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# EXISTENCE OF POSITIVE SOLUTIONS TO NONLINEAR ELLIPTIC PROBLEM IN THE HALF SPACE

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Abstract. This paper concerns nonlinear elliptic equations in the half space $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, $n \geq 2$, with a nonlinear term satisfying some conditions related to a certain Kato class of functions. We prove some existence results and asymptotic behaviour for positive solutions using a potential theory approach.

## 1. Introduction

In the present paper, we study the nonlinear elliptic equation

$$\Delta u + f(., u) = 0, \text{ in } \mathbb{R}^n_+ \tag{1.1}$$

in the sense of distributions, with some boundary values determined below (see problems (1.6), (1.11) and (1.12)). Here $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, $(n \geq 2)$.

Several results have been obtained for (1.1), in both bounded and unbounded domain $D \subset \mathbb{R}^n$ with different boundary conditions; see for example [2, 3, 4, 5, 7, 8, 10, 12, 13, 14, 16, 17] and the references therein. Our goal of this paper is to undertake a study of (1.1) when the nonlinear term $f(x, t)$ satisfies some conditions related to a certain Kato class $K^\infty(\mathbb{R}^n_+)$, and to answer the questions of existence and asymptotic behaviour of positive solutions.

Our tools are based essentially on some inequalities satisfied by the Green function $G(x, y)$ of $(-\Delta)$ in $\mathbb{R}^n_+$. This allows us to state some properties of functions in the class $K^\infty(\mathbb{R}^n_+)$ which was introduced in [2] for $n \geq 3$, and in [3] for $n = 2$.

**Definition 1.1.** A Borel measurable function $q$ in $\mathbb{R}^n_+$ belongs to the class $K^\infty(\mathbb{R}^n_+)$ if $q$ satisfies the following two conditions

$$\lim_{\alpha\to 0}\Big(\sup_{x\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+\cap B(x,\alpha)}\frac{y_n}{x_n}G(x, y)|q(y)|dy\Big) = 0\,, \tag{1.2}$$

$$\lim_{M\to\infty}\Big(\sup_{x\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+\cap(|y|\geq M)}\frac{y_n}{x_n}G(x, y)|q(y)|dy\Big) = 0. \tag{1.3}$$

The class $K^\infty(\mathbb{R}^n_+)$ is sufficiently rich. It contains properly the classical Kato class $K^\infty_n(\mathbb{R}^n_+)$, defined by Zhao [21], for $n \geq 3$ in unbounded domains $D$ as follows:

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**Definition 1.2.** A Borel measurable function $q$ on $D$ belongs to the Kato class $K_n^\infty(D)$ if $q$ satisfies the following two conditions

$$\lim_{\alpha\to 0}\sup_{x\in D}\int_{D\cap(|x-y|\le\alpha)}\frac{|\psi(y)|}{|x-y|^{d-2}}dy)=0\,,$$
$$\lim_{M\to\infty}\sup_{x\in D}\int_{D\cap(|y|\ge M)}\frac{|\psi(y)|}{|x-y|^{d-2}}dy)=0.$$

Typical examples of functions $q$ in the class $K^\infty(\mathbb{R}^n_+)$ are: $q\in L^p(\mathbb{R}^n_+)\cap L^1(\mathbb{R}^n_+)$, where $p>\frac{n}{2}$ and $n\ge 3$; and

$$q(x)=\frac{1}{(|x|+1)^{\mu-\lambda}x_n^\lambda},$$

where $\lambda<2<\mu$ and $n\ge 2$ (see [2, 3]).

We shall refer in this paper to the bounded continuous solution $Hg$ of the Dirichlet problem

$$\begin{gathered}\Delta u=0,\quad \text{in } \mathbb{R}^n_+\\ \lim_{x_n\to 0}u(x)=g(x'),\end{gathered}\tag{1.4}$$

where $g$ is a nonnegative bounded continuous function in $\mathbb{R}^{n-1}$ (see [1, p. 418]). We also refer to the potential of a measurable nonnegative function $f$, defined on $\mathbb{R}^n_+$ by

$$Vf(x)=\int_{\mathbb{R}^n_+}G(x,y)f(y)dy.$$

Our paper is organized as follows. Existence results are proved in sections 3, 4, and 5. In section 2, we collect some preliminary results about the Green function $G$ and the class $K^\infty(\mathbb{R}^n_+)$. We prove further that if $p>n/2$ and $a\in L^p(\mathbb{R}^n_+)$, then for $\lambda<2-\frac{n}{p}<\mu$, the function

$$x\mapsto\frac{a(x)}{(|x|+1)^{\mu-\lambda}x_n^\lambda},$$

is in $K^\infty(\mathbb{R}^n_+)$. In section 3, we establish an existence result for equation (1.1) where a singular term and a sublinear term are combined in the nonlinearity $f(x,t)$.

The pure singular elliptic equation

$$\Delta u+p(x)u^{-\gamma}=0,\quad \gamma>0,\ x\in D\subseteq\mathbb{R}^n$$

has been extensively studied for both bounded and unbounded domains. We refer to [7, 8, 10, 12, 13, 14] and references therein, for various existence and uniqueness results related to solutions for above equation.

For more general situations Mâagli and Zribi showed in [17] that the problem

$$\begin{gathered}\Delta u+\varphi(.,u)=0,\quad x\in D\\ u\big|_{\partial D}=0\\ \lim_{|x|\to\infty}u(x)=0,\quad \text{if } D \text{ is unbounded}\end{gathered}\tag{1.5}$$

admits a unique positive solution if $\varphi$ is a nonnegative measurable function on $(0,\infty)$, which is non-increasing and continuous with respect to the second variable and satisfies

(H0) For all $c>0$, $\varphi(.,c)\in K_n^\infty(D)$.

If $D = \mathbb{R}^n_+$, the result of Mâagli and Zribi [17] has been improved later by Bachar and Mâagli in [2], where they gave an existence and an uniqueness result for (1.5), with the more restrictive condition

(H0') For all $c > 0$, $\varphi(., c) \in K^\infty(\mathbb{R}^n_+)$.

On the other hand, (1.1) with a sublinear term $f(., u)$ have been studied in $\mathbb{R}^n$ by Brezis and Kamin in [5]. Indeed, the authors proved the existence and the uniqueness of a positive solution for the problem

$$\begin{gathered}\Delta u + \rho(x)u^\alpha = 0 \quad \text{in } \mathbb{R}^n,\\ \liminf_{|x|\to\infty} u(x) = 0,\end{gathered}$$

with $0 < \alpha < 1$ and $\rho$ is a nonnegative measurable function satisfying some appropriate conditions.

In this section, we combine a singular term and a sublinear term in the nonlinearity. Indeed, we consider the boundary value problem

$$\begin{gathered}\Delta u + \varphi(., u) + \psi(., u) = 0, \quad \text{in } \mathbb{R}^n_+\\ u > 0, \quad \text{in } \mathbb{R}^n_+\\ \lim_{x_n\to 0} u(x) = 0,\\ \lim_{|x|\to+\infty} u(x) = 0,\end{gathered} \tag{1.6}$$

in the sense of distributions, where $\varphi$ and $\psi$ are required to satisfy the following hypotheses:

(H1) $\varphi$ is a nonnegative Borel measurable function on $\mathbb{R}^n_+ \times (0, \infty)$, continuous and non-increasing with respect to the second variable.

(H2) For all $c > 0$, $x \mapsto \varphi(x, c\theta(x))$ belongs to $K^\infty(\mathbb{R}^n_+)$, where
$$\theta(x) = \frac{x_n}{(1 + |x|)^n}.$$

(H3) $\psi$ is a nonnegative Borel measurable function on $\mathbb{R}^n_+ \times (0, \infty)$, continuous with respect to the second variable such that there exist a nontrivial nonnegative function $p$ and a nonnegative function $q \in K^\infty(\mathbb{R}^n_+)$ satisfying for $x \in \mathbb{R}^n_+$ and $t > 0$,
$$p(x)h(t) \le \psi(x, t) \le q(x)f(t), \tag{1.7}$$
where $h$ is a measurable nondecreasing function on $[0, \infty)$ satisfying
$$\lim_{t\to 0^+} \frac{h(t)}{t} = +\infty \tag{1.8}$$
and $f$ is a nonnegative measurable function locally bounded on $[0, \infty)$ satisfying
$$\limsup_{t\to\infty} \frac{f(t)}{t} < \|Vq\|_\infty. \tag{1.9}$$

Using a fixed point argument, we shall state the following existence result.

**Theorem 1.3.** *Assume (H1)–(H3). Then the problem* (1.6) *has a positive solution* $u \in C_0(\mathbb{R}^n_+)$ *satisfying for each* $x \in \mathbb{R}^n_+$
$$a\theta(x) \le u(x) \le V(\varphi(., a\theta))(x) + bVq(x),$$
*where* $a, b$ *are positive constants.*

Note that Mâagli and Masmoudi studied in [16, 18] the case $\varphi \equiv 0$, under similar conditions to those in (H3). Indeed the authors gave an existence result for

$$\Delta u + \psi(., u) = 0 \text{ in } D, \tag{1.10}$$

with some boundary conditions, where $D$ is an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with compact nonempty boundary.

Typical examples of nonlinearities satisfying (H1)–(H3) are:

$$\varphi(x,t) = p(x)(\theta(x))^\gamma t^{-\gamma},$$

for $\gamma \geq 0$, and

$$\psi(x,t) = p(x)t^\alpha \log(1 + t^\beta),$$

for $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$, where $p$ is a nonnegative function in $K^\infty(\mathbb{R}^n_+)$.

In section 4, we consider the nonlinearity $f(x,t) = -t\varphi(x,t)$ and we use a potential theory approach to investigate an existence result for (1.1). Let $\alpha \in [0,1]$ and $\omega$ be the function defined on $\mathbb{R}^n_+$ by $\omega(x) = \alpha x_n + (1 - \alpha)$. We shall prove in this section the existence of positive continuous solutions for the following nonlinear problem

$$\begin{gathered}
\Delta u - u\varphi(., u) = 0, \quad \text{in } \mathbb{R}^n_+ \\
u > 0, \quad \text{in } \mathbb{R}^n_+ \\
\lim_{x_n \to 0} u(x) = (1 - \alpha)g(x'), \\
\lim_{x_n \to +\infty} \frac{u(x)}{x_n} = \alpha\lambda,
\end{gathered} \tag{1.11}$$

in the sense of distributions, where $\lambda$ is a positive constant, $g$ is a nontrivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$ and $\varphi$ satisfies the following hypotheses:

- (H4) $\varphi$ is a nonnegative measurable function on $\mathbb{R}^n_+ \times [0, \infty)$.
- (H5) For all $c > 0$, there exists a positive function $q_c \in K^\infty(\mathbb{R}^n_+)$ such that the map $t \mapsto t(q_c(x) - \varphi(x, t\omega(x)))$ is continuous and nondecreasing on $[0, c]$ for every $x \in \mathbb{R}^n_+$.

**Theorem 1.4.** *Under assumptions (H4) and (H5), problem* (1.11) *has a positive continuous solution $u$ such that for each $x \in \mathbb{R}^n_+$,*

$$c(\alpha\lambda x_n + (1 - \alpha)Hg(x)) \leq u(x) \leq \alpha\lambda x_n + (1 - \alpha)Hg(x),$$

*where $c \in (0, 1)$.*

Note that if $\alpha = 0$, then the solution $u$ satisfies $cHg(x) \leq u(x) \leq Hg(x)$, $c \in (0, 1)$. In particular, $u$ is bounded on $\mathbb{R}^n_+$. Our techniques are similar to those used by Mâagli and Masmoudi in [16, 18].

Section 5 deals with the question of existence of continuous bounded solutions for the problem

$$\begin{gathered}
\Delta u - \varphi(., u) = 0, \quad \text{in } \mathbb{R}^n_+ \\
u > 0, \quad \text{in } \mathbb{R}^n_+ \\
\lim_{x_n \to 0} u(x) = g(x'),
\end{gathered} \tag{1.12}$$

where $g$ is a nontrivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$. We also establish an uniqueness result for such solutions. Here the nonlinearity $\varphi$ satisfies the following conditions:

- (H6) $\varphi$ is a nonnegative measurable function on $\mathbb{R}^n_+ \times [0, \infty)$, continuous and nondecreasing with respect to the second variable.

(H7) $\varphi(.,0) = 0$.
(H8) For all $c > 0$, $\varphi(.,c) \in K^{\infty}(\mathbb{R}^n_+)$.

**Theorem 1.5.** *Under assumptions (H6)–(H8), problem* (1.12) *has a unique positive solution $u$ such that for each $x \in \mathbb{R}^n_+$,*

$$0 < u(x) \le Hg(x).$$

Note that if $q \in K^{\infty}(\mathbb{R}^n_+)$ and $\varphi(x,t) \le q(x)t$ locally on $t$, then the solution $u$ satisfies in particular $cHg(x) \le u(x) \le Hg(x)$, $c \in (0,1)$. This result follows the result in [4], where we studied the following polyharmonic problem, for every integer $m$,

$$\begin{gathered}(-\Delta)^m u + \varphi(.,u) = 0, \quad \text{in } \mathbb{R}^n_+\\ u > 0, \quad \text{in } \mathbb{R}^n_+\\ \lim_{x_n \to 0} \frac{u(x)}{x_n^{m-1}} = g(x')\end{gathered} \tag{1.13}$$

(in the sense of distributions). Here $\varphi$ is a nonnegative measurable function on $\mathbb{R}^n_+ \times (0,\infty)$, continuous and non-increasing with respect to the second variable and satisfies some conditions related to a certain Kato class appropriate to the $m$-polyharmonic case. In fact in [4], we proved that for a fixed positive harmonic function $h_0$ in $\mathbb{R}^n_+$, if $g \ge (1+c)h_0$, for $c > 0$, then the problem (1.13) has a positive continuous solution $u$ satisfying $u(x) \ge x_n^{m-1} h_0(x)$ for every $x \in \mathbb{R}^n_+$. Thus a natural question to ask is for $t \to \varphi(x,t)$ nondecreasing, whether or not (1.13) has a solution, which we aim to study in the case $m = 1$.

**Notation.** To simplify our statements, we define the following symbols.
$\mathbb{R}^n_+ := \{x = (x_1,...,x_n) = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, $n \ge 2$.
$\overline{x} = (x', -x_n)$, for $x \in \mathbb{R}^n_+$.
Let $\mathcal{B}(\mathbb{R}^n_+)$ denote the set of Borel measurable functions in $\mathbb{R}^n_+$ and $\mathcal{B}^+(\mathbb{R}^n_+)$ the set of nonnegative functions in this space.
$C_b(\mathbb{R}^n_+) = \{w \in C(\mathbb{R}^n_+) : w \text{ is bounded in } \mathbb{R}^n_+\}$
$C_0(\mathbb{R}^n_+) = \{w \in C(\mathbb{R}^n_+) : \lim_{x_n \to 0} w(x) = 0 \text{ and } \lim_{|x| \to \infty} w(x) = 0\}$
$C_0(\overline{\mathbb{R}^n_+}) = \{w \in C(\overline{\mathbb{R}^n_+}) : \lim_{|x| \to \infty} w(x) = 0\}$.

Note that $C_b(\mathbb{R}^n_+)$, $C_0(\mathbb{R}^n_+)$ and $C_0(\overline{\mathbb{R}^n_+})$ are three Banach spaces with the uniform norm $\|w\|_\infty = \sup_{x \in \mathbb{R}^n_+} |w(x)|$.

For any $q \in \mathcal{B}(\mathbb{R}^n_+)$, we put

$$\|q\| := \sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{y_n}{x_n} G(x,y)|q(y)|dy.$$

Recall that the potential $Vf$ of a function $f \in \mathcal{B}^+(\mathbb{R}^n_+)$, is lower semi-continuous in $\mathbb{R}^n_+$. Furthermore, for each function $q \in \mathcal{B}^+(\mathbb{R}^n_+)$ such that $Vq < \infty$, we denote by $V_q$ the unique kernel which satisfies the following resolvent equation (see [15, 19]):

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{1.14}$$

For each $u \in \mathcal{B}(\mathbb{R}^n_+)$ such that $V(q|u|) < \infty$, we have

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u. \tag{1.15}$$

Let $f$ and $g$ be two positive functions on a set $S$. We call $f \sim g$, if there is $c > 0$ such that

$$\frac{1}{c} g(x) \le f(x) \le c g(x) \quad \text{for all } x \in S.$$

We call $f \preceq g$, if there is $c > 0$ such that

$$f(x) \le cg(x) \quad \text{for all } x \in S.$$

The following properties will be used in this article: For $x, y \in \mathbb{R}^n_+$, note that $|x-\overline{y}|^2 - |x-y|^2 = 4x_n y_n$. So we have

$$|x-\overline{y}|^2 \sim |x-y|^2 + x_n y_n \tag{1.16}$$

$$x_n + y_n \le |x-\overline{y}|. \tag{1.17}$$

Let $\lambda, \mu > 0$ and $0 < \gamma \le 1$, then for $t \ge 0$ we have

$$\log(1+\lambda t) \sim \log(1+\mu t), \tag{1.18}$$

$$\log(1+t) \preceq t^\gamma. \tag{1.19}$$

## 2. Properties of the Green function and the Kato class $K^\infty(\mathbb{R}^n_+)$

In this section, we briefly recall some estimates on the Green function $G$ and we collect some properties of functions belonging to the Kato class $K^\infty(\mathbb{R}^n_+)$, which are useful at stating our existence results. For $x, y \in \mathbb{R}^n_+$, we set

$$G(x,y) = \begin{cases} \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}}\big[\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-\overline{y}|^{n-2}}\big], & \text{if } n \ge 3 \\ \frac{1}{4\pi}\log\big(1 + \frac{4x_2y_2}{|x-y|^2}\big), & \text{if } n = 2, \end{cases}$$

the Green function of $(-\Delta)$ in $\mathbb{R}^n_+$ (see [1, p. 92]). Then we have the following estimates and inequalities whose proofs can be found in [2] for $n \ge 3$ and in [3] for $n = 2$.

**Proposition 2.1.** *For $x, y \in \mathbb{R}^n_+$, we have*

$$G(x,y) \sim \begin{cases} \frac{x_n y_n}{|x-y|^{n-2}|x-\overline{y}|^2} & \text{if } n \ge 3, \\ \frac{x_2 y_2}{|x-\overline{y}|^2}\log(1 + \frac{|x-\overline{y}|^2}{|x-y|^2}) & \text{if } n = 2. \end{cases} \tag{2.1}$$

**Corollary 2.2.** *For $x, y \in \mathbb{R}^n_+$, we have*

$$\frac{x_n y_n}{(|x|+1)^n(|y|+1)^n} \preceq G(x,y). \tag{2.2}$$

**Theorem 2.3** (3G-Theorem). *There exists $C_0 > 0$ such that for each $x, y, z \in \mathbb{R}^n_+$, we have*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C_0\big[\frac{z_n}{x_n}G(x,z) + \frac{z_n}{y_n}G(y,z)\big]. \tag{2.3}$$

Let us recall in the following properties of functions in the class $K^\infty(\mathbb{R}^n_+)$. The proofs of these propositions can be found in [2, 3].

**Proposition 2.4.** *Let $q$ be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$. Then we have: (i) $\|q\| < \infty$; (ii) The function $x \mapsto \frac{x_n}{(|x|+1)^n}q(x)$ is in $L^1(\mathbb{R}^n_+)$. and (iii)*

$$\frac{x_n}{(|x|+1)^n} \preceq Vq(x). \tag{2.4}$$

For a fixed nonnegative function $q$ in $K^\infty(\mathbb{R}^n_+)$, we put

$$\mathcal{M}_q := \{\varphi \in B(\mathbb{R}^n_+), \ \ |\varphi| \preceq q\}.$$

**Proposition 2.5.** *Let $q$ be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$, then the family of functions*
$$V(\mathcal{M}_q) = \{V\varphi : \ \varphi \in \mathcal{M}_q\}$$
*is relatively compact in $C_0(\mathbb{R}^n_+)$.*

**Proposition 2.6.** *Let $q$ be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$, then the family of functions*
$$\mathcal{N}_q = \Big\{ \int_{\mathbb{R}^n_+} \frac{y_n}{x_n} G(x,y)|\varphi(y)|dy : \varphi \in \mathcal{M}_q \Big\}$$
*is relatively compact in $C_0(\overline{\mathbb{R}^n_+})$.*

In the sequel, we use the following notation
$$\alpha_q := \sup_{x,y\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{G(x,z)G(z,y)}{G(x,y)}|q(z)|dz.$$

**Lemma 2.7.** *Let $q$ be a function in $K^\infty(\mathbb{R}^n_+)$. Then we have*
$$\|q\| \le \alpha_q \le 2C_0\|q\|,$$
*where $C_0$ is the constant given in* (2.3)*.*

*Proof.* By (2.3), we obtain easily that $\alpha_q \le 2C_0\|q\|$. On the other hand, we have by Fatou lemma that for each $x \in \mathbb{R}^n_+$
$$\int_{\mathbb{R}^n_+} \frac{z_n}{x_n} G(x,z)|q(z)|dz \le \liminf_{|\zeta|\to\infty} \int_{\mathbb{R}^n_+} \frac{z_n}{x_n}\frac{|x-\zeta|^n}{|z-\zeta|^n} G(x,z)|q(z)|dz.$$
Now since for each $x,z \in \mathbb{R}^n_+$ and $\zeta \in \partial\mathbb{R}^n_+$, we have
$$\lim_{y\to\zeta} \frac{G(z,y)}{G(x,y)} = \frac{z_n}{x_n}\frac{|x-\zeta|^n}{|z-\zeta|^n}.$$
Then by Fatou lemma we deduce that
$$\int_{\mathbb{R}^n_+} \frac{z_n}{x_n}\frac{|x-\zeta|^n}{|z-\zeta|^n} G(x,z)|q(z)|dz \le \liminf_{y\to\zeta} \int_{\mathbb{R}^n_+} G(x,z)\frac{G(z,y)}{G(x,y)}|q(z)|dz \le \alpha_q.$$
We derive obviously that $\|q\| \le \alpha_q$. □

**Proposition 2.8.** *Let $q$ be a function in $K^\infty(\mathbb{R}^n_+)$ and $v$ be a nonnegative superharmonic function in $\mathbb{R}^n_+$. Then for each $x \in \mathbb{R}^n_+$, we have*
$$\int_{\mathbb{R}^n_+} G(x,y)v(y)|q(y)|dy \le \alpha_q v(x). \tag{2.5}$$

*Proof.* Let $v$ be a nonnegative superharmonic function in $\mathbb{R}^n_+$, then there exists (see [20, Theorem 2.1]) a sequence $(f_k)_k$ of nonnegative measurable functions in $\mathbb{R}^n_+$ such that the sequence $(v_k)_K$ defined on $\mathbb{R}^n_+$ by $v_k := Vf_k$ increases to $v$. Since for each $x,z \in \mathbb{R}^n_+$, we have
$$\int_{\mathbb{R}^n_+} G(x,y)G(y,z)|q(y)|\,dy \le \alpha_q G(x,z),$$
it follows that
$$\int_{\mathbb{R}^n_+} G(x,y)v_k(y)|q(y)|\,dy \le \alpha_q v_k(x).$$
Hence, the result holds from the monotone convergence theorem. □

**Corollary 2.9.** *Let $q$ be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$ and $v$ be a nonnegative superharmonic function in $\mathbb{R}^n_+$, then for each $x \in \mathbb{R}^n_+$ such that $0 < v(x) < \infty$, we have*

$$\exp(-\alpha_q)v(x) \leq (v - V_q(qv))(x) \leq v(x).$$

*Proof.* The upper inequality is trivial. For the lower one, we consider the function $\gamma(\lambda) = v(x) - \lambda V_{\lambda q}(qv)(x)$ for $\lambda \geq 0$. The function $\gamma$ is completely monotone on $[0,\infty)$ and so $\log \gamma$ is convex in $[0,\infty)$. This implies that

$$\gamma(0) \leq \gamma(1)\exp(-\frac{\gamma'(0)}{\gamma(0)}).$$

That is

$$v(x) \leq (v - V_q(qv))(x)\exp(\frac{V(qv)(x)}{v(x)}).$$

So, the result holds by (2.5). $\square$

We close this section by giving a class of functions included in $K^\infty(\mathbb{R}^n_+)$. We need the following key Lemma. For the proof we can see [4].

**Lemma 2.10.** *For $x, y$ in $\mathbb{R}^n_+$, we have the following properties:*

(1) *If $x_n y_n \leq |x-y|^2$ then $(x_n \vee y_n) \leq \frac{\sqrt{5}+1}{2}|x-y|$.*
(2) *If $|x-y|^2 \leq x_n y_n$ then $\frac{3-\sqrt{5}}{2}x_n \leq y_n \leq \frac{3+\sqrt{5}}{2}x_n$ and $\frac{3-\sqrt{5}}{2}|x| \leq |y| \leq \frac{3+\sqrt{5}}{2}|x|$.*

In what follows we will use the following notation

$$D_1 := \{y \in \mathbb{R}^n_+ : x_n y_n \leq |x-y|^2\},$$
$$D_2 := \{y \in \mathbb{R}^n_+ : |x-y|^2 \leq x_n y_n\}.$$

We point out that $D_2 = \overline{B}(\widetilde{x}, \frac{\sqrt{5}}{2}x_n)$, where $\widetilde{x} = (x_1, \dots, x_{n-1}, \frac{3}{2}x_n)$ and consequently $D_1 = B^c(\widetilde{x}, \frac{\sqrt{5}}{2}x_n)$

**Proposition 2.11.** *Let $p > n/2$ and $a$ be a function in $L^p(\mathbb{R}^n_+)$. Then for $\lambda < 2 - \frac{n}{p} < \mu$, the function $\varphi(x) = \frac{a(x)}{(|x|+1)^{\mu-\lambda}x_n^\lambda}$ is in $K^\infty(\mathbb{R}^n_+)$.*

*Proof.* Let $p > n/2$ and $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $a$ be a function in $L^p(\mathbb{R}^n_+)$ and $\lambda < 2 - \frac{n}{p} < \mu$. First, we claim that the function $\varphi$ satisfies (1.2). Let $0 < \alpha < 1$. Since $x_n \leq (1 + |x|)$ and $y_n \leq (1 + |y|)$, then we remark that if $|x-y| \leq \alpha$, then $(|x|+1) \sim (|y|+1)$ and consequently

$$|x - \overline{y}| \preceq (|y|+1), \quad \text{for } y \in B(x,\alpha). \tag{2.6}$$

Put $\lambda^+ = \max(\lambda, 0)$. So to show the claim, we use the Hölder inequality and we distinguish the following two cases:

*Case 1.* $n \geq 3$. Using (2.1), (1.17) and the fact that $|x-y| \leq |x-\overline{y}|$ and $y_n \leq (1+|y|)$, we deduce that

$$\begin{aligned}
&\int_{B(x,\alpha)\cap\mathbb{R}^n_+} \frac{y_n}{x_n} G(x,y)\varphi(y)dy \\
&\leq \|a\|_p \Big(\int_{B(x,\alpha)\cap\mathbb{R}^n_+} \frac{y_n^{2q}}{|x-y|^{(n-2)q}|x-\overline{y}|^{2q} y_n^{\lambda q}(|y|+1)^{(\mu-\lambda)q}} dy\Big)^{\frac{1}{q}} \\
&\leq \|a\|_p \Big(\int_{B(x,\alpha)\cap\mathbb{R}^n_+} \frac{dy}{|x-y|^{(n-2+\lambda^+)q}}\Big)^{\frac{1}{q}} \\
&\preceq \alpha^{2-\frac{n}{p}-\lambda^+},
\end{aligned}$$

which tends to zero as $\alpha \to 0$.

*Case 2.* $n = 2$. Using (2.1), (2.6), (1.17) and taking $\gamma \in (\frac{\lambda^+}{2}, \frac{1}{q})$ in (1.19), we obtain that

$$\begin{aligned}
&\int_{B(x,\alpha)\cap\mathbb{R}^2_+} \frac{y_2}{x_2} G(x,y)\varphi(y)dy \\
&\leq \|a\|_p \Big(\int_{B(x,\alpha)\cap\mathbb{R}^2_+} \frac{y_2^{(2-\lambda)q}}{|x-\overline{y}|^{2q}(|y|+1)^{(\mu-\lambda)q}} \big(\log(1+\frac{|x-\overline{y}|^2}{|x-y|^2})\big)^q dy\Big)^{\frac{1}{q}} \\
&\leq \|a\|_p \Big(\int_{B(x,\alpha)\cap\mathbb{R}^2_+} \frac{|x-\overline{y}|^{(2\gamma-\lambda^+)q}}{(|y|+1)^{(\mu-\lambda^+)q}|x-y|^{2\gamma q}} dy\Big)^{\frac{1}{q}} \\
&\leq \|a\|_p \Big(\int_{B(x,\alpha)\cap\mathbb{R}^2_+} \frac{1}{|x-y|^{2\gamma q}} dy\Big)^{\frac{1}{q}} \\
&\preceq \alpha^{2-2\gamma q}
\end{aligned}$$

which tends to zero as $\alpha \to 0$.
Now, we claim that the function $\varphi$ satisfies (1.3). Let $M > 1$ and put $\Omega := \{y \in \mathbb{R}^n_+ : (|y| \geq M) \cap (|x-y| \geq \alpha)\}$ and

$$I(x,M) := \int_\Omega \frac{y_n}{x_n} G(x,y)\varphi(y)dy.$$

By the above argument, to show the claim we need only to prove that $I(x,M) \longrightarrow 0$, as $M \longrightarrow \infty$, uniformly on $x \in \mathbb{R}^n_+$. So we use the Hölder inequality and we distinguish the following two cases:
*Case 1.* $y \in D_1$. From (2.1), it is clear that $G(x,y) \preceq \frac{x_n y_n}{|x-y|^n}$. Then we have

$$\int_{\Omega\cap D_1} \frac{y_n}{x_n} G(x,y)\varphi(y)dy \preceq \|a\|_p \Big(\int_{\Omega\cap D_1} \frac{y_n^{(2-\lambda)q}}{|y|^{(\mu-\lambda)q}|x-y|^{nq}} dy\Big)^{1/q}.$$

Now we write that $2-\lambda = (2-\lambda-\frac{n}{p}) + \frac{n}{p}$ and we put $\gamma = \mu - 2 + \frac{n}{p}$. Hence, using the fact that $y_n \leq \max(|y|, |x-y|)$, we deduce that

$$\int_{\Omega\cap D_1} \frac{y_n}{x_n} G(x,y)\varphi(y)dy \preceq \|a\|_p \Big(\int_{\Omega\cap D_1} \frac{dy}{|x-y|^n |y|^{\gamma q}}\Big)^{1/q}.$$

On the other hand

$$\begin{aligned}
&\int_{\Omega\cap D_1} |x-y|^{-n}|y|^{-\gamma q}dy\\
&\preceq \sup_{|x|\leq \frac{M}{2}} \int_{\Omega\cap\mathbb{R}^n_+} |x-y|^{-n}|y|^{-\gamma q}dy\\
&\quad + \sup_{|x|\geq \frac{M}{2}} \int_{(\max(M,\frac{|x|}{2})\leq|y|\leq 2|x|)\cap\mathbb{R}^n_+\cap(|x-y|\geq\alpha)} |x-y|^{-n}|y|^{-\gamma q}dy\\
&\quad + \sup_{|x|\geq \frac{M}{2}} \int_{(|y|\geq 2|x|)\cap\mathbb{R}^n_+\cap(|x-y|\geq\alpha)} |x-y|^{-n}|y|^{-\gamma q}dy\\
&\quad + \sup_{|x|\geq 2M} \int_{(M\leq|y|\leq\frac{|x|}{2})\cap\mathbb{R}^n_+\cap(|x-y|\geq\alpha)} |x-y|^{-n}|y|^{-\gamma q}dy\\
&\preceq \int_{(|y|\geq M)} \frac{1}{|y|^{n+\gamma q}}\,dy + \sup_{|z|\geq\frac{M}{2}} \frac{\log(\frac{3|z|}{\alpha})}{|z|^{\gamma q}}\\
&\preceq \frac{1}{M^{\gamma q}} + \sup_{|z|\geq\frac{M}{2}} \frac{\log(\frac{3|z|}{\alpha})}{|z|^{\gamma q}}.
\end{aligned}$$

*Case 2.* $y\in D_2$. From Lemma 2.10, we have that $|y|\sim|x|$, $y_n\sim x_n\sim|x-\overline{y}|$. This implies:

If $n\geq 3$, then by (2.1), we deduce that

$$\begin{aligned}
\int_{\Omega\cap D_2} \frac{y_n}{x_n}G(x,y)\varphi(y)dy &\preceq \|a\|_p \frac{1}{x_n^{\lambda}|x|^{\mu-\lambda}}\Big(\int_{\Omega\cap B(x,cx_n)} \frac{dy}{|x-y|^{(n-2)q}}\Big)^{\frac1q}\\
&\preceq \|a\|_p \frac{x_n^{2-\lambda-\frac{n}{p}}}{|x|^{\mu-\lambda}}\\
&\preceq \|a\|_p \frac{1}{M^{\mu-2+\frac{n}{p}}}.
\end{aligned}$$

If $n=2$, then from (2.1) and (1.18) it follows that

$$\begin{aligned}
\int_{\Omega\cap D_2} \frac{y_2}{x_2}G(x,y)\varphi(y)dy &\preceq \|a\|_p \frac{1}{x_2^{\lambda}|x|^{\mu-\lambda}}\Big(\int_{\Omega\cap B(x,cx_n)} (\log(1+\frac{x_2^2}{|x-y|^2}))^q dy\Big)^{\frac1q}\\
&\preceq \|a\|_p \frac{x_n^{\frac{2}{q}-\lambda}}{|x|^{\mu-\lambda}}\\
&\preceq \|a\|_p \frac{1}{M^{\mu-2+\frac{2}{p}}}.
\end{aligned}$$

Hence we conclude that $I(x,M)$ converges to zero as $M\to\infty$ uniformly on $x\in\mathbb{R}^n_+$. This completes the proof. $\square$

## 3. Proof of Theorem 1.3

Recall that $\theta(x)=\frac{x_n}{(|x|+1)^n}$ on $\mathbb{R}^n_+$.

*Proof of Theorem 1.3.* Assuming (H1)–(H3), we shall use the Schauder fixed point theorem. Let $K$ be a compact of $\mathbb{R}^n_+$ such that, using (H3), we have

$$0<\alpha:=\int_K \theta(y)p(y)dy<\infty.$$

We put $\beta := \min\{\theta(x) : x \in K\}$. We note that by (2.2) there exists a constant $\alpha_1 > 0$ such that for each $x, y \in \mathbb{R}^n_+$
$$\alpha_1 \theta(x)\theta(y) \leq G(x, y). \tag{3.1}$$
Then from (1.8), we deduce that there exists $a > 0$ such that
$$\alpha_1 \alpha h(a\beta) \geq a. \tag{3.2}$$
On the other hand, since $q \in K^\infty(\mathbb{R}^n_+)$, then by Proposition 2.5 we have that $\|Vq\|_\infty < \infty$. So taking $0 < \delta < \frac{1}{\|Vq\|_\infty}$ we deduce by (1.9) that there exists $\rho > 0$ such that for $t \geq \rho$ we have $f(t) \leq \delta t$. Put $\gamma = \sup_{0 \leq t \leq \rho} f(t)$. So we have that
$$0 \leq f(t) \leq \delta t + \gamma, \quad t \geq 0. \tag{3.3}$$
Furthermore by (2.4), we note that there exists a constant $\alpha_2 > 0$ such that
$$\alpha_2 \theta(x) \leq Vq(x), \quad \forall x \in \mathbb{R}^n_+, \tag{3.4}$$
and from (H2) and Proposition 2.5, we have that $\|V\varphi(., a\theta)\|_\infty < \infty$. Let
$$b = \max\Big\{\frac{a}{\alpha_2}, \frac{\delta\|V\varphi(., a\theta)\|_\infty + \gamma}{1 - \delta\|Vq\|_\infty}\Big\}$$
and consider the closed convex set
$$\Lambda = \{u \in C_0(\mathbb{R}^n_+) : a\theta(x) \leq u(x) \leq V\varphi(., a\theta)(x) + bVq(x), \forall x \in \mathbb{R}^n_+\}.$$
Obviously, by (3.4) we have that the set $\Lambda$ is nonempty. Define the integral operator $T$ on $\Lambda$ by
$$Tu(x) = \int_{\mathbb{R}^n_+} G(x, y)[\varphi(y, u(y)) + \psi(y, u(y))]dy, \quad \forall x \in \mathbb{R}^n_+.$$
Let us prove that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$ and $x \in \mathbb{R}^n_+$, then by (3.3) we have
$$\begin{aligned}
Tu(x) &\leq V\varphi(., a\theta)(x) + \int_{\mathbb{R}^n_+} G(x, y)q(y)f(u(y))dy \\
&\leq V\varphi(., a\theta)(x) + \int_{\mathbb{R}^n_+} G(x, y)q(y)[\delta u(y) + \gamma]dy \\
&\leq V\varphi(., a\theta)(x) + \int_{\mathbb{R}^n_+} G(x, y)q(y)[\delta(\|V\varphi(., a\theta)\|_\infty + b\|Vq\|_\infty) + \gamma]dy \\
&\leq V\varphi(., a\theta)(x) + bVq(x).
\end{aligned}$$
Moreover from the monotonicity of $h$, (3.1) and (3.2), we have
$$\begin{aligned}
Tu(x) &\geq \int_{\mathbb{R}^n_+} G(x, y)\psi(y, u(y))dy \\
&\geq \alpha_1\theta(x) \int_{\mathbb{R}^n_+} \theta(y)p(y)h(a\theta(y))dy \\
&\geq \alpha_1\theta(x)h(a\beta) \int_K \theta(y)p(y)dy \\
&\geq \alpha_1\alpha h(a\beta)\theta(x) \\
&\geq a\theta(x).
\end{aligned}$$
On the other hand, we have that for each $u \in \Lambda$,
$$\varphi(., u) \leq \varphi(., a\theta) \quad \text{and} \quad \psi(., u) \leq [\delta(\|V\varphi(., a\theta)\| + b\|Vq\|_\infty) + \gamma]q. \tag{3.5}$$

This implies by Proposition (2.5) that $T\Lambda$ is relatively compact in $C_0(\mathbb{R}^n_+)$. In particular, we deduce that $T\Lambda \subset \Lambda$.

Next, we prove the continuity of $T$ in $\Lambda$. Let $(u_k)_k$ be a sequence in $\Lambda$ which converges uniformly to a function $u$ in $\Lambda$. Then since $\varphi$ and $\psi$ are continuous with respect to the second variable, we deduce by the dominated convergence theorem that

$$\forall x \in \mathbb{R}^n_+,\ Tu_k(x) \to Tu(x) \quad \text{as } k \to \infty.$$

Now, since $T\Lambda$ is relatively compact in $C_0(\mathbb{R}^n_+)$, then we have the uniform convergence. Hence $T$ is a compact operator mapping from $\Lambda$ to itself. So the Schauder fixed point theorem leads to the existence of a function $u \in \Lambda$ such that

$$u(x) = \int_{\mathbb{R}^n_+} G(x,y)[\varphi(y,u(y)) + \psi(y,u(y))]dy, \quad \forall x \in \mathbb{R}^n_+. \tag{3.6}$$

Finally, since $q$ and $\varphi(.,a\theta)$ are in $K^\infty(\mathbb{R}^n_+)$, we deduce by (3.5) and Proposition (2.4), that $y \mapsto \varphi(y,u(y)) + \psi(y,u(y)) \in L^1_{loc}(\mathbb{R}^n_+)$. Moreover, since $u \in C_0(\mathbb{R}^n_+)$, we deduce from (3.6), that $V(\varphi(.,u) + \psi(.,u)) \in L^1_{loc}(\mathbb{R}^n_+)$. Hence $u$ satisfies in the sense of distributions the elliptic equation

$$\Delta u + \varphi(.,u) + \psi(.,u) = 0, \text{ in } \mathbb{R}^n_+$$

and so it is a solution of the problem (1.6). □

**Example 3.1.** Let $\alpha, \beta \geq 0$ such that $0 \leq \alpha + \beta < 1$ and $p \in K^\infty(\mathbb{R}^n_+)$. Then the problem

$$\begin{gathered} \Delta u + p(x)[(u(x))^{-\gamma}(\theta(x))^{\gamma} + (u(x))^{\alpha}\log(1 + (u(x))^{\beta})] = 0, \quad \text{in } \mathbb{R}^n_+ \\ u > 0, \quad \text{in } \mathbb{R}^n_+ \end{gathered} \tag{3.7}$$

has a solution $u \in C_0(\mathbb{R}^n_+)$ satisfying $a\theta(x) \leq u(x) \leq bVp(x)$, where $a, b > 0$.

**Remark 3.2.** Taking in Example 3.1 the function $p(x) = \frac{1}{x_n^\lambda(1+|x|)^{\mu-\lambda}}$, for $\lambda < 2 < \mu$, we deduce from [2, 3] that the solution of (3.7) has the following behaviour

(i) $u(x) \preceq \frac{x_n^{2-\lambda}}{(1+|x|)^{n+2-2\lambda}}$, if $1 < \lambda < 2$ and $\mu \geq n + 2 - \lambda$
(ii) $u(x) \preceq \theta(x)\log(\frac{2(1+|x|)^2}{x_n})$, if $\lambda = 1$ and $\mu \geq n + 1$ or $\lambda < 1$ and $\mu = n + 1$.
(iii) $u(x) \preceq \theta(x)$, if $\lambda < 1$ and $\mu > n + 1$
(iv) $u(x) \preceq \frac{x_n^{\mu-n}}{(1+|x|)^{2\mu-n-2}}$, if $n < \mu < \min(n + 1, n + 2 - \lambda)$.

## 4. Proof of Theorem 1.4

In this section, we are interested in the existence of continuous solutions for the problem (1.11). We recall that $\omega(x) = \alpha x_n + (1 - \alpha)$, $x \in \mathbb{R}^n_+$, where $\alpha \in [0, 1]$. We aim to prove Theorem 1.4. So we need the following lemma

**Lemma 4.1.** *Let $q$ be a nonnegative function in $K^\infty(\mathbb{R}^n_+)$, then the family of functions*

$$\Big\{ \int_{\mathbb{R}^n_+} \frac{\omega(y)}{\omega(x)} G(x,y)|\varphi(y)|dy \ : \ \varphi \in \mathcal{M}_q \Big\}$$

*is relatively compact in $C_0(\overline{\mathbb{R}^n_+})$.*

*Proof.* We remark that

$$\frac{\omega(y)}{\omega(x)} = \frac{\alpha y_n + (1-\alpha)}{\alpha x_n + (1-\alpha)} \le \max(1, \frac{y_n}{x_n}) \le 1 + \frac{y_n}{x_n}.$$

So the result holds from Propositions 2.5 and 2.6. $\square$

*Proof of Theorem 1.4.* Let $\lambda > 0$ and $c := \sup\{\lambda, \|g\|_\infty\}$. Then by (H5), there exists a nonnegative function $q := q_c \in K^\infty(\mathbb{R}^n_+)$, such that the map

$$t \mapsto t(q(x) - \varphi(x, t\omega(x))) \tag{4.1}$$

is continuous and nondecreasing on $[0, c]$. We denote by $h(x) = \alpha\lambda x_n + (1-\alpha)Hg(x)$. Let

$$\Lambda := \left\{u \in \mathcal{B}^+(\mathbb{R}^n_+) : \exp(-\alpha_q)h \le u \le h\right\}.$$

Note that since for $u \in \Lambda$, we have $u \le h \le c\,\omega$, then (4.1) implies in particular that for $u \in \Lambda$

$$0 \le \varphi(., u) \le q. \tag{4.2}$$

We define the operator $T$ on $\Lambda$ by

$$Tu(x) := h(x) - V_q(qh)(x) + V_q[(q - \varphi(., u))u](x).$$

First, we claim that $\Lambda$ is invariant under $T$. Indeed, for each $u \in \Lambda$ we have

$$Tu(x) \le h(x) - V_q(qh)(x) + V_q(qu)(x) \le h(x).$$

Moreover, by (4.2) and Corollary 2.9, we obtain

$$Tu(x) \ge h(x) - V_q(qh)(x) \ge \exp(-\alpha_q)h(x).$$

Next, we prove that the operator $T$ is nondecreasing on $\Lambda$. Let $u, v \in \Lambda$ such that $u \le v$, then from (4.1) we have

$$Tv - Tu = V_q[(q - \varphi(., v))v - (q - \varphi(., u))u] \ge 0.$$

Now, we consider the sequence $(u_j)$ defined by $u_0 = h - V_q(qh)$ and $u_{j+1} = Tu_j$ for $j \in \mathbb{N}$. Then since $\Lambda$ is invariant under $T$, we obtain obviously that $u_1 = Tu_0 \ge u_0$ and so from the monotonicity of $T$, we deduce that

$$u_0 \le u_1 \le \dots \le u_j \le h.$$

Hence by (4.1) and the dominated convergence theorem, we deduce that the sequence $(u_j)$ converges to a function $u \in \Lambda$, which satisfies

$$u(x) = h(x) - V_q(qh)(x) + V_q[(q - \varphi(., u))u](x).$$

Or, equivalently

$$u - V_q(qu) = (h - V_q(qh)) - V_q(u\varphi(., u)).$$

Applying the operator $(I + V(q.))$ on both sides of the above equality and using (1.14), we deduce that $u$ satisfies

$$u = h - V(u\varphi(., u)). \tag{4.3}$$

Finally, we need to verify that $u$ is a positive continuous solution for the problem (1.11). Indeed, from (4.2), we have

$$u\varphi(., u) \le qh \le cq\omega. \tag{4.4}$$

This implies by Proposition 2.4 that either $u$ and $u\varphi(., u)$ are in $L^1_{\rm loc}(\mathbb{R}^n_+)$. Furthermore, from (4.4), we have that $\frac{1}{\omega}u\varphi(., u) \in \mathcal{M}_q$. Which implies by Lemma 4.1 that $\frac{1}{\omega}V(u\varphi(., u)) \in C_0(\overline{\mathbb{R}^n_+})$. In particular, we have $V(u\varphi(., u)) \in L^1_{\rm loc}(\mathbb{R}^n_+)$.

Hence, by (4.3), we obtain that $u$ is continuous on $\mathbb{R}^n_+$ and satisfies (in the sense of distributions) the elliptic differential equation

$$\Delta u - u\varphi(., u) = 0 \text{ in } \mathbb{R}^n_+.$$

On the other hand, since $\frac{1}{\omega}V(u\varphi(., u)) \in C_0(\overline{\mathbb{R}^n_+})$ and $Hg(x)$ is bounded on $\mathbb{R}^n_+$ and satisfies $\lim_{x_n\to 0} Hg(x) = g(x')$, we deduce easily that $\lim_{x_n\to 0} u(x) = (1-\alpha)g(x')$ and $\lim_{x_n\to+\infty} \frac{u(x)}{x_n} = \alpha\lambda$. This completes the proof. $\square$

**Example 4.2.** Let $\gamma > 1$, $\alpha \in [0, 1]$, $\beta > 0$ and $\lambda < 2 < \mu$. Let $g$ be a nontrivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$ and $p \in B^+(\mathbb{R}^n_+)$ satisfying

$$p(x) \preceq \frac{1}{(|x|+1)^{\mu-\lambda} x_n^{\lambda} (x_n+1)^{\gamma-1}}.$$

Then the problem

$$\begin{gathered}\Delta u - p(x)u^{\gamma}(x) = 0, \quad \text{in } \mathbb{R}^n_+\\ \lim_{x_n\to 0} u(x) = (1-\alpha)g(x'),\\ \lim_{x_n\to+\infty} \frac{u(x)}{x_n} = \alpha\beta,\end{gathered}$$

(in the sense of distributions) has a continuous positive solution $u$ satisfying

$$u(x) \sim \alpha\beta x_n + (1-\alpha)Hg(x).$$

## 5. Proof of Theorem 1.5

In this section, we need the following standard Lemma. For $u \in B(\mathbb{R}^n_+)$, put $u^+ = \max(u, 0)$.

**Lemma 5.1.** *Let $\varphi$ and $\psi$ satisfy (H6)–(H8). Assume that $\varphi \le \psi$ on $\mathbb{R}^n_+ \times \mathbb{R}_+$ and there exist continuous functions $u, v$ on $\mathbb{R}^n_+$ satisfying*

(a) $\Delta u - \varphi(., u^+) = 0 = \Delta v - \psi(., v^+)$ *in* $\mathbb{R}^n_+$
(b) $u, v \in C_b(\mathbb{R}^n_+)$
(c) $u \ge v$ *on* $\partial\mathbb{R}^n_+$.

*Then $u \ge v$ in $\mathbb{R}^n_+$.*

*Proof of Theorem 1.5.* An immediate consequence of the comparison principle in Lemma 5.1 is that problem (1.12) has at most one solution in $\mathbb{R}^n_+$. The existence of a such solution is assured by the Schauder fixed point Theorem. Indeed, to construct the solution, we consider the convex set

$$\Lambda = \{u \in C_b(\mathbb{R}^n_+) : u \le \|g\|_\infty\}.$$

We define the integral operator $T$ on $\Lambda$ by

$$Tu(x) = Hg(x) - V(\varphi(., u^+))(x).$$

Since $Hg(x) \le \|g\|_\infty$, for $x \in \mathbb{R}^n_+$, we deduce that for each $u \in \Lambda$,

$$Tu \le \|g\|_\infty, \text{ in } \mathbb{R}^n_+.$$

Furthermore, putting $q = \varphi(., \|g\|_\infty)$, we have by (H8) that $q \in K^\infty(\mathbb{R}^n_+)$. So by (H6), we deduce that $V(\varphi(., u^+)) \in V(\mathcal{M}_q)$. This together with the fact that $Hg \in C_b(\mathbb{R}^n_+)$ imply by Proposition 2.5 that $T\Lambda$ is relatively compact in $C_b(\mathbb{R}^n_+)$ and in particular $T\Lambda \subset \Lambda$.

From the continuity of $\varphi$ with respect to the second variable, we deduce that $T$ is continuous in $\Lambda$ and so it is a compact operator from $\Lambda$ to itself. Then by the Schauder fixed point Theorem, we deduce that there exists a function $u \in \Lambda$ satisfying

$$u(x) = Hg(x) - V(\varphi(., u^+))(x).$$

This implies, using Proposition 2.4 and the fact that $V(\varphi(., u^+)) \in C_0(\mathbb{R}^n_+)$, that $u$ satisfies in the sense of distributions

$$\begin{gathered}\Delta u - \varphi(., u^+) = 0\\ \lim_{x_n \to 0} u(x) = g(x').\end{gathered}$$

Hence by (H7) and Lemma 5.1, we conclude that $u \geq 0$ in $\mathbb{R}^n_+$. This completes the proof. □

**Corollary 5.2.** *Let $\varphi$ satisfying (H6)–(H8) and $g$ be a nontrivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$. Suppose that there exists a function $q \in K^\infty(\mathbb{R}^n_+)$ such that*

$$0 \leq \varphi(x,t) \leq q(x)t \quad \text{on } \mathbb{R}^n_+ \times [0, \|g\|_\infty]. \tag{5.1}$$

*Then the solution $u$ of* (1.12) *given by Theorem 1.5 satisfies*

$$e^{-\alpha_q} Hg(x) \leq u(x) \leq Hg(x).$$

*Proof.* Since $u$ satisfies the integral equation

$$u(x) = Hg(x) - V(\varphi(., u))(x),$$

using (1.15), we obtain

$$\begin{aligned}u - V_q(qu) &= (Hg - V_q(qHg)) - (V(\varphi(., u)) - V_q(qV(\varphi(., u))) \\ &= (Hg - V_q(qHg)) - V_q(\varphi(., u)).\end{aligned}$$

That is,

$$u = (Hg - V_q(qHg)) + V_q(qu - \varphi(., u)).$$

Now since $0 < u \leq \|g\|_\infty$ then by (5.1), the result follows from Corollary 2.9. □

**Example 5.3.** Let $g$ be a nontrivial nonnegative bounded continuous function in $\mathbb{R}^{n-1}$. Let $\sigma > 0$ and $q \in K^\infty(\mathbb{R}^n_+)$. Put $\varphi(x,t) = q(x)t^\sigma$. Then the problem

$$\begin{gathered}\Delta u - q(x)u^\sigma = 0, \quad \text{in } \mathbb{R}^n_+\\ \lim_{x_n \to 0} u(x) = g(x')\end{gathered}$$

(in the sense of distributions) has a positive bounded continuous solution $u$ in $\mathbb{R}^n_+$ satisfying

$$0 \leq Hg(x) - u(x) \leq \|g\|^\sigma_\infty Vq(x).$$

Furthermore, if $\sigma \geq 1$, we have by Corollary 5.2 that for each $x \in \mathbb{R}^n_+$

$$e^{-\alpha_q} Hg(x) \leq u(x) \leq Hg(x).$$

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## References


[1] Armitage, D. H., Gardiner, S. J.; Classical potentiel theory, Springer Verlag, 2001.

[2] Bachar, I., Mâagli, H.; Estimates on the Green's function and existence of positive solutions of nonlinear singular elliptic equations in the half space, to appear in Positivity (Articles in advance).

[3] Bachar, I., Mâagli, H., Mâatoug, L.; Positive solutions of nonlinear elliptic equations in a half space in $\mathbb{R}^2$, Electronic. J. Differential Equations. **2002**(2002) No. 41, 1-24 (2002).

[4] Bachar, I., Mâagli, H., Zribi, M.; Estimates on the Green function and existence of positive solutions for some polyharmonic nonlinear equations in the half space, Manuscripta math. **113**, 269-291 (2004).

[5] Brezis, H., Kamin, S.; Sublinear elliptic equations in $\mathbb{R}^n$, Manus. Math. **74**, 87-106 (1992).

[6] Chung, K.L., Zhao, Z.; From Brownian motion to Schrödinger's equation, Springer Verlag (1995).

[7] Crandall, M. G., Rabinowitz, P. H., Tartar, L.,; On a Dirichlet problem with a singular nonlinearity. Comm. Partial Differential Equations **2** 193-222 (1977).

[8] Diaz, J. I., Morel, J. M., Oswald, L.; An elliptic equation with singular nonlinearity, Comm. Partial Differential Equations **12**, 1333-1344 (1987).

[9] Dautray, R., Lions,J.L. et al.; Analyse mathématique et calcul numérique pour les sciences et les techniques, Coll.C.E.A Vol 2, L'opérateur de Laplace, Masson (1987).

[10] Edelson, A.; Entire solutions of singular elliptic equations, J. Math. Anal. appl. **139**, 523-532 (1989).

[11] Helms, L. L.; Introduction to Potential Theory. Wiley-Interscience, New york (1969).

[12] Kusano, T., Swanson, C.A.; Entire positive solutions of singular semilinear elliptic equations, Japan J. Math. **11** 145-155 (1985).

[13] Lair, A.V., Shaker, A.W.; Classical and weak solutions of a singular semilinear ellptic problem, J. Math. Anal. Appl. **211** 371-385 (1997).

[14] Lazer, A. C., Mckenna, P. J.; On a singular nonlinear elliptic boundary-value problem, Proc. Amer. Math. Soc. **111** 721-730 (1991).

[15] Mâagli, H.; Perturbation semi-linéaire des résolvantes et des semi-groupes, Potential Ana. **3**, 61-87 (1994).

[16] Mâagli, H., Masmoudi, S.; Positive solutions of some nonlinear elliptic problems in unbounded domain, Ann. Aca. Sci. Fen. Math. **29**, 151-166 (2004).

[17] Mâagli, H., Zribi, M., Existence and estimates of solutions for singular nonlinear elliptic problems, J. Math. Anal. Appl. **263** 522-542 (2001).

[18] Masmoudi, S.; On the existence of positive solutions for some nonlinear elliptic problems in unbounded domain in $\mathbb{R}^2$ to appear in Nonlinear.Anal.

[19] Neveu, J.; Potentiel markovian reccurent des chaînes de Harris, Ann. Int. Fourier **22 (**2), 85-130 (1972).

[20] Port, S. C., Stone, C. J.; Brownian motion and classical Potential theory, Academic Press (1978).

[21] Z. Zhao; On the existence of positive solutions of nonlinear elliptic equations. A probabilistic potential theory approach, Duke Math. J. 69 (1993) 247-258.



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