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DERIVATION OF MODELS OF COMPRESSIBLE MISCIBLE DISPLACEMENT IN PARTIALLY FRACTURED RESERVOIRS

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Abstract. We derive rigorously homogenized models for the displacement of one compressible miscible fluid by another in fractured porous media. We denote by ϵ the characteristic size of the heterogeneity in the medium. A parameter $\alpha \in [0,1]$ characterizes the cracking degree of the rock. We carefully define an adapted microscopic model which is scaled by appropriate powers of ϵ. We then study its limit as $\epsilon \to 0$. Assuming a totally fractured or a partially fractured medium, we obtain two effective macroscopic limit models. The first one is a double porosity model. The second one is of single porosity type but it still contains some effects due to the partial storage in the matrix part. The convergence is shown using two-scale convergence techniques.

1. Introduction

Most of the natural reservoirs are characterized by the existence of a system of highly conductive fissures together with a large number of matrix blocks. Because the fractures can rapidly distribute pollution over vast areas, they are perceived as controlling the water quality of the entire aquifer, although their storativity is usually significantly smaller than that of the surrounding matrix. Therefore, fractured aquifers are considered vulnerable to pollution process. One can assume that such reservoirs possess two distinct porous structures. The problem of flow through a fractured environment is thus primarily a problem of flow through a dual-porosity system. Its two components are hydraulically interconnected and can not be treated separately. The degree of interconnection between these two flow systems defines the character of the entire flow domain and is a function of the hydraulic properties of each of them.

Two main approaches are usually used for the modelling of such reservoirs. The first one treats the system as a global porous medium with averaged porosity and permeability. It leads to a single porosity model. The second one uses the concept of double porosity introduced by Barenblatt et al [5]. Water in the matrix is considered practically immobile. The less permeable part of the rock contributes as global sink or source terms for the transported solutes in the fracture. These two types of model have been rigorously derived using homogenization tools for

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some displacement problems. Assuming that the reservoir is a porous medium with highly oscillating porosity and permeability with respect to a parameter ϵ and passing to the limit $\epsilon \to 0$, one obtains a single porosity model (see for instance [9]). Modelling the reservoir with a periodic structure controlled by a parameter $\epsilon > 0$ wich represents the size of each block of the matrix, scaling the equations of the flow in the matrix by appropriate powers of ϵ to represent the discontinuities of the flow, and letting $\epsilon \to 0$, one gets a double porosity model (see for instance [4, 10]).

By the way, the former derivation is based on the assumption of a totally fissured medium, ignoring the heterogeneity of a natural reservoir. The matrix of cells may also be connected so that some flow occurs directly within the cell matrix. This is the case of a *partially fissured medium* considered in [14] and [13]. The authors introduce two flows in the microscopic model for the matrix. The first one is the slow scale flow usually introduced for the double porosity model. The second one is a global flow within the matrix. The homogenization process then leads to a macroscopic model containing both single porosity and double porosity characteristics.

The authors in [14] and [13] consider the flow of single phase fluid. The aim of the present paper is to perform the same work for a miscible and compressible displacement in a partially fractured medium. In Section 2, we thus derive a new microscopic model with two flows in the matrix. A special attention is devoted to the respect of the miscibility assumption. This model is consistent with the totally fractured case and with the non fractured case. The proportion between the rapidly varying and the slowly varying part of the flow is specified by a parameter $\alpha \in [0, 1]$. We derive in Section 3 uniform estimates on the microscopic solutions. Choosing $\alpha = 0$ or $\alpha > 0$, we let the scaling parameter ϵ tend to zero in Sections 4 and 5 and we get the associated macroscopic models. We mainly use two-scale convergence arguments.

Contrary to the case studied in [14, 13], we show that the double porosity part of the model almost disappears as soon as a direct flow occurs in the matrix. It emphasizes in particular the role of the dispersion tensor which models all the velocities heterogeneity at the microscopic level. It is characteristic of a miscible flow.

From a physical viewpoint this contribution aims to give a better understanding of the transport processes in a more or less fissured medium. From a more mathematical viewpoint, it gives a unified approach for the homogenization in a fractured and in a non fractured medium.

2. The microscopic system

2.1. **Decomposition of the flows.** We consider the displacement of two miscible compressible species in a fractured porous reservoir Ω. The domain Ω thus consists of two subdomains, the fissures Ω_f and the matrix Ω_m.

We begin by describing the displacement in the fracture Ω_f where we do not decompose the flow. We follow the lines of [6, 19, 15]. We assume identical compressibilities for the species. We thus consider that the density of the fluid is of the form

$$\rho = \rho_o \, e^{p_f}, \quad \rho_o > 0,$$

where p_f is the pressure in Ω_f. Let f_i be the concentration of one of the two species of the mixture and let v_i be its velocity in the flow. The conservation of mass of this component is expressed by the following equation in Ω_f,

$$\partial_t(\phi_f \rho f_i) + \operatorname{div}(\rho f_i v_i) = \rho q \hat{f}_i,$$

where ϕ_f is the porosity of the fracture and $q\hat{f}_i$ is a source term such that $\hat{f}_1 + \hat{f}_2 = 1$. We then define the average velocity of the flow $\underline{v}_f = (f_1 v_1 + f_2 v_2)/(f_1 + f_2) = f_1 v_1 + f_2 v_2$. Due to the Darcy law $\underline{v}_f$ is given by

$$\underline{v}_f = -(k_f/\mu(f_1))\nabla p_f.$$

We neglect the gravitational terms for sake of clarity in the estimates below. The permeability of the fracture is k_f. The viscosity μ is a nonlinear function depending on one of the two concentrations in the mixture. We cite for instance the Koval model [17] where μ is defined for $c \in (0,1)$ by $\mu(c) = \mu(0)(1 + (M^{1/4} - 1)c)^{-4}$, the constant $M = \mu(0)/\mu(1)$ being the mobility ratio. Setting $j_i = \rho f_i(v_i - \underline{v}_f)$, the latter equation becomes

$$\partial_t(\phi_f \rho f_i) + \operatorname{div}(\rho f_i \underline{v}_f) + \operatorname{div}(j_i) = \rho q \hat{f}_i. \tag{2.1}$$

The tensor j_i models the effects of the heterogeneity of the velocities in the mixture. Analogous to Fick's law this dispersive flux is considered proportional to the concentration gradient $j_i = \rho \mathcal{D}(\underline{v}_f)\nabla f_i$, where the dispersion tensor is

$$\mathcal{D}(\underline{u}) = \phi_f\big(D_m Id + D_p(\underline{u})\big) = \phi_f\big(D_m Id + |\underline{u}|\left(\alpha_l \mathcal{E}(\underline{u}) + \alpha_t(Id - \mathcal{E}(\underline{u}))\right)\big), \tag{2.2}$$

where $\mathcal{E}(\underline{u})_{ij} = \underline{u}_i \underline{u}_j/|\underline{u}|^2$, α_l and α_t are the longitudinal and transverse dispersion constants and D_m is the molecular diffusion. For the usual rates of flow, these reals are such that $\alpha_l \geq \alpha_t \geq D_m > 0$. Such a dispersive tensor is characteristic of a miscible model. Using the definition of the density ρ in (2.1), dividing by $\rho > 0$, and using the classical assumption of weak compressibility to neglect the terms containing $\underline{v}_f \cdot \nabla p_f$, we get

$$\phi_f \partial_t f_i + \phi_f f_i \partial_t p_f + \operatorname{div}(f_i \underline{v}_f) - \operatorname{div}(\mathcal{D}(\underline{v}_f)\nabla f_i) = q\hat{f}_i. \tag{2.3}$$

Bearing in mind $f_1 + f_2 = 1$ and summing up the later relation for $i = 1, 2$, we model the conservation of the total mass by

$$\phi_f \partial_t p_f + \operatorname{div}(\underline{v}_f) = q, \quad \underline{v}_f = -\frac{k_f}{\mu(f_1)}\nabla p_f. \tag{2.4}$$

The flow in Ω_f is then completely modelled by (2.4) coupled with

$$\phi_f \partial_t f_1 + \underline{v}_f \cdot \nabla f_1 - \operatorname{div}(\mathcal{D}(\underline{v}_f)\nabla f_1) = q(\hat{f}_1 - f_1). \tag{2.5}$$

We now perform a similar calculation in the matrix part of the domain. But, following [14], we assume that the flow of each component is made of two parts. The first component accounts for the global diffusion in the pore system. The second one corresponds to high frequency spatial variations which lead to local storage in the matrix. The ith concentration in the matrix m_i is then given by

$$m_i = \alpha c_i + \beta C_i, \quad 0 \leq \alpha < 1,\ \alpha + \beta = 1.$$

Note that the value of the parameter α may be for instance given by experimental data on samples of porous media and by stochastic reconstruction (see [7] and the references therein). The partial concentrations c_i and C_i satisfy

$$\partial_t(\phi\rho c_i) + \operatorname{div}(\rho c_i v_i) = \rho q \hat{c}_i, \quad \partial_t(\phi\rho C_i) + \operatorname{div}(\rho C_i V_i) = \rho q \hat{C}_i, \tag{2.6}$$

where ϕ is the porosity of the matrix, $\rho = \rho_o e^p$ is the density of the mixture and p is the pressure in the matrix. We also define two Darcy velocities $\underline{v} = (c_1 v_1 + c_2 v_2)/(c_1 + c_2)$ and $\underline{V} = (C_1 V_1 + C_2 V_2)/(C_1 + C_2)$. We assume that they both obey a Darcy law depending on the pressure p. The rapidly varying and the slowly varying components are distinguished by two different permeabilities k and K satisfying

$$\underline{v} = -(k/\mu(m_1))\nabla p, \quad \underline{V} = -(K/\mu(m_1))\nabla p. \tag{2.7}$$

Introducing these Darcy velocities in (2.6), we obtain

$$\begin{gathered} \partial_t(\phi\rho c_i) + \operatorname{div}(\rho c_i \underline{v}) + \operatorname{div}(\rho c_i (v_i - \underline{v})) = \rho q \hat{c}_i, \\ \partial_t(\phi\rho C_i) + \operatorname{div}(\rho C_i \underline{V}) + \operatorname{div}(\rho C_i (V_i - \underline{V})) = \rho q \hat{C}_i. \end{gathered} \tag{2.8}$$

We now use the classical dispersive characteristic of a miscible displacement. To this aim, we define a global average velocity $\underline{\mathcal{V}} = (\alpha(c_1+c_2)\underline{v} + \beta(C_1+C_2)\underline{V})/(\alpha(c_1 + c_2) + \beta(C_1 + C_2))$, that is

$$\underline{\mathcal{V}} = \alpha(c_1 + c_2)\underline{v} + \beta(C_1 + C_2)\underline{V}. \tag{2.9}$$

The relative velocities are assumed to satisfy a Fick's law given by $c_i(v_i - \underline{v}) + c_i(\underline{v} - \underline{\mathcal{V}}) = c_i(v_i - \underline{\mathcal{V}}) = \mathcal{D}(\underline{\mathcal{V}})\nabla c_i$ and $C_i(V_i - \underline{V}) + c_i(\underline{V} - \underline{\mathcal{V}}) = C_i(V_i - \underline{\mathcal{V}}) = \mathcal{D}(\underline{\mathcal{V}})\nabla C_i$, the tensor $\mathcal{D}$ being of the form (2.2). Using the definition of ρ and neglecting the quadratic velocities terms, we then write (2.8) in the following form for $i = 1, 2$.

$$\phi\partial_t d_i + \phi d_i \partial_t p + \operatorname{div}(\underline{\mathcal{V}} d_i) - \operatorname{div}(\mathcal{D}(\underline{\mathcal{V}})\nabla d_i) = q\hat{d}_i, \quad d_i = c_i \text{ or } C_i. \tag{2.10}$$

Finally we use $m_1 + m_2 = 1$ to get a relation for the conservation of the total mass $\phi\partial_t p + \operatorname{div}(\underline{\mathcal{V}}) = q$. The flow in the matrix part is thus governed by the following set of equations:

$$\phi\partial_t p + \operatorname{div}(\underline{\mathcal{V}}) = q, \quad \underline{\mathcal{V}} = \alpha(c_1 + c_2)\underline{v} + (1 - \alpha(c_1 + c_2))\underline{V}, \tag{2.11}$$

$$\underline{v} = -\frac{k}{\mu(m_1)}\nabla p, \quad \underline{V} = -\frac{K}{\mu(m_1)}\nabla p, \tag{2.12}$$

$$\phi\partial_t C_1 + \underline{\mathcal{V}} \cdot \nabla C_1 - \operatorname{div}(\mathcal{D}(\underline{\mathcal{V}})\nabla C_1) = q(\hat{C}_1 - C_1), \tag{2.13}$$

$$\phi\partial_t c_i + \underline{\mathcal{V}} \cdot \nabla c_i - \operatorname{div}(\mathcal{D}(\underline{\mathcal{V}})\nabla c_i) = q(\hat{c}_i - c_i), \quad i = 1, 2. \tag{2.14}$$

Note that choosing $\alpha = 1$ and thus $\beta = 0$ and $k = K$, this model is exactly the same as the one derived in the fracture.

2.2. **The scaled microscopic system.** Since we neglect the gravitational terms, we describe the far-field repository by a domain $\Omega \subset \mathbb{R}^2$ with a periodic structure, controlled by a parameter $\epsilon > 0$ which represents the size of each block of the matrix. Note that all the results of the paper remain true in a domain Ω of $\mathbb{R}^3$ (see Remark 3.3 below for the minor modifications of the proof). The $\mathcal{C}^1$ boundary of Ω is Γ and ν is the corresponding exterior normal. As in [13], the standard period ($\epsilon = 1$) is a cell Y consisting of a matrix block Y_m of external $\mathcal{C}^1$ boundary ∂Y_m

and of a fracture domain Y_f. We assume that $|Y| = 1$. The ϵ-reservoir consists of copies ϵY covering Ω. The two subdomains of Ω are defined by

$$\Omega_f^\epsilon = \Omega \cap \big\{\cup_{\xi\in\mathcal{A}}\epsilon(Y_f + \xi)\big\}, \quad \Omega_m^\epsilon = \Omega \cap \big\{\cup_{\xi\in\mathcal{A}}\epsilon(Y_m + \xi)\big\},$$

where $\mathcal{A}$ is an appropriate infinite lattice. The fracture-matrix interface is denoted by $\Gamma_{fm}^\epsilon = \partial\Omega_f^\epsilon \cap \partial\Omega_m^\epsilon \cap \Omega$ and ν_{fm} is the corresponding unit normal pointing out Ω_f^ϵ. An example of unit cell in the case of a partially fractured medium is given in Fig. 1. See [1] for a more complicated admissible 2D-structure.

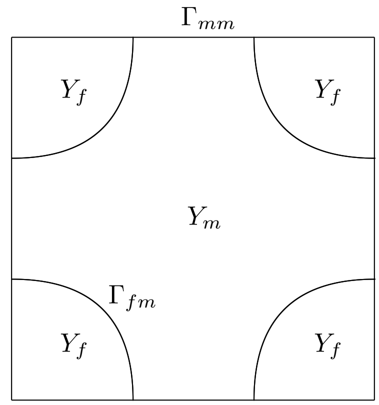

Figure 1. Unit cell of a partially fractured medium

The major difference between a partially fractured structure and a totally fractured one is that the matrix block Y_m is not completely surrounded by the fracture domain Y_f. Let us denote by $J = (0,T)$ the time interval of interest. To homogenize the reservoir, we shall let tend to zero the size ϵ of the cells.

Our starting point consists of the equations derived in the latter subsection. As we assume a periodic structure in the reservoir, the porosities $(\phi_f^\epsilon(x), \phi^\epsilon(x)) = (\phi_f(\frac{x}{\epsilon}), \phi(\frac{x}{\epsilon}))$ and the permeabilities $(k_f^\epsilon(x), k^\epsilon(x)) = (k_f(\frac{x}{\epsilon}), k(\frac{x}{\epsilon}))$ of the fracture and of the matrix are periodic of period $(\epsilon Y_f, \epsilon Y_m)$. These quantities are assumed to be smooth and bounded, but globally they are discontinuous across Γ_{fm}^ϵ. A slowly varying flow and a rapidly varying flow occur in the matrix Ω_m^ϵ. The equations for the rapidly varying flow will be scaled by appropriate powers of ϵ to conserve the flow between the matrix and the fractures as $\epsilon \to 0$ (*cf* [4, 14]). Scaling (2.4)-(2.5) and (2.11)-(2.14), we get

$$\phi_f^\epsilon \partial_t f_1^\epsilon + \underline{v}_f^\epsilon \cdot \nabla f_1^\epsilon - \operatorname{div}(\mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon) = q(\hat{f}_1 - f_1^\epsilon) \quad \text{in } \Omega_f^\epsilon \times J, \tag{2.15}$$

$$\phi_f^\epsilon \partial_t p_f^\epsilon + \operatorname{div}(\underline{v}_f^\epsilon) = q, \ \underline{v}_f^\epsilon = -\frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon \quad \text{in } \Omega_f^\epsilon \times J, \tag{2.16}$$

$$\phi^\epsilon \partial_t C_1^\epsilon + \underline{\mathcal{V}}^\epsilon \cdot \nabla C_1^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon) = q(\hat{C}_1 - C_1^\epsilon) \quad \text{in } \Omega_m^\epsilon \times J, \tag{2.17}$$

$$\phi^\epsilon \partial_t c_1^\epsilon + \underline{\mathcal{V}}^\epsilon \cdot \nabla c_1^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon) = q(\hat{c}_1 - c_1^\epsilon) \quad \text{in } \Omega_m^\epsilon \times J, \tag{2.18}$$

$$\phi^\epsilon \partial_t c_2^\epsilon + \underline{\mathcal{V}}^\epsilon \cdot \nabla c_2^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_2^\epsilon) = q(\hat{c}_2 - c_2^\epsilon) \quad \text{in } \Omega_m^\epsilon \times J, \tag{2.19}$$

$$\phi^\epsilon \partial_t p^\epsilon + \operatorname{div}(\underline{\mathcal{V}}^\epsilon) = q, \quad \underline{\mathcal{V}}^\epsilon = \underline{\mathcal{V}}_s^\epsilon + \epsilon\underline{\mathcal{V}}_h^\epsilon, \quad \text{in } \Omega_m^\epsilon \times J, \tag{2.20}$$

$$\underline{\mathcal{V}}_s^\epsilon = -\alpha(c_1^\epsilon + c_2^\epsilon)\frac{k^\epsilon}{\mu(m_1^\epsilon)}\nabla p^\epsilon, \ \underline{\mathcal{V}}_h^\epsilon = -(1 - \alpha(c_1^\epsilon + c_2^\epsilon))\frac{\epsilon k^\epsilon}{\mu(m_1^\epsilon)}\nabla p^\epsilon, \tag{2.21}$$

where $m_1^\epsilon = \alpha c_1^\epsilon + \beta C_1^\epsilon$, $0 \le \alpha < 1$, $\alpha + \beta = 1$. The flow in the fissures is described by (2.15)-(2.16). The matrix behavior is described by (2.17)-(2.21). In particular, (2.17) governs the slowly varying component while (2.18)-(2.19) governs the high frequency varying ones. The source term q is a nonnegative function of $L^2(\Omega \times J)$ and

$$\alpha \hat{c}_1 + \beta \hat{C}_1 = \hat{f}_1, \quad 0 \le \hat{f}_1 \le 1, \quad \hat{c}_1 + \hat{c}_2 = 1.$$

We assume that the porosities and the symmetric permeability tensors satisfy

$$0 < \phi_- \le \phi_f(x),\ \phi(x) \le \phi_-^{-1}, \quad k_-|\xi|^2 \le k_f(x)\xi \cdot \xi,\ k(x)\xi \cdot \xi \le k_-^{-1}|\xi|^2,$$

$k_- > 0$, a.e. in Ω, for all $\xi \in \mathbb{R}^2$. The viscosity $\mu \in W^{1,\infty}(0,1)$ is such that

$$0 < \mu_- \le \mu(x) \le \mu_+ \quad \forall x \in (0,1).$$

The tensor $\mathcal{D}$ is already defined in (2.2). The tensor $\mathcal{D}^\epsilon$ has a similar structure but its diffusive part $(\alpha + \beta\epsilon^2)D_m Id$ contains the same proportions of slowly and rapidly varying flows than the matrix. The main property of these tensors is

$$\begin{gathered} \mathcal{D}(\underline{v}_f^\epsilon)\xi \cdot \xi \ge \phi_-(D_m + \alpha_t|\underline{v}_f^\epsilon|)|\xi|^2, \quad \forall \xi \in \mathbb{R}^2, \\ \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\xi \cdot \xi \ge \phi_-(D_m(\alpha + \beta\epsilon^2) + \alpha_t|\underline{\mathcal{V}}_s^\epsilon + \epsilon^2 \underline{\mathcal{V}}_h^\epsilon|)|\xi|^2, \quad \forall \xi \in \mathbb{R}^2. \end{gathered} \tag{2.22}$$

The model is completed by the following boundary and initial conditions. We begin by the continuity relations across the interface $\Gamma_{fm}^\epsilon \times J$.

$$\beta \mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nu_{fm} = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon \cdot \nu_{fm}, \tag{2.23}$$

$$\alpha \mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nu_{fm} = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon \cdot \nu_{fm}, \tag{2.24}$$

$$\alpha \mathcal{D}(\underline{v}_f^\epsilon)\nabla(1 - f_1^\epsilon) \cdot \nu_{fm} = -\alpha \mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nu_{fm} = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_2^\epsilon \cdot \nu_{fm}, \tag{2.25}$$

$$f_1^\epsilon = \alpha c_1^\epsilon + \beta C_1^\epsilon, \tag{2.26}$$

$$\underline{v}_f^\epsilon \cdot \nu_{fm} = \underline{\mathcal{V}}^\epsilon \cdot \nu_{fm}, \quad p_f^\epsilon = p^\epsilon. \tag{2.27}$$

We add a zero flux condition out of the full domain Ω.

$$\mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon \cdot \nu = 0 \text{ on } \partial\Omega_f^\epsilon \cap \Gamma, \tag{2.28}$$

$$\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon \cdot \nu = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon \cdot \nu = \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_2^\epsilon \cdot \nu = 0 \text{ on } \partial\Omega_m^\epsilon \cap \Gamma, \tag{2.29}$$

$$\underline{v}_f^\epsilon \cdot \nu = 0 \text{ on } \partial\Omega_f^\epsilon \cap \Gamma,\ \underline{\mathcal{V}}^\epsilon \cdot \nu = 0 \text{ on } \partial\Omega_m^\epsilon \cap \Gamma. \tag{2.30}$$

The initial conditions in Ω are the following.

$$(f_1^\epsilon(x,0), C_1^\epsilon(x,0), c_1^\epsilon(x,0), c_2^\epsilon(x,0)) = (\chi_f^\epsilon f_1^o(x), C_1^o(x), c_1^o(x), c_2^o(x)), \tag{2.31}$$

$$p_f^\epsilon(x,0) = \chi_f^\epsilon(x)p^o(x),\ p^\epsilon(x,0) = \chi_m^\epsilon(x)p^o(x). \tag{2.32}$$

We assume that p^o belongs to $H^1(\Omega)$, and that $(f_1^o, C_1^o, c_1^o, c_2^o) \in L^\infty(\Omega) \times (L^2(\Omega_m^\epsilon))^3$ satisfies

$$0 \le f_1^o(x) \le 1 \text{ a.e. in } \Omega, \tag{2.33}$$

$$\alpha c_1^o(x) + \beta C_1^o(x) = \chi_m^\epsilon f_1^o(x),\ 0 \le C_\alpha \le c_1^o(x) + c_2^o(x) \le 1 \text{ a.e. in } \Omega_m^\epsilon. \tag{2.34}$$

The constant $0 \le C_\alpha < 1$ is introduced to prevent a degeneration in the limit study of the pressure equation in Ω_m^ϵ (see Section 5).

2.3. **Variational formulation and existence for the microscopic model.** We now state an existence result for the problem (2.15)-(2.21), (2.23)-(2.32). The usual equations modelling a miscible and compressible displacement in porous media are of the form (2.4)-(2.5). The existence of weak solutions for this problem is proved in [11]. In the present paper, the decomposition of the flow in the matrix part of the domain induces two additional difficulties. The first one is a new coupling between concentrations and pressure due to the term $\alpha(c_1^\epsilon + c_2^\epsilon)$ in (2.21). But from a mathematical viewpoint this difficulty is similar to the one due to the concentration dependent viscosity in the Darcy law. The second novelty occurs at the interface Γ_{fm}^ϵ. One has to link one concentration f_1^ϵ in the fracture with three concentrations $(C_1^\epsilon, c_1^\epsilon, c_2^\epsilon)$ in the matrix. Thus, following [13], we introduce appropriate concentrations spaces for the problem. Let H^ϵ be the Hilbert space $H^\epsilon = L^2(\Omega_f^\epsilon) \times L^2(\Omega_m^\epsilon) \times L^2(\Omega_m^\epsilon)$ with the inner product

$$\begin{aligned}&\big([u_f, u_m, U_m], [\psi_f, \psi_m, \Psi_m]\big)_{H^\epsilon}\\ &= \int_{\Omega_f^\epsilon} u_f(x)\,\psi_f(x)\,dx + \int_{\Omega_m^\epsilon} u_m(x)\,\psi_m(x)\,dx + \int_{\Omega_m^\epsilon} U_m(x)\,\Psi_m(x)\,dx.\end{aligned}$$

Let $\gamma_j^\epsilon : H^1(\Omega_j^\epsilon) \to L^2(\partial\Omega_j^\epsilon)$ be the usual trace map and χ_j^ϵ be the characteristic function associated with Ω_j^ϵ, $j = f, m$. Let V^ϵ be the following Banach space

$$\begin{aligned}V^\epsilon = H^\epsilon \cap \big\{&(u_f, u_m, U_m) \in H^1(\Omega_f^\epsilon) \times H^1(\Omega_m^\epsilon) \times H^1(\Omega_m^\epsilon);\\ &\gamma_f^\epsilon u_f = \alpha\gamma_m^\epsilon u_m + \beta\gamma_m^\epsilon U_m \text{ on } \Gamma_{fm}^\epsilon\big\}\end{aligned}$$

endowed with the norm

$$\begin{aligned}\|(u_f, u_m, U_m)\|_{V^\epsilon} &= \|\chi_f^\epsilon u_f\|_{L^2(\Omega)} + \|\chi_m^\epsilon u_m\|_{L^2(\Omega)} + \|\chi_m^\epsilon U_m\|_{L^2(\Omega)}\\ &\quad + \|\chi_f^\epsilon \nabla u_f\|_{(L^2(\Omega))^2} + \|\chi_m^\epsilon \nabla u_m\|_{(L^2(\Omega))^2} + \|\chi_m^\epsilon \nabla U_m\|_{(L^2(\Omega))^2}.\end{aligned}$$

The introduction of similar spaces for the pressure is useless because we only use one pressure variable in the matrix part of the domain. Note that the pair of spaces (H^ϵ, V^ϵ) possesses the same "good" properties as $(L^2(\Omega), H^1(\Omega))$. In particular, we have the compact embedding $V^\epsilon \subset H^\epsilon$. Thus, adapting the proof of [11] to the present piecewise structure, one can state the following existence result.

Theorem 2.1. *Let $0 < \epsilon < 1$. There exists a solution $(p_f^\epsilon, p^\epsilon, f_1^\epsilon, c_1^\epsilon, C_1^\epsilon, c_2^\epsilon)$ of Problem (2.15)-(2.21), (2.23)-(2.32) in the following sense.*
(i) The pressure part $(p_f^\epsilon, p^\epsilon)$ belongs to $L^2(J; H^1(\Omega_f^\epsilon)) \times L^2(J; H^1(\Omega_m^\epsilon))$ and is a weak solution of (2.16), (2.20)–(2.21), (2.27), (2.32). Indeed, for any function $\psi \in \mathcal{C}^1(J; H^1(\Omega))$,

$$\begin{aligned}&-\int_{\Omega\times J} (\chi_f^\epsilon \phi_f^\epsilon p_f^\epsilon + \chi_m^\epsilon \phi^\epsilon p^\epsilon)\partial_t\psi\\ &+\int_{\Omega\times J} \Big(\chi_f^\epsilon \frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon + \chi_m^\epsilon(\alpha(c_1^\epsilon + c_2^\epsilon)(1-\epsilon^2) + \epsilon^2)\frac{k^\epsilon}{\mu(m_1^\epsilon)}\nabla p_\epsilon\Big)\cdot\nabla\psi \qquad (2.35)\\ &= -\int_{\Omega} (\chi_f^\epsilon \phi_f^\epsilon + \chi_m^\epsilon \phi^\epsilon) p^o \psi(x,0) + \int_{\Omega\times J} q\psi.\end{aligned}$$

(ii) The concentration part $(f_1^\epsilon, c_1^\epsilon, C_1^\epsilon, c_2^\epsilon)$ is such that $(f_1^\epsilon, c_1^\epsilon, C_1^\epsilon) \in L^2(J; V^\epsilon) \cap H^1(J; (V^\epsilon)')$ and $c_2^\epsilon \in L^2(J; H^1(\Omega_m^\epsilon)) \cap H^1(J; (H^1(\Omega_m^\epsilon))')$. It satisfies for any

$(d_f, d_1, D_1) \in L^2(J; V^\epsilon)$ and any $d_2 \in L^2(J; H^1(\Omega_m^\epsilon))$ the following relations.

$$\begin{aligned}
&\int_{\Omega_f^\epsilon \times J} \phi_f^\epsilon \partial_t f_1^\epsilon \, d_f + \int_{\Omega_m^\epsilon \times J} \phi^\epsilon \partial_t c_1^\epsilon \, d_1 + \int_{\Omega_m^\epsilon \times J} \phi^\epsilon \partial_t C_1^\epsilon D_1 + \int_{\Omega_f^\epsilon \times J} (\underline{v}_f^\epsilon \cdot \nabla f_1^\epsilon) \, d_f \\
&+ \int_{\Omega_m^\epsilon \times J} \underline{\mathcal{V}}^\epsilon \cdot (d_1 \nabla c_1^\epsilon + D_1 \nabla C_1^\epsilon) + \int_{\Omega_f^\epsilon \times J} \mathcal{D}(\underline{v}_f^\epsilon) \nabla f_1^\epsilon \cdot \nabla d_f \\
&+ \int_{\Omega_m^\epsilon \times J} \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon) \nabla c_1^\epsilon \cdot \nabla d_1 + \int_{\Omega_m^\epsilon \times J} \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon) \nabla C_1^\epsilon \cdot \nabla D_1 \\
&= \int_{\Omega_f^\epsilon \times J} q\, (\hat{f}_1 - f_1^\epsilon) \, d_f + \int_{\Omega_m^\epsilon \times J} q\, (\hat{c}_1 - c_1^\epsilon) \, d_1 + \int_{\Omega_m^\epsilon \times J} q\, (\hat{C}_1 - C_1^\epsilon) \, D_1,
\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}
&\int_{\Omega_m^\epsilon \times J} \phi^\epsilon \partial_t c_2^\epsilon \, d_2 + \int_{\Omega_m^\epsilon \times J} (\underline{\mathcal{V}}^\epsilon \cdot \nabla c_2^\epsilon) \, d_2 \\
&+ \int_{\Omega_m^\epsilon \times J} \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon) \nabla c_2^\epsilon \cdot \nabla d_2 - \int_{\partial\Omega_m^\epsilon \times J} (\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon) \nabla c_2^\epsilon \cdot \nu_m) \, \gamma_m^\epsilon d_2 \\
&= \int_{\Omega_m^\epsilon \times J} q\, (1 - c_2^\epsilon) \, d_2.
\end{aligned} \tag{2.37}$$

Furthermore, the following maximum principles hold:

$$0 \le f_1^\epsilon(x,t) \le \hat{f}_1 \quad \text{a.e. in } \Omega_f^\epsilon \times J,$$

$$0 \le m_1^\epsilon(x,t) \le \hat{f}_1 \quad \text{a.e. in } \Omega_m^\epsilon \times J, \tag{2.38}$$

$$0 \le C_\alpha \le c_1^\epsilon(x,t) + c_2^\epsilon(x,t) \le 1 \quad \text{a.e. in } \Omega_m^\epsilon \times J. \tag{2.39}$$

Proof. The proof of this existence result follows the lines of [11] and is based on a fixed point approach. The necessary a priori estimates are the same as the uniform ones derived in Section 3 below. We thus do not detail the proof in the present paper. We only give the details for the crucial maximum principles (2.38)-(2.39). In view of the method developed in [11], we can assume in what follows that the Darcy velocities are regularized so that $\underline{v}_f^\epsilon$ and $\underline{\mathcal{V}}^\epsilon$ belong to $L^\infty(\Omega \times J)$. We first consider the problem of the left-hand sides of estimates (2.38). We note that the function $\alpha c_1^\epsilon + \beta C_1^\epsilon = m_1^\epsilon$ satisfies the following system in $\Omega_m^\epsilon \times J$.

$$\phi^\epsilon \partial_t m_1^\epsilon + \underline{\mathcal{V}}^\epsilon \cdot \nabla m_1^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon) \nabla m_1^\epsilon) = q(\hat{f}_1 - m_1^\epsilon), \tag{2.40}$$

$$m_1^\epsilon = f_1^\epsilon, \quad \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon) \nabla m_1^\epsilon \cdot \nu_{fm} = (\alpha^2 + \beta^2) \mathcal{D}(\underline{v}_f^\epsilon) \nabla f_1^\epsilon \cdot \nu_{fm} \quad \text{on } \Gamma_{fm}^\epsilon \times J, \tag{2.41}$$

$$\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon) \nabla m_1^\epsilon \cdot \nu = 0 \quad \text{on } (\partial\Omega_m^\epsilon \cap \Gamma) \times J, \tag{2.42}$$

$$m_1^\epsilon(x,0) = \alpha c_1^o(x) + \beta C_1^o(x) = \chi_m^\epsilon(x) f_1^o(x) \quad \text{in } \Omega_m^\epsilon. \tag{2.43}$$

For any function f, we denote by $(f)^-$ the function $(f)^- = \sup(0, -f)$. We now multiply (2.15) by $(\alpha^2 + \beta^2)(f_1^\epsilon)^-$ (respectively (2.40) by $(m_1^\epsilon)^-$) and we integrate over Ω_f^ϵ (respectively Ω_m^ϵ). Noting that $(f_1^\epsilon)^- = (m_1^\epsilon)^-$ on Γ_{fm}^ϵ, we sum up the resulting relations to kill the non zero boundary terms. We get

$$\begin{aligned}
&\frac{d}{2dt}\int_\Omega \big(\chi_f^\epsilon(\alpha^2+\beta^2)\phi_f^\epsilon|(f_1^\epsilon)^-|^2+\chi_m^\epsilon\phi^\epsilon|(m_1^\epsilon)^-|^2\big)-\int_{\Omega_m^\epsilon}(\underline{\mathcal{V}}^\epsilon\cdot\nabla m_1^\epsilon)\,(m_1^\epsilon)^-\\
&-(\alpha^2+\beta^2)\int_{\Omega_f^\epsilon}(\underline{v}_f^\epsilon\cdot\nabla f_1^\epsilon)\,(f_1^\epsilon)^- -(\alpha^2+\beta^2)\int_{\Omega_f^\epsilon}\mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon\cdot\nabla(f_1^\epsilon)^-\\
&-\int_{\Omega_m^\epsilon}\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla m_1^\epsilon\cdot\nabla(m_1^\epsilon)^- +\int_\Omega q\hat f_1\big(\chi_f^\epsilon(\alpha^2+\beta^2)(f_1^\epsilon)^-+\chi_m^\epsilon(m_1^\epsilon)^-\big)\\
&+\int_\Omega q\,\big(\chi_f^\epsilon(\alpha^2+\beta^2)|(f_1^\epsilon)^-|^2+\chi_m^\epsilon|(m_1^\epsilon)^-|^2\big)=0.
\end{aligned}\tag{2.44}$$

Using $-\nabla f\cdot\nabla(f)^-=\nabla(f)^-\cdot\nabla(f)^-$ and the basic properties (2.22) of the tensors $(\mathcal{D},\mathcal{D}^\epsilon)$, we note that

$$\begin{gathered}
-\int_{\Omega_f^\epsilon}\mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon\cdot\nabla(f_1^\epsilon)^-\,dx\ge\int_{\Omega_f^\epsilon}\phi_-(D_m+\alpha_t|\underline{v}_f^\epsilon|)\,|\nabla(f_1^\epsilon)^-|^2\,dx,\\
-\int_{\Omega_m^\epsilon}\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla m_1^\epsilon\cdot\nabla(m_1^\epsilon)^-\,dx\\
\ge\int_{\Omega_m^\epsilon}\phi_-(D_m(\alpha+\beta\epsilon^2)+\alpha_t|\underline{\mathcal{V}}_s^\epsilon+\epsilon^2\underline{\mathcal{V}}_h^\epsilon|)\,|\nabla(m_1^\epsilon)^-|^2\,dx.
\end{gathered}$$

The convective terms in (2.44) are then estimated as follows using the Cauchy-Schwarz and Young inequalities.

$$\begin{gathered}
\Big|\int_{\Omega_f^\epsilon}(\underline{v}_f^\epsilon\cdot\nabla f_1^\epsilon)\,(f_1^\epsilon)^-\Big|\le\int_{\Omega_f^\epsilon}\frac{\alpha_t}{2}|\underline{v}_f^\epsilon|\,|\nabla(f_1^\epsilon)^-|^2+C\|\underline{v}_f^\epsilon\|_\infty\int_{\Omega_f^\epsilon}|(f_1^\epsilon)^-|^2,\\
\Big|\int_{\Omega_m^\epsilon}(\underline{\mathcal{V}}^\epsilon\cdot\nabla m_1^\epsilon)\,(m_1^\epsilon)^-\Big|\le\int_{\Omega_m^\epsilon}\frac{\alpha D_m}{2}\,|\nabla(m_1^\epsilon)^-|^2+\frac{C}{\alpha}\|\underline{\mathcal{V}}^\epsilon\|_\infty^2\int_{\Omega_m^\epsilon}|(m_1^\epsilon)^-|^2.
\end{gathered}$$

All the terms of (2.44) containing the source q are nonnegative. The previous estimates then lead to

$$\begin{aligned}
&\frac{\phi_-}{2}\frac{d}{dt}\int_\Omega\big((\alpha^2+\beta^2)\chi_m^\epsilon|(f_1^\epsilon)^-|^2+\chi_m^\epsilon|(m_1^\epsilon)^-|^2\big)\,dx+(\alpha^2+\beta^2)\phi_-\int_{\Omega_f^\epsilon}(D_m\\
&+\frac{\alpha_t}{2}|\underline{v}_f^\epsilon|)\,|\nabla(f_1^\epsilon)^-|^2dx+\phi_-\int_{\Omega_m^\epsilon}(D_m(\frac{\alpha}{2}+\beta\epsilon^2)+\alpha_t|\underline{\mathcal{V}}_s^\epsilon+\epsilon^2\underline{\mathcal{V}}_h^\epsilon|)\,|\nabla(m_1^\epsilon)^-|^2dx\\
&\le C\int_\Omega\big((\alpha^2+\beta^2)\chi_f^\epsilon|(f_1^\epsilon)^-|^2+\chi_m^\epsilon|(m_1^\epsilon)^-|^2\big)\,dx.
\end{aligned}$$

Using the Gronwall lemma with Hypotheses (2.33)-(2.34) on the initial data, we conclude that $(f_1^\epsilon)^-(x,t)=0$ a.e. in $\Omega_f^\epsilon\times J$ and $(m_1^\epsilon)^-(x,t)=0$ a.e. in $\Omega_m^\epsilon\times J$, that is the first part of (2.38). The second part is proved similarly, multiplying (2.15) by $(\alpha^2+\beta^2)(\hat f_1-f_1^\epsilon)^-$ (respectively (2.40) by $(\hat f_1-m_1^\epsilon)^-$).

We now justify (2.39). To this aim, we consider the problem satisfied by $(c_1^\epsilon+c_2^\epsilon)$ in $\Omega_m^\epsilon\times J$:

$$\phi^\epsilon\partial_t(c_1^\epsilon+c_2^\epsilon)+\underline{\mathcal{V}}^\epsilon\cdot\nabla(c_1^\epsilon+c_2^\epsilon)-\operatorname{div}(\mathcal{D}^\epsilon\nabla(c_1^\epsilon+c_2^\epsilon))=q(1-c_1^\epsilon-c_2^\epsilon),\tag{2.45}$$

$$\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla(c_1^\epsilon+c_2^\epsilon)\cdot\nu=0\quad\text{on }\partial\Omega_m^\epsilon\times J,\tag{2.46}$$

$$(c_1^\epsilon+c_2^\epsilon)(x,0)=c_1^o(x)+c_2^o(x)\quad\text{in }\Omega.\tag{2.47}$$

Multiplying (2.45) by $(c_1^\epsilon + c_2^\epsilon - C_\alpha)^-$, integrating over Ω_m^ϵ and using the same tools as in the previous part of this proof, we state that $c_1^\epsilon(x,t) + c_2^\epsilon(x,t) \geq C_\alpha \geq 0$ almost everywhere in $\Omega_m^\epsilon \times J$. We now detail the proof of the second part of (2.39). We multiply (2.45) by $(1 - c_1^\epsilon - c_2^\epsilon)^-$ and we integrate over Ω_m^ϵ. Using the basic properties (2.22) of the tensor $\mathcal{D}^\epsilon$ we get

$$\begin{aligned}
&\frac{\phi_-}{2}\frac{d}{dt}\int_{\Omega_m^\epsilon} |(1 - c_1^\epsilon - c_2^\epsilon)^-|^2 + \phi_- \int_{\Omega_m^\epsilon} D_m(\alpha + \beta\epsilon^2)|\nabla(1 - c_1^\epsilon - c_2^\epsilon)^-|^2\\
&+ \int_{\Omega_m^\epsilon} (\underline{\mathcal{V}}^\epsilon \cdot \nabla(c_1^\epsilon + c_2^\epsilon))\,(1 - c_1^\epsilon - c_2^\epsilon)^- - \int_{\Omega_m^\epsilon} q(1 - c_1^\epsilon - c_2^\epsilon)\,(1 - c_1^\epsilon - c_2^\epsilon)^- \leq 0.
\end{aligned}$$

The fourth term of the left-hand side is nonnegative. The convective term is estimated with the Cauchy-Schwarz and Young inequalities as follows.

$$\begin{aligned}
&\Big|\int_{\Omega_m^\epsilon} (\underline{\mathcal{V}}^\epsilon \cdot \nabla(c_1^\epsilon + c_2^\epsilon))\,(1 - c_1^\epsilon - c_2^\epsilon)^-\Big|\\
&= \Big|\int_{\Omega_m^\epsilon} (\underline{\mathcal{V}}^\epsilon \cdot \nabla(1 - c_1^\epsilon - c_2^\epsilon)^-)\,(1 - c_1^\epsilon - c_2^\epsilon)^-\Big|\\
&\leq \int_{\Omega_m^\epsilon} \frac{\phi_-\alpha D_m}{2}\,|\nabla(1 - c_1^\epsilon - c_2^\epsilon)^-|^2 + C\|\underline{\mathcal{V}}^\epsilon\|_\infty^2 \int_{\Omega_m^\epsilon} |(1 - c_1^\epsilon - c_2^\epsilon)^-|^2.
\end{aligned}$$

Using these estimates in the first relation, we get

$$\begin{aligned}
&\frac{\phi_-}{2}\frac{d}{dt}\int_{\Omega_m^\epsilon} |(1 - c_1^\epsilon - c_2^\epsilon)^-|^2\,dx + \phi_- \int_{\Omega_m^\epsilon} D_m(\frac{\alpha}{2} + \beta\epsilon^2)|\nabla(1 - c_1^\epsilon - c_2^\epsilon)^-|^2\,dx\\
&\leq C\int_{\Omega_m^\epsilon} |(1 - c_1^\epsilon - c_2^\epsilon)^-\,dx|^2\,dx.
\end{aligned}$$

The latter relation and the Gronwall lemma combined with Assumption (2.34) let us conclude that $(1 - c_1^\epsilon - c_2^\epsilon)^-(x,t) = 0$ and thus $c_1^\epsilon(x,t) + c_2^\epsilon(x,t) \leq 1$ a.e. in $\Omega_m^\epsilon \times J$. This completes the proof of the theorem. $\square$

3. Uniform estimates

We begin by stating the following properties of the pressure solutions of the problem (2.16), (2.20)-(2.21), (2.27), (2.30), (2.32).

Lemma 3.1. *The pressure satisfies the following uniform estimates*

$$\begin{gathered}
\|p_f^\epsilon\|_{L^\infty(J;L^2(\Omega_f^\epsilon))} + \|p_f^\epsilon\|_{L^2(J;H^1(\Omega_f^\epsilon))} \leq C,\\
\|\underline{v}_f^\epsilon\|_{(L^2(J;L^2(\Omega_f^\epsilon)))^2} \leq C,\\
\|p^\epsilon\|_{L^\infty(J;L^2(\Omega_m^\epsilon))} \leq C,\\
\|\alpha^{1/2}(c_1^\epsilon + c_2^\epsilon)^{1/2}\nabla p^\epsilon\|_{(L^2(J;L^2(\Omega_m^\epsilon)))^2} + \|\epsilon\nabla p^\epsilon\|_{(L^2(J;L^2(\Omega_m^\epsilon)))^2} \leq C,\\
\|\underline{\mathcal{V}}_s^\epsilon\|_{(L^2(J;L^2(\Omega_m^\epsilon)))^2} \leq C,\\
\|\underline{\mathcal{V}}_h^\epsilon\|_{(L^2(J;L^2(\Omega_m^\epsilon)))^2} \leq C.
\end{gathered}$$

Furthermore the time derivative $(\chi_f^\epsilon\phi_f^\epsilon\partial_t p_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\partial_t p^\epsilon)$ *is uniformly bounded in* $L^2(J;(H^1(\Omega))')$.

Proof. The estimates are derived from integration by parts. We multiply (2.16) by p_f^ϵ and integrate over $\Omega_f^\epsilon \times J$. We multiply (2.20) by p^ϵ and integrate over $\Omega_m^\epsilon \times J$. Summing up the resulting relations, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_f^\epsilon} \phi_f^\epsilon\,|p_f^\epsilon|^2\,dx + \frac{1}{2}\int_{\Omega_m^\epsilon} \phi^\epsilon\,|p^\epsilon|^2\,dx + \int_{\Omega_f^\epsilon\times J} \frac{k_f^\epsilon}{\mu(f_1^\epsilon)}\nabla p_f^\epsilon\cdot\nabla p_f^\epsilon\,dx\,dt\\
&+\int_{\Omega_m^\epsilon\times J}\big(\alpha(c_1^\epsilon+c_2^\epsilon)(1-\epsilon^2)+\epsilon^2\big)\,\frac{k^\epsilon}{\mu(m_1^\epsilon)}\nabla p^\epsilon\cdot\nabla p^\epsilon\,dx\,dt\\
&=\frac{1}{2}\int_\Omega (\chi_f^\epsilon\phi_f^\epsilon(x)+\chi_m^\epsilon\phi^\epsilon(x))\,|p^o(x)|^2\,dx + \int_{\Omega\times J} q\,(\chi_f^\epsilon p_f^\epsilon + \chi_m^\epsilon p^\epsilon)\,dx\,dt.
\end{aligned}$$

Applying the Cauchy-Schwarz and Young inequalities with the properties of ϕ_f^ϵ, ϕ^ϵ, k_f^ϵ, k^ϵ and μ in the latter relation, we get

$$\begin{aligned}
&\frac{\phi_-}{2}\int_{\Omega_f^\epsilon}|p_f^\epsilon|^2\,dx + \frac{\phi_-}{2}\int_{\Omega_m^\epsilon}|p^\epsilon|^2\,dx + \frac{k_-}{\mu_+}\int_{\Omega_f^\epsilon\times J}|\nabla p_f^\epsilon|^2\,dx\,dt\\
&+\frac{k_-}{\mu_+}\int_{\Omega_m^\epsilon\times J}\big(\alpha(c_1^\epsilon+c_2^\epsilon)\,|\nabla p^\epsilon|^2+\epsilon^2\,(1-\alpha(c_1^\epsilon+c_2^\epsilon))\,|\nabla p^\epsilon|^2\big)\,dx\,dt\\
&\le C\big(\|p^o\|_{L^2(\Omega)},\|q\|_{L^2(\Omega\times J)}\big) + \int_{\Omega_f^\epsilon\times J}|p_f^\epsilon|^2\,dx\,dt + \int_{\Omega_m^\epsilon\times J}|p^\epsilon|^2\,dx\,dt.
\end{aligned}$$

Using the Gronwall lemma, we prove the desired estimates. The result on the time derivatives then follows straightforward from (2.16), (2.20)-(2.21). □

We can now establish the following results concerning the concentrations functions $(f_1^\epsilon, C_1^\epsilon, c_1^\epsilon, c_2^\epsilon)$.

Lemma 3.2. (i) *The functions $(f_1^\epsilon, C_1^\epsilon, c_1^\epsilon, c_2^\epsilon)$ are uniformly bounded in the space $L^\infty(J;L^2(\Omega_f^\epsilon))\times(L^\infty(J;L^2(\Omega_f^\epsilon)))^3$ and are such that*

$$\begin{gathered}
0\le f_1^\epsilon(x,t)\le \hat f_1\le 1 \quad \text{almost everywhere in } \Omega_f^\epsilon\times J,\\
0\le \alpha c_1^\epsilon(x,t)+\beta C_1^\epsilon(x,t)\le \hat f_1\le 1 \quad \text{almost everywhere in } \Omega_m^\epsilon\times J\\
0\le c_1^\epsilon(x,t)+c_2^\epsilon(x,t)\le 1 \quad \text{almost everywhere in } \Omega_m^\epsilon\times J;
\end{gathered}$$

(ii) *the sequence $((D_m^{1/2}+\alpha_t^{1/2}|\underline{v}_f^\epsilon|^{1/2})\nabla f_1^\epsilon)$ is uniformly bounded in $(L^2(\Omega_f^\epsilon\times J))^2$;*

(iii) *for $i=1,2$, the diffusive terms $\alpha^{1/2}(1+(c_1^\epsilon+c_2^\epsilon)^{1/2}|\nabla p^\epsilon|^{1/2})\nabla c_i^\epsilon$ and $\epsilon(1+|\epsilon\nabla p^\epsilon|^{1/2})\nabla c_i^\epsilon$ are uniformly bounded in $(L^2(\Omega_m^\epsilon\times J))^2$. The same estimates hold for C_1^ϵ.*

Proof. The maximum principles of *(i)* are a direct consequence of the construction of the solution $(f_1^\epsilon, C_1^\epsilon, c_1^\epsilon, c_2^\epsilon)$ in Theorem 2.1. We write the variational formulation

(2.36) with the test function $(d_f, d_1, D_1) = (f_1^\epsilon, c_1^\epsilon, C_1^\epsilon)$. We get

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_f^\epsilon} \phi_f^\epsilon |f_1^\epsilon|^2\,dx + \frac{1}{2}\int_{\Omega_m^\epsilon} \phi^\epsilon(|c_1^\epsilon|^2 + |C_1^\epsilon|^2)\,dx + \int_{\Omega_f^\epsilon\times J} \mathcal{D}(\underline{v}_f^\epsilon)\nabla f_1^\epsilon\cdot\nabla f_1^\epsilon\,dx\,dt \\
&+ \int_{\Omega_m^\epsilon\times J} (\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla c_1^\epsilon\cdot\nabla c_1^\epsilon + \mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla C_1^\epsilon\cdot\nabla C_1^\epsilon)\,dx\,dt \\
&+ \int_{\Omega_f^\epsilon\times J} (\underline{v}_f^\epsilon\cdot\nabla f_1^\epsilon)\,f_1^\epsilon\,dx\,dt + \int_{\Omega_m^\epsilon\times J} \underline{\mathcal{V}}^\epsilon\cdot(c_1^\epsilon\,\nabla c_1^\epsilon + C_1^\epsilon\,\nabla C_1^\epsilon)\,dx\,dt \\
&+ \int_{\Omega\times J} q\,(\chi_f^\epsilon|f_1^\epsilon|^2 + \chi_m^\epsilon\,(|c_1^\epsilon|^2 + |C_1^\epsilon|^2))\,dx\,dt \\
&= \int_{\Omega\times J} q\hat{f}_1 f_1^\epsilon\,dx\,dt + \int_{\Omega_m^\epsilon\times J} q(\hat{c}_1 c_1^\epsilon + \hat{C}_1 C_1^\epsilon)\,dx\,dt \\
&\quad + \frac{1}{2}\int_\Omega \big(\phi_f^\epsilon|f_1^o|^2 + \phi^\epsilon(|c_1^o|^2 + |C_1^o|^2)\big)dx.
\end{aligned}
\tag{3.1}
$$

The convective terms in (3.1) are estimated as follows using the Cauchy-Schwarz and Young inequalities. In the fractured part, we write

$$
\Big|\int_{\Omega_f^\epsilon\times J} (\underline{v}_f^\epsilon\cdot\nabla f_1^\epsilon)\,f_1^\epsilon\,dx\Big| \le \int_{\Omega_f^\epsilon\times J} \frac{\alpha_t}{2}|\underline{v}_f^\epsilon|\,|\nabla f_1^\epsilon|^2\,dx + C\|f_1^\epsilon\|_\infty^2 \int_{\Omega_f^\epsilon} |\underline{v}_f^\epsilon|\,dx,
$$

where $0 \le f_1^\epsilon(x,t) \le 1$ a.e. in $\Omega_f^\epsilon \times J$ and $\underline{v}_f^\epsilon$ is uniformly bounded in $(L^1(\Omega_f^\epsilon\times J))^2$ thanks to Lemma 3.1. In the matrix part, the work is more difficult because we do not have an estimate for c_1^ϵ and C_1^ϵ in $L^\infty(\Omega_m^\epsilon\times J)$. We thus get firstly

$$
\begin{aligned}
\Big|\int_{\Omega_m^\epsilon\times J} \underline{\mathcal{V}}^\epsilon\cdot(c_1^\epsilon\,\nabla c_1^\epsilon + C_1^\epsilon\,\nabla C_1^\epsilon)\Big| &\le \int_{\Omega_m^\epsilon\times J} \frac{\alpha_t}{2}|\underline{\mathcal{V}}_s^\epsilon + \epsilon^2\underline{\mathcal{V}}_h^\epsilon|\,(|\nabla c_1^\epsilon|^2 + |\nabla C_1^\epsilon|^2) \\
&\quad + C\int_{\Omega_f^\epsilon} (|\underline{\mathcal{V}}_s^\epsilon| + |\underline{\mathcal{V}}_h^\epsilon|)\,(|c_1^\epsilon|^2 + |C_1^\epsilon|^2)\,dx.
\end{aligned}
$$

In such a matrix domain, the Gagliardo-Nirenberg inequality reads

$$
\|u\|_{L^4(\Omega_m^\epsilon)} \le C\|u\|_{L^2(\Omega_m^\epsilon)}^{1/2}\|(\alpha+\beta\epsilon)u\|_{H^1(\Omega_m^\epsilon)}^{1/2}, \quad \forall u\in H^1(\Omega_m^\epsilon).
$$

The second term of the right-hand side of the latter relation is then treated as follows using the Gagliardo-Nirenberg inequality and Lemma 3.1.

$$
\begin{aligned}
&\int_{\Omega_f^\epsilon} |\underline{\mathcal{V}}^\epsilon|\,(|c_1^\epsilon|^2 + |C_1^\epsilon|^2) \\
&\le \frac{k_+}{\mu_-}\int_{\Omega_f^\epsilon} \big(\alpha(c_1^\epsilon + c_2^\epsilon)(1-\epsilon) + \epsilon\big)|\nabla p^\epsilon|\,\big(|c_1^\epsilon|^2 + |c_2^\epsilon|^2\big) \\
&\le C\Big(\int_{\Omega_f^\epsilon} \big(\alpha^2(c_1^\epsilon + c_2^\epsilon)^2 + \epsilon^2(1-\alpha(c_1^\epsilon + c_2^\epsilon))\big)^2|\nabla p^\epsilon|^2\Big)^{1/2} \\
&\quad \times \big(\|c_1^\epsilon\|_{L^4(\Omega_m^\epsilon)}^2 + \|C_1^\epsilon\|_{L^4(\Omega_m^\epsilon)}^2\big) \\
&\le C\big(\|c_1^\epsilon\|_{L^4(\Omega_m^\epsilon)}^2 + \|C_1^\epsilon\|_{L^4(\Omega_m^\epsilon)}^2\big) \\
&\le C\big(\|c_1^\epsilon\|_{L^2(\Omega_m^\epsilon)}\|(\alpha+\beta\epsilon)c_1^\epsilon\|_{H^1(\Omega_m^\epsilon)} + \|C_1^\epsilon\|_{L^2(\Omega_m^\epsilon)}\|(\alpha+\beta\epsilon)C_1^\epsilon\|_{H^1(\Omega_m^\epsilon)}\big) \\
&\le \frac{C}{\delta}\big(\|c_1^\epsilon\|_{L^2(\Omega_m^\epsilon)}^2 + \|C_1^\epsilon\|_{L^2(\Omega_m^\epsilon)}^2\big) + \phi_-\delta\int_{\Omega_f^\epsilon} (\alpha+\epsilon^2)D_m\big(|\nabla c_1^\epsilon|^2 + |\nabla C_1^\epsilon|^2\big),
\end{aligned}
$$

for any $\delta > 0$. The last term in the left-hand side of (3.1) is nonnegative. Using the latter estimates, the Cauchy-Schwarz and Young inequalities for the right-hand side source terms and the basic properties (2.22) of the tensors $\mathcal{D}$ and $\mathcal{D}^\epsilon$, it follows from (3.1) that

$$\begin{aligned}
&\frac{\phi_-}{2}\int_\Omega (\chi_f^\epsilon |f_1^\epsilon|^2 + \chi_m^\epsilon(|c_1^\epsilon|^2 + |C_1^\epsilon|^2))\,dx + \phi_- \int_{\Omega_f^\epsilon \times J} (D_m + \frac{\alpha_t}{2}|\underline{v}_f^\epsilon|)\,|\nabla f_1^\epsilon|^2\,dx\,dt\\
&+ \phi_- \int_{\Omega_m^\epsilon \times J} ((\alpha+\epsilon^2)(1-\delta)D_m + \frac{\alpha_t}{2}|\underline{\mathcal{V}}_s^\epsilon + \epsilon^2 \underline{\mathcal{V}}_h^\epsilon|)\big(|\nabla c_1^\epsilon|^2 + |C_1^\epsilon|^2\big)\,dx\,dt\\
&\le C + C\int_{\Omega_f^\epsilon \times J} |f_1^\epsilon|^2\,dx\,dt + \frac{C}{\delta}\int_{\Omega_m^\epsilon \times J} (|c_1^\epsilon|^2 + |C_1^\epsilon|^2)\,dx\,dt.
\end{aligned}$$

We choose $0 < \delta < 1$. Using the Gronwall lemma yields to the result for f_1^ϵ, c_1^ϵ and C_1^ϵ. Once we know the estimate for c_1^ϵ, we obtain similar ones for c_2^ϵ by multiplying (2.18) by c_1^ϵ, (2.19) by c_2^ϵ, integrating over Ω_m^ϵ and summing up the results to kill the terms on Γ_{fm}. Our claim is proved. □

Remark 3.3. This proof fails if we consider a domain $\Omega \subset \mathbb{R}^3$ instead of $\mathbb{R}^2$ because of the use of the Gagliardo-Nirenberg inequality. Nevertheless we can get the same result using minor modifications. The simplest way is to add a L^∞-estimate for c_1^ϵ and C_1^ϵ. To this aim, we note that the function $d_1^\epsilon = \frac{\alpha}{\beta}C_1^\epsilon - c_1^\epsilon$ is solution of the following problem in $\Omega_m^\epsilon \times J$.

$$\begin{gathered}
\phi^\epsilon \partial_t d_1^\epsilon + \underline{\mathcal{V}}^\epsilon \cdot \nabla d_1^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla d_1^\epsilon) = q(\frac{\alpha}{\beta}\hat{C}_1 - \hat{c}_1 - d_1^\epsilon),\\
\mathcal{D}^\epsilon(\underline{\mathcal{V}}^\epsilon)\nabla d_1^\epsilon \cdot \nu = 0 \quad \text{on } \partial\Omega_m^\epsilon \times J,\\
d_1^\epsilon(x,0) = \frac{\alpha}{\beta}C_1^o(x) - c_1^o(x) \quad \text{in } \Omega_m^\epsilon.
\end{gathered}$$

Similar arguments to the ones used in Theorem 2.1 together with an additional assumption on $\frac{\alpha}{\beta}C_1^o - c_1^o$ lead to a maximum principle for $\frac{\alpha}{\beta}C_1^\epsilon - c_1^\epsilon$. Combining it with Lemma 3.2 (ii), we get a L^∞-bound for c_1^ϵ and C_1^ϵ.

4. The macroscopic model: the case of a totally fractured media ($\alpha = 0$)

We assume in this section that $\alpha = 0$ and then $\beta = 1$. We thus do not consider the concentrations variables $(c_1^\epsilon, c_2^\epsilon)$. And we define global pressure and concentration functions θ^ϵ and ξ^ϵ by

$$\theta^\epsilon = \begin{cases} p_f^\epsilon & \text{in } \Omega_f^\epsilon \times J,\\ p^\epsilon & \text{in } \overline{\Omega_m^\epsilon} \times J, \end{cases} \qquad \xi^\epsilon = \begin{cases} f_1^\epsilon & \text{in } \Omega_f^\epsilon \times J,\\ C_1^\epsilon & \text{in } \overline{\Omega_m^\epsilon} \times J, \end{cases} \tag{4.1}$$

and the global porosity Φ^ϵ, tensor of permeability K^ϵ and new diffusion tensor $\mathcal{D}^\epsilon$ by $\Phi^\epsilon = \chi_f^\epsilon \phi_f^\epsilon + \chi_m^\epsilon \phi^\epsilon$, $K^\epsilon = \chi_f^\epsilon k_f^\epsilon + \chi_m^\epsilon \epsilon^2 k^\epsilon$ and $\mathcal{D}^\epsilon = \chi_f^\epsilon \mathcal{D} + \chi_m^\epsilon \mathcal{D}^\epsilon$.

4.1. **The limit double porosity model.**

The aim of this section is to derive rigorously the double porosity model described below. We obtain a macroscopic fracture system driven by equations in $\Omega \times J$, similar to the microscopic ones:

$$\overline{\phi_f}^{Y_f}\,\partial_t P_f - \operatorname{div}\underline{\mathcal{V}}_f = q - \int_{Y_m} \phi(y)\,\partial_t p\,dy, \quad \underline{\mathcal{V}}_f = -\frac{\overline{K}_f}{\mu(F_1)}\nabla P_f, \tag{4.2}$$

$$\begin{aligned} &\overline{\phi_f}^{Y_f}\,\partial_t F_1 + \underline{\mathcal{V}}_f\cdot\nabla F_1 - \operatorname{div}(\overline{\mathcal{D}}_f(\nabla P_f,\mu(F_1))\nabla F_1) + q\,|Y_f|\,F_1 \\ &= q\hat{f}_1 - q\int_{Y_m} C_1\,dy - \int_{Y_m}\phi(y)\,\partial_t C_1\,dy + \int_{Y_m}\frac{k(y)}{\mu(C_1)}\nabla_y p\cdot\nabla_y C_1\,dy. \end{aligned} \tag{4.3}$$

The matrix plays the role of a source and produces the additional right-hand side source-like terms. The homogenization process gives for this macroscopic level an effective porosity $\overline{\phi_f}^{Y_f}$, an effective rock permeability $\overline{K}_f$ and an homogenized tensor of diffusion $\overline{\mathcal{D}}_f$ defined by

$$\overline{\phi_f}^{Y_f} = \int_{Y_f}\phi_f(y)dy, \tag{4.4}$$

$$\overline{K}_{f_{ij}} = \int_{Y_f} k_f(y)(\nabla_y v^i(y)+e^i)\cdot(\nabla_y v^j(y)+e^j)dy \quad 1\le i,j\le 2, \tag{4.5}$$

$$\overline{\mathcal{D}}_{f_{ij}}(\nabla P_f,\mu(F_1)) = \int_{Y_f} D_f(\nabla_y w^i+e^i)\cdot(\nabla_y w^j+e^j)dy, \tag{4.6}$$

$D_f = \mathcal{D}\Big(\frac{K_f(y)}{\mu(F_1)}\nabla P_f\Big)$ and $K_f(y)_{ij} = k_f(y)\left(\nabla_y v^i(y)+e^i\right)\cdot\left(\nabla_y v^j(y)+e^j\right)$. The functions $(v^i)_{1\le i\le 2}$ and $(w^i)_{1\le i\le 2}$ are respectively solutions of the cell problems (4.7) and (4.8) below.

$$\begin{gathered} -\operatorname*{div}_y(k_f(y)(\nabla_y v^i(y)+e^i)) = 0 \quad \text{in } Y_f, \\ k_f(y)(\nabla_y v^i(y)+e^i)\cdot\nu_y = 0 \quad \text{on } \Gamma_{fm},\ y\mapsto v^i(x,y)\ Y\text{-periodic}, \end{gathered} \tag{4.7}$$

$$\begin{gathered} -\operatorname*{div}_y(D_f(x,y)(\nabla_y w^i(x,y)+e^i)) = 0 \quad \text{in } Y_f, \\ D_f(x,y)(\nabla_y w^i(x,y)+e^i)\cdot\nu_y = 0 \quad \text{on } \Gamma_{fm},\ y\mapsto w^i(x,y)\ Y\text{-periodic}, \end{gathered} \tag{4.8}$$

where e^j is the unit vector in the j-th direction. On the other hand, to each $x\in\Omega$ corresponds a matrix block, driven by equations in $\{x\}\times Y_m\times J$ which give the new source terms:

$$\phi(y)\,\partial_t p + \operatorname*{div}_y \underline{\mathcal{V}}_h = q, \quad \underline{\mathcal{V}}_h = -\frac{k(y)}{\mu(C_1)}\nabla_y p, \tag{4.9}$$

$$\phi(y)\,\partial_t C_1 + \underline{\mathcal{V}}_h\cdot\nabla_y C_1 - \operatorname*{div}_y(\mathcal{D}(y,\underline{\mathcal{V}}_h)\,\nabla_y C_1) + q\,C_1 = q\hat{f}_1. \tag{4.10}$$

The equations (4.2)-(4.3), (4.9)-(4.10) are provided with the following initial and boundary conditions

$$\overline{K}_f\nabla P_f\cdot\nu = 0, \quad \overline{\mathcal{D}}_f\nabla F_1\cdot\nu = 0 \quad \text{on } \Gamma\times J, \quad p = P_f,\ C_1 = F_1 \quad \text{on } \Gamma_{fm}, \tag{4.11}$$

$$P_f(x,0) = p(x,y,0) = p^o(x), \quad F_1(x,0) = C_1(x,y,0) = f_1^o(x) \quad \text{in } \Omega\times Y_m. \tag{4.12}$$

We claim and prove the following convergence result.

Theorem 4.1. *As the scaling parameter ϵ tends to zero, the microscopic model* (2.15)–(2.17)*,* (2.20)–(2.21)*,* (2.23)*,* (2.26)–(2.32) *with* $\alpha = 0$ *converges to the double porosity macroscopic model* (4.2)–(4.12).

We have captured the interactions between the local and the global scales. This model is consistent with the double porosity formulations of the engineering literature, *cf* [5]. But in [5], the exchanges between fractures and blocks are assumed quasi-stationary. Without this hypothesis, we obtain additional memory terms.

The proof of the homogenization process will be carried out by using the two-scale convergence introduced by G.Nguetseng in [18] and developed by Allaire in [2]. We recall the basic definition and properties of this concept.

Proposition 4.2. *A sequence of functions (v^ϵ) bounded in $L^2(\Omega\times J)$ two-scale converges to a limit $v^o(x,y,t)$ belonging to $L^2(\Omega\times Y\times J)$, $v^\epsilon \overset{2}{\rightharpoonup} v^o$, if*

$$\lim_{\epsilon\to 0}\int_{\Omega\times J} v^\epsilon(x,t)\,\Psi(x,x/\epsilon,t)\,dx\,dt = \int_{\Omega\times J}\int_Y v^o(x,y,t)\,\Psi(x,y,t)\,dx\,dy\,dt,$$

for any test function $\Psi(x,y,t)$, Y-periodic in the second variable, satisfying

$$\lim_{\epsilon\to 0}\int_{\Omega\times J} |\Psi(x,x/\epsilon,t)|^2\,dx\,dt = \int_{\Omega\times J}\int_Y |\Psi(x,y,t)|^2\,dx\,dy\,dt.$$

(i) From each bounded sequence (v^ϵ) in $L^2(\Omega\times J)$ one can extract a subsequence which two-scale converges.
(ii) Let (v^ϵ) be a bounded sequence in $L^2(J;H^1(\Omega))$ which converges weakly to v in $L^2(J;H^1(\Omega))$. Then $v^\epsilon \overset{2}{\rightharpoonup} v$ and there exists a function $v^1\in L^2(\Omega\times J;H^1_{per}(Y))$ such that, up to a subsequence, $\nabla v^\epsilon \overset{2}{\rightharpoonup} \nabla v(x,t)+\nabla_y v^1(x,y,t)$.
(iii) Let (v^ϵ) be a bounded sequence in $L^2(\Omega\times J)$ with $(\epsilon\nabla v^\epsilon)$ bounded in $(L^2(\Omega\times J))^2$. Then, there exists a function $v^o\in L^2(\Omega\times J;H^1_{per}(Y))$ such that, up to a subsequence, $v^\epsilon \overset{2}{\rightharpoonup} v^o$ and $\epsilon\nabla v^\epsilon \overset{2}{\rightharpoonup} \nabla_y v^o(x,y,t)$.

To exploit the *a priori* estimates obtained in the fractured part Ω_f^ϵ, we need to extend the functions p_f^ϵ and f_1^ϵ to the whole domain Ω. To this aim, following [1], we claim that there exists three constants $k_i=k_i(Q_f)>0$, $i=1,2,3$, and a linear and continuous extension operator $\Pi^\epsilon : H^1(\Omega_f^\epsilon)\to H^1_{loc}(\Omega)$ such that $\Pi^\epsilon v = v$ a.e. in Ω_f^ϵ and

$$\int_{\Omega(\epsilon k_1)} |\Pi^\epsilon v|^2\,dx \le k_2\int_{\Omega_f^\epsilon}|v|^2\,dx,\quad \int_{\Omega(\epsilon k_1)}|\nabla(\Pi^\epsilon v)|^2\,dx\le k_3\int_{\Omega_f^\epsilon}|\nabla v|^2\,dx$$

for all $v\in H^1(\Omega_f^\epsilon)$, with $\Omega(\epsilon k_1)=\{x\in\Omega \mid \operatorname{dist}(x,\Gamma)>\epsilon k_1\}$. To avoid dealing with boundary layers, we make the following additional assumption on the structure of the domain Ω:

$$\Omega_m^\epsilon = \Omega(\epsilon k_1)\cap\big\{\cup_{k\in\mathbb{Z}^2}\epsilon\,(Y_m+k)\big\}\quad\text{and}\quad \Omega_f^\epsilon=\Omega\setminus\overline{\Omega_m^\epsilon}.$$

For any subset $\Omega'\subset\subset\Omega$, we get with Lemmas 3.1 and 3.2 $\int_{\Omega'\times J}|\nabla(\Pi^\epsilon p_f^\epsilon)|^2dx\,dt\le C$ and $\int_{\Omega'\times J}|\nabla(\Pi^\epsilon f_1^\epsilon)|^2dx\,dt\le C$, if $\epsilon<\operatorname{dist}(\Omega',\Gamma)/k_1$. Thus we can state convergence results in any subset $\Omega'\subset\subset\Omega$ and conclude with a density argument. But, for sake of simplicity, we prefer assume that the blocks are removed in an ϵk_1-neighborhood of Γ. We then get

$$\int_{\Omega\times J}|\nabla(\Pi^\epsilon p_f^\epsilon)|^2\,dx\,dt\le C,\quad \int_{\Omega\times J}|\nabla(\Pi^\epsilon f_1^\epsilon)|^2\,dx\,dt\le C.$$

We ensure the existence of functions $P_f\in L^2(J;H^1(\Omega))$, $P_f^1\in L^2(\Omega\times J;H^1_{per}(Y))$, $p\in L^2(\Omega\times J;H^1_{per}(Y))$, $F_1\in L^2(J;H^1(\Omega))$, $F_1^1\in L^2(\Omega\times J;H^1_{per}(Y))$ and $C_1\in L^2(\Omega\times J;H^1_{per}(Y))$ such that, up to subsequences not relabeled for convenience, we have the following convergence.

$$\Pi^\epsilon p_f^\epsilon \rightharpoonup P_f \text{ weakly in } L^2(J;H^1(\Omega)),$$

$$\nabla(\Pi^\epsilon p_f^\epsilon) \stackrel{2}{\rightharpoonup} \nabla P_f(x,t) + \nabla_y P_f^1(x,y,t),$$
$$\theta^\epsilon \stackrel{2}{\rightharpoonup} p(x,y,t), \quad \epsilon\nabla\theta^\epsilon \stackrel{2}{\rightharpoonup} \nabla_y p(x,y,t),$$
$$\Pi^\epsilon f_1^\epsilon \rightharpoonup F_1 \quad \text{weakly in } L^2(J;H^1(\Omega)), \nabla(\Pi^\epsilon f_1^\epsilon) \stackrel{2}{\rightharpoonup} \nabla F_1(x,t) + \nabla_y F_1^1(x,y,t),$$
$$\xi^\epsilon \stackrel{2}{\rightharpoonup} C_1(x,y,t), \quad \epsilon\nabla\xi^\epsilon \stackrel{2}{\rightharpoonup} \nabla_y C_1(x,y,t).$$

We also assert that $\Phi^\epsilon \stackrel{2}{\rightharpoonup} \Phi(y) = \chi_f(y)\phi_f(y) + \chi_m(y)\phi(y)$,

$$K^\epsilon \stackrel{2}{\rightharpoonup} K(y) = \chi_f(y)k_f(y) + \chi_m(y)k(y),$$

and that we can consider that Φ^ϵ and K^ϵ are admissible test functions for the two-scale convergence. We give in the following lemma some additional compactness result.

Lemma 4.3. *The sequence $(\Pi^\epsilon f_1^\epsilon)$ is sequentially compact in $L^2(\Omega \times J)$.*

Proof. We begin by proving that the sequence $(\Phi^\epsilon \partial_t \xi^\epsilon)$ is uniformly bounded in $L^2(J;(H^2(\Omega))')$. Let $g \in L^2(J;H^2(\Omega))$. Equations (2.15) and (2.17) give

$$\begin{aligned}
&\langle \Phi^\epsilon \partial_t \xi^\epsilon, g\rangle_{L^2(J;(H^2(\Omega))')\times L^2(J;H^2(\Omega))} \\
&= \int_{\Omega\times J} (\frac{K^\epsilon}{\mu(\xi^\epsilon)} \nabla\theta^\epsilon \cdot \nabla\xi^\epsilon)\, g - \int_{\Omega\times J} \mathcal{D}^\epsilon \nabla\xi^\epsilon \cdot \nabla g + \int_{\Omega\times J} q\,(\hat{f}_1 - \xi^\epsilon)\, g.
\end{aligned}$$

In view of the previous lemmas, we have

$$\begin{aligned}
&\Big|\int_{\Omega\times J} (\frac{K^\epsilon}{\mu(\xi^\epsilon)} \nabla\theta^\epsilon \cdot \nabla\xi^\epsilon)\, g\, dx\, dt\Big| \\
&\le \frac{C}{\mu_-} \||K^\epsilon\nabla\theta^\epsilon|^{1/2}\nabla\xi^\epsilon\|_{(L^2(\Omega\times J))^2} \||K^\epsilon\nabla\theta^\epsilon|^{1/2}\|_{L^4(\Omega\times J)} \|g\|_{L^4(\Omega\times J)} \\
&\le C\|g\|_{L^2(J;H^2(\Omega))}, \\
&\Big|\int_{\Omega\times J} \mathcal{D}^\epsilon\nabla\xi^\epsilon \cdot \nabla g\, dx\, dt\Big| \le C\|\nabla g\|_{(L^2(J;L^4(\Omega)))^2} \le C\|g\|_{L^2(J;H^2(\Omega))},
\end{aligned}$$
$$\Big|\int_{\Omega\times J} q\,(\hat{f}_1 - \xi^\epsilon)\, g\, dx\, dt\Big| \le C\|g\|_{L^2(\Omega\times J)}.$$

Then $|\langle \Phi^\epsilon \partial_t \xi^\epsilon, g\rangle| \le C\|g\|_{L^2(J;H^2(\Omega))}$ and $(\phi_f^\epsilon \partial_t(\Pi^\epsilon f_1^\epsilon))$ is uniformly bounded in $L^2(J;(H^2(\Omega))')$. A compactness argument of Aubin's type ensures that $(\phi_f^\epsilon(\Pi^\epsilon f_1^\epsilon))$ is compact in $L^2(J;(H^1(\Omega))')$. We thus can pass to the limit in the product $\langle \phi_f^\epsilon(\Pi^\epsilon f_1^\epsilon), \Pi^\epsilon f_1^\epsilon\rangle_{L^2(J;(H^1(\Omega))')\times L^2(J;H^1(\Omega))}$. Since $\phi_f^\epsilon(x) \ge \phi_- > 0$ almost everywhere in Ω, it follows that $(\Pi^\epsilon f_1^\epsilon)$ is compact in $L^2(\Omega \times J)$. $\square$

Now, we have the first tools to study the behavior of the microscopic system as ϵ tends to zero. We begin by the pressure equations.

4.2. **The Pressure Problem.** We want now to pass to the limit in System (2.16), (2.20)–(2.21), (2.22), (2.30), (2.32) recalled below.

$$\Phi^\epsilon \partial_t \theta^\epsilon - \operatorname{div}((1/\mu(\xi^\epsilon))K^\epsilon\nabla\theta^\epsilon) = q \quad \text{in } \Omega \times J,$$
$$K^\epsilon\nabla\theta^\epsilon \cdot \nu = 0 \quad \text{on } \Gamma \times J, \quad \theta^\epsilon(x,0) = p^o(x) \quad \text{in } \Omega.$$

As in [2], we multiply the first equation by a test function in the form $\Psi(x,t) + \epsilon\Psi_1(x,x/\epsilon,t) + \psi(x,x/\epsilon,t)$, with $\Psi \in \mathcal{D}(\Omega \times J)$, $\Psi_1 \in \mathcal{D}(\Omega \times J; C^\infty_{per}(Y))$ and

$\psi \in \mathcal{D}(\Omega \times J; C^\infty_{per}(Y))$ such that $\psi(x,y,t) = 0$ for $y \in Y_f$. Integrating over $\Omega \times J$, we obtain

$$\begin{aligned}
&- \int_{\Omega\times J} \Phi^\epsilon(x)\, \theta^\epsilon \partial_t \big(\Psi(x,t) + \epsilon\, \Psi_1(x, x/\epsilon, t) + \psi(x, x/\epsilon, t)\big)\, dx\, dt \\
&+ \int_\Omega \Phi^\epsilon(x)\, p^o(x) \big(\Psi(x,0) + \epsilon \Psi_1(x, x/\epsilon, 0) + \psi(x, x/\epsilon, 0)\big)\, dx \\
&\quad + \int_{\Omega\times J} \frac{K^\epsilon(x)}{\mu(\xi^\epsilon)} \nabla\theta^\epsilon \cdot \big(\nabla\Psi + \epsilon \nabla_x \Psi_1^\epsilon + \nabla_y \Psi_1^\epsilon + \nabla_x \psi^\epsilon + \frac{1}{\epsilon}\nabla_y \psi^\epsilon\big)\, dx\, dt \\
&\quad = \int_{\Omega\times J} q\big(\Psi(x,t) + \epsilon\Psi_1(x, x/\epsilon, t) + \psi(x, x/\epsilon, t)\big)\, dx\, dt.
\end{aligned}$$

Letting $\epsilon \to 0$, we get

$$\begin{aligned}
&- \int_{\Omega\times J}\int_{Y_f} \phi_f(y)\, P_f\, \partial_t \Psi(x,t) - \int_{\Omega\times J}\int_{Y_m} \phi(y)\, p\, \partial_t(\Psi(x,t) + \psi(x,y,t)) \\
&+ \int_\Omega \int_Y (\chi_f(y)\phi_f(y) + \chi_m(y)\phi(y))\, p^o\, (\Psi(x,0) + \chi_m(y)\psi(x,y,0)) \\
&+ \int_{\Omega\times J}\int_{Y_f} \frac{k_f(y)}{\mu(F_1)} (\nabla P_f + \nabla_y P_f^1) \cdot (\nabla\Psi + \nabla_y \Psi_1) \\
&+ \lim_{\epsilon\to 0} \int_{\Omega_m^\epsilon \times J} \frac{\epsilon k^\epsilon}{\mu(C_1^\epsilon)} \nabla p^\epsilon \cdot \nabla_y \psi(x, \frac{x}{\epsilon}, t) \\
&= \int_{\Omega\times J\times Y} q\, (\Psi + \chi_m(y)\psi).
\end{aligned}$$

By density, this equality holds true for any functions $(\Psi, \Psi_1, \psi) \in H^1_o(\Omega\times J) \times L^2(\Omega \times J; H^1_{per}(Y)) \times L^2(\Omega\times J; H^1_{per}(Y_m))$. Another integration by parts shows (taking successively the test functions ψ, Ψ_1 and Ψ equal to zero) that it is a variational formulation of the following two-scale homogenized system in $\Omega \times J$:

$$\begin{gathered}
\overline{\phi_f}^{Y_f} \partial_t P_f - \operatorname{div}\Big(\frac{1}{\mu(F_1)} \int_{Y_f} k_f(y)(\nabla P_f + \nabla_y P_f^1) dy\Big) = q - \int_{Y_m} \phi(y)\, \partial_t p\, dy, \\
- \operatorname*{div}_y \Big(\frac{k_f(y)}{\mu(F_1)} (\nabla P_f + \nabla_y P_f^1)\Big) = 0 \quad \text{in } Y_f, \\
\phi(y)\, \partial_t p + \operatorname*{div}_y \underline{\mathcal{V}}_h = q \quad \text{in } Y_m, \quad \text{where } -\epsilon\, \frac{k^\epsilon}{\mu(C_1^\epsilon)} \nabla p^\epsilon \xrightharpoonup{2} \underline{\mathcal{V}}_h \\
k_f(y)(\nabla P_f + \nabla_y P_f^1) \cdot \nu_y = 0 \quad \text{on } \Gamma_{fm}, \\
k_f(y)(\nabla P_f + \nabla_y P_f^1) \cdot \nu = 0 \quad \text{on } \Gamma, \\
P_f(x,0) = p(x,y,0) = p^o(x) \quad \text{in } \Omega \times Y_m.
\end{gathered}$$

Now we eliminate the function P_f^1 in the former system. We use the solution $(v^i)_{1\le i\le 2}$ of the cell problem (4.7) and the homogenized permeability tensor $\overline{K}_f$ defined by (4.5). Through the relation $P_f^1(x,y,t) = \sum_{i=1}^2 \partial_{x_i} P_f(x,t)\, v^i(y)$, we recover the following homogenized system.

$$\overline{\phi_f}^{Y_f}\, \partial_t P_f - \operatorname{div}\Big(\frac{\overline{K}_f}{\mu(C_1)} \nabla P_f\Big) = q - \int_{Y_m} \phi\, \partial_t p\, dy \quad \text{in } \Omega \times J, \tag{4.13}$$

$$\phi\, \partial_t p + \operatorname*{div}_y \underline{\mathcal{V}}_h = q \quad \text{in } \Omega \times Y_m \times J, \tag{4.14}$$

$$\overline{K}_f \nabla P_f \cdot \nu = 0 \text{ on } \partial\Omega \times J,\ P_f(x,0) = p(x,y,0) = p^o(x) \text{ in } \Omega \times Y_m. \tag{4.15}$$

We now determine the nonexplicit limit $\underline{\mathcal{V}}_h$ using a dilation operator.

4.3. **Introduction of an Appropriate Dilation Operator.** Due to the nonlinearities and to the strong coupling of the problem, the two-scale convergence does not provide an explicit form for the source-like terms which model the influence of the matrix at the macroscopic level. To overcome this difficulty, we follow an idea of [4], that can also be compared with the unfolding method of Cioranescu et al [12]. Firstly, we recall that in this section we assume no direct flow between the matrix cells. We thus can assume that the component s of Ω_m^ϵ are strictly separated. For each $\epsilon > 0$, we then define a dilation operator $\widetilde{\cdot}$ mapping measurable functions on $\Omega_m^\epsilon \times J$ to measurable functions on $\Omega \times Y_m \times J$ by

$$\widetilde{u}(x,y,t) = u(c^\epsilon(x) + \epsilon\, y, t) \quad \text{for } y \in Y_m,\ (x,t) \in \Omega \times J,$$

where $c^\epsilon(x)$ denotes the lattice translation point of the ϵ-cell domain containing x. This dilation allows to reach directly the scale of the standard cell. Since $\Omega_m^\epsilon = \Omega(\epsilon k_1) \cap (\cup_{k\in\mathbb{Z}^2} \epsilon(Y_m + k))$, we recall that $c^\epsilon(x) = \epsilon k$ for $x \in \epsilon(Y_m + k)$. Thus the function $\widetilde{u}$ is constant in x on each block $\epsilon(Y_m + k)$, $k \in \mathbb{Z}^2$, of Ω. We extend this operator from Y_m to $\cup_k (Y_m + k)$ periodically. The dilation has the following properties (*cf* [4]). Any function $u \in L^2(J; H^1(\Omega_m^\epsilon))$ satisfies

$$\|\widetilde{u}\|_{L^2(\Omega_m^\epsilon \times J \times Y_m)} = \|u\|_{L^2(\Omega \times J)}, \quad \nabla_y \widetilde{u} = \epsilon\, \widetilde{\nabla_x u} \text{ a.e. in } \Omega \times J \times Y_m.$$

Moreover, if $(v,w) \in (L^2(0,T; H^1(\Omega_m^\epsilon)))^2$, then

$$\begin{gathered}
(\widetilde{v}, \widetilde{w})_{L^2(\Omega \times J \times Y_m)} = (v,w)_{L^2(\Omega_m^\epsilon \times J)}\,, \\
(\widetilde{v}, w)_{L^2(\Omega \times J \times Y)} = (v, \widetilde{w})_{L^2(\Omega \times J \times Y)}\,, \\
\|\nabla_y \widetilde{v}\|_{(L^2(\Omega \times J \times Y_m))^2} = \epsilon \big\|\widetilde{\nabla_x v}\big\|_{(L^2(\Omega_m^\epsilon \times J))^2}.
\end{gathered}$$

The following result makes the link between two-scale convergence and weak convergence of dilated sequences. We refer to [8] for its detailed proof.

Proposition 4.4. *If (v^ϵ) is a bounded sequence of $L^2(\Omega_m^\epsilon \times J)$ such that $\widetilde{v^\epsilon}$ converges weakly to v^o in $L^2(\Omega \times J; L^2_{per}(Y_m))$ and $\chi_m^\epsilon v^\epsilon$ two-scale converges to v, then we have $v^o = v$ a.e. in $\Omega \times J \times Y_m$.*

We now determine the limit behavior of the dilated solutions $\widetilde{C}_1^\epsilon$ and $\widetilde{p}^\epsilon$. The outline of this study is the following. We find the equations satisfied by the dilated solutions in $\Omega \times Y_m \times J$. Some estimates lead to the convergence of $(\widetilde{p}^\epsilon, \widetilde{C}_1^\epsilon)$ to (p, C_1) as $\epsilon \to 0$. On the other hand, for each fixed $k \in \mathbb{Z}^2$, we note that the restriction in $\epsilon(Y_m + k)$ of $(\widetilde{p}^\epsilon, \widetilde{C}_1^\epsilon)$ is independent of x. Thus we define corresponding functions $(\widetilde{p}_k^\epsilon(y,t), \widetilde{C}_{1k}^\epsilon(y,t))$. We get enough compactness results to find the equations satisfied by their limit (p_k, C_{1k}) for each $k \in \mathbb{Z}^2$. Then, by a density argument, the limit equations for (p, C_1) are deduced.

We begin with finding the equations satisfied by the dilated solutions $\widetilde{C}_1^\epsilon$ and $\widetilde{p}^\epsilon$. We define a test function $\hat{\psi}$ by

$$\hat{\psi}(x,z,t) = \begin{cases} \psi\big((z - c^\epsilon(x))/\epsilon, t\big) & \text{for } z \in \epsilon Y_m + c^\epsilon(x) \\ 0 & \text{for } z \notin \epsilon Y_m + c^\epsilon(x), \end{cases}$$

for any function $\psi \in L^2(J; H^1_o(Y_m))$. We multiply (2.20) by $\hat{\psi}$ and integrate over Ω^ϵ_m. Since $\Omega^\epsilon_m = \cup_{x\in\Omega} (\epsilon Y_m + c^\epsilon(x))$ and $(\epsilon Y_m + c^\epsilon(x_1)) \cap (\epsilon Y_m + c^\epsilon(x_2)) = \emptyset$ for $x_1 \neq x_2$, we get for almost every $x \in \Omega^\epsilon_m$

$$\begin{aligned}&\int_{\epsilon Y_m + c^\epsilon(x)} \big(\phi^\epsilon(z)\,\partial_t p^\epsilon)(z,t)\,\hat{\psi}(x,z,t) + \epsilon^2 \frac{k^\epsilon(z)}{\mu(C^\epsilon_1)} \nabla p^\epsilon(z,t)\cdot\nabla_z\hat{\psi}(x,z,t)\big)\,dz\\ &= \int_{\epsilon Y_m + c^\epsilon(x)} q\,\hat{\psi}(x,z,t)\,dz.\end{aligned}$$

With the change of variable $z \mapsto \epsilon y + c^\epsilon(x) = \epsilon(y+k)$, $k \in \mathbb{Z}^2$, we recover the variational formulation of the equation

$$\phi(y)\,\partial_t\widetilde{p^\epsilon} + \operatorname*{div}_y \widetilde{q^\epsilon} = \widetilde{q}, \quad \widetilde{q^\epsilon} = -(1/\mu(\widetilde{C^\epsilon_1}))k(y)\nabla_y\widetilde{p^\epsilon}. \tag{4.16}$$

We proceed in the same way for the concentration (2.17). We get

$$\phi(y)\,\partial_t\widetilde{C^\epsilon_1} + \widetilde{q^\epsilon}\cdot\nabla_y\widetilde{C^\epsilon_1} - \operatorname*{div}_y(\mathcal{D}(y,\widetilde{q^\epsilon})\nabla_y\widetilde{C^\epsilon_1}) + \widetilde{q}\,\widetilde{C^\epsilon_1} = \widetilde{q}\hat{f}_1. \tag{4.17}$$

This system is provided with the following boundary and initial conditions.

$$\widetilde{p^\epsilon} = \widetilde{P^\epsilon_f} \quad\text{and}\quad \widetilde{C^\epsilon_1} = \widetilde{f^\epsilon_1} \quad \text{in } H^{1/2}(\partial Y_m) \text{ for } (x,t)\in\Omega\times J, \tag{4.18}$$

$$\widetilde{p^\epsilon}(x,y,0) = \widetilde{p^o}(x,y), \quad \widetilde{C^\epsilon_1}(x,y,0) = \widetilde{C^o_1}(x,y) \quad \text{in } \Omega\times Y_m. \tag{4.19}$$

Using the basic properties of the dilation operator and the estimates of Section 3, we claim that, for subsequences not relabeled for convenience, the following convergence take place.

$$\begin{gathered}\widetilde{p^\epsilon} \rightharpoonup p,\ \widetilde{C^\epsilon_1} \rightharpoonup C_1 \quad \text{weakly in } L^2(\Omega\times J\times Y_m),\\ \nabla_y\widetilde{p^\epsilon} \rightharpoonup \nabla_y p,\ \nabla_y\widetilde{C^\epsilon_1} \rightharpoonup \nabla_y C_1 \quad \text{weakly in } (L^2(\Omega\times J\times Y_m))^2.\end{gathered}$$

We then choose a fixed $k \in \mathbb{Z}^2$. For each fixed $\epsilon > 0$, we define the functions $\widetilde{p^\epsilon_k}$ and $\widetilde{C^\epsilon_{1k}}$ in $Y_m \times J$ by

$$\begin{gathered}\widetilde{p^\epsilon_k}(y,t) = \begin{cases}\widetilde{p^\epsilon}(x,y,t)_{/x\in\epsilon(Y_m+k)} & \text{if } k \text{ is such that } \epsilon(Y_m+k)\cap\Omega\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases}\\ \widetilde{C^\epsilon_{1k}}(y,t) = \begin{cases}\widetilde{C^\epsilon_1}(x,y,t)_{/x\in\epsilon(Y_m+k)} & \text{if k is such that } \epsilon(Y_m+k)\cap\Omega\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases}\end{gathered}$$

Roughly speaking, $k = (k_1,k_2) \in \mathbb{Z}^2$ is such that $\epsilon(Y_m+k)\cap\Omega \neq \emptyset$ if $k_i < |\Omega|_i/\epsilon$, $i = 1,2$ (where $|\Omega|_i$ denotes the value of the measure of Ω in the i-th direction). For each $\epsilon > 0$ such that $\epsilon(Y_m+k)\cap\Omega \neq \emptyset$, $(\widetilde{p^\epsilon_k}, \widetilde{C^\epsilon_{1k}})$ is a solution of Pb. (4.16)-(4.19) in $Y_m \times J$. Furthermore, since any $f \in L^2(\Omega\times J)$ satisfies

$$\|\widetilde{f}_k\|_{L^2(Y_m\times J)} = \frac{1}{\epsilon|Y_m|}\|\widetilde{f}\|_{L^2(\epsilon(Y_m+k)\times Y_m\times J)} \leq \frac{1}{\epsilon|Y_m|}\|\widetilde{f}\|_{L^2(\Omega\times J\times Y_m)},$$

we have enough regularity properties to get with (4.16)-(4.19) some estimates for $\widetilde{p^\epsilon_k}$ and $\widetilde{C^\epsilon_{1k}}$. They are analogous to those obtained for p^ϵ_f and f^ϵ_1 in the fracture. We claim that

$$\begin{gathered}\|\widetilde{p^\epsilon_k}\|_{L^\infty(J;L^2_{per}(Y_m))\cap L^2(J;H^1_{per}(Y_m))} \leq C, \quad 0 \leq \widetilde{C^\epsilon_{1k}}(y,t) \leq 1 \quad \text{a.e. in } Y_m\times J,\\ \big\|\big(D_m + \alpha_t|\nabla_y\widetilde{p^\epsilon_k}|\big)^{1/2}\nabla_y\widetilde{C^\epsilon_{1k}}\big\|_{(L^2(J\times Y_m))^2} + \|\partial_t\widetilde{C^\epsilon_{1k}}\|_{L^2(J;(H^2_{per}(Y_m))')} \leq C,\end{gathered}$$

where C is a generic constant, independent of ϵ and k. Using in particular compactness arguments of Aubin's type for the concentration $\widetilde{C}^{\epsilon}_{1k}$, we deduce of these latter estimates that, up to subsequences not relabeled for convenience, the following convergence take place.

$$\widetilde{p}^{\epsilon}_k \rightharpoonup p_k \quad \text{weakly in } L^2(J; H^1_{per}(Y_m)),$$
$$\widetilde{C}^{\epsilon}_{1k} \rightharpoonup C_{1k} \quad \text{weakly in } L^2(J; H^1_{per}(Y_m)) \text{ and a.e. in } Y_m \times J,$$

for some limit functions $p_k \in L^2(J; H^1_{per}(Y_m))$ and $C_k \in L^2(J; H^1_{per}(Y_m))$. We can then easily pass to the limit $\epsilon \to 0$ in the pressure equation (4.16) satisfied by $\widetilde{p}^{\epsilon}_k$. We get

$$\phi(y)\,\partial_t p_k - \operatorname*{div}_y((1/\mu(C_{1k}))k(y)\nabla_y p_k) = q \quad \text{in } Y_m \times J. \tag{4.20}$$

This equation is satisfied for each $k \in \mathbb{Z}^2$. It reminds to make the link with the limit (p, C_1) of $(\widetilde{p}^{\epsilon}, \widetilde{C}^{\epsilon}_1)$ by density arguments. On the first hand, for each $k \in \mathbb{Z}^2$, there exists $\epsilon(k) > 0$ such that $\epsilon(Y_m+k)\cap\Omega \neq \emptyset$ and then $\widetilde{p}^{\epsilon}_k(y,t) = \widetilde{p}^{\epsilon}(x,y,t)_{/x\in\epsilon(Y_m+k)}$ for any $\epsilon < \epsilon(k)$. Since $\lim_{\epsilon\to0}|\epsilon(Y_m+k)| = 0$, the subset $\epsilon(Y_m+k)\cap\Omega$ tends when $\epsilon \to 0$ to a point $\{x_k\} \subset \Omega$. Then $p_k(y,t) = p(x_k,y,t)$. In the same way, we get $C_{1k}(y,t) = C_1(x_k,y,t)$. On the other hand, $\lim_{\epsilon\to0}|\Omega\setminus((\cup_{k\in\mathbb{Z}^2}\epsilon(Y_m+k))\cap\Omega)| = 0$. Thus the set $(\cup_{k\in\mathbb{Z}^2}\{x_k\})$ is dense in Ω. Since the functions $(p(x_k,y,t), C_1(x_k,y,t))$ satisfy (4.20) for any $k \in \mathbb{Z}^2$, we conclude by density that p is a solution of (4.9).

To end this subsection, we add the following result of strong convergence.

Lemma 4.5. *The sequence $\mu(C^{\epsilon}_1)^{-1/2}\epsilon\nabla p^{\epsilon}$ satisfies*

$$\lim_{\epsilon\to0}\|\mu(C^{\epsilon}_1)^{-1/2}\epsilon\nabla p^{\epsilon}\|_{(L^2(\Omega^{\epsilon}_m\times J))^2} = \|\mu(C_1)^{-1/2}\nabla_y p\|_{(L^2(\Omega\times J\times Y_m))^2}.$$

Using the terminology of [2], $\mu(C^{\epsilon}_1)^{-1/2}\epsilon\nabla p^{\epsilon}$ is said strongly two-scale converging to $\mu(C_1)^{-1/2}\nabla_y p$. For any bounded sequence v^{ϵ} of $(L^2(\Omega\times J))^2$ such that $v^{\epsilon} \overset{2}{\rightharpoonup} v$ and any test admissible function ψ, we can assert that

$$\begin{aligned}
&\lim_{\epsilon\to0}\int_{\Omega\times J}(\mu(C^{\epsilon}_1)^{-1/2}\epsilon\nabla p^{\epsilon}\cdot v^{\epsilon})\,\psi(x,x/\epsilon,t)\,dx\,dt\\
&= \int_{\Omega\times J}\int_{Y_m}(\mu(C_1)^{-1/2}\nabla_y p\cdot v)\,\psi(x,y,t)\,dx\,dy\,dt.
\end{aligned}$$

Proof. We multiply (4.16) by $\widetilde{p}^{\epsilon}$ and we integrate over $\Omega\times(0,t)\times Y_m = \Omega_t\times Y_m$, for $t \in J$. Letting $\epsilon \to 0$ and using (4.9), we write

$$\begin{aligned}
&\lim_{\epsilon\to0}\Big(\frac{d}{2dt}\int_{\Omega\times Y_m}\phi(y)\,|\widetilde{p}^{\epsilon}|^2 - \frac12\int_{\Omega\times Y_m}\phi(y)\,|\widetilde{p}^{o}|^2 + \int_{\Omega_t\times Y_m}\frac{k(y)}{\mu(\widetilde{C}^{\epsilon}_1)}\nabla_y\widetilde{p}^{\epsilon}\cdot\nabla_y\widetilde{p}^{\epsilon}\Big)\\
&= \lim_{\epsilon\to0}\int_{\Omega_t\times Y_m}\widetilde{q}\,\widetilde{p}^{\epsilon} = \int_{\Omega_t\times Y_m} q\,p\\
&= \frac{d}{2dt}\int_{\Omega\times Y_m}\phi(y)\,|p|^2 - \frac12\int_{\Omega\times Y_m}\phi(y)\,|\widetilde{p}^{o}|^2 + \int_{\Omega_t\times Y_m}\frac{k(y)}{\mu(C_1)}\nabla_y p\cdot\nabla_y p.
\end{aligned}$$

Bearing in mind that k is a symmetric definite positive tensor, we conclude in particular from the letter relation that

$$\lim_{\epsilon\to0}\Big\|\frac{1}{\mu(\widetilde{C}^{\epsilon}_1)^{1/2}}\nabla_y\widetilde{p}^{\epsilon}\Big\|_{(L^2(\Omega\times J\times Y_m))^2} = \Big\|\frac{1}{\mu(C_1)^{1/2}}\nabla_y p\Big\|_{(L^2(\Omega\times J\times Y_m))^2}$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\Big\|\frac{1}{\mu(\widetilde{C}_1^\epsilon)^{1/2}}\epsilon\widetilde{\nabla p^\epsilon}\Big\|_{(L^2(\Omega\times J\times Y_m))^2} = \lim_{\epsilon\to 0}\Big\|\widetilde{\frac{1}{\mu(C_1^\epsilon)^{1/2}}\epsilon\nabla p^\epsilon}\Big\|_{(L^2(\Omega\times J\times Y_m))^2} \\
&= \lim_{\epsilon\to 0}\Big\|\frac{1}{\mu(C_1^\epsilon)^{1/2}}\epsilon\nabla p^\epsilon\Big\|_{(L^2(\Omega_m^\epsilon\times J))^2}.
\end{aligned}
$$

Lemma 4.5 is proved. □

4.4. **Corrector for the flux function and concentration problem.** Now we study the behavior of the concentration problem (2.15)-(2.17) as ϵ tends to zero. We recall it is

$$
\begin{gathered}
\Phi^\epsilon\,\partial_t\xi^\epsilon - (1/\mu(\xi^\epsilon))K^\epsilon\nabla\theta^\epsilon\cdot\nabla\xi^\epsilon - \operatorname{div}(\mathcal{D}^\epsilon\nabla\xi^\epsilon) + q\,\xi^\epsilon = q\hat{f}_1, \\
\mathcal{D}^\epsilon\nabla\xi^\epsilon\cdot\nu = 0, \quad \xi^\epsilon(x,0) = f_1^o(x).
\end{gathered}
$$

The starting point of the asymptotic study is similar to the one used for the pressure problem in Subsection 4.1. We multiply the latter equation by a test function $\Psi(x,t) + \epsilon\Psi_1(x,x/\epsilon,t) + \psi(x,x/\epsilon,t)$ and we integrate over $\Omega\times J$. Letting $\epsilon\to 0$, we obtain the limit variational formulation corresponding to (2.15)-(2.17). The difficulty is due to the nonlinearities involving the Darcy velocity in the convective terms and in the dispersive ones.

We begin by deriving the matrix equation (4.10). Indeed we have obtained sufficient compactness results in Subsection 4.2 to pass to the limit via the dilated equation (4.17). On the one hand, using the a.e. convergence of $\widetilde{C}_{1k}^\epsilon$ to C_{1k}, we get by density

$$
\begin{aligned}
&\phi(y)\,\partial_t C_1 - \mu(C_1)^{-1/2}k(y)\big(\lim_{\epsilon\to 0}\mu(\widetilde{C}_1^\epsilon)^{-1/2}\nabla_y\widetilde{p}^\epsilon\big)\cdot\nabla_y C_1 \\
&- \operatorname*{div}_y\big(\mu(C_1)^{-1/2}\mathcal{D}\big(\lim_{\epsilon\to 0}\mu(\widetilde{C}_1^\epsilon)^{-1/2}\nabla_y\widetilde{p}^\epsilon\big)\nabla_y C_1\big) \\
&= q\,(\hat{f}_1 - C_1).
\end{aligned}
$$

On the other hand, we note that Lemma 4.5 gives the strong convergence of $\widetilde{\mu(C_1^\epsilon)^{-1/2}\epsilon\nabla p^\epsilon} = \mu(\widetilde{C}_1^\epsilon)^{-1/2}\nabla_y\widetilde{p}^\epsilon$ to $\mu(C_1)^{-1/2}\nabla_y p$ in $(L^2(\Omega\times J\times Y_m))^2$. It is sufficient to explicit all the limits in the former equation. We obtain the limit equation (4.10).

Passing to the limit in the fractured part is less obvious. We are going to derive a corrector for the Darcy velocity in the fracture. Our aim is to find a function $\underline{\mathcal{V}}_o(x,y,t)\in(L^2(\Omega\times J\times Y))^2$ such that $\underline{\mathcal{V}}_o(x,x/\epsilon,t)$ is an admissible test function for the two-scale convergence and such that

$$
\lim_{\epsilon\to 0}\|\chi_f^\epsilon\big(-(K^\epsilon/\mu(\xi^\epsilon))\nabla\theta^\epsilon - \underline{\mathcal{V}}_o(x,x/\epsilon,t)\big)\|_{(L^2(\Omega\times J))^2} = 0.
$$

Then we will pass to the limit in $\chi_f^\epsilon\underline{\mathcal{V}}_o(x,x/\epsilon,t)\cdot\nabla\xi^\epsilon$ instead of $\chi_f^\epsilon(K^\epsilon/\mu(\xi^\epsilon))\nabla\theta^\epsilon\cdot\nabla\xi^\epsilon$; respectively we will pass to the limit in $\chi_f^\epsilon\mathcal{D}(\underline{\mathcal{V}}_o(x,x/\epsilon,t))\nabla\xi^\epsilon$ instead of $\chi_f^\epsilon\mathcal{D}(K^\epsilon/\mu(\xi^\epsilon))\nabla\theta^\epsilon)\nabla\xi^\epsilon$. We begin with the following result.

Lemma 4.6. *The following convergence hold.*

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\Omega\times J}\frac{K^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\,dx\,dt \\
&= \int_{\Omega\times J}\int_{Y_f}\frac{k_f(y)}{\mu(F_1)}(\nabla P_f + \nabla_y P_f^1)\cdot(\nabla P_f
\end{aligned}
$$

$$+ \nabla_y P_f^1)\, dx\, dy\, dt + \int_{\Omega\times J}\int_{Y_m} \frac{k(y)}{\mu(C_1)} \nabla_y p \cdot \nabla_y p\, dx\, dy\, dt.$$

Proof. Let us consider the following energy equation for $t \in J$. We denote Ω_t the set $\Omega \times (0,t)$.

$$\frac{1}{2}\int_\Omega \Phi^\epsilon(\theta^\epsilon(x,t)^2 - p^o(x)^2)dx + \int_{\Omega_t} \frac{K^\epsilon}{\mu(\xi^\epsilon)} \nabla\theta^\epsilon \cdot \nabla\theta^\epsilon\, dx\, ds = \int_{\Omega_t} q\,\theta^\epsilon\, dx\, ds.$$

In view of the two-scale convergence of θ^ϵ we have

$$\lim_{\epsilon\to 0}\int_{\Omega_t} q\,\theta^\epsilon\ dx\, ds = \int_{\Omega_t}\int_Y q\,(P_f(x,t) + \chi_m(y)p(x,y,t))\, dx\, dy\ ds,$$

and so, in view of the variational formulation obtained in the former subsection,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\Big(\frac{1}{2}\int_\Omega \Phi^\epsilon\theta^\epsilon(x,t)^2\, dx + \int_{\Omega_t} \frac{K^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\ dx\, ds\Big)\\
&= \frac{1}{2}\int_\Omega\int_Y \Phi(P_f + \chi_m p)^2(x,y,t)\, dx\, dy + \int_{\Omega_t}\int_{Y_f} \frac{k_f(y)}{\mu(F_1)}(\nabla P_f + \nabla_y P_f^1)\cdot(\nabla P_f\\
&\quad + \nabla_y P_f^1)\, dx\, dyds + \int_{\Omega_t}\int_{Y_m} \frac{k(y)}{\mu(C_1)}\ \nabla_y p\cdot\nabla_y p\, dx\, dyds,
\end{aligned}$$

where we recall that $\Phi(y) = \chi_f(y)\phi_f(y) + \chi_m(y)\phi(y)$. The limit of each term in the left-hand side of the last relation is larger than the corresponding two-scale limit in the right-hand side. Thus equality holds for each contribution and Lemma 4.6 is proved. □

Now, comparing the results of Lemmas 4.5 and 4.6, we conclude that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\Omega\times J} \chi_f^\epsilon \frac{K^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\cdot\nabla\theta^\epsilon\, dx\, dt\\
&= \int_{\Omega\times J}\int_{Y_f} \frac{k_f(y)}{\mu(F_1)}(\nabla P_f + \nabla_y P_f^1)\cdot(\nabla P_f + \nabla_y P_f^1)\, dx\, dy\, dt.
\end{aligned}$$

We recall that K^ϵ is a symmetric definite positive tensor and that it is considered as an admissible test function for the two-scale convergence. Furthermore we have showed in Lemma 4.3 that $\chi_f^\epsilon(\Pi^\epsilon f_1^\epsilon)$ converges almost everywhere in $\Omega\times J$ to F_1. The latter relation then leads to

$$\lim_{\epsilon\to 0}\Big\|\chi_f^\epsilon \frac{K^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon\Big\|_{(L^2(\Omega\times J))^2} = \Big\|\chi_f(y)\frac{k_f(y)}{\mu(F_1)}(\nabla P_f + \nabla_y P_f^1)\Big\|_{(L^2(\Omega\times J\times Y))^2}.$$

This relation is sufficient to assert the following result.

Proposition 4.7. *Let us define*

$$\underline{\mathcal{V}}_o(x,y,t) = -\chi_f(y)\frac{k_f(y)}{\mu(F_1(x,t))}(\nabla P_f(x,t) + \nabla_y P_f^1(x,y,t)).$$

Assume that $\underline{\mathcal{V}}_o$ is an admissible test function for the two-scale convergence, that is $\lim_{\epsilon\to 0}\int_{\Omega\times J}|\underline{\mathcal{V}}_0(x,x/\epsilon,t)|^2dx\, dt \le \int_{\Omega\times J}\int_Y |\underline{\mathcal{V}}_0(x,y,t)|^2 dxdy\, dt$. The function $\underline{\mathcal{V}}_0(x,x/\epsilon,t)$ is a corrector for the flux function in the following sense.

$$\lim_{\epsilon\to 0}\Big\|-\chi_f^\epsilon\frac{K^\epsilon}{\mu(\xi^\epsilon)}\nabla\theta^\epsilon - \underline{\mathcal{V}}_0(x,\frac{x}{\epsilon},t)\Big\|_{(L^2(\Omega\times J))^2} = 0.$$

We then can substitute $\underline{\mathcal{V}}_0(x,x/\epsilon,t)$ to $\chi_f^\epsilon(K^\epsilon/\mu(\xi^\epsilon))\nabla\theta^\epsilon$ to pass to the limit in the different equations. We get (4.3). This completes the proof of Theorem 4.1.

5. The macroscopic model: the case of a partially fractured medium ($\alpha > 0$)

In the present section, we derive rigorously the homogenized model corresponding to the partially fractured model (2.15)-(2.21), (2.23)-(2.32) when $\alpha > 0$. The limit model is described in Theorem 5.5 at the end of the section.

We begin by recalling the estimates derived in Section 3.

$$\begin{gathered}
\|\chi_f^\epsilon p_f^\epsilon\|_{L^\infty(J;L^2(\Omega))} + \|\chi_f^\epsilon \nabla p_f^\epsilon\|_{(L^2(\Omega\times J))^2} \le C,\\
\|\chi_m^\epsilon p^\epsilon\|_{L^\infty(J;L^2(\Omega))} + \|\chi_m^\epsilon \alpha(c_1^\epsilon + c_2^\epsilon)\nabla p^\epsilon\|_{(L^2(\Omega\times J))^2} \le C,\\
\|\chi_f^\epsilon f_1^\epsilon\|_{L^\infty(\Omega\times J)} + \|\chi_f^\epsilon (1+|\nabla p_f^\epsilon|)\nabla f_1^\epsilon\|_{(L^2(\Omega\times J))^2} \le C,\\
\|\chi_m^\epsilon C_1^\epsilon\|_{L^\infty(J;L^2(\Omega))} + \|\chi_m^\epsilon (1+|\underline{\mathcal{V}}_s^\epsilon|)\nabla C_1^\epsilon\|_{(L^2(\Omega\times J))^2} \le C,\\
\|\chi_m^\epsilon c_i^\epsilon\|_{L^\infty(J;L^2(\Omega))} + \|\chi_m^\epsilon (1+|\underline{\mathcal{V}}_s^\epsilon|)\nabla c_i^\epsilon\|_{(L^2(\Omega\times J))^2} \le C,\ i=1,2.
\end{gathered}$$

For the rest of this article, we assume that the pressure equation (2.20) is not degenerate, that is the constant C_α defined in (2.34) satisfies

$$C_\alpha > 0.$$

The more technical degenerate case when $C_\alpha = 0$ is postponed to a forthcoming paper. In view of (2.39) in Theorem 2.1, we now have $0 < C_\alpha \le c_1^\epsilon(x,t) + c_2^\epsilon(x,t) \le 1$ a.e. in $\Omega \times J$ and $\|\chi_m^\epsilon \nabla p^\epsilon\|_{(L^2(\Omega\times J))^2} \le C$. Thus there exist functions $p \in L^\infty(J;L^2(\Omega)) \cap L^2(J;H^1(\Omega))$, $p^1 \in L^2(\Omega\times J; H^1_{per}(Y))$, $(f_1, C_1, c_1, c_2) \in (L^\infty(J;L^2(\Omega)) \cap L^2(J;H^1(\Omega)))^4$ and $(f_1^1, C_1^1, c_1^1, c_2^1) \in (L^2(\Omega\times J; H^1_{per}(Y)))^4$ such that, up to extracted subsequences, as $\varepsilon \to 0$,

$$\begin{gathered}
\theta^\epsilon = \chi_f^\epsilon p_f^\epsilon + \chi_m^\epsilon p^\epsilon \overset{2}{\rightharpoonup} p, \quad \nabla\theta^\epsilon \overset{2}{\rightharpoonup} \nabla p(x,t) + \nabla_y p^1(x,y,t)),\\
\chi_f^\epsilon f_1^\epsilon \overset{2}{\rightharpoonup} \chi_f(y) f_1, \quad \chi_f^\epsilon \nabla f_1^\epsilon \overset{2}{\rightharpoonup} \chi_f(y)(\nabla f_1(x,t) + \nabla_y f_1^1(x,y,t)),\\
\chi_m^\epsilon C_1^\epsilon \overset{2}{\rightharpoonup} \chi_m(y) C_1, \quad \chi_m^\epsilon \nabla C_1^\epsilon \overset{2}{\rightharpoonup} \chi_m(y)(\nabla C_1(x,t) + \nabla_y C_1^1(x,y,t)),\\
\chi_m^\epsilon c_i^\epsilon \overset{2}{\rightharpoonup} \chi_m(y) c_i, \quad \chi_m^\epsilon \nabla c_i^\epsilon \overset{2}{\rightharpoonup} \chi_m(y)(\nabla c_i(x,t) + \nabla_y c_i^1(x,y,t)),\ i=1,2.
\end{gathered}$$

We begin with the following result linking the limit concentrations f_1 and $m_1 = \alpha c_1 + \beta C_1$.

Lemma 5.1. *The concentrations $f_1(x,t)$ and $m_1(x,t) = \alpha c_1(x,t) + \beta C_1(x,t)$ are equal almost everywhere in $\Omega \times J$.*

Proof. Let $c^\epsilon = \chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon \in L^2(J;H^1(\Omega))$. It satisfies $\gamma_f^\epsilon c^\epsilon = \gamma_f^\epsilon f_1^\epsilon = \gamma_m^\epsilon m_1^\epsilon = \gamma_m^\epsilon c^\epsilon$ and $\epsilon\nabla c^\epsilon = \epsilon\chi_f^\epsilon \gamma f_1^\epsilon + \epsilon\chi_m^\epsilon m_1^\epsilon \in (L^2(\Omega\times J))^2$. We know that $c^\epsilon \overset{2}{\rightharpoonup} \chi_f(y) f_1 + \chi_m(y) m_1$ and $\epsilon\nabla c_\epsilon \overset{2}{\rightharpoonup} 0$. For any $\underline{\Psi} \in (\mathcal{C}_o^\infty(\Omega; \mathcal{C}_{per}^\infty(Y)))^2$ we write

$$\int_\Omega \epsilon\nabla c^\epsilon \cdot \underline{\Psi}(x, \frac{x}{\epsilon})\, dx = -\int_\Omega c^\epsilon \Big(\epsilon \operatorname*{div}_x \underline{\Psi}(x,\frac{x}{\epsilon}) + \operatorname*{div}_y \underline{\Psi}(x,\frac{x}{\epsilon})\Big)\, dx.$$

We take the two-scale limits on both sides. We get

$$\begin{aligned}
0 &= -\int_\Omega \int_Y (\chi_f(y) f_1(x,t) + \chi_m(y) m_1(x,t)) \operatorname*{div}_y \underline{\Psi}(x,y)\, dx\, dy\\
&= -\int_\Omega \int_{\partial Y_f} f_1(x,t)\underline{\Psi}(x,s)\cdot \nu_f\, dx\, ds - \int_\Omega \int_{\partial Y_m} m_1(x,t)\underline{\Psi}(x,s)\cdot \nu_m\, dx\, ds.
\end{aligned}$$

This proves that $f_1(x,t) = m_1(x,t)$ for $s \in \partial Y_f \cap \partial Y_m = \Gamma_{fm}$ and thus $f_1(x,t) = m_1(x,t)$ a.e. in $\Omega \times J$. $\square$

We then claim and prove the following compactness result.

Lemma 5.2. *The sequences $(\chi_f^\epsilon f_1^\epsilon)$, $(\chi_m^\epsilon m_1^\epsilon)$ and $(\chi_m^\epsilon(c_1^\epsilon + c_2^\epsilon))$ are sequentially compact in $L^2(\Omega \times J)$.*

Proof. On the one hand, let $\psi \in L^2(J; H^2(\Omega))$. We multiply (2.15) by $(\alpha^2+\beta^2)\chi_f^\epsilon\psi$ and (2.40) by $\chi_m^\epsilon\psi$. We integrate over $\Omega \times J$ and sum up the results. Using the same type of arguments than in the proof of Lemma 4.3, we conclude that the sequence $\partial_t(\phi_f^\epsilon(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon + \phi^\epsilon\chi_m^\epsilon m_1^\epsilon)$ is uniformly bounded in $L^2(J; (H^2(\Omega))')$. Since $(\phi_f^\epsilon(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon + \phi^\epsilon\chi_m^\epsilon m_1^\epsilon)$ is uniformly bounded in $L^\infty(\Omega\times J)$, a standard argument of Aubin's type proves that $(\phi_f^\epsilon(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon + \phi^\epsilon\chi_m^\epsilon m_1^\epsilon)$ lies in a compact subset of $L^2(J; (H^1(\Omega))')$. Therefore, there is $\xi \in L^2(J; (H^1(\Omega))')$, such that, up to an extracted subsequence,

$$\phi_f^\epsilon(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon + \phi^\epsilon\chi_m^\epsilon m_1^\epsilon \to \xi \quad \text{in } L^2(J; (H^1(\Omega))') \text{ as } \epsilon \to 0.$$

Two-scale convergence arguments show that

$$\xi = (\alpha^2+\beta^2)(\int_{Y_f} \phi_f(y)dy) f_1 + (\int_{Y_m} \phi(y)dy) m_1,$$

where $m_1 = \alpha c_1 + \beta C_1 = f_1$ by Lemma 5.1.

On the other hand, the sequence $(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon)$ is uniformly bounded in space $L^2(J; H^1(\Omega))$. We thus can pass to the limit in the product $\langle \phi_f(\alpha^2+\beta^2)\chi_f^\epsilon f_1^\epsilon + \phi^\epsilon\chi_m^\epsilon m_1^\epsilon, \chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon\rangle_{(H^1(\Omega))'\times H^1(\Omega)}$ as follows.

$$\begin{aligned}
&\lim_{\epsilon\to 0}\Big(\big\langle \chi_f^\epsilon(\alpha^2+\beta^2)\phi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon\phi^\epsilon m_1^\epsilon, \chi_f^\epsilon f_1^\epsilon\big\rangle + \big\langle \chi_f^\epsilon(\alpha^2+\beta^2)\phi_f^\epsilon f_1^\epsilon \\
&+ \chi_m^\epsilon\phi^\epsilon m_1^\epsilon, \chi_m^\epsilon m_1^\epsilon\big\rangle\Big) = \big\langle\big((\alpha^2+\beta^2)\int_{Y_f}\phi_f(y)dy + \int_{Y_m}\phi(y)dy\big)f_1, |Y_f| f_1\big\rangle \\
&+ \big\langle\big((\alpha^2+\beta^2)\int_{Y_f}\phi_f(y)dy + \int_{Y_m}\phi(y)dy\big)f_1, |Y_m| m_1\big\rangle \\
&= \big\langle\big((\alpha^2+\beta^2)\int_{Y_f}\phi_f(y)dy + \int_{Y_m}\phi(y)dy\big)f_1, f_1\big\rangle.
\end{aligned}$$

As a consequence we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\big\langle\big((\alpha^2+\beta^2)\chi_f^\epsilon\phi_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\big)(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon - f_1), \chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon - f_1\big\rangle \\
&= \lim_{\epsilon\to 0}\Big(\big\langle\big((\alpha^2+\beta^2)\chi_f^\epsilon\phi_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\big)(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon), \chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon\big\rangle \\
&\quad - 2\big\langle\big((\alpha^2+\beta^2)\chi_f^\epsilon\phi_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\big)(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon), f_1\big\rangle \\
&\quad + \big\langle\big((\alpha^2+\beta^2)\chi_f^\epsilon\phi_f^\epsilon + \chi_m^\epsilon\phi^\epsilon\big)f_1, f_1\big\rangle\Big) = 0.
\end{aligned}$$

Since $\alpha^2+\beta^2 > 0$, $\phi_f, \phi^\epsilon \geq \phi_- > 0$, this shows that $(\chi_f^\epsilon f_1^\epsilon + \chi_m^\epsilon m_1^\epsilon)$ strongly converges to f_1 in $L^2(\Omega\times J)$. A similar calculation using (2.45) gives the result for $\chi_m^\epsilon(c_1^\epsilon + c_2^\epsilon)$. The proof of the Lemma is complete. $\square$

We now have the tools to pass to the limit in the pressure equation. The structure of the problem satisfied by θ^ϵ is similar with the one of the pressure problem in the fractured part in the totally fractured setting (Section 4). We thus do not detail the details of the convergence. We claim that the limit homogenized pressure problem in $\Omega \times J$ is the following.

$$
\begin{gathered}
\overline{\phi}\,\partial_t p - \operatorname{div}\Big(\int_Y \Big(\chi_f(y)\frac{k_f(y)}{\mu(f_1)} + \chi_m(y)\alpha(c_1+c_2)\frac{k(y)}{\mu(m_1)}\Big)(\nabla p + \nabla_y p^1)\,dy\Big) = q,\\
-\operatorname{div}_y\Big(\Big(\chi_f(y)\frac{k_f(y)}{\mu(f_1)} + \chi_m(y)\alpha(c_1+c_2)\frac{k(y)}{\mu(m_1)}\Big)(\nabla p + \nabla_y p^1)\Big) = 0 \quad \text{in } Y,\\
\Big(\chi_f(y)\frac{k_f(y)}{\mu(f_1)} + \chi_m(y)\alpha(c_1+c_2)\frac{k(y)}{\mu(m_1)}\Big)(\nabla p + \nabla_y p^1)\cdot\nu = 0 \quad \text{on } \Gamma\times J,
\end{gathered}
$$

where $\overline{\phi}$ is defined by

$$
\overline{\phi} = \int_{Y_f}\phi_f(y)\,dy + \int_{Y_m}\phi(y)\,dy = \overline{\phi_f}^{Y_f} + \overline{\phi}^{Y_m}. \tag{5.1}
$$

Let $v^j(x,y,t)$ be the Y-periodic solution of the cell-problem

$$
\begin{gathered}
-\operatorname{div}_y\big((\chi_f(y)k_f(y) + \chi_m(y)\alpha(c_1+c_2)k(y))(\nabla_y v^j + e^j)\big) = 0 \quad \text{in } Y,\\
\int_Y v^j\,dy = 0.
\end{gathered} \tag{5.2}
$$

Defining an homogenized tensor of permeability $\overline{K} = (\overline{K}_{ij})_{1\le i,j\le 2}$ in Ω by

$$
\begin{aligned}
\overline{K}_{ij}(x,t) = \int_Y &\big(\chi_f(y)k_f(y) + \chi_m(y)\alpha(c_1+c_2)k(y)\big)\\
&\times(\nabla_y v^i(x,y,t) + e^i)(\nabla_y v^j(x,y,t) + e^j)\,dy,
\end{aligned} \tag{5.3}
$$

and setting

$$
p^1(x,t,y) = \sum_{j=1}^{2} v^j(x,y,t)\,\partial_j p(x,t),
$$

we claim the following result.

Proposition 5.3. *The homogenized pressure problem is*

$$
\overline{\phi}\,\partial_t p + \operatorname{div}\underline{v} = q, \quad \underline{v} = -\frac{1}{\mu(f_1)}\overline{K}\nabla p \quad \text{in } \Omega\times J, \tag{5.4}
$$

$$
\underline{v}\cdot\nu\big|_{\Gamma\times J} = 0, \quad p\big|_{t=0} = p^o, \tag{5.5}
$$

where $\overline{\phi}$ (respectively $\overline{K}$) is the homogenized porosity (respectively permeability tensor) given in (5.1) *(respectively in* (5.3)*).*

As in Section 4, we have to introduce a corrector for the Darcy velocity in order to pass to the limit in the concentration equation. We denote

$$
\begin{gathered}
\underline{v}_o(x,t,y) = -(\chi_f(y)\frac{k_f(y)}{\mu(f_1)} + \chi_m(y)\alpha(c_1+c_2)\frac{k(y)}{\mu(f_1)})(\nabla p + \nabla_y p^1),\\
\underline{v}_o^\varepsilon(x,t) = \underline{v}_o(x,t,x/\varepsilon), \quad (x,t)\in\Omega\times J.
\end{gathered}
$$

Following the lines of Subsection 4.4, we can prove the following corrector result.

Lemma 5.4. *We have, for a subsequence,*

$$
\int_{\Omega\times J} |\chi_f^\epsilon \underline{v}_f^\epsilon + \chi_m^\epsilon \underline{\mathcal{V}}^\epsilon - \underline{v}_o^\varepsilon|^2\,dx\,dt \to 0 \quad \text{as } \varepsilon\to 0.
$$

Let us now turn to the concentration problem. Once again, the technical difficulties are not greater than the ones of Subsection 4.4 in the fractured part. We thus only give the outline of the convergence study. At the limit, we need to characterize the behavior of two global concentrations, $f_1 = m_1$ and $(c_1 + c_2)$.

We begin with f_1. We multiply $(\alpha^2 + \beta^2)(2.15) + (2.40)$ by a test function $\psi(x,t) + \epsilon\psi_1(x, x/\epsilon, t)$, with $\psi \in \mathcal{D}(\Omega \times J)$, $\psi_1 \in \mathcal{D}(\Omega \times J; \mathcal{C}^\infty_{per}(Y))$. The terms on Γ_{fm} are equal to zero when we integrate by parts. Noting that $\chi^\epsilon_f(\alpha^2+\beta^2)k^\epsilon_f \nabla p^\epsilon_f + \chi^\epsilon_m k^\epsilon \nabla p^\epsilon = (\alpha^2+\beta^2)(\chi^\epsilon_f k^\epsilon_f \nabla p^\epsilon_f + \chi^\epsilon_m k^\epsilon \nabla p^\epsilon) + (1-\alpha^2-\beta^2)\chi^\epsilon_m k^\epsilon \nabla p^\epsilon$ and passing to the limit, we get the following homogenized problem in $\Omega \times J$.

$$
\begin{aligned}
&\big((\alpha^2+\beta^2)\overline{\phi_f}^{Y_f} + \overline{\phi}^{Y_m}\big)\,\partial_t f_1 - \frac{(\alpha^2+\beta^2)}{\mu(f_1)}\overline{K}\nabla p\cdot\nabla f_1\\
&+ (1-\alpha^2-\beta^2)\int_{Y_m}\chi_m(y)\underline{v}_o(x,t,y)\cdot(\nabla f_1 + \nabla_y f_1^1)\,dy - \operatorname{div}(\overline{\mathcal{D}}(\nabla p,\mu(f_1))\nabla f_1)\\
&= \big((\alpha^2+\beta^2)|Y_f| + |Y_m|\big)q\,(\hat{f}_1 - f_1),
\end{aligned}
\tag{5.6}
$$

$$
\overline{\mathcal{D}}(\nabla p,\mu(f_1))\nabla f_1\cdot\nu\big|_{\Gamma\times J} = 0,\quad f_1\big|_{t=0} = f_1^o. \tag{5.7}
$$

The function f_1^1 and the tensor of diffusion $\overline{\mathcal{D}} = (\overline{\mathcal{D}}_{ij})_{1\le i,j\le 2}$ are defined by

$$
f_1^1(x,t,y) = \sum_{j=1}^2 w^j(x,t,y)\,\partial_j f_1, \tag{5.8}
$$

$$
\begin{aligned}
&\overline{\mathcal{D}}_{ij}(\nabla p,\mu(f_1))\\
&= \int_Y \big((\alpha^2+\beta^2)\chi_f(y) + \chi_m(y)\big)\mathcal{D}(\underline{v}_o)\,\big(\nabla_y w^i(x,y) + e^i\big)\cdot\big(\nabla_y w^j(x,y) + e^j\big)dy,
\end{aligned}
\tag{5.9}
$$

where $w^j(x,t,y)$ is the Y-periodic solution of the following cell-problem for $(x,t) \in \Omega\times J$:

$$
\begin{gathered}
-\operatorname*{div}_y\Big(\big((\alpha^2+\beta^2)\chi_f(y) + \chi_m(y)\big)\mathcal{D}(\underline{v}_o)(\nabla_y w^j + e^j)\Big) = 0 \quad \text{in } Y,\\
\int_Y w^j(x,t,y)\,dy = 0,\quad j = 1,2.
\end{gathered}
\tag{5.10}
$$

For the asymptotic study of $(c_1^\epsilon + c_2^\epsilon)$ we exploit Problem (2.45)-(2.47). It leads to the following homogenized problem

$$
\begin{aligned}
&\overline{\phi}^{Y_m}\,\partial_t(c_1+c_2) + \int_{Y_m}\chi_m(y)\underline{v}_o(x,t,y)\cdot(\nabla(c_1+c_2) + \nabla_y(c_1^1+c_2^1))\,dy\\
&- \operatorname{div}(\overline{\mathcal{D}}_m(\nabla p,\mu(f_1))\nabla(c_1+c_2))\\
&= |Y_m|q\,(1-c_1-c_2) \quad \text{in } \Omega\times J,
\end{aligned}
\tag{5.11}
$$

$$
\overline{\mathcal{D}}_m\nabla(c_1+c_2)\cdot\nu\big|_{\Gamma\times J} = 0,\quad (c_1+c_2)\big|_{t=0} = c_1^o + c_2^o. \tag{5.12}
$$

The functions c_1^1, c_2^1 and the homogenized tensor of diffusion $\overline{D}_m = (\overline{D}_{m_{ij}})_{1\le i,j\le 2}$ are defined by

$$
(c_1^1 + c_2^1)(x,t,y) = \sum_{j=1}^2 w_m^j(x,t,y)\,\partial_j(c_1+c_2), \tag{5.13}
$$

$$\overline{\mathcal{D}}_{m_{ij}}(\nabla p, \mu(f_1)) = \int_{Y_m} \mathcal{D}(\underline{v}_o)(\nabla_y w_m^i + e^i) \cdot (\nabla_y w_m^j + e^j)\, dy, \tag{5.14}$$

where $w_m^j(x,t,y)$ is the Y-periodic solution of the following cell-problem for $(x,t) \in \Omega \times J$:

$$\begin{gathered} -\operatorname*{div}_y(\mathcal{D}(\underline{v}_o)(x,t,y)(\nabla_y w_m^j(x,t,y) + e^j)) = 0 \quad \text{in } Y_m, \\ \mathcal{D}(\underline{v}_o)(\nabla_y w_m^j + e^j)) \cdot \nu_y = 0 \quad \text{on } \Gamma_{fm}, \quad j = 1,2. \end{gathered} \tag{5.15}$$

We summarize the results of the present section in the following theorem.

Theorem 5.5. *The homogenized partially fractured model in $\Omega \times J$ is:*

$$\begin{gathered} (\overline{\phi_f}^{Y_f} + \overline{\phi}^{Y_m})\, \partial_t p + \operatorname{div} \underline{v} = q, \quad \underline{v} = -\frac{1}{\mu(f_1)} \overline{K} \nabla p, \\ \underline{v} \cdot \nu\big|_{\Gamma \times J} = 0, \quad p\big|_{t=0} = p^o, \\ p^1(x,t,y) = \sum_{j=1}^{2} v^j(x,y,t)\, \partial_j p(x,t), \\ \underline{v}_o(x,t,y) = -(\chi_f(y) \frac{k_f(y)}{\mu(f_1)} + \chi_m(y) \alpha (c_1 + c_2) \frac{k(y)}{\mu(f_1)})(\nabla p + \nabla_y p^1), \end{gathered}$$

$$\begin{aligned} &\big((\alpha^2 + \beta^2) \overline{\phi_f}^{Y_f} + \overline{\phi}^{Y_m}\big)\, \partial_t f_1 + (\alpha^2 + \beta^2) \underline{v} \cdot \nabla f_1 \\ &+ (1 - \alpha^2 - \beta^2) \int_{Y_m} \chi_m(y) \underline{v}_o(x,t,y) \cdot (\nabla f_1 + \nabla_y f_1^1)\, dy - \operatorname{div}(\overline{\mathcal{D}}(\nabla p, \mu(f_1)) \nabla f_1) \\ &= \big((\alpha^2 + \beta^2)|Y_f| + |Y_m|\big) q\, (\hat{f}_1 - f_1), \end{aligned}$$

$$\begin{gathered} f_1^1(x,t,y) = \sum_{j=1}^{2} w^j(x,t,y)\, \partial_j f_1, \\ \overline{\mathcal{D}}(\nabla p, \mu(f_1)) \nabla f_1 \cdot \nu\big|_{\Gamma \times J} = 0, \quad f_1\big|_{t=0} = f_1^o, \end{gathered}$$

$$\begin{aligned} &\overline{\phi}^{Y_m}\, \partial_t (c_1 + c_2) + \int_{Y_m} \chi_m(y) \underline{v}_o(x,t,y) \cdot (\nabla(c_1 + c_2) + \nabla_y (c_1^1 + c_2^1))\, dy \\ &- \operatorname{div}(\overline{\mathcal{D}}_m(\nabla p, \mu(f_1)) \nabla(c_1 + c_2)) \\ &= |Y_m| q\, (1 - c_1 - c_2), \end{aligned}$$

$$\begin{gathered} (c_1^1 + c_2^1)(x,t,y) = \sum_{j=1}^{2} w_m^j(x,t,y)\, \partial_j (c_1 + c_2), \\ \overline{\mathcal{D}}_m \nabla(c_1 + c_2) \cdot \nu\big|_{\Gamma \times J} = 0, \quad (c_1 + c_2)\big|_{t=0} = c_1^o + c_2^o. \end{gathered}$$

The auxiliary functions v^j, w^j, w_m^j are defined by the cell problems (5.2), (5.10), (5.15). *The homogenized permeability is given in* (5.3)*, while the diffusion tensors $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}_m$ are given in* (5.9) *and* (5.14).

Remark 5.6. The latter model is consistent with the one corresponding to a non-fractured porous medium. Indeed, letting $\alpha \to 1$ and $|Y_m| \to 0$, one gets

$$\begin{gathered} \overline{\phi}^Y \partial_t p + \operatorname{div} \underline{v} = q, \quad \underline{v} = -\frac{1}{\mu(f_1)} \overline{K} \nabla p, \\ \underline{v} \cdot \nu\big|_{\Gamma \times J} = 0, \quad p\big|_{t=0} = p^o, \end{gathered}$$

$$p^1(x,t,y) = \sum_{j=1}^{2} v^j(x,y,t)\,\partial_j p(x,t), \quad \underline{v}_o(x,t,y) = -\frac{k(y)}{\mu(f_1)}(\nabla p + \nabla_y p^1),$$
$$\overline{\phi}^Y \partial_t f_1 + \underline{v}\cdot\nabla f_1 - \operatorname{div}(\overline{\mathcal{D}}(\nabla p,\mu(f_1))\nabla f_1) = q\,(\hat{f}_1 - f_1),$$
$$\overline{\mathcal{D}}(\nabla p,\mu(f_1))\nabla f_1\cdot\nu\big|_{\Gamma\times J} = 0, \quad f_1\big|_{t=0} = f_1^o.$$

Note that this model is the homogenized form corresponding to the following microscopic model in a non-fractured porous medium (see [9] for a rigorous derivation):

$$\phi^\epsilon\partial_t p^\epsilon + \operatorname{div}(\underline{v}^\epsilon) = q, \quad \underline{v}^\epsilon = -\frac{k^\epsilon}{\mu(f_1^\epsilon)}\nabla p^\epsilon \quad \text{in } \Omega\times J,$$
$$\phi^\epsilon\partial_t f_1^\epsilon + \underline{v}^\epsilon\cdot\nabla f_1^\epsilon - \operatorname{div}(\mathcal{D}(\underline{v}^\epsilon)\nabla f_1^\epsilon) = q(\hat{f}_1 - f_1^\epsilon) \quad \text{in } \Omega\times J,$$
$$\underline{v}^\epsilon\cdot\nu\big|_{\Gamma\times J} = 0, \quad \mathcal{D}(\underline{v}^\epsilon)\nabla f_1^\epsilon\cdot\nu\big|_{\Gamma\times J} = 0, \quad p_f^\epsilon\big|_{t=0} = p^o, \quad f_1^\epsilon\big|_{t=0} = f_1^o.$$

Remark 5.7. Let us add some "physical" comments on the homogenized model of miscible displacement in partially fractured media described in Theorem 5.5. Contrary to the one-component flow considered in [13], it seems that the double porosity characteristics (that is a system of equations in the homogenized fracture coupled with one in a microscopic matrix block) disappear as soon as a direct flow occurs in the matrix part, that is as soon as $\alpha > 0$. This corresponds to some experimental data ([3] and references therein). However, the model of Theorem 5.5 is also not a single porosity model as the one described in Remark 5.6. It really contains some matrix effects. We recall in particular that the smaller α is, the more important is the storage in the matrix. And in the model of Theorem 5.5, the homogenized permeability is concentration ($\alpha(c_1+c_2)$) dependent, and the convective effects in the homogenized fracture depend on α. One can compare this effects with some models where the permeability is concentration dependent: propagation in clays (see [16] and the references therein) or blood flow in micro vessels (see [20] and the references therein) for instance.

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