} f_4(s)A(s)\{\frac{\Delta(s-x_0)^2-(s-x_0)^3}{4\Delta}\}ds;\\
f_1 &= (x-x_0)-f_3\Delta-\int_{x_0}^{x_1} f_4(s)A(s)(s-x_0)ds;\\
f_0 &= 1-f_3-\int_{x_0}^{x_1} f_4(s)A(s)ds, \quad \Delta=x_1-x_0.
\end{aligned} \tag{7.3}$$

Substituting parts 1 and 2 of (7.3) into part 1 of (7.2),

$$\begin{aligned}
&f_4(\zeta)+\int_{\zeta}^{x_1} f_4(s)A(s)\frac{(s-\zeta)^3}{6}ds\\
&+\frac{(x_1-\zeta)^3}{6}\int_{x_0}^{x_1} f_4(s)A(s)\{\frac{(s-x_0)^3-3\Delta(s-x_0)^2}{2\Delta^3}\}ds\\
&+(x_1-\zeta)\int_{x_0}^{x_1} f_4(s)A(s)\{\frac{\Delta(s-x_0)^2-(s-x_0)^3}{4\Delta}\}ds\\
&=\frac{(x-\zeta)}{6}H(x-\zeta)-\frac{(x_1-\zeta)^3}{6}(\frac{3\Delta(x-x_0)^2-(x-x_0)^3}{2\Delta^3})\\
&\quad-(x_1-\zeta)(\frac{(x-x_0)^3-\Delta(x-x_0)^2}{4\Delta}), \quad \zeta\in G.
\end{aligned} \tag{7.4}$$

That is, the first element $f_4(\zeta,x)$ of the Green functional

$$f(x)=(f_4(\zeta,x),f_3(x),f_2(x)f_1(x),f_0(x))$$

of problem (7.1) becomes the solution of the independent integral equation (7.4); the latter four elements $f_3(x)$, $f_2(x)$, $f_1(x)$ and $f_0(x)$ of $f(x)$ can be obtained by (7.3). The equation (7.4) has a unique solution $f_4(\zeta,x)\in L_q(G)$ (for given $x\in\bar G$) if and only if $Vu=0$ has only the trivial solution (Theorem 6.2). If $Vu=0$ has a nontrivial solution, then the Green functional does not exist.

In order to compare the Green functional with the classical type Green function, equation (7.4) is considered. Assume that $A(x)$ is absolutely continuous on $\bar G$. If a function $f_4(\zeta)=f_4(\zeta,x)\in L_q(G)$ is the solution of (7.4), then $f_4(\zeta,x)$ is absolutely continuous on $\bar G$ with respect to ζ (for a given $x\in\bar G$). Therefore, by differentiating (7.4) with respect to ζ, it is obtained that $f_4'''(\zeta)$ becomes absolutely continuous on $[x_0,x]$ and $[x,x_1]$ with respect to ζ. Therefore,

$$(V_4^\star f_4)(\zeta)\equiv\frac{d^4f_4(\zeta)}{d\zeta^4}+A(\zeta)f_4(\zeta)=0,\ \zeta\in(x_0,x)\cup(x,x_1). \tag{7.5}$$

The boundary conditions of (7.5) can be obtained from (7.4) as

$$\begin{gathered}
f_4(x_0)=f_4(x_1)=0,\\
f_4(x+0)=f_4(x-0),\\
f_4'(x+0)=f_4'(x-0),\\
f_4''(x+0)=f_4''(x-0),\\
f_4'''(\zeta)|_{\zeta=x+0}=f_4'''(\zeta)|_{\zeta=x-0}+1.
\end{gathered} \tag{7.6}$$

That is, the solution of (7.4) is equivalent to the solution of problem (7.5)-(7.6). In other words, $f_4(\zeta)$ is the solution of problem (7.5)-(7.6). Therefore, $f_4(\zeta, x)$ as a function of ζ is the classical Green function for the corresponding traditional adjoint problem given by $(V_4^\star f_4)(\zeta) = \psi_4(\zeta)$, $\zeta \in G$, and $f_4(x_0) = f_4'(x_0) = f_4''(x_1) = f_4(x_1) = 0$, where $\psi_4 \in L_1(G)$ is a given function. It can be easily proven that the function $f_4(\zeta, x)$ as a function of x is the classical Green function for the equation $(7.1)_1$ with $u(x_0) = u'(x_0) = u''(x_1) = u(x_1) = 0$ (see [19, p.200]).

Let us considered some simple cases. Let $A_j = 0$, $j = 0, 1, 2, 3$ and $x_0 = 0$, $x_1 = l$ in the equations (6.3). If it is taken $\beta = 0$, $\alpha_2 = 1$ and $|\alpha_1|$ is sufficiently small, then the system of equations (6.3) has unique solution. Some simple results are given below.

(i) If $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = c = \mu/EI_x$ are taken in the equation $(6.3)_1$, then the Green function of an elastic beam having two ends which are fixed support and elastic support is obtained as

$$\begin{aligned} f_4(x,\zeta) = \frac{(x-\zeta)^3}{6} H(x-\zeta) - \frac{(l-\zeta)^3}{6}\Big[\frac{x^2}{l^2} - \frac{2(1+cl)(x^3 - x^2 l)}{l^2(4l + cl^2)}\Big] \\ - \Big(l - \zeta + \frac{c(l-\zeta)^2}{2}\Big)\Big\{\frac{(x^3 - 4x^2 l)}{(4l + cl^2)}\Big\}, \end{aligned} \tag{7.7}$$

where μ, I_x, E are elastic material constants [12].

(ii) $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 1$ are taken in part 1 of (6.3), then the Green function of an elastic beam having both ends fixed is obtained as

$$f_4(x,\zeta) = \frac{(x-\zeta)^3}{6} H(x-\zeta) - \frac{(l-\zeta)^3}{6}\frac{x^2}{l^2}\Big(3 - \frac{2x}{l}\Big) - \frac{x^2}{l^2}(x-l)\frac{(l-\zeta)^2}{2}. \tag{7.8}$$

Note that (7.4) is a Fredholm's equation of the second kind for a given $x \in \bar{G}$. Therefore, it can be solved approximately by a known method [5, 8]. Thus, (6.3) can also be used for the approximate calculations of the Green functional and solution. The present Green function concept can also be used to investigate some classes of nonlinear equations associated with linear non-local conditions [9, 15, 16]. Thus, the nonlinear problem can be reduced to equivalent nonlinear integral equations.

References

[1] Akhiev, S. S.; Representations of the solutions of some linear operator equations, *Soviet Math. Dokl.*, **21**(2) (1980), 555-558.

[2] Akhiev, S. S.; Fundamental solutions of functional differential equations and their representations, *Soviet Math. Dokl.*, **29**(2) (1984), 180- 184.

[3] Akhiev, S. S. and Oruçoğlu, K.; Fundamental Solutions of Some Linear Operator Equations and Applications, *Acta Applicandae Mathematicae*, **71** (2002), 1-30.

[4] Akhiev, S. S.; A New Fundamental Solution Concept and Application to Some Local and Nonlocal Problems, *Bul. Tech. Univ. Istanbul*, **47**(3) (1994), 93-99.

[5] Bakhvalov, N. S., Jidkov, N. P. and Kobel'kov, G. M.; *Numerical Methods*, Nauka, Moscow, 1987 (in Russian).

[6] Bellman, R. and Cooke, K. L.; *Differential Difference Equations*, Academic Press, 1963.

[7] Brown, A. L. and Page, A.; *Elements of Functional Analysis*, New York, 1970.

[8] Collatz, L.; *The numerical treatment of differential equations*, Springer-Verlag, Berlin, 1966.

[9] Gyulov, T. and Tersian, S.; Existence of Trivial and Nontrivial Solutions of A Fourth-Order Differential Equations, *Electronic Journal of Differential Equations*, **2004** (2004), No. 41, 1-14.

[10] Halanay , A.; *Differential Equations*, Academic Press, New York, 1966.

[11] Hörmander, L.; *Linear Partial Differential Operators*, Springer- Verlag, New York, 1976.

[12] İnan, M.; *Strenght of Materials* I.T.U. Foundation Press, 8. Edition, 2001 (in Turkish).

[13] Kantorovich, L. V. and Akilov, G. P.; *Functional Analysis* (2nd ed, translated by Howard L. Silcock), Pergamon Press, New York, 1982.

[14] Krein, S. G.; *Linear equations in Banach space*, Nauka, Moscow, 1971 (in Russian).

[15] Liu, X., Jiang, W. and Guo, Y.; Multi-Point Boundary Value Problems For Higher Order Differential Equations, *Applied Mathematics E-Notes*, **4** (2004), 106-113.

[16] Ma, R.; Existence Theorems for a Second order m-Point Boundary Value Problem, *J. Math. Analysis and Applications*, **211**(2) (1997), 545-555.

[17] Naimark, M. A.; *Linear Differential operators*, Nauka, Moscow, 1969 (in Russian).

[18] Shilov, G. E.; *Mathematical analysis:* Second special course, Nauka, Moscow, 1965; English transl., Generalized functions and partial differential equations, Gordon and Breach, New York, 1968.

[19] Stakgold, I.; *Green' s Functions and Boundary Value Problems*, Wiley Interscience, New York, 1979.

[20] Tikhonov, A. N., Vasil'eva, A. B. and Sveshnikov, A. G.; *Differential Equations*, Nauka, Moscow, 1980 (in Russian).

Istanbul Technical University, Faculty of Science, Maslak 34469, Istanbul, Turkey

E-mail address: koruc@itu.edu.tr