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On the ellipticity and solvability of an abstract second-order differential equation *

Abdelillah El Haial & Rabah Labbas

Abstract

In this work we give some new results on a complete abstract second-order differential equation of elliptic type in the non-homogeneous case. Existence, uniqueness, and maximal regularity of the strict solution are proved under some natural assumptions which imply the ellipticity of the differential equation.

1 Introduction

In this paper, we study the second-order abstract differential equation

$$u''(t) + 2Bu'(t) + Au(t) = f(t), \quad t \in (0, 1) \tag{1}$$

under the non-homogeneous boundary conditions

$$u(0) = \varphi, \quad u(1) = \psi, \tag{2}$$

where φ, ψ and $f(t)$ belong to a complex Banach space E, and A, B are two closed linear operators with domains $D(A)$, $D(B)$. We are interested in the existence, uniqueness, and maximal regularity of the strict solution u when f is regular (for instance f is Hölder continuous function). We recall that $u \in C([0, 1]; E)$ is a strict solution of (1)-(2) if and only if

$$u \in C^2([0, 1]; E) \cap C^1([0, 1]; D(B)) \cap C([0, 1]; D(A))$$

and u satisfies (1) and (2).

Throughout this paper we assume the following hypotheses:

H1) Operator B generates a strongly continuous group on E

H2) There exists $K > 0$ such that for all $\lambda \geqslant 0$, $\|(A - B^2 - \lambda I)^{-1}\|_{L(E)} \leqslant K/(1 + \lambda)$

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H3) For all $\mu \in \mathbb{R}$, and all $\lambda \geqslant 0$, $(A - B^2 - \lambda I)^{-1}(B - \mu I)^{-1} - (B - \mu I)^{-1}(A - B^2 - \lambda I)^{-1} = 0$

H4) There exists $K > 0$ such that for all $\lambda \geqslant 0$,
i) $\|B(A - B^2 - \lambda I)^{-1}\|_{L(E)} \leqslant K/(1 + \sqrt{\lambda})$,
ii) $\|B^2(A - B^2 - \lambda I)^{-1}\|_{L(E)} \leqslant K$,
iii) $\|A(A - B^2 - \lambda I)^{-1}\|_{L(E)} \leqslant K$.

Note that H1 implies $\overline{D(B)} = E$, but the domain $D(A - B^2)$ may be not dense. On the other hand, it is not difficult to see that assumptions H1, H2, H3 permit us to apply the Da Prato and Grisvard [1] sum theory and deduce that, necessarily,

$$(A - \lambda I)^{-1} \in L(E) \quad \text{and} \quad \|(A - \lambda I)^{-1}\|_{L(E)} \leqslant K/(1 + \lambda), \text{ for all } \lambda \geqslant 0.$$

(See the proof of Theorem 3.3, the end of statement i)).

Several authors have studied equation (1) when it is regarded as an abstract Cauchy problem, that is under the following initial data $u(0) = \varphi$, $u'(0) = \psi$. See, for instance, Favini [2], Neubrander [10], Liang and Xiao [8]. The techniques used in these papers are based on the parabolicity of (1), that is the parabolicity of the operator pencil defined by

$$P(\lambda) : D(A) \cap D(B) \to E; \quad x \mapsto P(\lambda)x = (\lambda^2 I + 2\lambda B + A)x.$$

In this paper we try to provide an unified treatment of some class of second order partial differential equations when they are regarded as equations of elliptic type. The principal part of these equations can be written in the form

$$\mathcal{P}u = \partial_t^2 u + 2b(.)\partial_{xt}^2 u + a(.)\partial_x^2 u,$$

or more generally

$$\mathcal{P}u = \partial_t^2 u + \mathcal{B}(x, \partial_x)\partial_t u + \mathcal{A}(x, \partial_x)\partial_x^2 u,$$

where $\mathcal{B}(x, \partial_x)$ and $\mathcal{A}(x, \partial_x)$ are linear differential operators in $\mathbb{R}^n$ with some natural assumptions on their coefficients of order m and l.

So, our objective is to express the ellipticity of these concrete operators in general abstract form, find a representation of the solution of the problem (1)-(2) and study its maximal regularity.

Some particular equations of type (1) can be used to describe the singularities of the solutions in elliptic problems on singular domains. Let us give the following example of Laplacian operator

$$\begin{aligned} -\Delta u &= g, \quad g \in L^p(Q) \\ u &= 0 \quad \text{on } \partial Q, \end{aligned}$$

in a conical domain $Q = \{\rho\sigma \;:\; \rho > 0, \sigma \in G\} \subset \mathbb{R}^n$, where G is an open subset of the sphere S^{n-1}. In the three dimensional domain, the boundary ∂Q has a vertex at O and also edges depending on whether the boundary ∂G has corners or not. Let us assume that the variational solution u of this Dirichlet problem exists. So, we would have $u = u_r + u_s$, with a regular part $u_r \in W^{2,p}(Q)$ and a singular part u_s in explicit form. Then, one performs the change of variable $\rho = e^t$ which maps Q onto the infinite cylinder $\mathbb{R} \times G$. The Laplacian equation in polar coordinates is

$$D_\rho^2 u + \frac{n-1}{\rho} D_\rho u + \frac{1}{\rho^2}\Delta' u = g$$

where Δ' denotes the Laplace-Beltrami operator on S^{n-1}. The above mentioned change of variable leads us to

$$D_t^2 u + D_t u + \Delta' u = e^{2t} g = g_1,$$

then by puttting $v(t)(\sigma) = e^{\varpi t} u(e^t \sigma)$ and $g_2(t)(\sigma) = e^{\varpi t} g_1$ we have

$$\begin{gathered} D_t^2 v + (n-2-2\varpi) D_t v + \varpi(\varpi - n + 2) v + \Delta' v = g_2 \\ v = 0 \quad \text{on } \partial(\mathbb{R} \times G) \end{gathered}$$

where $\varpi = -2 + n/p$, precisely the opposite of the Sobolev exponent of $W^{2,p}$. This final equation can be regarded as a particular form of our abstract general problem (1). See, for details, Labbas-Moussaoui-Najmi [7].

Equations (1)-(2) can be illustrated, for instance, by the following simple model differential problem

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2}(t,x) + 2b \frac{\partial^2 u}{\partial x \partial t}(t,x) + a \frac{\partial^2 u}{\partial x^2}(t,x) = f(t,x), \quad (t,x) \in \Sigma, \\ u(0,x) = \varphi(x), \quad u(1,x) = \psi(x), \quad x \in \mathbb{R}, \end{gathered} \tag{3}$$

where $\Sigma =]0,1[\times \mathbb{R}$ and

$$a - b^2 > 0. \tag{4}$$

Then we can choose in $E = L^p(\mathbb{R})$, $1 < p < \infty$, the operators A, B defined by

$$\begin{gathered} D(B) = W^{1,p}(\mathbb{R}), \quad (Bu)(x) = bu'(x), \ \forall u \in D(B) \\ D(A) = W^{2,p}(\mathbb{R}), \quad (Au)(x) = au''(x), \ \forall u \in D(A). \end{gathered}$$

In our study, hypotheses H2 and H4 express the ellipticity of (1) and generalize the one used in Krein [5] in the case $B = 0$. In example (3) these two assumptions are equivalent to (4). So we cannot reduce equations (1)-(2) to some first order system. When $B = 0$, Labbas [6] has studied problem (1)-(2) and gives necessary and sufficient conditions on φ, ψ, $f(0)$ and $f(1)$ for having a unique strict solution when f is Hölder continuous function. In this work we generalize these results since all the hypotheses considered here, coincide with those used in [6] in the case $B = 0$. Note that assumptions H1, H2, H3 allow us to apply the sum theory of linear operators as in Da Prato-Grisvard [1] and give $D_A(\theta; +\infty) = D_L(\theta; +\infty) \cap D_{B^2}(\theta; +\infty)$ where $L = A - B^2$ and, for $\theta \in]0,1[$

$$D_L(\theta; +\infty) = \left\{ \varphi \in E : \sup_{r>0} r^\theta \| L(L - rI)^{-1} \varphi \|_E < \infty \right\},$$

(see Grisvard [4]).

In section 2, we build the natural representation of the solution of (1)-(2) by using the operational calculus and the Dunford integral.

In section 3, we prove an essential lemma, which allow us to justify and to study the optimal smoothness of the previous representation; we then give necessary and sufficient conditions on φ, ψ, $f(0)$ and $f(1)$ for having a strict solution when f is Hölder continuous function.

Finally, in section 4, we give an example, to which our abstract results can be applied.

2 Construction of the solution

If A and B are two scalars such that $B^2 - A = -\lambda$ is strictly positive, then the solution of (1) is given by

$$u(t) = \frac{\sinh\sqrt{-\lambda}(1-t)}{\sinh\sqrt{-\lambda}}e^{-tB}\varphi + \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}e^{(1-t)B}\psi - \int_0^1 G_{\sqrt{-\lambda}}(t,s)f(s)ds,$$

where

$$G_{\sqrt{-\lambda}}(t,s) = \begin{cases} G^+_{\sqrt{-\lambda}}(t,s) = \frac{\sinh\sqrt{-\lambda}(1-t)\sinh\sqrt{\lambda}s}{\sqrt{-\lambda}\sinh\sqrt{-\lambda}}, & 0 \leqslant s \leqslant t \\ G^-_{\sqrt{-\lambda}}(t,s) = \frac{\sinh\sqrt{-\lambda}(1-s)\sinh\sqrt{\lambda}t}{\sqrt{-\lambda}\sinh\sqrt{-\lambda}}, & t \leqslant s \leqslant 1. \end{cases}$$

Now, it is well known that H2 implies the existence of $\delta_0 \in]0, \pi/2[$ and $\varepsilon_0 > 0$ such that the resolvent set of $A - B^2$ contains the following sector of the complex plane

$$S(\delta_0, \varepsilon_0) = \{z \in \mathbb{C} : |\arg(z)| \leqslant \delta_0\} \cup B(0, \varepsilon_0),$$

where $B(0, \varepsilon_0)$ is the open ball of radius ε_0. If γ denotes the sectorial boundary curve of $S(\delta_0, \varepsilon_0)$ oriented positively, then the natural representation of the solution of (1)-(2), in the abstract case, is given by the Dunford integral

$$\begin{aligned} u(t) \quad = \quad & \frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi d\lambda \\ & +\frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}\psi d\lambda \\ & -\frac{1}{2\pi i}\int_\gamma\int_0^1 G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}(L-\lambda I)^{-1}f(s)ds\, d\lambda, \end{aligned} \tag{5}$$

where $L = A - B^2$, $D(L) = D(A) \cap D(B^2)$,

$$g_{\sqrt{-\lambda}}(t) = \frac{\sinh\sqrt{-\lambda}(1-t)}{\sinh\sqrt{\lambda}} \quad t \in (0,1), \lambda \in \gamma.$$

Here $\sqrt{-\lambda}$ is the analytic determination defined by $Re\sqrt{-\lambda} > 0$. From writing

$$\left|g_{\sqrt{-\lambda}}(t)\right| = \left|e^{-\sqrt{-\lambda}t}\Big(\frac{1-e^{-2\sqrt{-\lambda}(1-t)}}{1-e^{-2\sqrt{-\lambda}}}\Big)\right|,$$

we deduce that there exists two constants c_0 and K_0 such that

$$\begin{aligned} \left|g_{\sqrt{-\lambda}}(t)\right| &\leqslant K_0 e^{-c_0|\lambda|^{1/2}t} \quad \forall\lambda\in\gamma,\ \forall t\in]0,1], \\ \left|g_{\sqrt{-\lambda}}(1-t)\right| &\leqslant K_0 e^{-c_0|\lambda|^{1/2}(1-t)} \quad \forall\lambda\in\gamma,\ \forall t\in[0,1[, \end{aligned} \tag{6}$$

where $c_0 = \cos(\pi/2 - \delta_0/2)$ and $K_0 = 2/\big(1-\exp(-2\mathbf{c}_0\sqrt{\varepsilon_0})\big)$. On the other hand, for any $f \in C([0,1];E)$, we can see that

$$\Big\|\int_0^1 G_{\sqrt{-\lambda}}(t,s)f(s)ds\Big\|_E \leqslant \frac{1}{|\lambda|\cos(\delta_0/2)}\|f\|_{C(E)}, \quad \forall\lambda\in\gamma. \tag{7}$$

According to H1, H2 and estimates (6)-(7), all the integrals in (5) converge absolutely for every $t\in]0,1[$.

3 Smoothness of the solution

Let us set, for φ and ψ in E,

$$\begin{aligned} U(t,-B,L)\varphi &= \frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi d\lambda \quad t\in]0,1], \\ U(1-t,B,L)\psi &= \frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}\psi d\lambda \quad t\in[0,1[, \end{aligned}$$

then we have the essential lemma

Lemma 3.1 *Under assumptions H1 and H2*

i) there exists a positive constant $K = K(\varepsilon_0,\delta_0)$ such that

$$\begin{aligned} \|U(t,-B,L)\varphi\|_E &\leqslant K\|\varphi\|_E, \quad \forall\varphi\in E, \forall t\in]0,1], \\ \|U(1-t,B,L)\psi\|_E &\leqslant K\|\psi\|_E, \quad \forall\psi\in E, \forall t\in[0,1[, \end{aligned}$$

ii) $U(.,-B,L)\varphi \in C([0,1];E)$ if and only if $\varphi\in\overline{D(L)}$,

iii) $U(1-.,B,L)\psi \in C([0,1];E)$ if and only if $\psi\in\overline{D(L)}$.

Proof. i) For $t>0$ we can write

$$\begin{aligned} &U(t,-B,L)\varphi \\ &= \frac{e^{-tB}}{2\pi i}\int_{\gamma_+^t} g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi d\lambda + \frac{e^{-tB}}{2\pi i}\int_{\gamma_-^t} g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi d\lambda \end{aligned}$$

$$= \quad I_+ + I_-,$$

where

$$\gamma_+^t = \left\{z \in \gamma;\ |z| \geqslant 1/t^2\right\}, \quad \gamma_-^t = \left\{z \in \gamma;\ |z| \leqslant 1/t^2\right\}. \tag{8}$$

Then we have

$$\|I_+\|_E \leqslant K_0 (\int_{1/t^2}^{+\infty} \frac{e^{-c_0|\lambda|^{1/2}t}}{|\lambda|} d\,|\lambda|) \|\varphi\|_E \leqslant 2K_0 \int_1^{+\infty} \frac{e^{-c_0\sigma}}{\sigma} d\sigma \|\varphi\|_E \leqslant K\|\varphi\|_E,$$

and

$$\begin{aligned}
I_- \quad &= \quad \frac{e^{-tB}}{2\pi i} \int_{\gamma_-^t} (g_{\sqrt{-\lambda}}(t) - g_{\sqrt{-\lambda}}(0))(L-\lambda I)^{-1}\varphi d\lambda \\
&\quad + \frac{e^{-tB}}{2\pi i} \int_{\gamma_-^t} (L-\lambda I)^{-1}\varphi d\lambda \\
&= \quad -\frac{e^{-tB}}{2\pi i} \int_{\gamma_-^t} (\int_0^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(1-s)}{\sinh\sqrt{\lambda}} ds)(L-\lambda I)^{-1}\varphi d\lambda \\
&\quad - \frac{e^{-tB}}{2\pi i} \int_{C_{1/t^2}} (L-\lambda I)^{-1}\varphi d\lambda \\
&= \quad I'_- + I''_-,
\end{aligned}$$

where

$$C_{1/t^2} = \left\{z : |\arg z| \leqslant \delta_0 \text{ and } |z| = 1/t^2\right\}.$$

Then

$$\|I'_-\|_E \leqslant \frac{K_0}{2\pi} \int_{\varepsilon_0}^{1/t^2} \frac{|\lambda|^{1/2}\,t}{|\lambda|} d\,|\lambda|\,.\|\varphi\|_E \leqslant K\|\varphi\|_E,$$

$$\|I''_-\|_E \leqslant \frac{1}{2\pi} \int_{-\delta_0}^{+\delta_0} \|(L - \frac{e^{i\theta}}{t^2} I)^{-1}\varphi\|_E \frac{d\theta}{t^2} \leqslant K\|\varphi\|_E.$$

Similarly we obtain

$$\|U(1-t,B,L)\psi\|_E \leqslant K\|\psi\|_E, \quad \forall\psi \in E\ ,\ \forall t \in [0,1[.$$

ii) Fix $\varepsilon > 0$ and let $\varphi \in \overline{D(L)}$, then there exists $y \in D(L)$ such that

$$\|\varphi - y\| \leqslant \varepsilon. \tag{9}$$

Using the identity

$$(L-\lambda I)^{-1}y = \frac{(L-\lambda I)^{-1}Ly}{\lambda} - \frac{y}{\lambda},$$

we have

$$U(t,-B,L)y = \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{\lambda}}(t) \frac{(L-\lambda I)^{-1}}{\lambda} Ly d\lambda,$$

which gives

$$\|U(t,-B,L)y\|_E \leqslant \frac{1}{2\pi}\int_{\varepsilon_0}^{\infty}\frac{K}{|\lambda|^2}d\,|\lambda|\,\|Ly\|_E \leqslant K\|Ly\|_E \ \forall t\in[0,1]. \tag{10}$$

Now, from the equality

$$U(t,-B,L)\varphi-\varphi = U(t,-B,L)\varphi - U(t,-B,L)y + U(t,-B,L)y - y + y - \varphi,$$

and the estimates (9), (10) we deduce that

$$U(t,-B,L)\varphi-\varphi \to 0 \quad \text{as} \quad t\to 0^+.$$

The continuity in $t>0$ is easily verified. Conversely, assume $U(.,-B,L)\varphi \in C([0,1];E)$, then $\lim_{t\to 0^+}U(t,-B,L)\varphi$ exists and is necessarily equal to φ; however

$$g_{\sqrt{-\lambda}}(t)e^{-tB}(L-\lambda I)^{-1}\varphi \in D(L),$$

which implies that $U(t,-B,L)\varphi \in \overline{D(L)}$.

For statement iii) it is enough to substitute $1-t$ for t. □

Let us consider, for $\theta\in\,]0,1[$, the well known real interpolation space between $D(L)$ and E characterized by

$$D_L(\theta;+\infty) = \big\{\varphi\in E : \sup_{r>0} r^{\theta}\|L(L-rI)^{-1}\varphi\|_E < \infty\big\}.$$

(See Grisvard[3]).

Lemma 3.2 *Under assumptions H1, H2, H3 and for* $\theta\in]0,1/2[$ *we have*

$$U(.,-B,L)\varphi\in C^{2\theta}([0,1];E) \quad \textit{if and only if} \quad \varphi\in D_L(\theta;+\infty).$$

Proof. Let $\varphi\in D_L(\theta;+\infty)$ and $\tau,t\in]0,1[$ such that $\tau<t$, then

$$\begin{aligned}
&U(t,-B,L)\varphi - U(\tau,-B,L)\varphi\\
&= \frac{e^{-tB}}{2\pi i}\int_{\gamma}(g_{\sqrt{-\lambda}}(t)-g_{\sqrt{-\lambda}}(\tau))\frac{L(L-\lambda I)^{-1}}{\lambda}\varphi d\lambda\\
&\quad+\frac{(e^{-tB}-e^{-\tau B})}{2\pi i}\int_{\gamma} g_{\sqrt{-\lambda}}(\tau)\frac{L(L-\lambda I)^{-1}}{\lambda}\varphi d\lambda\\
&= I_1+I_2.
\end{aligned}$$

I_1 may be written as

$$\begin{aligned}
I_1 &= \frac{e^{-tB}}{2\pi i}\int_{\gamma_+^{t-\tau}}(g_{\sqrt{-\lambda}}(t)-g_{\sqrt{-\lambda}}(\tau))\frac{L(L-\lambda I)^{-1}}{\lambda}\varphi d\lambda\\
&\quad+\frac{e^{-tB}}{2\pi i}\lim_{\gamma_-^{t-\tau}}(g_{\sqrt{-\lambda}}(t)-g_{\sqrt{-\lambda}}(\tau))\frac{L(L-\lambda I)^{-1}}{\lambda}\varphi d\lambda
\end{aligned}$$

$$= \quad I_1' + I_1'',$$

(where $\gamma_+^{t-\tau}$ and $\gamma_-^{t-\tau}$ are defined in (8)), and

$$\|I_1'\|_E \leqslant 2K_0 \int_{\gamma_+^{t-\tau}} \frac{d\,|\lambda|}{|\lambda|^{\theta+1}} \|\varphi\|_{D_L(\theta;+\infty)},$$

$$\|I_1''\|_E \leqslant K_0 \lim_{\gamma_-^{t-\tau}} \frac{|t-\tau|}{|\lambda|^{\theta+1/2}} d\,|\lambda|\, \|\varphi\|_{D_L(\theta;+\infty)},$$

from which it follows

$$\|I_1\|_E \leqslant K(t-\tau)^{2\theta} \|\varphi\|_{D_L(\theta;+\infty)}.$$

We write I_2 as

$$I_2 = (t-\tau)^{2\theta} e^{-\tau B} \Big(\frac{e^{-(t-\tau)B}\Phi_1 - \Phi_1}{(t-\tau)^{2\theta}} \Big),$$

with

$$\Phi_1 = \frac{1}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(\tau)(L-\lambda I)^{-1}\varphi d\lambda,$$

then, for $\forall r > 0$, the classical operational calculus gives

$$L(L-rI)^{-1}\Phi_1 = \frac{1}{2\pi i} \int_\gamma g_{\sqrt{\lambda}}(\tau) \frac{L(L-\lambda I)^{-1}\varphi}{r-\lambda} d\lambda,$$

which implies that

$$\|L(L-rI)^{-1}\Phi_1\|_E \leqslant \frac{K}{r^\theta} \|\varphi\|_{D_L(\theta;+\infty)};$$

therefore,

$$\Phi_1 \in D_L(\theta;+\infty) \subset D_{B^2}(\theta;+\infty) = D_B(2\theta;+\infty)\,.$$

The last equality holds by using the well known reiteration theorem in interpolation theory (see, Lions-Peetre [9]). So, we obtain

$$\|I_2\|_E \leqslant K(t-\tau)^{2\theta} \|\Phi_1\|_{D_B(2\theta;+\infty)}.$$

The main result in this work is the following

Theorem 3.3 *Assume H1,H2,H3, and H4. Let* $\varphi \in D(L), \psi \in D(L)$ *and* $f \in C^{2\theta}([0,1];E)$ *with* $\theta \in]0,1/2[$ *. Then* u *given in (5) satisfies*

i) $u(t) \in D(A)$ *for all* $t \in [0,1]$

ii) $Au(.)$ *belongs to* $C([0,1];E)$ *if and only if* $A\varphi, A\psi, f(0)$ *and* $f(1)$ *belong to* $\overline{D(L)}$

iii) $Au(.)$ *belongs to* $C^{2\theta}([0,1];E)$ *if and only if* $A\varphi$ *,*$A\psi$*,* $f(0)$ *and* $f(1)$ *belong to* $D_L(\theta;+\infty)$*.*

Furthermore, if

$$B\varphi \in D_L(\theta + 1/2; +\infty) \text{ and } B\psi \in D_L(\theta + 1/2; +\infty), \tag{11}$$

then

iv) $u'(t) \in D(B)$ *for all* $t \in [0,1]$

v) $Bu'(.) \in C([0,1]; E)$ *if and only if* $B^2\varphi$ *and* $B^2\psi$ *belong to* $\overline{D(L)}$

vi) $Bu'(.) \in C^{2\theta}([0,1]; E)$ *if and only if* $B^2\varphi$, $B^2\psi$ *belong to* $D_L(\theta; +\infty)$.

Proof. Statement i). The study of the first and second integral in representation (5) is identical since t and $(1-t)$ have the same role. So we can assume that $\psi = 0$. Moreover in the third integral in (5) we write $f(s) = (f(s)-f(t))+f(t)$, then, after two integration by parts, we get

$$\begin{aligned}
u(t) &= \frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi d\lambda \\
&\quad -\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad -\frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad -\frac{1}{2\pi i}\int_\gamma\int_0^1 G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}(L-\lambda I)^{-1}(f(s)-f(t))ds\, d\lambda \\
&\quad +\frac{1}{2\pi i}\int_\gamma (B^2+\lambda I)^{-1}(L-\lambda I)^{-1}f(t)d\lambda \\
&= \sum_{i=1}^{5} a_i .
\end{aligned}$$

All these integrals converge absolutely. From Lemma 3.1 and H4, we deduce that the first integral a_1 belongs to $D(A)$. Writing $A = L+B^2 = (L-\lambda I)+(B^2+\lambda I)$ and using H4 we obtain

$$A(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t) = (L-\lambda I)^{-1}f(t) + (B^2+\lambda I)^{-1}f(t),$$

which implies, as for the first integral, that a_2 and a_3 belong to $D(A)$. For a_4, we use iii) of H5 which gives

$$\begin{aligned}
&\Big\|\frac{1}{2\pi i}\int_\gamma\int_0^1 G_{\sqrt{\lambda}}(t,s)e^{-(t-s)B}A(L-\lambda I)^{-1}(f(s)-f(t))ds\, d\lambda\Big\|_E \\
&\leqslant K\int_{\varepsilon_0}^{\infty}(\sup 0 \leqslant t \leqslant 1 \int_0^1 \left|G_{\sqrt{-\lambda}}(t,s)\right| |t-s|^{2\theta}\, ds) d\,|\lambda|\, \|f\|_{C^{2\theta}(E)} \\
&\leqslant K(\int_{\varepsilon_0}^{\infty}\frac{1}{|\lambda|^{1+\theta}}d\,|\lambda|)\|f\|_{C^{2\theta}(E)},
\end{aligned}$$

the last estimate follows in virtue of Hölder's inequality. For a_5 we apply the Da Prato-Grisvard's [1] sums theory to B^2 and L. In fact, from H1 we deduce that B^2 generates a bounded holomorphic semigroup in E (see Stone [11]); then (B^2+L) is closable and

$$(\overline{B^2+L})^{-1} = S = \frac{1}{2\pi i}\int_\gamma (B^2+\lambda I)^{-1}(L-\lambda I)^{-1}d\lambda,$$

which implies that $Sx = A^{-1}x$ for all $x \in E$; therefore, $a_5 = A^{-1}f(t)$.

Statements ii) and iii). We have

$$\begin{aligned}
Au(t) &= \frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}A\varphi d\lambda \\
&\quad -\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad -\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}f(t)d\lambda \\
&\quad -\frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad -\frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}f(t)d\lambda \\
&\quad -\frac{1}{2\pi i}\int_\gamma\int_0^1 G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&\quad +f(t) \\
&= \sum_{i=1}^{6} v_i(t) + f(t).
\end{aligned}$$

As in lemmas 3.1 and 3.2, $v_1(.) \in C([0,1];E)$ if and only if $A\varphi \in \overline{D(L)}$ and $v_1(.) \in C^{2\theta}([0,1];E)$ if and only if $A\varphi \in D_L(\theta;+\infty)$. Let us write for $0 \leqslant \tau < t \leqslant 1$

$$\begin{aligned}
v_2(t)-v_2(\tau) &= -\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(B^2+\lambda I)^{-1}(f(t)-f(\tau))d\lambda \\
&\quad -\frac{e^{-\tau B}}{2\pi i}\int_\gamma (g_{\sqrt{-\lambda}}(t)-g_{\sqrt{-\lambda}}(\tau))(B^2+\lambda I)^{-1}(f(\tau)-f(0))d\lambda \\
&\quad -\frac{(e^{-tB}-e^{-\tau B})}{2\pi i}\int_\gamma g_{\sqrt{\lambda}}(t)(B^2+\lambda I)^{-1}(f(\tau)-f(0))d\lambda \\
&\quad +\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(B^2+\lambda I)^{-1}f(0)d\lambda \\
&\quad -\frac{e^{-\tau B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(\tau)(B^2+\lambda I)^{-1}f(0)d\lambda
\end{aligned}$$

$$= \sum_{i=1}^{5} b_i,$$

then, as in the proof of Lemma 3.1, we have

$$\|b_1\| \leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)},$$

and writing b_2 as

$$b_2 = -\frac{e^{-\tau B}}{2\pi i}\int_\tau^t\int_\gamma \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(1-s)}{\sinh\sqrt{-\lambda}}(B^2+\lambda I)^{-1}(f(\tau)-f(0))d\lambda ds,$$

we get

$$\begin{aligned}
\|b_2\|_E &\leqslant K\int_\tau^t\int_{\varepsilon_0}^{+\infty}\frac{e^{-|\lambda|^{1/2}c_0 s}\tau^{2\theta}}{|\lambda|^{1/2}}d|\lambda|\,ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_\tau^t\int_{\sqrt{\varepsilon_0}s}^{+\infty}\frac{e^{-\sigma c_0}\tau^{2\theta}}{s}d\sigma ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_\tau^t(\int_0^{+\infty}e^{-\sigma c_0}d\sigma)s^{2\theta-1}ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t^{2\theta}-\tau^{2\theta})\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)};
\end{aligned}$$

for b_3, we have

$$b_3 = \frac{1}{2\pi i}\int_\tau^t\int_\gamma g_{\sqrt{\lambda}}(t)e^{-sB}B(B^2+\lambda I)^{-1}(f(\tau)-f(0))d\lambda ds,$$

and

$$\begin{aligned}
\|b_3\|_E &\leqslant K\int_\tau^t\int_{\varepsilon_0}^{+\infty}\frac{e^{-|\lambda|^{1/2}c_0 t}\tau^{2\theta}}{|\lambda|^{1/2}}d|\lambda|\,ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_\tau^t\int_{\varepsilon_0}^{+\infty}\frac{e^{-|\lambda|^{1/2}c_0 s}\tau^{2\theta}}{|\lambda|^{1/2}}d|\lambda|\,ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.
\end{aligned}$$

The sum b_4+b_5 can be estimated by the same method used in lemma 2; it follows that

$$\|b_4+b_5\|_E \leqslant K(t-\tau)^{2\theta}\|f(0)\|_{D_{B^2}(\theta;+\infty)},$$

if and only if $f(0)\in D_{B^2}(\theta;+\infty)$. Similarly $v_3(.)$ belongs to $C^{2\theta}([0,1];E)$ if and only if $f(0)$ belongs to $D_L(\theta;+\infty)$ and $v_4(.)+v_5(.)$ belongs to $C^{2\theta}([0,1];E)$ if and only if $f(1)$ belongs to $D_L(\theta;+\infty)$.

Finally,

$v_6(t)-v_6(\tau)$

$$
\begin{aligned}
&= -\frac{1}{2\pi i}\int_\gamma \int_\tau^t G^+_{\sqrt{\lambda}}(t,s)e^{-(t-s)B}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&- \frac{1}{2\pi i}\int_\gamma \int_0^\tau (G^+_{\sqrt{-\lambda}}(t,s)-G^+_{\sqrt{-\lambda}}(\tau,s))e^{-(t-s)B}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&- \frac{1}{2\pi i}\int_\gamma \int_0^\tau G^+_{\sqrt{\lambda}}(t,s)(e^{-(t-s)B}-e^{-(\tau-s)B})A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&- \frac{1}{2\pi i}\int_\gamma \int_\tau^1 G^-_{\sqrt{\lambda}}(t,s)(e^{(s-t)B}-e^{(s-\tau)B})A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&+ \frac{1}{2\pi i}\int_\gamma \int_\tau^t G^-_{\sqrt{\lambda}}(t,s)e^{(s-t)B}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&- \frac{1}{2\pi i}\int_\gamma \int_\tau^1 (G^-_{\sqrt{-\lambda}}(t,s)-G^-_{\sqrt{-\lambda}}(\tau,s))e^{(s-t)B}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&+ \frac{e^{-\tau B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(\tau)A(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}(f(t)-f(\tau))d\lambda \\
&+ \frac{e^{(1-\tau)B}}{2\pi i}\int_\gamma g_{\sqrt{\lambda}}(1-\tau)A(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}(f(t)-f(\tau))d\lambda \\
&- (f(t)-f(\tau)) \\
&= \sum_{i=1}^{8} c_i - (f(t)-f(\tau)).
\end{aligned}
$$

Using the fact that in the first integral (since $0 \leqslant \tau < s < t \leqslant 1$)

$$
\left| G^+_{\sqrt{-\lambda}}(t,s) \right| \leqslant K\frac{e^{-|\lambda|^{1/2}c_0(t-s)}}{|\lambda|^{1/2}}\ ,
$$

we have

$$
\begin{aligned}
\|c_1\|_E &\leqslant K\int_\tau^t \int_{\varepsilon_0}^{+\infty} \frac{e^{-|\lambda|^{1/2}c_0(t-s)}}{|\lambda|^{1/2}}(t-s)^{2\theta}d\,|\lambda|\,ds\|f\|_{C^{2\theta}(E)} \\
&\leqslant K\int_\tau^t (t-s)^{2\theta-1}(\int_{\sqrt{\varepsilon_0}(t-s)}^{+\infty} e^{-\sigma c_0}d\sigma)ds\|f\|_{C^{2\theta}(E)} \\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.
\end{aligned}
$$

Writing c_2 as

$$
\begin{aligned}
c_2 &= -\frac{1}{2\pi i}\int_0^\tau \int_\gamma \int_\tau^t \frac{\cosh\sqrt{-\lambda}(1-\xi)\sinh\sqrt{-\lambda}s}{\sinh\sqrt{-\lambda}} \\
&\quad \times e^{-(\tau-s)B}A(L-\lambda I)^{-1}(f(s)-f(t))d\xi d\lambda ds,
\end{aligned}
$$

we obtain

$$
\|c_2\|_E \leqslant K\int_0^\tau \int_{\varepsilon_0}^{+\infty} \int_\tau^t e^{-|\lambda|^{1/2}c_0(\xi-s)}(t-s)^{2\theta}d\xi d\,|\lambda|\,ds\|f\|_{C^{2\theta}(E)}
$$

$$\begin{aligned}
&\leqslant K\int_0^{\tau}\int_{\tau-s}^{t-s}\int_{\varepsilon_0}^{+\infty} e^{-|\lambda|^{1/2}c_0\eta}(t-s)^{2\theta}d\,|\lambda|\,d\eta ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_0^{\tau}\int_{\tau-s}^{t-s}\frac{(t-s)^{2\theta}}{\eta^2}(\int_{\sqrt{\varepsilon_0}\eta}^{+\infty}\sigma e^{-\sigma c_0}d\sigma)d\eta ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_0^{\tau}(t-s)^{2\theta}(\int_{\tau-s}^{t-s}\frac{d\eta}{\eta^2})ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_0^{\tau}(t-s)^{2\theta}(\frac{1}{\tau-s}-\frac{1}{t-s})ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)\int_0^{\tau}\frac{|t-s|^{2\theta-1}}{\tau-s}ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}(\int_0^{+\infty}\frac{z^{2\theta-1}}{1+z}dz)\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.
\end{aligned}$$

For c_3+c_4, we write

$$c_3+c_4=e^{-\tau B}(e^{-(t-\tau)B}\Phi_2-\Phi_2)$$

where

$$\begin{aligned}
\Phi_2 \;=\;& \frac{-1}{2\pi i}\int_\gamma\int_0^{\tau} G^+_{\sqrt{-\lambda}}(t,s)e^{sB}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda\\
&-\frac{1}{2\pi i}\int_\gamma\int_\tau^{1} G^-_{\sqrt{\lambda}}(t,s)e^{sB}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda\,;
\end{aligned}$$

using the fact that for $r>0$ and $\lambda\in\gamma$

$$L(L-rI)^{-1}A(L-\lambda I)^{-1}=\frac{r}{r-\lambda}A(L-rI)^{-1}-\frac{\lambda}{r-\lambda}A(L-\lambda I)^{-1}$$

we get

$$\begin{aligned}
&L(L-rI)^{-1}\Phi_2\\
&= \frac{1}{2\pi i}\int_\gamma\int_0^{\tau} G^+_{\sqrt{\lambda}}(t,s)e^{sB}\frac{\lambda}{r-\lambda}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda\\
&\quad+\frac{1}{2\pi i}\int_\gamma\int_\tau^{1} G^-_{\sqrt{\lambda}}(t,s)e^{sB}\frac{\lambda}{r-\lambda}A(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\|L(L-rI)^{-1}\Phi_2\|_E\\
&\leqslant K\int_{\varepsilon_0}^{+\infty}(\sup_{0\leqslant t\leqslant 1}\int_0^1\left|G_{\sqrt{-\lambda}}(t,s)\right||s-t|^{2\theta}\,ds)\frac{|\lambda|}{|r-\lambda|}d\,|\lambda|\,\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(\int_{\varepsilon_0}^{+\infty}\frac{d\,|\lambda|}{|\lambda|^{\theta}\,|r-\lambda|})\|f\|_{C^{2\theta}(E)}
\end{aligned}$$

$$\leqslant \frac{K}{r^\theta}\|f\|_{C^{2\theta}(E)};$$

therefore, $\Phi_2 \in D_L(\theta;+\infty) \subset D_{B^2}(\theta;+\infty) = D_B(2\theta;+\infty)$. Thus we have

$$\begin{aligned}\|c_3+c_4\|_E &\leqslant K(t-\tau)^{2\theta}\sup_{t-\tau>0}\|\frac{e^{-(t-\tau)B}\Phi_2-\Phi_2}{(t-\tau)^{2\theta}}\|_E \\ &\leqslant K(t-\tau)^{2\theta}\|\Phi_2\|_{D_B(2\theta;+\infty)}.\end{aligned}$$

By reasoning in the same manner for the last integral we obtain

$$\|v_6(t)-v_6(\tau)\|_E = O((t-\tau)^{2\theta}).$$

Statement iv). We have

$$\begin{aligned}u'(t) &= \frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{\lambda}}(t)(L-\lambda I)^{-1}\varphi d\lambda \\ &\quad -\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}B\varphi d\lambda \\ &\quad -\frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\ &\quad +\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)B(B^2+\lambda I)^{-1}(L-\lambda I)^{-1}f(t)d\lambda \\ &\quad -\frac{e^{(1-t)B}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\ &\quad +\frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)B(B^2+\lambda I)^{-1}(L-\lambda I)^{-1}f(t)d\lambda \\ &\quad +\frac{1}{2\pi i}\int_\gamma\int_0^1 G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}B(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\ &\quad -\frac{1}{2\pi i}\int_\gamma\int_0^1 \partial_t G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\ &= \sum_{i=1}^8 w_i(t).\end{aligned}$$

The derivative of the last integral in (5) is obtained by differentiating the kernel and then writing $f(s) = (f(s)-f(t))+f(t)$ and carrying out two integrations by parts. We verify that all these integrals converge absolutely. The first integral $w_1(t) \in D(B)$. In fact, we write

$$\frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}B\varphi d\lambda = \frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)\frac{L(L-\lambda I)^{-1}B\varphi}{\lambda}d\lambda,$$

then we have

$$\|\frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)\frac{L(L-\lambda I)^{-1}B\varphi}{\lambda}d\lambda\|_E$$

$$
\begin{aligned}
&\leqslant K\Big(\int_{\varepsilon_0}^{+\infty} \frac{e^{-|\lambda|^{1/2}c_0 t}}{|\lambda|^{1+\theta}} d\,|\lambda|\Big)\|B\varphi\|_{D_L(\theta+1/2;+\infty)} \\
&\leqslant K\|B\varphi\|_{D_L(\theta+1/2;+\infty)}.
\end{aligned}
$$

Due to Lemma 3.1 and H4, we have $w_2(t) \in D(B)$. Moreover, we can write

$$
\begin{aligned}
&w_3(t) + w_4(t) \\
&= -\frac{e^{-tB}}{2\pi i}\int_\gamma (\sqrt{-\lambda}g_{\sqrt{-\lambda}}(t) - g'_{\sqrt{-\lambda}}(t))(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad +\frac{e^{-tB}}{2\pi i}\int_\gamma \sqrt{-\lambda}g_{\sqrt{\lambda}}(t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad +\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)B(B^2+\lambda I)^{-1}(L-\lambda I)^{-1}f(t)d\lambda \\
&= \frac{e^{-tB}}{\pi i}\int_\gamma \frac{\sqrt{-\lambda}e^{-\sqrt{-\lambda}(2-t)}}{1-e^{-2\sqrt{-\lambda}}}(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad +\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(B-\sqrt{-\lambda}I)^{-1}(L-\lambda I)^{-1}f(t)d\lambda,
\end{aligned}
$$

which implies that $w_3(t) + w_4(t) \in D(B)$. Similarly, we show that the sum $w_5(t)+w_6(t) \in D(B)$. As in the proof of statement i), we also obtain that $w_7(t)$ and $w_8(t)$ belong to $D(B)$.

Statements v) and vi). We have

$$
\begin{aligned}
Bu'(t) &= \frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{\lambda}}(t)(L-\lambda I)^{-1}B\varphi d\lambda \\
&\quad -\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}B^2\varphi d\lambda \\
&\quad -\frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)B(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad +\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)B^2(B^2+\lambda I)^{-1}(L-\lambda I)^{-1}f(t)d\lambda \\
&\quad -\frac{e^{(1-t)B}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(1-t)B(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)d\lambda \\
&\quad +\frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)B^2(B^2+\lambda I)^{-1}(L-\lambda I)^{-1}f(t)d\lambda \\
&\quad +\frac{1}{2\pi i}\int_\gamma\int_0^1 G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}B^2(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&\quad -\frac{1}{2\pi i}\int_\gamma\int_0^1 \partial_t G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}B^2(L-\lambda I)^{-1}(f(s)-f(t))ds\,d\lambda \\
&= \sum_{i=1}^{8} x_i(t),
\end{aligned}
$$

and for $0 \leqslant \tau < t \leqslant 1$, we get

$$\begin{aligned} x_1(t) - x_1(\tau) &= -\frac{e^{-tB}}{2\pi i}\int_\tau^t \int_\gamma g_{\sqrt{-\lambda}}(s) L(L-\lambda I)^{-1} B\varphi d\lambda ds \\ &\quad + \frac{(e^{-tB} - e^{-\tau B})}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(\tau)\frac{L(L-\lambda I)^{-1}B\varphi}{\lambda} d\lambda \\ &= I_1 + I_2, \end{aligned}$$

then

$$\begin{aligned} \|I_1\|_E &\leqslant K_0 \int_\tau^t \int_{\varepsilon_0}^{+\infty} \frac{e^{-|\lambda|^{1/2} c_0 s}}{|\lambda|^{\theta+1/2}} d\,|\lambda|\, ds \|B\varphi\|_{D_L(\theta+1/2;+\infty)} \\ &\leqslant K_0 \int_\tau^t \int_{\sqrt{\varepsilon_0} s}^{+\infty} \frac{2\sigma e^{-\sigma c_0}}{(\frac{\sigma}{s})^{2\theta+1} s^2} d\sigma ds \|B\varphi\|_{D_L(\theta+1/2;+\infty)} \\ &\leqslant K \int_\tau^t s^{2\theta-1}(\int_0^{+\infty} \frac{e^{-\sigma c_0}}{\sigma^{2\theta}} d\sigma) ds \|B\varphi\|_{D_L(\theta+1/2;+\infty)} \\ &\leqslant K(t-\tau)^{2\theta}\|B\varphi\|_{D_L(\theta+1/2;+\infty)}. \end{aligned}$$

Writing I_2 in the form

$$I_2 = e^{-\tau B}(e^{-(t-\tau)B}\Phi_3 - \Phi_3),$$

with

$$\Phi_3 = \frac{1}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(\tau)\frac{L(L-\lambda I)^{-1}B\varphi}{\lambda} d\lambda,$$

we obtain

$$\|L(L-rI)^{-1}\Phi_3\|_E \leqslant \frac{K}{r^\theta}\|B\varphi\|_{D_L(\theta+1/2;+\infty)} \quad \forall r > 0.$$

Therefore, $\Phi_3 \in D_L(\theta;+\infty) \subset D_B(2\theta;+\infty)$. Thus we have

$$\begin{aligned} \|I_2\|_E &\leqslant K(t-\tau)^{2\theta} \sup_{t-\tau>0} \|\frac{e^{-(t-\tau)B}\Phi_3 - \Phi_3}{(t-\tau)^{2\theta}}\|_E \\ &\leqslant K(t-\tau)^{2\theta}\|\Phi_3\|_{D_B(2\theta;+\infty)}. \end{aligned}$$

The techniques for the rest of the proof is quite similar to those used in the precedent statements. □

Remark In the same manner we have a similar result to Theorem 3.3 when the second member f has a spatial smoothness, that is for all $t \in [0,1]$, $f(t) \in D_L(\sigma,\infty)$ and $\sup_{t\in(0,1)} \|f(t)\|_{D_L(\sigma,\infty)} < \infty$, with $\sigma \in]0,1[$. We omit to prove this assertion in this work.

Remark We can consider other hypotheses instead of (11) involving a comparison of $D(B)$ and $D((-L)^\alpha)$ for some $\alpha \in]0,1[$.

Example

Here we exhibit an example which can give an idea of other possible applications. Consider $E = L^p(\mathbb{R})$ with $1 < p < \infty$ and

$$D(A) = W^{2m,p}(\mathbb{R}), \quad A\psi = \mathcal{A}(x,\partial_x)\psi = \sum_{0 \leqslant k \leqslant 2m} a_k \psi^{(k)},$$
$$D(B) = W^{m,p}(\mathbb{R}), \quad B\psi = \mathcal{B}(x,\partial_x)\psi = \sum_{0 \leqslant k \leqslant m} b_k \psi^{(k)},$$

where $(a_{2m} - b_m^2) > 0$. Then we know that

$$\begin{aligned} D_L(\theta;+\infty) &= (D(L);E)_{1-\theta,\infty} \\ &= (W^{2m,p}(\mathbb{R});L^p(\mathbb{R}))_{1-\theta,\infty} \\ &= \mathcal{B}^{2m\theta}_{p,\infty}(\mathbb{R}). \end{aligned}$$

Where this last space is a Besov space defined, for instance, in Grisvard [4]. It is not difficult to see that H1, H2, H3, and H4 are satisfied. Applying Theorem 3.3 we have the following statement.

Theorem 3.4 *Let $f \in C^{2\theta}([0,1];L^p(\mathbb{R}))$ such that for $j = 0,1$ the mappings $x \mapsto f(j,x)$ belong to $\mathcal{B}^{2m\theta}_{p,\infty}(\mathbb{R})$ and assume that φ, ψ belong to $W^{2m,p}(\mathbb{R})$ and $\varphi^{(2m)}$, $\psi^{(2m)}$ belong to $\mathcal{B}^{2m\theta}_{p,\infty}(\mathbb{R})$. Then the problem*

$$\begin{gathered} \partial_t^2 u(t,x) + \mathcal{B}(x,\partial_x)\partial_t u(t,x) + \mathcal{A}(x,\partial_x)\partial_x^2 u(t,x) = f(t,x) \\ u(0,x) = \varphi(x), \quad u(1,x) = \psi(x), \end{gathered}$$

has a unique solution u satisfying

i) $u \in C^2([0,1];L^p(\mathbb{R})) \cap C^1([0,1],W^{m,p}(\mathbb{R})) \cap C([0,1],W^{2m,p}(\mathbb{R}))$,

ii) $\partial_x^{2m}u$, $\partial_x^m\partial_t u$ *and* $\partial_t^2 u$ *belong to* $C^{2\theta}([0,1];L^p(\mathbb{R}))$.

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Abdelillah El Haiai
Université du Havre, I.U.T,
B.P 4006, 76610 Le Havre, France

Rabah Labbas
Laboratoire de Mathématiques, Faculté des Sciences et Techniques,
Université du Havre, B.P 540, 76058 Le Havre Cedex, France
e-mail: Labbas@univ-lehavre.fr