Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 131, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ACOUSTIC SCATTERING FROM OPEN CAVITIES IN THE TIME DOMAIN

BO CHEN, FUMING MA, YUKUN GUO

Abstract. This article concerns acoustic scattering from two-dimensional open cavities in the time domain. A transparent boundary condition is provided to reformulate the problem as an equivalent initial boundary value problem in the interior region of the open cavity. The well-posedness, that is, the existence, uniqueness and stability of the solution to the reduced problem, is proved via a "Laplace domain" to time domain analysis. Moreover, time domain boundary integral equations for the reduced problem are established.

1. Introduction

Scattering problems of acoustic waves have attracted extensive attention because of their significance in industry and medical equipment. Roughly speaking, acoustic scattering problems can be formulated in two scenarios: frequency domain problems with time-harmonic or nearly time-harmonic wave fields and time domain problems with time-dependent non-harmonic wave fields. Frequency domain problems, which mainly deal with the Helmholtz equation, can be taken as a simplification of time domain problems [11, 17, 28]. Analytical methods of time domain problems are in diverse forms such as the direct analysis [27] and the analysis related to the so-called "Laplace domain" problem given by the Fourier-Laplace transform [7, 24]. Since time domain scattering problems arise more naturally in diverse application areas, in recent years, time domain scattering and inverse scattering problems have attracted a lot of attention [12, 13, 14, 16].

We are mainly concerned about the scattering of acoustic waves in homogeneous and isotropic medium. In this article, the scatterers are chosen as two-dimensional open cavities embedded in the half-plane. The corresponding frequency domain problems have been studied in [1, 6, 20, 21]. In recent years, the time domain electromagnetic scattering from open cavities has been studied in [10, 18, 19, 26]. However, to the best of our knowledge, there is no rigorous mathematical analysis of the time domain acoustic scattering from open cavities in the literature.

In comparison to the scattering problem with a bounded scatterer, the scattering from open cavities is more challenging because of the inherent unboundedness. To overcome this difficulty, we develop a transparent boundary condition (TBC) to

2010 *Mathematics Subject Classification.* 35L20, 35B35, 35A22.
Key words and phrases. Time domain acoustic scattering; open cavity;
transparent boundary condition; well-posedness; boundary integral equation.

Submitted March 8, 2019. Published December 16, 2019.

reformulate the original scattering problem with unbounded scatterers in an equivalent initial boundary value problem in the interior of the open cavity. Then the well-posedness of the solution to the reduced problem is proved via the so-called "Laplace domain" problem given by the Fourier-Laplace transform of the time domain problem. Finally, the retarded potential boundary integral equation (RPBIE) method [14] is used to solve the reduced problem and the convolution quadrature (CQ) method is used for the calculation of the reduced problem. Detailed analysis of the single and double layer potentials and the boundary integral equations can be found in [2, 8, 15, 25]. Our work is inspired by the corresponding frequency domain investigations and the related time domain analyses of the Maxwell equations and the acoustic waves [3, 5, 24].

The outline of this article is as follows. The model scattering problem and the relevant spaces are shown in Section 2. Then a TBC is developed to reformulate the time domain scattering problem into a reduced initial boundary value problem in a bounded domain in Section 3. In Section 4, the well-posedness of the reduced problem and the equivalence of the two time domain scattering problems are proved. In Section 5, RPBIEs for the reduced time domain problem and the CQ method for the computation of the reduced problem are established.

2. Problem setting

2.1. **Model problem.** Consider the scattering of transient acoustic waves by an open cavity embedded in the ground plane. The outer space is filled with homogeneous background medium. The ground plane and the lower boundary of the cavity are assumed to be sound-soft. Adopting Cartesian coordinates (x_1, x_2, x_3), the cavity and the incident field are both assumed to be invariant with respect to x_3. Thus the three-dimensional scattering problem can be simplified to the two-dimensional case.

The incident field is chosen as the cylindrical wave emitted from a line source parallel to the x_3-axis in the half-space $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0\}$. Denote by $y_{3d} := (y, y_3)$ the coordinate of a source point with $y := (y_1, y_2) \in \mathbb{R}^2$. Consider that a causal signal $\lambda(t)$ (that is, $\lambda(t) = 0$ for $t < 0$) is simultaneously emitted from all the source points on the excitation line. Then the incident field has the form (see, e.g., [24])

$$u^i(t, x; y) := k(t, x; y) * \lambda(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^2 \backslash \{y\},$$

where $k * \lambda$ denotes the time convolution of k and λ and

$$k(t, x; y) := \frac{H(t - c^{-1}|x - y|)}{2\pi\sqrt{t^2 - c^{-2}|x - y|^2}}$$

is the Green's function of the operator $c^{-2}\partial_{tt} - \Delta$ in the free space $\mathbb{R} \times \mathbb{R}^2$. In this paper, Δ is the Laplacian in $\mathbb{R}^2$, $\partial_t = \partial/\partial t$, $\partial_{tt} = \partial^2/\partial t^2$, H is the Heaviside step function and c is the constant wave speed of the homogeneous background medium. For the sake of simplicity, we choose $c \equiv 1$ throughout the rest of this paper.

For the two-dimensional scattering problem, denote by $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ the upper half-plane and $\mathbb{R}^2_0 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$ the x_1-axis. The source point y is assumed to be located in the upper half-plane $\mathbb{R}^2_+$. For $x = (x_1, x_2)$, define $x^\rho := (x_1, -x_2)$. Then the reflected field is

$$u^\rho(t, x; y) := -k(t, x; y^\rho) * \lambda(t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^2 \backslash \{y^\rho\}.$$

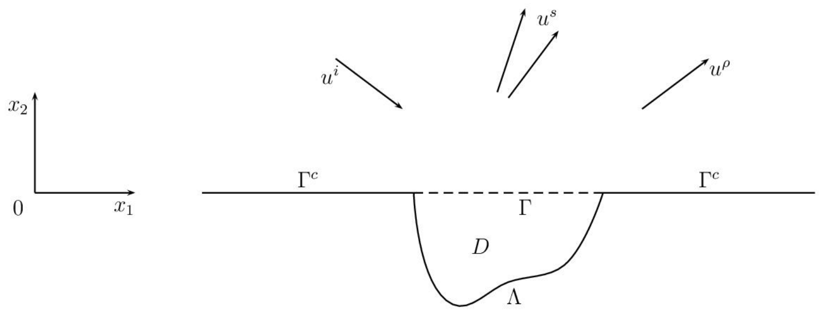

FIGURE 1. Sketch of the two dimensional scattering problem.

A geometrical illustration of the model scattering problem is shown in Figure 1. Denote by D the cavity with the boundary $\partial D = \Gamma \cup \Lambda$. The ground line is $\mathbb{R}_0^2 = \Gamma \cup \Gamma^c$. The lower boundary Λ of the cavity is assumed to be C^2-smooth and $\mathbb{R}_0^2$ is not a tangent to Λ.

The total field u, which is divided into the incident field u^i, the reflected field u^ρ and the scattered field u^s, satisfies

$$u_{tt} - \Delta u = f \quad \text{in } \mathbb{R} \times (D \cup \mathbb{R}_+^2), \tag{2.1}$$

$$u = 0 \quad \text{on } \mathbb{R} \times (\Gamma^c \cup \Lambda), \tag{2.2}$$

$$u(0, \cdot) = u_t(0, \cdot) = 0 \quad \text{in } D \cup \mathbb{R}_+^2, \tag{2.3}$$

where the source term f satisfies $\operatorname{supp}(f) \subset \mathbb{R}_+ \times \mathbb{R}_+^2$.

2.2. **Space-time Sobolev spaces.** We recall some notation concerning Sobolev spaces (refer to [9, 24] for details). Given a generic Lipschitz domain $\Omega \in \mathbb{R}^d$, we define

$$(u, v)_\Omega := \int_\Omega uv, \quad (\nabla u, \nabla v)_\Omega := \int_\Omega \nabla u \cdot \nabla v.$$

On this basis, we define

$$\|u\|_\Omega := [(u, \overline{u})_\Omega]^{1/2},$$

where $\overline{u}$ denotes the complex conjugate of u. Then the $H^1(\Omega)$-norm is defined as

$$\|u\|_{H^1(\Omega)} := \left(\|u\|_\Omega^2 + \|\nabla u\|_\Omega^2\right)^{1/2}.$$

For $c > 0$, we define $\underline{c} := \min\{1, c\}$ and

$$\|\!|u|\!\|_{c,\Omega} := \left(\|\nabla u\|_\Omega^2 + c^2 \|u\|_\Omega^2\right)^{1/2}.$$

We define the space

$$H_\Delta^1(\Omega) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$$

with the norm

$$\|u\|_\Delta := \left(\|u\|_\Omega^2 + \|\nabla u\|_\Omega^2 + \|\Delta u\|_\Omega^2\right)^{1/2}.$$

Then we consider the trace spaces on Γ. Denote by

$$\langle \xi, \eta \rangle_\Gamma := \int_\Gamma \xi \eta \, \mathrm{d}_\Gamma$$

the $L^2(\Gamma)$ inner product on Γ. We define

$$\tilde{H}^{1/2}(\Gamma) := \{V \in H^{1/2}(\Gamma) : E_0(V) \in H^{1/2}(\mathbb{R}_0^2)\}.$$

where E_0 is the extension operator from $H^{1/2}(\Gamma)$ to $H^{1/2}(\mathbb{R}_0^2)$ defined by

$$E_0(V)(x) := \begin{cases} V(x), & x \in \Gamma, \\ 0, & x \in \mathbb{R}_0^2 \backslash \Gamma. \end{cases}$$

In fact, $\tilde{H}^{1/2}(\Gamma)$ is the dual of $H^{-1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product (refer to [1, 15]). We define the spaces

$$\begin{gathered} H_E^1(D) := \{V \in H^1(D) : V|_\Lambda = 0 \text{ and } V|_\Gamma \in \tilde{H}^{1/2}(\Gamma)\}, \\ H_{\Delta,E}^1(D) := \{V \in H_\Delta^1(D) : V|_\Lambda = 0 \text{ and } V|_\Gamma \in \tilde{H}^{1/2}(\Gamma)\}, \\ H_{\Delta,E}^1(D \cup \mathbb{R}_+^2) := \{V \in H_\Delta^1(D \cup \mathbb{R}_+^2) : V = 0 \text{ on } \Gamma^c \cup \Lambda\} \end{gathered}$$

with the norm of $H^1(D)$, $H_\Delta^1(D)$ and $H^1(D \cup \mathbb{R}_+^2)$, respectively.

We denote $\mathbb{C}_\sigma := \{\omega \in \mathbb{C} : \operatorname{Im}(\omega) \geq \sigma > 0\}$ and in particular $\mathbb{C}_+ := \{\omega \in \mathbb{C} : \operatorname{Im}(\omega) > 0\}$. The Fourier-Laplace transform is defined by

$$\mathcal{L}[f](\omega) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} f(t)\, \mathrm{d}t, \quad \omega \in \mathbb{C}_\sigma. \tag{2.4}$$

Correspondingly, the inversion formula is

$$\mathcal{L}^{-1}[\varphi](t) := \frac{1}{2\pi} \int_{-\infty+\mathrm{i}\sigma}^{\infty+\mathrm{i}\sigma} \mathrm{e}^{-\mathrm{i}\omega t} \varphi(\omega)\, \mathrm{d}\omega. \tag{2.5}$$

To analyze the time domain scattering problem, we recall some notation concerning space-time Sobolev spaces. For a Hilbert space X, denote by $\mathcal{D}'(X)$ and $\mathcal{S}'(X)$ the space of X-valued distributions and tempered distributions on the real line, respectively. For $\sigma \in \mathbb{R}$, we define

$$\begin{gathered} \mathcal{L}'_\sigma(\mathbb{R}, X) := \{f \in \mathcal{D}'(X) : \mathrm{e}^{-\sigma t} f \in \mathcal{S}'(X)\}, \\ \mathcal{L}'_\sigma(\mathbb{R}_+, X) := \{f \in \mathcal{L}'_\sigma(\mathbb{R}, X) : f(t) = 0,\ \forall t < 0\}. \end{gathered}$$

For $\sigma \in \mathbb{R}$ and $p \in \mathbb{R}$, define the space

$$H_\sigma^p(\mathbb{R}, X) := \Big\{f \in \mathcal{L}'_\sigma(\mathbb{R}, X) : \int_{-\infty+\mathrm{i}\sigma}^{\infty+\mathrm{i}\sigma} |\omega|^{2p} \|\mathcal{L}[f](\omega)\|_X^2\, \mathrm{d}\omega < \infty\Big\}$$

with the norm

$$\|f\|_{H_\sigma^p(\mathbb{R},X)} := \Big(\int_{-\infty+\mathrm{i}\sigma}^{\infty+\mathrm{i}\sigma} |\omega|^{2p} \|\mathcal{L}[f](\omega)\|_X^2\, \mathrm{d}\omega\Big)^{1/2}. \tag{2.6}$$

Taking into consideration the causality, we define the space

$$H_\sigma^p(\mathbb{R}_+, X) := \big\{f \in H_\sigma^p(\mathbb{R}, X) : f(t) = 0,\ \forall t < 0\big\}$$

with the norm of $H_\sigma^p(\mathbb{R}, X)$.

3. Reduced problem

In this section, a TBC is proposed to get an equivalent initial boundary problem of (2.1)–(2.3) in the bounded domain D. Since the source points are separated from the unbounded scatterer, the scattered field $u^s := u - u^i - u^\rho$ satisfies the homogeneous wave equation [24]

$$u^s_{tt} - \Delta u^s = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^2_+. \tag{3.1}$$

We define $u^0 := u|_{\mathbb{R}\times\mathbb{R}^2_0}$. Note that $u^i + u^\rho = 0$ on $\mathbb{R} \times \mathbb{R}^2_0$. Then

$$u^s = u^0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2_0. \tag{3.2}$$

For the corresponding frequency domain problem, the Fourier transform with respect to x_1 is employed to get the differential equations of x_2 (refer to [1]). However, since an additional variable t is involved for time domain problems, an additional integral transform is needed for the analysis. After careful consideration, we find the Fourier-Laplace transform to be effective and befitting the well-posedness analysis.

3.1. TBC in the "Laplace domain".

Formally, taking the Fourier-Laplace transform of (3.1) and (3.2) with respect to t yields

$$\Delta U^s(\omega,\cdot) + \omega^2 U^s(\omega,\cdot) = 0 \quad \text{in } \mathbb{R}^2_+, \tag{3.3}$$

$$U^s(\omega,\cdot) = U^0(\omega,\cdot) \quad \text{on } \mathbb{R}^2_0, \tag{3.4}$$

where $\omega \in \mathbb{C}_+$, U^s and U^0 are respectively the Fourier-Laplace transform of u^s and u^0 with respect to t.

Furthermore, taking the Fourier transform of (3.3) and (3.4) with respect to x_1 yields

$$\big(\partial^2_{x_2} + (\omega^2 - \xi_1^2)\big)\,\mathcal{F}_{x_1}[U^s](\omega,\xi_1,x_2) = 0, \quad x_2 > 0, \tag{3.5}$$

$$\mathcal{F}_{x_1}[U^s](\omega,\xi_1,x_2) = \mathcal{F}_{x_1}[U^0](\omega,\xi_1,x_2), \quad x_2 = 0. \tag{3.6}$$

The causality of the time domain problem implies the finite energy of the acoustic wave at each time (refer to [24]). Then $\omega \in \mathbb{C}_+$ and the finite energy imply

$$\mathcal{F}_{x_1}[U^s] = \mathrm{e}^{\mathrm{i}\omega\sqrt{1-\xi_1^2/\omega^2}x_2}\mathcal{F}_{x_1}[U^0].$$

Here $\sqrt{a}$ is the principle square root of $a \in \mathbb{C}$, that is, $\mathrm{Re}(\sqrt{a}) \ge 0$. For $\omega \in \mathbb{C}_+$, set $\omega = \eta + \mathrm{i}\sigma$, $\sigma > 0$. Note that

$$\eta\,\mathrm{Im}\,\big(1 - \frac{\xi_1^2}{\omega^2}\big) = \frac{2\xi_1^2\eta^2\sigma}{(\eta^2+\sigma^2)^2} \ge 0.$$

Moreover, the definition of the principle square root implies

$$\mathrm{Im}\,\big(1 - \frac{\xi_1^2}{\omega^2}\big)\,\mathrm{Im}\,\Big((\sqrt{1 - \frac{\xi_1^2}{\omega^2}}\Big) \ge 0.$$

Thus

$$\mathrm{Re}\,\Big(\mathrm{i}\omega\sqrt{1 - \frac{\xi_1^2}{\omega^2}}\Big) = -\sigma\,\mathrm{Re}\,\Big(\sqrt{1 - \frac{\xi_1^2}{\omega^2}}\Big) - \eta\,\mathrm{Im}\,\Big(\sqrt{1 - \frac{\xi_1^2}{\omega^2}}\Big) \le 0.$$

The inverse Fourier transform gives

$$U^s(\omega,x_1,x_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi_1 x_1}\mathrm{e}^{\mathrm{i}\omega\sqrt{1-\xi_1^2/\omega^2}x_2}\mathcal{F}_{x_1}[U^0](\omega,\xi_1,0)\,\mathrm{d}\xi_1.$$

Then we have

$$\frac{\partial U^s}{\partial \nu}\Big|_{x_2=0} = \mathcal{T}^F U^0, \tag{3.7}$$

where the operator $\mathcal{T}^F$ is defined by

$$(\mathcal{T}^F V)(\omega, x_1, x_2) := \frac{\mathrm{i}\omega}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi_1 x_1}\sqrt{1-\xi_1^2/\omega^2}\,\mathcal{F}_{x_1}[E_0(V)](\omega,\xi_1,0)\,\mathrm{d}\xi_1.$$

We define

$$G(\omega,\cdot) := \frac{\partial(U^i(\omega,\cdot)+U^\rho(\omega,\cdot))}{\partial\nu} \quad \text{on } \Gamma,$$

where U^i and U^ρ are respectively the Fourier-Laplace transform of u^i and u^ρ with respect to t. Then we have the boundary condition

$$\frac{\partial U(\omega,\cdot)}{\partial\nu} = \mathcal{T}^F(\omega,\cdot)U(\omega,\cdot) + G(\omega,\cdot) \quad \text{on } \Gamma,$$

which is a TBC in the so-called "Laplace domain".

3.2. **TBC in the time domain.** Then the inverse Fourier-Laplace transform is needed to formulate a TBC in the time domain. Note that there are restrictions to use the strong inversion formula (2.5). Consider the inverse Fourier-Laplace transform of $F(\omega)$, $\mathrm{Im}(\omega)=\sigma>0$. Assume that $F(\omega)$ satisfies

$$|F(\omega)| \le C_F(\sigma)|\omega|^\mu, \tag{3.8}$$

in which $\mu\in\mathbb{R}$ and $C_F:\mathbb{R}_+\to\mathbb{R}_+$ is a non-increasing function such that

$$C_F(\sigma) \le \frac{M}{\sigma^\iota}, \quad \forall\sigma\in(0,1),$$

where ι and M are positive constants.

Again, we set $\omega=\eta+\mathrm{i}\sigma$. For $\mu<-1$, the inverse Fourier-Laplace transform is defined as

$$f(t)=\mathcal{L}^{-1}[F](t) := \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\sigma}^{\infty+\mathrm{i}\sigma} \mathrm{e}^{-\mathrm{i}\omega t}F(\omega)\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\sigma t}\mathrm{e}^{-\mathrm{i}\eta t}F(\eta+\mathrm{i}\sigma)\mathrm{d}\eta.$$

Let

$$\chi = \frac{\sigma^2}{\sigma^2+\eta^2}.$$

Assumption (3.8) implies that $f(t)$ is well defined for $t\in\mathbb{R}$ and

$$|f(t)| \le \frac{1}{2\pi}C_F(\sigma)\sigma^{1+\mu}\mathrm{e}^{\sigma t}B\Big(\frac{1}{2}, -\frac{\mu+1}{2}\Big),$$

where B is the Euler beta function,

$$B(x,y) = \int_0^1 \chi^{x-1}(1-\chi)^{y-1}\,\mathrm{d}\chi.$$

A contour integration argument implies that $f(t)$ is independent of $\sigma>0$. Taking the limit $\sigma\to\infty$ implies that $f(t)=0$, $\forall t<0$. Moreover, set $\sigma=t^{-1}$ for $t>1$, we have the estimation

$$|f(t)| \le Mt^{\iota-(\mu+1)}B\left(\frac{1}{2}, -\frac{\mu+1}{2}\right).$$

Then $f(t)$ is a causal function with polynomial growth. Obviously we have $f(t)\in\mathcal{L}'_\sigma(\mathbb{R}_+, X)$.

For $\mu \geq -1$, we denote $F(\omega) = (-\mathrm{i}\omega)^m F_m(\omega)$ with the integer $m > \mu + 1$. Then

$$|F_m(\omega)| \leq C_F(\sigma)|\omega|^{\mu-m}$$

with $\mu - m < -1$. Then there exists a causal function $f_m(t)$ with polynomial growth such that $f_m(t) = \mathcal{L}^{-1}[F_m](t)$. On account of $\mathcal{L}[f_m^{(m)}](\omega) = (-\mathrm{i}\omega)^m \mathcal{L}[f_m](\omega)$, the inverse Fourier-Laplace transform of $F(\omega)$ is $f(t) = f_m^{(m)}(t)$. Then we also have $f(t) \in \mathcal{L}'_\sigma(\mathbb{R}_+, X)$ for this case. For more details, see the analysis of Lubich [23] and Sayas [24].

Based on the above analysis, there is always an inverse Fourier-Laplace transform of $F(\omega)$ with or without the strong inversion formula (2.5). For simplification, assume that the strong inversion formula can be used throughout this paper. Back to time domain, we can get the following definition of the boundary operator and the TBC.

Definition 3.1. The boundary operator $\mathcal{T} : \mathcal{L}'_\sigma(\mathbb{R}, \tilde{H}^{1/2}(\Gamma)) \to \mathcal{L}'_\sigma(\mathbb{R}, H^{-1/2}(\Gamma))$ is defined as

$$\mathcal{T}v := \frac{1}{4\pi^2} \int_{-\infty+\mathrm{i}\sigma}^{\infty+\mathrm{i}\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t + i\xi_1 x_1} \mathrm{i}\omega \sqrt{1 - \xi_1^2/\omega^2} \mathcal{L} \circ \mathcal{F}_{x_1}[E_0(v)](\omega, \xi_1, 0)\, \mathrm{d}\xi_1 \mathrm{d}\omega.$$

Definition 3.2. The transparent boundary condition (TBC) of the time domain scattering problem (2.1)–(2.3) is defined as

$$\frac{\partial u}{\partial \nu}\Big|_\Gamma = \mathcal{T}u + g.$$

where $\mathcal{T}$ is the boundary operator and

$$g := \frac{\partial(u^i + u^\rho)}{\partial \nu}\Big|_{x_2=0} = 2\frac{\partial u^i}{\partial \nu}\Big|_{x_2=0} \quad \text{on } \mathbb{R} \times \Gamma.$$

Then we have a new time domain scattering problem: Find u such that

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R} \times D, \tag{3.9}$$

$$u = 0 \quad \text{on } \mathbb{R} \times \Lambda, \tag{3.10}$$

$$\frac{\partial u}{\partial \nu} = \mathcal{T}u + g \quad \text{on } \mathbb{R} \times \Gamma, \tag{3.11}$$

$$u(0, \cdot) = u_t(0, \cdot) = 0 \quad \text{in } D. \tag{3.12}$$

4. Well-posedness

This section concerns the well-posedness of problem (3.9)–(3.12), and the equivalence between the scattering problems (2.1)–(2.3) and (3.9)–(3.12). The corresponding generalized equations are also considered here.

For the time domain scattering problem (2.1)–(2.3), we study the set of solutions $u \in \mathcal{L}'_\sigma(\mathbb{R}_+, H^1_{\Delta,E}(D \cup \mathbb{R}^2_+))$ such that

$$\ddot{u} - \Delta u = f \quad \text{in } \mathbb{R} \times (D \cup \mathbb{R}^2_+), \tag{4.1}$$

where $\ddot{u}$ is the generalized second order derivative of u with respect to t.

Similarly, for (3.9)–(3.12), we study the set of solutions $u \in \mathcal{L}'_\sigma(\mathbb{R}_+, H^1_{\Delta,E}(D))$ such that

$$\ddot{u} - \triangle u = 0 \quad \text{in } \mathbb{R} \times D, \tag{4.2}$$

$$\frac{\partial u}{\partial \nu} = \mathcal{T}u + g \quad \text{on } \mathbb{R} \times \Gamma. \tag{4.3}$$

Let $u \in \mathcal{L}'_\sigma(\mathbb{R}_+, H^1_{\Delta,E}(D))$ be a solution of problem (4.2)–(4.3). The Fourier-Laplace transform implies that $U(\omega,\cdot) \in H^1_{\Delta,E}(D)$ satisfies

$$\triangle U(\omega,\cdot) + \omega^2 U(\omega,\cdot) = 0 \quad \text{in } D, \tag{4.4}$$

$$\frac{\partial U(\omega,\cdot)}{\partial \nu} = \mathcal{T}^F U(\omega,\cdot) + G(\omega,\cdot) \quad \text{on } \Gamma. \tag{4.5}$$

Firstly, we give the following property of the operator $\mathcal{T}^F$.

Lemma 4.1. *Let $\omega \in \mathbb{C}$. The operator $\mathcal{T}^F : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ satisfies*

$$\operatorname{Im}\Big(\overline{\omega}\langle \mathcal{T}^F V(\omega,\cdot), \overline{V(\omega,\cdot)}\rangle_\Gamma\Big) \geq 0.$$

Proof. For $V(\omega,\cdot) \in C_0^\infty(\Gamma)$, we have

$$\begin{aligned}
&\langle \mathcal{T}^F V(\omega,\cdot), \overline{V(\omega,\cdot)}\rangle_\Gamma \\
&= \frac{\mathrm{i}\omega}{2\pi}\int_{\mathbb{R}} \sqrt{1-\xi_1^2/\omega^2}\,\mathcal{F}_{x_1}[E_0(V(\omega,\cdot))]\int_\Gamma \mathrm{e}^{\mathrm{i}\xi_1 x_1}\overline{V(\omega,\cdot)}\mathrm{d}s_x \mathrm{d}\xi_1 \\
&= \frac{\mathrm{i}\omega}{2\pi}\int_{\mathbb{R}} \sqrt{1-\xi_1^2/\omega^2}\,\mathcal{F}_{x_1}[E_0(V(\omega,\cdot))]\overline{\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi_1 x_1}E_0(V(\omega,\cdot))\mathrm{d}x_1}\mathrm{d}\xi_1 \\
&= \frac{\mathrm{i}\omega}{2\pi}\int_{\mathbb{R}} \sqrt{1-\xi_1^2/\omega^2}\,\left|\mathcal{F}_{x_1}[E_0(V(\omega,\cdot))]\right|^2 \mathrm{d}\xi_1.
\end{aligned}$$

Note that $V(\omega,\cdot) \in C_0^\infty(\Gamma)$ is dense in $\tilde{H}^{1/2}(\Gamma)$. Therefore, for $V(\omega,\cdot) \in \tilde{H}^{1/2}(\Gamma)$, we also have

$$\langle \mathcal{T}^F V(\omega,\cdot), \overline{V(\omega,\cdot)}\rangle_\Gamma = \frac{\mathrm{i}\omega}{2\pi}\int_{\mathbb{R}} \sqrt{1-\xi_1^2/\omega^2}\,\left|\mathcal{F}_{x_1}[E_0(V(\omega,\cdot))]\right|^2 \mathrm{d}\xi_1.$$

Thus

$$\begin{aligned}
\operatorname{Im}\Big(\overline{\omega}\langle \mathcal{T}^F V(\omega,\cdot), \overline{V(\omega,\cdot)}\rangle_\Gamma\Big) &= \operatorname{Re}\Big(\frac{|\omega|^2}{2\pi}\int_{\mathbb{R}} \sqrt{1-\xi_1^2/\omega^2}|\mathcal{F}_{x_1}[E_0(V(\omega,\cdot))]|^2 \mathrm{d}\xi_1\Big) \\
&\geq 0.
\end{aligned}$$

This completes the proof. □

Moreover, we need the following property of the norms given by [24]. Let $\omega \in \mathbb{C}$ and $\operatorname{Im}(\omega) = \sigma > 0$. The norms $\|\!|\cdot|\!\|_{|\omega|,D}$ and $\|\cdot\|_{H^1(D)}$ satisfy

$$\underline{\sigma}\|V\|_{H^1(D)} \leq \|\!|V|\!\|_{|\omega|,D} \leq \frac{|\omega|}{\underline{\sigma}}\|V\|_{H^1(D)}. \tag{4.6}$$

Then we give the following theorem about the well-posedness of problem (4.4)–(4.5).

Proposition 4.2. *Let $\omega \in \mathbb{C}$, $\operatorname{Im}(\omega) = \sigma > 0$ and $G(\omega,\cdot) \in H^{-1/2}(\Gamma)$. There exists a unique solution $U(\omega,\cdot) \in H^1_{\Delta,E}(D)$ of problem (4.4)–(4.5). Moreover, there exists a constant $C_{\sigma,D}$ depending only on σ and D such that*

$$\|U(\omega,\cdot)\|_{H^1_{\Delta,E}(D)} \leq C_{\sigma,D}|\omega|^2\|G(\omega,\cdot)\|_{H^{-1/2}(\Gamma)}.$$

Proof. Consider the variational form associate with problem (4.4)–(4.5). $U(\omega,\cdot) \in H^1_E(D)$ solves (4.4)–(4.5) if and only if

$$\begin{aligned}
&A\Big(U(\omega,\cdot), \overline{V(\omega,\cdot)}\Big) \\
&:= \Big(\nabla U(\omega,\cdot), \nabla\overline{V(\omega,\cdot)}\Big)_D - \omega^2\Big(U(\omega,\cdot), \overline{V(\omega,\cdot)}\Big)_D - \langle \mathcal{T}^F(\omega)U(\omega,\cdot), \overline{V(\omega,\cdot)}\rangle_\Gamma
\end{aligned}$$

$$= \langle G(\omega,\cdot), \overline{V(\omega,\cdot)}\rangle_\Gamma, \quad \forall V(\omega,\cdot) \in H^1_E(D).$$

It follows from Lemma 4.1 and (4.6) that

$$\begin{aligned}
&\operatorname{Im}\Big(-\overline{\omega}A\Big(U(\omega,\cdot), \overline{U(\omega,\cdot)}\Big)\Big)\\
&= \operatorname{Im}\Big(- \overline{\omega}\Big(\nabla U(\omega,\cdot), \nabla\overline{U(\omega,\cdot)}\Big)_D + \overline{\omega}\omega^2\Big(U(\omega,\cdot), \overline{U(\omega,\cdot)}\Big)_D\\
&\quad + \overline{\omega}\langle \mathcal{T}^F(\omega)U(\omega,\cdot), \overline{U(\omega,\cdot)}\rangle_\Gamma\Big)\\
&\geq \sigma\|\nabla U(\omega,\cdot)\|_D^2 + \sigma|\omega|^2\|U(\omega,\cdot)\|_D^2\\
&= \sigma|\!|\!|U(\omega,\cdot)|\!|\!|^2_{|\omega|,D}.
\end{aligned}$$

Then the unique solvability of (4.4)–(4.5) in $H^1_E(D)$ follows from a Lax-Milgram argument. Moreover, it follows from the trace theorem and (4.6) that

$$\begin{aligned}
|\!|\!|U(\omega,\cdot)|\!|\!|^2_{|\omega|,D} &\leq \frac{1}{\sigma}\operatorname{Im}\Big(-\overline{\omega}\langle G(\omega,\cdot), \overline{U(\omega,\cdot)}\rangle_\Gamma\Big)\\
&\leq \frac{|\omega|}{\sigma}\|G(\omega,\cdot)\|_{H^{-1/2}(\Gamma)}\|U(\omega,\cdot)\|_{\tilde{H}^{1/2}(\Gamma)}\\
&\leq C_D\frac{|\omega|}{\sigma}\|G(\omega,\cdot)\|_{H^{-1/2}(\Gamma)}\|U(\omega,\cdot)\|_{H^1_E(D)}\\
&\leq C_D\frac{|\omega|}{\sigma\underline{\sigma}}\|G(\omega,\cdot)\|_{H^{-1/2}(\Gamma)}|\!|\!|U(\omega,\cdot)|\!|\!|_{|\omega|,D},
\end{aligned}$$

where C_D is a constant depending only on D. Thus

$$|\!|\!|U(\omega,\cdot)|\!|\!|^2_{|\omega|,D} \leq C_D\frac{|\omega|}{\sigma\underline{\sigma}}\|G(\omega,\cdot)\|_{H^{-1/2}(\Gamma)}.$$

Then (4.6) implies

$$\|U(\omega,\cdot)\|^2_{H^1_E(D)} \leq C_D\frac{|\omega|}{\sigma\underline{\sigma}^2}\|G(\omega,\cdot)\|_{H^{-1/2}(\Gamma)}.$$

We have proved that there exists a unique solution $U(\omega,\cdot) \in H^1_E(D)$ of problem (4.4)–(4.5). Then (4.4) implies $\Delta U(\omega,\cdot) = -\omega^2 U(\omega,\cdot) \in H^1_E(D) \subset L^2(D)$. Thus $U(\omega,\cdot) \in H^1_{\Delta,E}(D)$, which means there exists a unique solution $U(\omega,\cdot) \in H^1_{\Delta,E}(D)$ of (4.4)–(4.5). Moreover, the definition of the norm $|\!|\!|\cdot|\!|\!|_{c,\omega}$ implies

$$\begin{aligned}
\|\Delta U(\omega,\cdot)\|_D &= \|\omega^2 U(\omega,\cdot)\|_D \leq |\omega|^2\|U(\omega,\cdot)\|_D\\
&\leq |\omega|\,|\!|\!|U(\omega,\cdot)|\!|\!|_{|\omega|,D}\\
&\leq C_D\frac{|\omega|^2}{\sigma\underline{\sigma}}\|G(\omega,\cdot)\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|U(\omega,\cdot)\|^2_{H^1_{\Delta,E}(D)} &= \|U(\omega,\cdot)\|^2_{H^1_E(D)} + \|\Delta U(\omega,\cdot)\|^2_D\\
&\leq C_D^2\frac{|\omega|^2}{\sigma^2\underline{\sigma}^2}\Big(\frac{1}{\underline{\sigma}^2} + |\omega|^2\Big)\|G(\omega,\cdot)\|^2_{H^{-1/2}(\Gamma)}\\
&\leq 2C_D^2\frac{|\omega|^4}{\sigma^2\underline{\sigma}^6}\|G(\omega,\cdot)\|^2_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

Thus

$$\|U(\omega,\cdot)\|_{H^1_{\Delta,E}(D)} \leq C_{\sigma,D}|\omega|^2\|G(\omega,\cdot)\|_{H^{-1/2}(\Gamma)},$$

where $C_{\sigma,D} = \sqrt{2}C_D/(\sigma\underline{\sigma}^3)$. This completes the proof. □

Consider the time domain scattering problem (4.2)–(4.3). We need the following lemma for the time domain analysis.

Lemma 4.3 ([23]). *Let $p, r \in \mathbb{R}$, $\omega \in \mathbb{C}$ and $\operatorname{Im}(\omega) > \sigma_0 > 0$. Let $F(\omega) \in \mathcal{B}(X, Y)$ be a bounded operator between the Hilbert spaces X and Y. Define $f(t) := \mathcal{L}^{-1}[F(\omega)](t)$ and $F_t g := \int_{-\infty}^{\infty} f(t)g(\cdot - t)\mathrm{d}t$, and assume that*

$$\|F(\omega)\|_{\mathcal{B}(X,Y)} \le C|\omega|^r, \quad \operatorname{Im}(\omega) > \sigma_0.$$

Then, for $\sigma > \sigma_0$, F_t is a bounded operator from $H_\sigma^{p+r}(\mathbb{R}_+, X)$ to $H_\sigma^p(\mathbb{R}_+, Y)$.

Then we have the following results in the time domain.

Theorem 4.4. *Let $\sigma > \sigma_0 > 0$, $p \in \mathbb{R}$ and $g \in H_\sigma^{p+2}(\mathbb{R}_+, H^{-1/2}(\Gamma))$. Then there exists a unique solution $u \in H_\sigma^p(\mathbb{R}_+, H^1_{\Delta,E}(D))$ of the scattering problem (4.2)–(4.3). Moreover, there exists a constant $C'_{\sigma_0,D}$ depending only on σ_0 and D such that*

$$\|u\|_{H_\sigma^p(\mathbb{R}_+, H^1_{\Delta,E}(D))} \le C'_{\sigma_0,D}\|g\|_{H_{\sigma_0}^{p+2}(\mathbb{R}_+, H^{-1/2}(\Gamma))}.$$

Proof. For $\operatorname{Im}(\omega) > \sigma_0 > 0$, it follows from Proposition 4.2 that there exists a unique solution $U(\omega, \cdot) \in H^1_{\Delta,E}(D)$ of problem (4.4)–(4.5) and

$$\|U(\omega, \cdot)\|_{H^1_{\Delta,E}(D)} \le C_{\sigma_0,D}|\omega|^2\|G(\omega, \cdot)\|_{H^{-1/2}(\Gamma)},$$

where $C_{\sigma_0,D} = 2C_D/(\sigma_0\underline{\sigma_0}^3)$, and C_D is the same constants in the proof of Proposition 4.2. Denote by $F(\omega, \cdot) \in \mathcal{B}(H^{-1/2}(\Gamma), H^1_E(D))$ the solution operator of (4.4)–(4.5) such that $U(\omega, \cdot) = F(\omega, \cdot)G(\omega, \cdot)$. Then

$$\|F(\omega, \cdot)\|_{\mathcal{B}(H^{-1/2}(\Gamma), H^1_{\Delta,E}(D))} \le C_{\sigma_0,D}|\omega|^2, \quad \operatorname{Im}(\omega, \cdot) > \sigma_0.$$

Using Proposition 4.2 and Lemma 4.3, an inverse Fourier-Laplace argument implies that $u = f * g \in H_\sigma^p(\mathbb{R}_+, H^1_E(D))$ is the unique solution of the scattering problem (4.2)–(4.3), in which f and g are the inverse Fourier-Laplace transform of $F(\omega, \cdot)$ and $G(\omega, \cdot)$, respectively. Moreover, Lemma 4.3 implies

$$\|u\|_{H_\sigma^p(\mathbb{R}_+, H^1_{\Delta,E}(D))} \le C'_{\sigma_0,D}\|g\|_{H_{\sigma_0}^{p+2}(\mathbb{R}_+, H^{-1/2}(\Gamma))}.$$

where $C'_{\sigma_0,D}$ is a constant depending only on σ_0 and D. □

Next we provide a proposition for the equivalence between the time domain scattering problems (4.1) and (4.2)–(4.3).

Proposition 4.5. *Let $\sigma > 0$, $p \in \mathbb{R}$ and $g \in H_\sigma^{p+2}(\mathbb{R}_+, H^{-1/2}(\Gamma))$. If $u_1 \in H_\sigma^p(\mathbb{R}_+, H^1_{\Delta,E}(D \cup \mathbb{R}^2_+))$ is the solution of the scattering problem (4.1) and $u_2 \in H_\sigma^p(\mathbb{R}_+, H^1_{\Delta,E}(D))$ is the solution of the scattering problem (4.2)–(4.3). Then*

$$u_1 = u_2 \quad \text{in } \mathbb{R} \times D.$$

Proof. If $u_1 \in H_\sigma^p(\mathbb{R}_+, H^1_{\Delta,E}(D \cup \mathbb{R}^2_+))$ is the solution of (4.1), combining $\operatorname{supp}(f) \subset \mathbb{R}_+ \times \mathbb{R}^2_+$ with the analysis of the TBC (4.3), we obtain that $u_1|_{\mathbb{R}\times D} \in H_\sigma^p(\mathbb{R}_+, H^1_{\Delta,E}(D))$ is the solution of (4.2)–(4.3).

Using Theorem 4.4, the unique solvability of the scattering problem (4.2)–(4.3) implies

$$u_1 = u_2 \quad \text{in } \mathbb{R} \times D.$$

The proof is complete. □

5. Boundary integral equations

In this section, we will show the process for solving the time domain scattering problem (3.9)–(3.12) and the CQ method to turn the calculation of the time domain problem into that of the corresponding frequency domain problems.

The retarded single layer potential on ∂D is defined as (see [9])

$$(SL_{\partial D}\phi)(t,x) := \int_{\partial D}\int_0^t k(t-\tau,|x-y|)\phi(y,\tau)\,\mathrm{d}\tau\mathrm{d}s_y, \quad t\in\mathbb{R},\ x\in\mathbb{R}^2\backslash\partial D.$$

The retarded double layer potential on ∂D is

$$(DL_{\partial D}\phi)(t,x) := \int_{\partial D}\int_0^t \frac{\partial k(t-\tau,|x-y|)}{\partial\nu(y)}\phi(y,\tau)\,\mathrm{d}\tau\mathrm{d}s_y, \quad t\in\mathbb{R},\ x\in\mathbb{R}^2\backslash\partial D,$$

where $\partial k/\partial\nu(y)$ is the normal derivative of k on ∂D with respect to y. Also of importance are the single and double layer operators on ∂D, which are defined as

$$(S_{\partial D}\phi)(t,x) := \int_{\partial D}\int_0^t k(t-\tau,|x-y|)\phi(y,\tau)\,\mathrm{d}\tau\mathrm{d}s_y, \quad t\in\mathbb{R},\ x\in\partial D$$

and

$$(K_{\partial D}\phi)(t,x) := \int_{\partial D}\int_0^t \frac{\partial k(t-\tau,|x-y|)}{\partial\nu(y)}\phi(y,\tau)\,\mathrm{d}\tau\mathrm{d}s_y, \quad t\in\mathbb{R},\ x\in\partial D,$$

respectively.

Denote by $\gamma^- u$ and $\gamma^+ u$ the restriction of u to ∂D from the interior and exterior, and by $\partial_\nu^- u$ and $\partial_\nu^+ u$ the normal derivatives on ∂D from the interior and exterior, respectively. The jumps are defined as

$$[\![u]\!] := \gamma^- u - \gamma^+ u, \quad [\![\partial_\nu u]\!] := \partial_\nu^- u - \partial_\nu^+ u.$$

The Kirchhoff's formula [24] for the solution of the wave equation is

$$u = SL_{\partial D}[\![\partial_\nu u]\!] - DL_{\partial D}[\![u]\!] \quad \text{in } \mathbb{R}\times D. \tag{5.1}$$

On the boundary ∂D, we have

$$\frac{1}{2}u = S_{\partial D}[\![\partial_\nu u]\!] - K_{\partial D}[\![u]\!] \quad \text{on } \mathbb{R}\times\partial D.$$

We are concerned with the time domain scattering problem in the bounded domain D. Assume that $u\equiv 0$ in $\mathbb{R}\times(\mathbb{R}^2\backslash\overline{D})$. Then

$$[\![\partial_\nu u]\!] = \partial_\nu^- u, \quad [\![u]\!] = \gamma^- u \quad \text{on } \mathbb{R}\times\partial D.$$

For the sake of simplicity, we write $\partial_\nu u = \partial_\nu^- u$, $u = \gamma^- u$ on ∂D. Then we have the following RPBIEs for the scattering problem (3.9)–(3.12):

$$\frac{1}{2}u = S_\Gamma(\mathcal{T}u + g) - K_\Gamma u + S_\Lambda\partial_\nu u \quad \text{on } \mathbb{R}\times\Gamma, \tag{5.2}$$

$$0 = S_\Gamma(\mathcal{T}u + g) - K_\Gamma u + S_\Lambda\partial_\nu u \quad \text{on } \mathbb{R}\times\Lambda. \tag{5.3}$$

We can get $u|_{\mathbb{R}\times\Gamma}$ and $\partial_\nu u|_{\mathbb{R}\times\Lambda}$ by solving (5.2)–(5.3). Then $\partial_\nu u|_{\mathbb{R}\times\Gamma}$ is given by (3.11) and $u|_{\mathbb{R}\times D}$ is given by the Kirchhoff's formula (5.1).

We recall the CQ method (refer to [4, 22]) for the time discretization of the RPBIEs (5.2)–(5.3). The time discretization is implemented in $[0,T]$. The terminal time T is chosen such that the energy of the scattered data inside the interested domain is negligible when $t>T$. We have the discretization

$$t_j = j\kappa,\ j=0,1,\dots,N,\ \kappa = T/N.$$

To solve an integral equation with convolution structure such as

$$S_\Gamma u = h, \quad \text{on } \mathbb{R}\times\Gamma,$$

the CQ method leads to the decoupled problems [4]

$$S_\Gamma^F \hat{u}_l(\omega_l, x) = \hat{h}_l(\omega_l, x), \quad x \in \Gamma,\ l = 0, 1, \dots, N,$$

where S_Γ^F is the Fourier-Laplace transform of the operator S_Γ, $\hat{u}_l(\cdot)$ and $\hat{h_l}$ are, respectively, the discrete Fourier transform of $u_j(\cdot) := u(t_j, \cdot)$ and $h_j := h(t_j)$ with respect to j, $\omega_l \in \mathbb{C}$ are constants depending on the time discretization. We choose

$$\omega_l = \frac{\mathrm{i}}{2\kappa}(\xi_l^2 - 4\xi_l + 3),$$

where

$$\xi_l = \gamma \mathrm{e}^{-\mathrm{i}\frac{2\pi l}{N_T}}, \quad l = 0, \dots, N_T - 1.$$

In this paper, we suggest to use the same strategy as that in [4] for choosing the stability parameter γ.

Note that

$$\begin{aligned} \mathcal{T}v &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi_1 x_1}\mathcal{L}^{-1}\Big[\mathrm{i}\omega\sqrt{1-\xi_1^2/\omega^2}\,\mathcal{L}\circ\mathcal{F}_{x_1}[E_0(v)](\omega,\xi_1,0)\Big]\mathrm{d}\xi_1 \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi_1 x_1}\mathcal{L}^{-1}\Big[\mathrm{i}\omega\sqrt{1-\xi_1^2/\omega^2}\Big] * \mathcal{F}_{x_1}[E_0(v)](t,\xi_1,0)\mathrm{d}\xi_1 \end{aligned}$$

and

$$g = 2\frac{\partial u^i}{\partial \nu} = 2\partial_\nu k(t, |x-z|) * \lambda(t), \quad x \in \mathbb{R}_0^2,$$

where $z := (z_1, z_2)$ is the source point. Then we obtain

$$S_\Gamma \mathcal{T} v = \frac{1}{2\pi}\int_\Gamma\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi_1 x_1} k(t,|x-y|) * \mathcal{L}^{-1}\Big[\mathrm{i}\omega\sqrt{1-\xi_1^2/\omega^2}\Big] * \mathcal{F}_{x_1}[E_0(v)](t,\xi_1,0)\mathrm{d}\xi_1\mathrm{d}s_y$$

and

$$S_\Gamma g = 2\int_\Gamma k(t,|x-y|) * \partial_\nu k(t,|y-z|) * \lambda(t)\mathrm{d}s_y.$$

To solve the integral equations (5.2)–(5.3), the CQ method leads to the decoupled problems

$$\frac{1}{2}\hat{u}_l(\omega_l,\cdot) = S_\Gamma^F\{\mathcal{T}^F\hat{u}_l + G^F\hat{\lambda}_l\}(\omega_l,\cdot) - K_\Gamma^F\hat{u}_l(\omega_l,\cdot) + S_\Lambda^F\partial_\nu\hat{u}_l(\omega_l,\cdot) \quad \text{on } \Gamma, \tag{5.4}$$

$$0 = S_\Gamma^F\{\mathcal{T}^F\hat{u}_l + G^F\hat{\lambda}_l\}(\omega_l,\cdot) - K_\Gamma^F\hat{u}_l(\omega_l,\cdot) + S_\Lambda^F\partial_\nu\hat{u}_l(\omega_l,\cdot) \quad \text{on } \Lambda, \tag{5.5}$$

where $l = 0, 1, \dots, N$, $\hat{u}_l(\cdot)$ and $\hat{\lambda}_l(\cdot)$ are, respectively, the discrete Fourier transform of $u_j(\cdot) := u(t_j, \cdot)$ and $\lambda_j := \lambda(t_j)$ with respect to j. The operators are

$$\begin{gathered} (S_\Gamma^F\varphi)(\omega,x) = \frac{\mathrm{i}}{4}\int_\Gamma H_0^{(1)}(\omega|x-y|)\varphi(\omega,y)\mathrm{d}s_y, \\ (K_\Gamma^F\varphi)(\omega,x) = \frac{\mathrm{i}}{4}\int_\Gamma \partial_{\nu(y)}H_0^{(1)}(\omega|x-y|)\varphi(\omega,y)\mathrm{d}s_y, \\ (\mathcal{T}^F\varphi)(\omega,x) = \frac{\mathrm{i}\omega}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi_1 x_1}\sqrt{1-\xi_1^2/\omega^2}\mathcal{F}_{x_1}(\varphi)(\omega,\xi_1,x_2)\mathrm{d}\xi_1, \\ (G^F\varphi)(\omega,x) = -\frac{\mathrm{i}\omega}{2}H_1^{(1)}(\omega|x-z|)\frac{x_2-z_2}{|x-z|}\varphi(\omega), \end{gathered}$$

where $H_n^{(1)}$ is the Hankel function of the first kind of order n.

Finally, we just need to solve the Helmholtz problems (5.4)–(5.5) instead of the time domain scattering problem (5.2)–(5.3).

Conclusion

We have analyzed the time domain acoustic scattering from open cavities. A transparent boundary condition has been developed to get an equivalent initial boundary value problem. The well-posedness of the reduced problem has been proved. Moreover, retarded potential boundary integral equations (RPBIEs) have been established to solve the reduced problem.

Our future work will include the analysis of the existence and uniqueness of the solutions for the RPBIEs and the iteration method for the inverse scattering problem.

Acknowledgements. Bo Chen was supported by the NSFC [No. 11671170], and by the Fundamental Research Funds for the Central Universities (Special Project for Civil Aviation University of China, No. 3122018L009). Fuming Ma was supported by the NSFC [No. 11771180]. Yukun Guo was supported by the NSFC [Nos.11971133, 11601107, 41474102 and 11671111].

References

[1] H. Ammari, G. Bao, A. W. Wood; *An integral equation method for the electromagnetic scattering from cavities*, Math. Meth. Appl. Sci., **23**(2000), 1057–1072.
[2] H. Ammari, G. Bao, A. W. Wood; *Analysis of the electromagnetic scattering from a cavity*, Jpn. J. Ind. Appl. Math., **19**(2002), 301–310.
[3] L. Banjai, C. Lubich, F. J. Sayas; *Stable numerical coupling of exterior and interior problems for the wave equation*, Numer. Math., **129**(2013), 611–646.
[4] L. Banjai, S. Sauter; *Rapid solution of the wave equation in unbounded domains*, SIAM J. Numer. Anal., **47**(2008), 227–249.
[5] L. Banjai, M. Schanz; *Wave Propagation Problems Treated with Convolution Quadrature and BEM*, Springer Berlin Heidelberg, Berlin, 2012.
[6] G. Bao, K. H. Yun, Z. Zhou; *Stability of the scattering from a large electromagnetic cavity in two dimensions*, SIAM J. Math. Anal., **44**(2012), 383–404.
[7] B. Chen, F. Ma, Y. Guo; *Time domain scattering and inverse scattering problems in a locally perturbed half-plane*, Appl. Anal., **96**(2017), 1303–1325.
[8] B. Chen, Y. Sun, Z. Zhuang; *Method of fundamental solutions for a Cauchy problem of the Laplace equation in a half-plane*, Bound. Value Probl., **2019**(2019), 34:1-14.
[9] Q. Chen, H. Haddar, A. Lechleiter, P. Monk; *A sampling method for inverse scattering in the time domain*, Inverse Probl., **26**(2010), 85001–85017.
[10] Z. Chen, J. C. Nédélec; *On Maxwell equations with the transparent boundary condition*, J. Comput. Math., **26**(2008), 284–296.
[11] D. Colton, R. Kress; *Integral Equation Methods in Scattering Theory*, John Wiley, New York, 1983.
[12] Y. Gao, P. Li, B. Zhang; *Analysis of transient acoustic-elastic interaction in an unbounded structure*, SIAM J. Math. Anal., **49**(2016), 3951–3972.
[13] Y. Guo, P. Monk, D. Colton; *Toward a time domain approach to the linear sampling method*, Inverse Probl., **29**(2013), 95016–95032.
[14] T. Ha-Duong; *On retarded potential boundary integral equations and their discretisation*, In *Computational Methods in Wave Propagation*, volume 31, pages 301–336, Springer, Heidelberg, 2003.
[15] G. C. Hsiao, W. L. Wendland; *Boundary Integral Equations*, Springer, Berlin, 2008.
[16] A. R. Laliena, F. J. Sayas; *Theoretical aspects of the application of convolution quadrature to scattering of acoustic waves*, Numer. Math., **112**(2009), 637–678.
[17] J. Li, H. Liu, J. Zou; *Locating multiple multiscale acoustic scatterers*, SIAM Multiscale Model. Simul., **12**(2014), 927–952.

[18] P. Li; *A survey of open cavity scattering problems*, J. Comput. Math., **36**(2018), 1–16.
[19] P. Li, W. Lilian, A. Wood; *Analysis of transient electromagnetic scattering from a three-dimensional open cavity*, SIAM J. Appl. Math., **75**(2015), 1675–1699.
[20] P. Li, A. Wood; *A two-dimensional Helmhotlz equation solution for the multiple cavity scattering problem*, J. Comput. Phys., **240**(2013), 100–120.
[21] P. Li, H. Wu, W. Zheng; *An overfilled cavity problem for Maxwell's equations*, J. Differ. Equ., **252**(2012), 3209–3225.
[22] C. Lubich; *Convolution quadrature and discretized operational calculus. I*, Numer. Math., **52**(1988), 129–145.
[23] C. Lubich; *On the multistep time discretization of linear initial-boundary value problems and their boundary integral equations*, Numer. Math., **67**(1994), 365–389.
[24] F. J. Sayas; *Retarded Potentials and Time Domain Boundary Integral Equations*, Springer Series in Computational Mathematics, Switzerland, 2016.
[25] Y. Sun; *Indirect boundary integral equation method for the Cauchy problem of the Laplace equation*, J. Sci. Comput., **71**(2017), 469–498.
[26] T. Van, A. Wood; *Analysis of transient electromagnetic scattering from overfilled cavities*, SIAM J. Appl. Math., **64**(2006), 688–708.
[27] X. Wang, Y. Guo, J. Li, H. Liu; *Mathematical design of a novel input/instruction device using a moving emitter*, Inverse Probl., **33**(2017), 105009(19pp).
[28] H. Zhang, B. Zhang; *A novel integral equation for scattering by locally rough surfaces and application to the inverse problem*, SIAM J. Appl. Math., **73**(2013), 1811–1829.

Bo Chen
College of Science, Civil Aviation University of China, Tianjin 300300, China
Email address: `charliecb@163.com`

Fuming Ma
Institute of Mathematics, Jilin University, Changchun 130012, China
Email address: `mfm@jlu.edu.cn`

Yukun Guo
School of Mathematics, Harbin Institute of Technology, Harbin 150001, China
Email address: `ykguo@hit.edu.cn`