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SOLVABILITY OF SOME INTEGRAL EQUATIONS IN HILBERT SPACES

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Abstract. We consider an integral equation of Fredholm and Volterra type with spectral parameter depending on time. Conditions of solvability are established when the initial value of the parameter coincides with an eigenvalue of Fredholm operator.

1. Introduction

Let H be a Hilbert space. We consider the integral equation

$$\int_0^t K(t,s)u(s)ds + Au(t) - \lambda(t)u(t) = f(t), \quad t > 0, \tag{1.1}$$

where $u : \mathbb{R}_+ \to H$ is unknown, $A : H \to H$ is a linear bounded self-adjoint operator, $K : Q \to \mathbb{R}$ is the kernel, $f(t)$ is a given function, and $\lambda(t)$ is a function which we may interpret as spectral parameter. Here $\mathbb{R}_+ = \{t \in R : t \geq 0\}$ is the positive half-line, and

$$Q = \{(t,s) \in R^2 : 0 \leq s \leq t < \infty\}. \tag{1.2}$$

The following equation is a typical example of this integral equation:

$$\int_0^t K(t,s)P(x,s)ds - \int_0^a R(x,y)P(y,t)dy - \lambda(t)P(x,t) = f(x,t), \tag{1.3}$$

where $0 \leq x \leq a$ and $t > 0$. We consider in Hilbert space $H = L_2(0,a)$, where $a > 0$. Equations of this type arise in the theory of elasticity [2, 3]. The kernels $K(t,s)$ and $R(x,y)$ are connected with some elastic creeping base and $\lambda(t)$ is the given value which describes the elastic properties of deformable body. We may reference also to work [4], where the similar integral equation was considered. In case when $\lambda(t)$ is constant, (1.1) was considered in [1].

2. Results

We assume that $\lambda(t)$ is a continuous function for $t \geq 0$. We denote the range of $\lambda(s)$ on the interval $[0,t]$ by

$$\Lambda(t) = \{\lambda(s),\ 0 \leq s \leq t\}. \tag{2.1}$$

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It is clear that $\Lambda(t)$ is a closed subset of $\mathbb{R}$.

We consider function $u : \mathbb{R}_+ \to H$ in an abstract setting, i. e., $u(t) \in H$ for $t \geq 0$. Assuming that $u(t)$ is bounded on each interval $[0 \leq t \leq T]$, we denote

$$\|u\|_t = \sup_{0\leq s\leq t} \|u(s)\|.$$

We Assume also that the kernel $K(t, s)$ is continuous on the set Q, as defined by (1.2). Denote

$$M(t) = \sup_{0\leq s\leq \tau\leq t} |K(\tau, s)|. \tag{2.2}$$

It is clear that $M(t)$ is a monotonically increasing function.

We denote by $\sigma(A)$ the spectrum of a bounded self-adjoint operator A. When parameter λ is outside this spectrum, i.e. $\lambda \notin \sigma(A)$, there exists the resolvent $R_\lambda = (A - \lambda I)^{-1}$ of the operator A.

We consider the auxilary equation

$$\int_0^t K(t, s)R_{\lambda(s)}v(s)ds + v(t) = f(t), \quad t > 0. \tag{2.3}$$

Assuming that we have a solution $v(t)$ of this equation, by putting $u(t) = R_{\lambda(t)}v(t)$ or $v(t) = (A - \lambda(t)I)u(t)$, we obtain

$$\int_0^t K(t, s)u(s)ds + (A - \lambda(t)I)u(t) = f(t), \quad t > 0.$$

This implies $u(t)$ is a solution of the equation (1.1).

At first we assume that $\lambda(t)$ does not intersect the spectrum of A, i.e. $\Lambda(t) \cap \sigma(A) = \emptyset$. Then obviously the norms of resolvent $\|R_\lambda\|$ are bounded for every $\lambda \in \Lambda(t)$.

Denote

$$B(t) = \sup_{\lambda\in\Lambda(t)} \|R_\lambda\|. \tag{2.4}$$

The following simple result shows that outside the spectrum of the operator A there is the solvability of the equation (1.1) without any additional condition.

Theorem 2.1. *Let $\Lambda(t) \cap \sigma(A) = \emptyset$ for all $t \geq 0$. Then there exists the unique solution $u(t)$ of equation* (1.1) *and*

$$\|u(t)\| \leq \|f\|_t B(t) \exp\{tM(t)B(t)\}..$$

To prove this theorem it is sufficient to show that there exists the unique solution of the auxiliary equation (2.3). This fact is evident from the following lemma.

Lemma 2.2. *Let $v_0(t) = f(t)$ and*

$$v_k(t) = -\int_0^t K(t, s)R_{\lambda(s)}v_{k-1}(s)ds, \quad k = 1, 2, \dots \tag{2.5}$$

Then

$$\|v_k\|_t \leq \|f\|_t \frac{[M(t)B(t)]^k}{k!} t^k.$$

Proof. We shall use induction. The estimate is trivial for $k=0$. Assume that it is true for some k and then prove it for $k+1$. It follows from the equality $v_{k+1}(t) = -\int_0^t K(t,s)R_{\lambda(s)}v_k(s)ds$ that

$$\begin{aligned}\|v_{k+1}(t)\| &\leq \int_0^t M(t)\|R_{\lambda(s)}\|\,\|f\|_s \frac{[M(s)B(s)]^k}{k!}s^k ds \\ &\leq M(t)B(t)\|f\|_t \frac{[M(t)B(t)]^k}{k!}\int_0^t s^k ds \\ &= \|f\|_t \frac{[M(t)B(t)]^{k+1}}{(k+1)!}t^{k+1}.\end{aligned}$$

□

Using this lemma, it is easy to show that the function

$$v(t) = \sum_{k=0}^{\infty}(-1)^{k+1}v_k(t)$$

is the unique solution of the equation (2.3) and it proves Theorem 2.1.

The problem is more complicated when $\Lambda(t)$ has a common point with spectrum of A and that is the main idea of our consideration. Suppose that $\lambda(0)$ coincides with one of the isolated points of the spectrum of the operator $A = A^*$. We suppose also that $\lambda(t) \notin \sigma(A)$ for all $t > 0$. It will be proved below that in this case the following quantity is finite:

$$B_1(t) = \sup_{0<s\leq t} s\|R_{\lambda(s)}\| < \infty, \quad t > 0. \tag{2.6}$$

Remember that the set Q is defined by equality (1.2). We introduce the class of kernels, which vanish at the point $(0,0)$ with some order $\alpha > 0$.

We say that $K(t,s) \in M^{\alpha}(Q)$ if $K(t,s) \in C(Q)$, $K(0,0) = 0$ and

$$|K(t,s)| \leq K_{\alpha}t^{\alpha}, \quad 0 < s < t, \tag{2.7}$$

with some constant $K_{\alpha} > 0$.

Analogously we say that $f \in N^{\alpha}(\mathbb{R}_+)$ if $f : \mathbb{R}_+ \to H$ is continuous on the half-line $t \geq 0$, $f(0) = 0$ and

$$\|f\|_{(\alpha)} = \sup_{t>0} t^{-\alpha}\|f(t)\| < \infty. \tag{2.8}$$

Theorem 2.3. *Let $K(t,s) \in M^{\alpha}(Q)$, $0 < \alpha < 1$. Suppose that $\lambda(t)$ is continuously differentiable function on the half-line $t \geq 0$. Let $\lambda(0)$ be an isolated point of the spectrum $\sigma(A)$ and $\lambda(t) \notin \sigma(A)$ for all $t > 0$. If*

$$\lambda'(0) \neq 0\,, \tag{2.9}$$

then for an arbitrary function $f \in N^{\alpha}(\mathbb{R}_+)$ there exists a solution $u(t)$ of (1.1) such that

$$\|u(t)\| \leq \|f\|_{(\alpha)}t^{\alpha}\exp[C(t)t^{\alpha}] \tag{2.10}$$

where $C(t) = K_{\alpha}B_1(t)/\alpha$. The function $B_1(t)$ and constant K_{α} are defined by equalities (2.6) and (2.7) respectively.

As we will show in Example 1, condition (2.9) is essential. This condition means that spectral parameter $\lambda(t)$ has to move away from the point $\lambda(0)$ with non-zero velocity.

First we prove some lemmas.

Lemma 2.4. *Suppose that $\lambda(t)$ is a continuously differentiable function for $t \geq 0$. Let $\lambda(0) = \lambda_0$ be an isolated point of the spectrum $\sigma(A)$ and $\lambda(t) \notin \sigma(A)$ for all $t > 0$. If condition (2.9) holds for every $T > 0$, then*

$$|\lambda(t) - \lambda_0| \geq c(T)t, \quad 0 \leq t \leq T\,. \tag{2.11}$$

Proof. We divide the interval $[0, T]$ into two parts: $[0, \delta)$ and $[\delta, T]$.
(1) According to the conditions of this lemma, for every $\delta > 0$ we have $\lambda(t) \notin \sigma(A)$ if $\delta \leq t \leq T$. Hence $\lambda(t) - \lambda_0 \neq 0$ on this interval. Then because of continuity,

$$|\lambda(t) - \lambda_0| \geq c_1(T, \delta) \geq c_1(T, \delta)\frac{t}{T}, \quad \delta \leq t \leq T. \tag{2.12}$$

(2) Further, since $\lambda'(0) \neq 0$ then for small enough $\delta > 0$ we have $|\lambda'(t)| \geq c_2(\delta) > 0$, $0 \leq t < \delta$. Then using Lagrange formulae we get

$$|\lambda(t) - \lambda_0| = |\lambda(t) - \lambda(0)| = |\lambda'(\tau)|(t - 0) \geq c_2(\delta)t, \quad 0 \leq t \leq \delta. \tag{2.13}$$

The required estimate (2.11) obviously follows from (2.12) and (2.13) with $c(T) = \min\{\frac{1}{T}c_1(T, \delta), c_2(\delta)\}$. □

The following lemma proves the correctness of the definition (2.6).

Lemma 2.5. *Under the conditions of Lemma 2.4 for an arbitrary $T > 0$,*

$$t\|R_{\lambda(t)}\| \leq C(T) < \infty, \quad 0 < t \leq T\,. \tag{2.14}$$

Proof. We use the well known estimate (see, e.g. [5])

$$\|R_\lambda(A)\| \leq 1/\operatorname{dist}\big(\lambda, \sigma(A)\big) \tag{2.15}$$

which is fulfilled for any self-adjoint operator A. We denote the distance from the point λ to the spectrum $\sigma(A)$ of operator A by

$$\operatorname{dist}\big(\lambda, \sigma(A)\big) = \inf_{\mu \in \sigma(A)} |\lambda - \mu|.$$

It follows from estimate (2.15) that in order to get (2.14) it is enough to prove the inequality

$$\operatorname{dist}\big(\lambda, \sigma(A)\big) \geq Ct, \quad C > 0, \quad 0 < t \leq T. \tag{2.16}$$

Denote

$$d(t) = \operatorname{dist}\big(\lambda, \sigma(A)\big). \tag{2.17}$$

On each interval $[0, T]$, obviously,

$$|\lambda(t) - \lambda_0| \leq C_0(T), \quad 0 \leq t \leq T. \tag{2.18}$$

Since $\lambda(0) = \lambda_0$ is an isolated point of $\sigma(A)$, the distance from point λ_0 to other points of $\sigma(A)$ is positive:

$$\rho = \operatorname{dist}\big(\lambda(0), \sigma(A)\backslash\lambda_0\big) > 0. \tag{2.19}$$

(1) According to conditions of lemma $d(0) = \operatorname{dist}\big(\lambda(0), \sigma(A)\big) = 0$. Then it is clear that for small enough $\delta > 0$ $d(t) \leq \rho/2$, $0 \leq t \leq \delta$.

Hence for these t the nearest point of $\sigma(A)$ to $\lambda(t)$ is

$$\lambda_0 : d(t) = |\lambda(t) - \lambda_0| \quad \text{if } 0 \leq t \leq \delta. \tag{2.20}$$

(2) Further, on compact interval $[\delta, T]$ and according to condition $\lambda(t) \notin \sigma(A)$, we have $d(t) > 0$ and hence

$$d(t) \geq d_0(T, \delta) > 0 \quad \text{for } \delta \leq t \leq T.$$

Taking into consideration this inequality and (2.18), we obtain

$$d(t) \geq d_0(T,\delta) \geq \frac{d_0(T,\delta)}{C_0(T)}|\lambda(t)-\lambda_0| = C_1|\lambda(t)-\lambda_0|, \quad \delta \leq t \leq T. \tag{2.21}$$

(3) From (2.20) and (2.21) it follows that

$$d(t) \geq C_2(T)|\lambda(t)-\lambda_0|, \quad 0 \leq t \leq T, \tag{2.22}$$

In this case the required estimate (2.16) follows from (2.22), definition (2.17) and Lemma 2.4. □

Lemma 2.6. *Let $v_k(t)$ be as defined by (2.5) and $v_0(t) = f(t)$. If all of conditions of Theorem 2.3 are fulfilled then*

$$\|v_k\|_t \leq \|f\|_{(\alpha)}[C(t)]^k \frac{t^{\alpha(k+1)}}{k!}, \tag{2.23}$$

where $C(T) = K_\alpha B_1(t)/\alpha$.

Proof. The estimate (2.23) is obviously true for $k = 0$. Assuming that it is true for some k we get

$$\|v_{k+1}(t)\| \leq K_\alpha t^\alpha \|f\|_{(\alpha)} \int_0^t \|R_{\lambda(s)}\| [C(s)]^k \frac{s^{\alpha(k+1)}}{k!} ds.$$

Further we apply estimate (2.14),

$$\begin{aligned}\|v_{k+1}(t)\| &\leq \|f\|_{(\alpha)} K_\alpha B_1(T)[C(t)]^k t^\alpha \frac{1}{k!}\int_0^t s^{\alpha(k+1)-1} ds \\ &= \|f\|_{(\alpha)}[C(t)]^{k+1}\frac{t^{\alpha(k+2)}}{(k+1)!}.\end{aligned}$$

□

Using this lemma we complete the proof of Theorem 2.3 as in the proof of Theorem 2.1.

As mentioned above, the condition (2.9) in Theorem 2.3 is important as confirmed in the following example.

Example. Consider equation

$$\int_0^t tu(s,x)ds + 3x\int_0^1 yu(t,y)dy - (1+t^2)u(t,x) = tx, \tag{2.24}$$

which is of the same type as equation (1.1). In this case $K(t,s) = t$, $R(x,y) = -3xy$, $f(t,x) = tx$, $\lambda(t) = 1+t^2$. Consider the operator

$$Au(x) = 3x\int_0^1 yu(y)dy.$$

and look for an eigenvalue λ and eigenfunction v such that $Av = \lambda v$. Obviously,

$$R(A) = \{v \in L_2[0,1] : v(x) = Cx,\ C \in R\}$$

and in as much as $v \in R(A)$ we get

$$(A-\lambda)v(x) = (A-\lambda)Cx = 3Cx\int_0^1 y^2 dy - \lambda Cx = Cx(1-\lambda) = 0.$$

This implies $\lambda = 1$ is an eigenvalue and hence it is isolated point of spectrum of compact operator A. Then all of conditions of Theorem 2.3 are fulfilled except for (2.9), because $\lambda(0) = 1 \in \sigma(A)$, but $\lambda'(0) = 0$.

Assume that there exists a solution $u(t, x)$ of (2.24). Denote

$$g(t, x) = u(t, x) - a(t)x, \quad with \quad a(t) = 3\int_0^1 xu(t, x)dx.$$

Obviously, the function $f(x) = x$ is orthogonal to $g(t, x)$ in the Hilbert space $L_2[0, 1]$:

$$\int_0^1 xg(t, x)dx = \int_0^1 xu(t, x)dx - a(t)\int_0^1 x^2dx = a(t)/3 - a(t)/3 = 0.$$

According to the definition $u(t, x) = g(t, x) + a(t)x$, and putting this representation in (2.24), we get

$$\begin{aligned} &t\int_0^t g(s, x)ds + 3x\int_0^1 yg(t, y)dy - (1 + t^2)g(t, x) \\ &+xt\int_0^t a(s)ds + 3xa(t)\int_0^1 y^2dy - (1 + t^2)a(t)x = tx. \end{aligned}$$

Note that because of orthogonality of the functions x and $g(t, x)$ this equality is equivalent to the following two equalities:

$$t\int_0^t g(s, x)ds - (1 + t^2)g(t, x) = 0, \tag{2.25}$$

$$\int_0^t a(s)ds - ta(t) = 1 - \frac{3}{t}\int_0^1 yg(t, y)dy\,. \tag{2.26}$$

After differentiating (2.25), we get

$$\int_0^t g(s, x)ds + tg(t, x) - 2tg(t, x) - (1 + t^2)g_t(t, x) = 0.$$

Hence, $g_t(0, x) = 0$. Further, we may rewrite (2.25) as

$$\int_0^t g(s, x)ds - (t + 1/t)g(t, x) = 0$$

and after differentiating this equation, we get

$$g(t, x) - g(t, x) + \frac{1}{t^2}g(t, x) - (t + 1/t)g_t = 0,$$

or

$$t(t^2 + 1)g_t(t, x) - g(t, x) = 0.$$

Now we find the solution of this ordinary differential equation

$$g(t, x) = C(x)\frac{t}{\sqrt{t^2 + 1}}.$$

Then

$$g_t(t, x) = \frac{C(x)}{(t^2 + 1)^{3/2}}$$

and because of the condition $g_t(0,x) = 0$, we have $g_t(0,x) = C(x) = 0$ Consequently, $g(t,x) \equiv 0$ Then equation (2.26) takes the form

$$\int_0^t a(s)ds - ta(t) = 1,$$

and when we put $t = 0$ we obtain $0 = 1$. This contradiction proves that there is no solution to (2.24) because $\lambda'(0) = 0$.

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