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ESTIMATES FOR SOLUTIONS TO NONLINEAR BOUNDARY-VALUE PROBLEMS IN CONIC DOMAINS

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ABSTRACT. We obtain sharp estimates on the solution and its derivative near the conic points. In particular, we show that the solution satisfies $|u(x)| \le C|x|^\lambda$ where lambda is an eigenvalue of the Sturm-Liouville problem. Also we prove that the solution has square summable weighted second generalized derivatives.

1. INTRODUCTION AND PRELIMINARIES

We consider mixed boundary-value problems in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$ for the equation

$$\sum_{i=1}^{n} \frac{d}{dx_i} a_i(x,u,u_x) + a(x,u,u_x) = 0, \quad x \in \Omega \tag{1.1}$$

This study includes equations such as $-\operatorname{div}(k + |\nabla u|^{p-2}) + \mu_1 |u|^\beta + u^2\phi(x)$, where $p > 1$ and $k \ge 0$.

The domain Ω is assumed to satisfy the isoperimetric inequalities defined in [8]. The boundary of the domain is decomposed as $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Then Dirichlet conditions are given on Γ_1, and Neumann conditions on Γ_2.

Our aim is to obtain sharp estimates on the solution and its derivative near the conic points. Also to obtain estimates for $|u|$ and $|\nabla u(x)|$ which correspond to $\varepsilon = 0$ in [2], but not obtained there. For the Dirichlet problem, these equations were considered in [5]. For the Dirichlet problem with linear equations, estimates on conical domains were considered in [6]. The mixed boundary-value problem for linear equations on conical domains was considered in [11]. Here we study a non-linear case.

Let us set some notation. $B_d(0)$ is ball of radius d with the center at the point 0. $\Omega_0^d = \Omega \cap B_d(0)$ is cone in $\mathbb{R}^n$; i.e., for sufficiently small d

$$\Omega_0^d = \{(r,\omega) : 0 < r < d,\ \omega = (\omega_1, \omega_2, \dots, \omega_{n-1}) \in G\},$$

where (r,ω) are spherical coordinates. G is a domain on a unit sphere S^{n-1} with infinitely differentiable boundary ∂G,

$$\Gamma_0^d = \{(r,\omega) : 0 < r < d;,\ \omega \in \partial G\} = \Gamma_{0,1}^d \cup \Gamma_{0,2}^d \subset \partial\Omega$$

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is the lateral surface of the cone Ω_0^d, $G_\rho = \Omega_0^d \cap \{|x| = \rho\}$, $0 < \rho < d$. $dx = r^{n-1} dr d\omega$, $d\Omega_\rho = \rho^{n-1} d\omega$, $d\omega$ is an element of area of the unit sphere, $|\nabla u|^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2} |\nabla_\omega u|^2$, where $|\nabla_\omega u|$ is projection of vector ∇u on tangent plane to the sphere S^{n-1} at the point ω,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{n} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_\omega u.$$

Here $\Delta_\omega u$ is the Laplace-Beltrami operator on a unit sphere.

Denote by $W^m_{\alpha,0}(\Omega)$ the space of functions having generalized derivatives up to order m in Ω with norm

$$\|u\|^2_{W^m_{\alpha,0}(\Omega)} = \sum_{|k|^m=0} \int_\Omega r^{\alpha-2(m-k)} \Big| \frac{\partial^{|k|} u}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \Big|^2 dx\,.$$

The function that are continuously differentiable in $\overline{\Omega}$ and vanishing on Γ_1 form a dense subset. In particular

$$\|u\|^2_{W^2_{\alpha,0}(\Omega)} = \in_\Omega \big(r^\alpha u^2_{xx} + r^{\alpha-2} |\nabla u|^2 + r^{\alpha-4} u^2 \big) dx\,.$$

By $W^1_{2,0}(\Omega)$ we denote the subset of the Sobolev space $W^1_2(\Omega)$ consisting of continuously differentiable functions in $\overline{\Omega}$ vanishing on Γ_1. (This is as dense subset of functions).

We shall use Hardy inequalities and some of its implications. For any function $u \in W^1_{2,0}(\Omega_0^d)$, we have

$$\int_{\Omega_0^d} r^{\alpha-4} u^2 dx \le \frac{4}{(4-n-\alpha)^2} \int_{\Omega_0^d} r^{\alpha-2} u_r^2 dx\,, \quad \alpha < 4-n, \tag{1.2}$$

which follows by integration with respect to $\omega \in G$ the correspondent Hardy inequality [4].

Allowing isoperimetricity for the domain Ω, we consider the eigenvalue problem

$$\begin{gathered} \Delta_\omega u + \lambda(\lambda + n - 2) u = 0\,, \quad \omega \in G, \\ u\big|_{\gamma_0} = 0, \quad \frac{\partial u}{\partial u}\big|_{\gamma_1} = 0, \end{gathered} \tag{1.3}$$

where $\partial G \in \gamma_0 \cup \gamma_1$. In [1], it was shown that this problem has at least one positive eigenvalue $\lambda = \lambda(G)$. Then by the variational principle for all $u \in W^1_{2,0}(G)$,

$$\int\limits_G u^2 d\omega \le \frac{1}{\lambda^2 + \lambda(n-2)} \int\limits_G |\nabla_\omega u|^2 d\omega. \tag{1.4}$$

Note that constants in inequalities (1.2) and (1.4) are the best possible.

When we multiply inequality (1.4) by $1/r$ and integrate with respect to $r \in (0, d)$, we have that for any function

$$u \in V = \big\{ v \in W^1_2(\Omega) : v(x) = 0,\ x \in \Gamma^d_{0,1},\ \frac{\partial v}{\partial n} = 0,\ x \in \Gamma^d_{0,2} \big\},$$

$$\int_{\Omega_0^d} r^{-n} u^2 dx \le \frac{1}{\lambda^2 + \lambda(n-2)} \int_{\Omega_0^d} r^{2-n} |\nabla u|^2 dx\,. \tag{1.5}$$

For any function $u \in V$,

$$\int_{\Omega_0^d} r^{\alpha-4} u^2 dx \le \big[(2 - \frac{n+\alpha}{2})^2 + \lambda(\lambda + n - 2) \big]^{-1} \int_{\Omega_0^d} r^{\alpha-2} |\nabla u|^2 dx, \tag{1.6}$$

whenever the integral in the right-hand side is finite. Here $\alpha \le 4-n$. To obtain this inequality we multiply inequality (1.4) by $1/r$ and integrate with respect to $r \in (0,d)$. Then

$$\int_{\Omega_0^d} r^{\alpha-4}u^2 dx \le \frac{1}{\lambda^2+\lambda(n-2)} \int_{\Omega_0^d} r^{\alpha-4}|\nabla_\omega u|^2 dx. \tag{1.7}$$

If $\alpha < 4-n$ inequality (1.6) is obtained by adding (1.2) and (1.7). If $\alpha = 4-n$ inequality (1.6) coincides with (1.5).

By a generalized solution of the mixed boundary-value problem for equation (1.1), we mean a function $u(x)$ in $W^1_{2,0}(\Omega)$ such that

$$\int_\Omega [a_i(x,u,u_x)\eta_{x_i} + a(x,u,u_x)\eta(x)]dx = 0, \quad \forall \eta(x) \in W^1_{2,0}(\Omega)\,. \tag{1.8}$$

In this paper, we use the repeated index convention; this is, the summation of terms with repeated indices.

On the coefficient we require the following conditions: The functions $a_i(x,u,p)$ are measurable at any $x \in \Omega$, $u \in \mathbb{R}$, $p \in \mathbb{R}^n$; differentiable with respect to p_j $(j = 1,\dots,n)$; and satisfy

$$\upsilon(|u|)\xi^2 \le \frac{\partial a_i(x,u,p)}{\partial p_j}\xi_i\xi_j \le \mu(|u|)\xi^2, \quad \forall \xi \in \mathbb{R}^n, \tag{1.9}$$

$$\frac{\partial a_i(0,0,p)}{\partial p_j} = \delta_i^j\,, \quad i,j = \overline{1,n}, \tag{1.10}$$

$$\Big[\sum_{i=1}^n a_i^2(x,u,p)\Big]^{1/2} \le \mu_1(|u|)(|p| + g(x))\,, \quad 0 \le g(x) \in L_q(\Omega), \tag{1.11}$$

where δ_i^j is the Kronecker symbol, $q > n$, $g(0) < \infty$.

The function $a(x,u,p)$ is measurable at $x \in \Omega$, $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ satisfies

$$|a(x,u,p)| \le \mu_2(|u|)(|p|^2 + f(x)), \tag{1.12}$$

where $0 \le f(x)$, $f \in L_{q/2}(\Omega)$, $q > n$, $v(t)[\mu(t), \mu_1(t), \mu_2(t)]$ is positive nondecreasing function (positive non-increasing) at $t \ge 0$, $\mu, v > 0$, $\mu_1, \mu_2 \ge 0$.

In [3] the boundedness and Hölder continuity of generalized solution of (1.8) was proved under the conditions (1.9)–(1.12). Assuming that the vrai max M of $|u(x)|$ is known, there exists $\gamma > 0$, $C_0 > 0$ dependent only on $M, n, q, \mu, \mu_1, \mu_2, v, \Omega$ such that

$$|u(x)| = |u(x) - u(0)| \le C_0|x|^\gamma, \quad |x| < d.$$

For continuous functions vrai max is the same as the max over the domain on which the function is defined.

2. Main results

Theorem 2.1. *Let $u(x)$ be a generalized solution of* (1.8). *Assume* (1.9)–(1.12) *and that for any $k > 0$ there exists $d_0 > 0$ such that for $p \in \mathbb{R}^n$, $|x| + |u| < d_0$, $0 \le h(x) \in L_q$, and $q > n$ we have*

$$\Big(\sum_{i=1}^n [a_i(x,u,p) - a_i(0,0,p)]^2\Big)^{1/2} \le K|p| + h(x)\,. \tag{2.1}$$

Also assume that $g(x) \in W^0_{\alpha-2}(\Omega)$, $h(x) \in W^0_{\alpha-2,0}(\Omega)$, $f(x) \in W^0_{\alpha,0}(\Omega)$, $\alpha \le 4-n$, *and*

$$\lambda > 2 - (n+\alpha)/2\,. \tag{2.2}$$

Then

$$\int_\Omega r^{\alpha-2}|\nabla u|^2 dx \quad \le C(1 + \|g\|_{W^0_{\alpha-2}(\Omega)} + \|f\|_{q/2,\Omega} + \|h\|_{W^0_{\alpha-2,0}(\Omega)} + \|f\|^2_{W^0_{\alpha,0}(\Omega)}), \tag{2.3}$$

where C is constant depending on $M, v, \mu_1, \mu_2, \mu, \alpha, n, \lambda, q, \operatorname{meas}\Omega, \operatorname{meas} G$.

Proof. For any $\delta \in (0,d)$ if r is the radius vector of the point $x \in \overline{\Omega}$ then $r_\delta = |r - \delta l| \ne 0$, for all $x \in \overline{\Omega}$, where for the fixed point $z \in S^{n-1}\backslash\overline{G}$ and unit radius vector $l = \overrightarrow{0z} = (l_1, \dots, l_n)$, the vector δl does not belong to Ω^d_0. Therefore, the function $\eta(x) = r_\delta^{\alpha-2} u(x)$ is admissible in identity (1.8). We obtain

$$\begin{aligned} &\int_\Omega r_\delta^{\alpha-2} a_i(x,u,u_x) u_{x_i} dx + \int_\Omega r_\delta^{\alpha-2} u(x) a(x,u,u_x) dx \\ &+ \int_\Omega (\alpha-2) u(x) r_\delta^{\alpha-4} a_i(x,u,u_x)(x_i - \delta l_i) dx = 0. \end{aligned} \tag{2.4}$$

Since $a_i(x,u,p) = p_j \int_0^1 \frac{\partial a_i(x,u,\tau p)}{\partial(\tau p_j)} d\tau + a_i(x,u,0)$, by (1.10) we have

$$\begin{gathered} a_i(0,0,p) = p_i + a_i^0\,, \quad a_i^0 \equiv a_i(0,0,0), \quad i = \overline{1,n} \\ a_i(x,u,p)p_i = |p|^2 + a_i^0 p_i + [a_i(x,u,p) - a_i(0,0,p)]p_i. \end{gathered} \tag{2.5}$$

Taking this into account, choosing some small number d and dividing the domain Ω into two subdomains Ω^d_0 and $\Omega\backslash\Omega^d_0$ we estimate the obtained integrals in each of subdomains separately. Then we apply inequality (1.6), use estimates from [7] and the fact that $u(x)$ is Hölder continuous. Finally, using conditions of the theorem passing to the limit as $\delta \to +0$ we obtain the required estimate. $\square$

Remark. Let $n = 2$, $0 \in \partial\Omega$ be a corner point, $G = (0,\omega_0), \omega_0$ is size of the angle in the neighbourhood of 0, $\Omega^d_0 = (0,d) \times (0,\omega_0)$. In this case eigenvalues problem (1.3) has the form

$$\begin{gathered} u'' + \lambda^2 u = 0, \quad u = u(\omega), \quad \omega \in G, \\ u(\omega)\Big|_{\omega=0} = 0, \quad \frac{\partial u}{\partial n}\Big|_{\omega=\omega_0} = 0\,. \end{gathered} \tag{2.6}$$

Here, the least positive eigenvalue of this problem is $\lambda = \pi/(2\omega_0)$ and condition (2.2) takes the form

$$\frac{\pi}{\omega_0} > 2 - \alpha, \quad \alpha \le 2.$$

Before, estimating $|u(x)|$, we prove the following lemma.

Lemma 2.2. *Let $u(x)$ be a generalized solution of* (1.1) *and let conditions* (1.9)–(1.12) *be satisfied. Then for any function*

$$v(x) \in V = \{v \in W^1_2(\Omega^\rho_0) : v(x) = 0,\ x \in \Gamma^\rho_{0,1};\ \frac{\partial v}{\partial n} = 0,\ x \in \Gamma^\rho_{0,2}\}$$

and almost all $\rho \in (0,d)$ the following equality holds

$$\int_{\Omega^\rho_0} [a_i(x,u,u_x) v_{x_i} + a(x,u,u_x) v(x)] dx = \int_{G_\rho} a_i(x,u,u_x) v(x) \cos(r, x_i) dG_\rho \tag{2.7}$$

To prove it we substitute $\eta(x) = v(x)(\chi_\rho)_h(x)$, for $v \in W^1_{2,0}(\Omega)$ into the integral identity (1.8), where $\chi_\rho(x)$ is characteristic function of the set Ω_0^ρ and $(\chi_\rho)_h$ is its Sobolev averaging. Such η is admissible by virtue of Theorem 2.1. Passing to the limit as $h \to 0$ we obtain (2.7). Passage to the limit is justified by the use of properties of mean functions [9, theorem 3.10, p.113] and Theorem 2.1.

Theorem 2.3. *Let $u(x)$ be a generalized solution of* (1.1)*. Assume conditions* (1.9)–(1.12) *and that*

$$\Big(\sum_{i=1}^{n}[a_i(x,u,p)-a_i(0,0,p)]^2\Big)^{1/2} \le \delta(|x|)|p| + h(x), \tag{2.8}$$

for any $x \in \Omega_0^d$, $u \in R, p \in \mathbb{R}^n$, where $\delta(r)$ is a nondecreasing positive function satisfying the Diny condition $\int_0^d \frac{\delta(r)}{r}dr < \infty$. In addition we assume that

$$\begin{gathered} a_i(x,u,p)p_i \ge v_0|p|^2 - \mu_3|u|^\beta - u^2\varphi(x); \\ a(x,u,p)u \le \mu_0|p|^2 + \mu_3|u|^\beta + u^2\varphi(x), \end{gathered} \tag{2.9}$$

where $2n/(n-2) > \beta > 2$, $0 \le \varphi(x) \in L_{q/2}(\Omega)$, $q > n$, $v_0 > 0$, $\mu_0, \mu_3 \ge 0$; $g(x) \in W^0_{2-n}(\Omega)$, $h(x) \in W^0_{2-n,0}(\Omega)$, $f(x) \in W^0_{4-n,0}(\Omega)$, and

$$\rho^2\int_G g^2(\rho,\omega)d\omega + \rho^2\int_G h^2(\rho,\omega)d\omega + \int_{\Omega_0^\rho} r^{4-n}f^2(x)dx \le k\rho^s,$$

with $s > 2\lambda(G)$, $0 < \rho < d$. Then

$$|u(x)| \le C|x|^{\lambda(G)}\,, \tag{2.10}$$

where $\lambda(G)$ is the least positive eigenvalue of (1.3) *and the constant C depends only on the known quantities of the problem.*

Proof. Substitute $v(x) = r^{2-n}u(x)$ in identity (2.7). Such a function is admissible by virtue of (1.5) and Theorem 2.1. Taking into account (2.5) and estimating integrals with multipliers a_i^0 and expression $u\,u_{x_0}$ we obtain

$$\begin{aligned} &\int_{\Omega_0^\rho} r^{2-n}|\nabla u|^2dx \\ &\le \frac{n-2}{2}\int_G u^2d\omega \\ &\quad + \int_{\Omega_0^\rho}[a_i(x,u,u_x)-a_i(0,0,u_x)]|\big[r^{2-n}|u_{x_i}| + (2-n)r^{-n}|x_i||u(x)|\big]dx \\ &\quad + \int_{\Omega_0^\rho} r^{2-n}|u|\,|a(x,u,u_x)|dx \\ &\quad + \rho\int_G |u(x)|[a_i(x,u,u_x)-a_i(0,0,u_x)]|\cos(r,x_i)|_{r=\rho}d\omega \\ &\quad + C_9\rho^{-\varepsilon}\|g\|_{W^0_{2-n}(\Omega)} + \rho^{2-\varepsilon}\int_G g^2(\rho,\omega)d\omega + \rho\int_G uu_\rho d\omega \end{aligned}$$

Denoting $v(\rho) = \int_0^\rho dr\int_G(ru_r^2 + \frac{1}{r}|\nabla_\omega u|^2)d\omega$ and estimating integrals in the right-hand side by means of inequalities (1.4), (1.5), Cauchy inequality with $\varepsilon > 0$, and Hölder property of $u(x)$, we obtain

$$v(\rho) \le c\rho^{\,2\lambda}\,, \quad 0 < \rho < d \tag{2.11}$$

where constant C depends on $M, d, v, \mu_1, \mu_2, \mu, n, \lambda, q, \operatorname{meas} G, \operatorname{meas}\Omega, \|g\|_{q,\Omega}$,

$$\|h\|_{W^0_{2-n,0}(\Omega)}, \|g\|_{W^0_{2-n}(\Omega)}, \|f\|_{W^0_{4-n,0}(\Omega)}, \|f\|_{q/2,\Omega}, \int_0^d \frac{\delta(r)}{r}dr, k, s$$

Consider the function

$$z(x') = \rho^{-\lambda(G)}u(\rho x'), \quad 0 < \rho < d \tag{2.12}$$

in layer $Q' = \{x' : 1/2 < |x'| < 1\}$, $u \equiv 0$ out of Ω, and use inequalities from [7, ch.2, inequality (2.22)]. Taking into account estimate (2.11), we obtain

$$\int_{\rho/2<|x|<\rho} |u|^q dx \le C\rho^{n+q\lambda}, \quad 2 \le q \le 2n/(n-2),\ n > 2. \tag{2.13}$$

Then taking into consideration results from [7, ch.4, theorem 7.6], by the assumption of this theorem, we obtain

$$|u(x)| \le M_2\rho^{\lambda(G)} \tag{2.14}$$

where $x \in \Omega_0^d \cap \{\rho/2 < |x| < \rho < d\}$ and M_2 is a constant depending on the known quantities. Taking that $|x| = 2\rho/3$ we obtain the required estimate (2.10) and the proof is complete. □

Theorem 2.4. *Let $u(x)$ be a generalized solution of* (1.1) *and assumptions of theorem 2.1 be satisfied. Assume that for $x \in \overline{\Omega}$ and $u, p \in \mathbb{R}^n$ the functions $a_i(x,u,p)$, $i = \overline{1,n}$ and $a(x,u,p)$ be differentiable with respect to their arguments and the following inequalities hold:*

$$\begin{gathered} a_i(x,u,p)p_i \ge v_0|p|^2 - \varphi_0(x) \\ \Big[\sum_{i=1}^n \big(\big|\frac{\partial a_i}{\partial u}\big|^2 + \big|\frac{\partial a}{\partial x_i}\big|^2\big)\Big]^{1/2} + \Big(\sum_{i,j=1}^n \big|\frac{\partial a_i}{\partial x_j}\big|^2\Big)^{1/2} \le \mu_4(|u|)(|p| + \varphi_1(x)) \\ \Big(\big|\frac{\partial a}{\partial u}\big|^2 + \sum_{i=1}^n \big|\frac{\partial a}{\partial x_i}\big|^2\Big)^{1/2} \le \mu_5(|u|)\big(|p|^2 + \varphi_2(x)\big), \end{gathered} \tag{2.15}$$

where $\varphi_i(x)$, $i = 0,1,2$ are nonnegative functions. Also assume that $\varphi_0(x), \varphi_2(x) \in L_{q/2}(\Omega)$, $\varphi_1(x) \in L_q(\Omega)$, $q > n$. Then $u(x) \in W^2_{\alpha,0}(\Omega)$ and

$$\begin{aligned} &\|u\|^2_{W^2_{\alpha,0}(\Omega)} \\ &\le c_1(1 + \|f\|_{q,\Omega} + \|f\|_{q/2,\Omega} + \|\varphi_0\|_{q/2,\Omega} + \|\varphi_2\|_{q/2,\Omega} + \|\varphi_1\|_{q,\Omega} \\ &\quad + \|h\|^2_{W^2_{\alpha-2,0}(\Omega)} + \|g\|^2_{W^0_{\alpha-2}(\Omega)} + \|f\|^2_{W^0_{\alpha,0}(\Omega)} \\ &\quad + c_2\Big\{\int_\Omega r^{(\alpha+h)q/4-n}[\varphi_0^{q/2}(x) + \varphi_1^q(x) + \varphi_2^{q/2}(x) + f^{q/2}(x) + g^q(x)]\Big\}^{4/q}, \end{aligned}$$

where $\alpha \le 4-n$. Provided that the last integral is finite, the constat $c_1, c_2 > 0$ depends on the known parameters.

To proof this theorem we considered a sequence of domains $\Omega_{k,\rho}$, which are intersections of Ω_0^d and some layers. Making some transformations and using an estimate from [7] and summing all the obtained inequalities over $k = 1, 2, \dots$. Using Theorem 2.1 we obtain the following corollary.

Corollary 2.5. *Let the conditions of Theorem 2.4, except for* (2.2)*, be fulfilled. Then generalized solution $u(x)$ of problem* (1.1) *is in $W^2(\Omega)$, for the following cases:*

(1) $n \geq 4$;
(2) $n = 2$ *and* $0 < \omega_0 < \frac{\pi}{2}$;
(3) $n = 3$ *and* $G \subset G_0 = \{\omega = (\theta; \varphi) : 0 < |\theta| < \omega_0 < \pi,\ 0 < \varphi < 2\pi\}$, *where* ω_0 *is solution of equation* $p_{1/2}(\cos\omega_0) = 0$ *for Legendre functions.*

Proof. (1) According to theorem 2.4 $u(x) \in W^2_{4-n,0}(\Omega)$. Condition (2.2) is trivial if $\alpha = 4 - n$ because $\lambda = \lambda(G) > 0$. Now the statement follows from inequality

$$\int_{\Omega_0^d} u^2_{xx} dx \leq d^{n-4} \int_{\Omega_0^d} r^{4-n} u^2_{xx} dx \leq \text{const}.$$

(2) Suppose $\alpha = 0$ in Theorem 2.4 then condition (2.1) is trivial. If $n = 2$ the statement follows from the remark.
(3) Condition (2.2) becomes $\lambda(G) > 1/2$. Let $\Omega_0 \subset S^2$ be a domain in which the eigenvalue problem (1.3) is solvable for $\lambda(G) = 1/2$ and $\partial\Omega_0 = \partial^1\Omega_0 \cup \partial^2\Omega_0$:

$$\begin{gathered} \Delta_\omega u + (1/2)(1 + 1/2)u = 0, \quad \omega \in \Omega_0 \\ u\Big|_{\partial^1\Omega_0} = 0, \quad \frac{\partial u}{\partial u}\Big|_{\partial^2\Omega_0} = 0 \end{gathered} \tag{2.16}$$

The condition $\lambda > 1/2$ implies $\Omega \subset \Omega_0$; see [3]. We are seeking of solution problem (2.16) of the form $u = v(\theta)$. Then for $v(\theta)$ we obtain

$$\begin{gathered} \frac{1}{\sin\theta} \frac{d}{d\theta}\big(\sin\theta \frac{dv}{d\theta}\big) + \frac{1}{2}\big(1 + \frac{1}{2}\big)v = 0, \quad 0 < |\theta| < \omega_0, \\ v(-\omega_0) = 0 \quad \frac{\partial v}{\partial n}(\omega_0) = 0\,. \end{gathered} \tag{2.17}$$

The solution to this equation is a Legendre function of the first genus $v(\theta) = p_{1/2}(\cos\theta)$, which has exactly one zero in the interval $0 < \theta < \pi$ which we denote by ω_0 (see [7]). Therefore, the corollary is proved. ☐

Theorem 2.6. *Let* $u(x)$ *be a generalized solution of* (1.1). *Let functions* $a_i(x, u, p)$, $a(x, u, p)$ *be differentiable with respect to their arguments and conditions* (1.9)–(1.12), (2.15) *with* $q = \infty$ *be satisfied. Under the assumptions in Theorem 2.3,*

$$|\nabla u(x)| \leq c|x|^{\lambda(G)-1} \tag{2.18}$$

where $\lambda(G)$ *is the least positive eigenvalue of* (1.3), *and constant* c *depends only on the known quantities.*

Proof. As in the proof of Theorem 2.3 consider function $z(x') = \rho^{-\lambda(G)}u(\rho x')$, $0 < \rho < d$ in the layer $Q' = \{x' : 1/2 < |x'| < 1\}$ assuming that $u \equiv 0$ outside of Ω. Under our conditions, the theorem from [4] on boundedness of modulus of gradient of solution inside of domain and near smooth pieces of boundary is valid:

$$\operatorname{vrai\,max}_{Q'} |\nabla' z| \leq M_3 \tag{2.19}$$

where $M_3 > 0$ depends on $v, v_0, \mu, \mu_1, \mu_2$ $\operatorname{vrai\,max}_{Q'} |z(x')|$. Then for the function $u(x)$ we obtain

$$|\nabla u(x)| \leq M_1 \rho^{\lambda(G)-1}, \quad x \in \Omega_0^d \cap \{\rho : 2 < |x| < \rho < d\}. \tag{2.20}$$

Taking $|x| = 2\rho/3$, we obtain the required estimate. ☐

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March 18, 2005. Addendum

In response to the editor's request, we want to add a reference that should have been included in the original bibliography.

[13] Borsuk, M. V.; Behavior of generalized solutions of the Dirichlet problem for second-order quasilinear elliptic equations of divergence type near a conical point. (Russian) Sibirsk. Mat. Zh. 31 (1990), no. 6, 25–38; translation in Siberian Math. J. 31 (1990), no. 6, 891–904 (1991)

Also we want to compare this reference with our article. The two articles have the same structure and visual appearance. Both articles follow the ideas presented by Condratyev [6], and have the same components: Weight inequalities, investigation of a corresponding spectral problem, and study of Holder continuity of solutions.

However, these two articles are different: [13] studies a Dirichlet boundary problem, while our article studies a mixed boundary problem.

1. The weight inequalities (1.4)-(1.7) require isoperimetric conditions on the domain, which are not needed for the Dirichlet problem.

2. The study of the spectrum for problem (1.3) in our article follows the method in [1]. In the Dirichlet case, the study follows the work by Mikhlen (see [13]). The mixed boundary problem has smallest eigenvalue $\lambda/(2\omega_0)$ and critical point $\pi/(2\omega_0)$, while the Dirichlet problem has smallest eigenvalue λ/ω_0 and critical point π/ω_0.

3. The study of Holder continuity of solutions for the mixed problem follows ideas in [3]. Meanwhile for the Dirichlet problem, the study follows ideas in [7].

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