Electronic Journal of Differential Equations, Vol. **2000**(2000), No. 24, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

Periodic and almost periodic solutions for multi-valued differential equations in Banach spaces *

E. Hanebaly & B. Marzouki

Abstract

It is known that for ω-periodic differential equations of monotonous type, in uniformly convex Banach spaces, the existence of a bounded solution on $\mathbb{R}^+$ is equivalent to the existence of an ω-periodic solution (see Haraux [5] and Hanebaly [7, 10]). It is also known that if the Banach space is strictly convex and the equation is almost periodic and of monotonous type, then the existence of a continuous solution with a precompact range is equivalent to the existence of an almost periodic solution (see Hanebaly [8]). In this note we want to generalize the results above for multi-valued differential equations.

1 Preliminaries

Let X and Y be Banach spaces, and 2^Y denote the collection of subsets of Y. For a multi-valued map $F : X \to 2^Y$ we define the following conditions:

F is **upper semi-continuous** (u.s.c.) in X if for every x_0 in X and every open set $G \subset Y$ with $Fx_0 \subset G$ there exists a neighborhood U of x_0 such that $Fx_0 \subset G$ for all $x \in U$. In practice F is u.s.c. at x_0 means that $Fx \subset Fx_0 + B_\varepsilon(0)$ for all x sufficiently close to x_0 and for ε sufficiently small.

F is **bounding** if it maps bounded subsets of X into bounded subsets of Y.

F is **dissipative** if $X = Y$ and

$$\langle Fx - Fy, x - y\rangle_- \leq 0 \quad \forall x \in X, \forall y \in X\,.$$

This implies that for all $x_1 \in Fx$ and all $y_1 \in Fy$,

$$\langle x_1 - y_1, x - y\rangle_- \leq 0\,,$$

where the lower semi-inner product on X introduced by Lumer [11] is defined as

$$\langle x, y\rangle_- = \|y\| \lim_{h\to 0^-} \frac{\|y + hx\| - \|y\|}{h}\,.$$

Mathematics Subject Classifications: 34A60, 34C25, 34C27, 47H10.
Key words and phrases: Multi-valued differential equation, Hyper-accretive, Almost periodicity, Banach space.

Submitted July 16, 1999. Published March 30, 2000.

F is **accretive** if $\langle Fx - Fy, x-y\rangle_+ \geq 0$ where the upper semi-inner product on X is defined as

$$\langle x, y\rangle_+ = \|y\| \lim_{h\to 0^+} \frac{\|y+hx\| - \|y\|}{h}\,.$$

We denote by $\rightharpoonup$ the convergence for the weak topology $\sigma(X, X^*)$. Recall that $x : J \subset \mathbb{R} \to X$ is said to be absolutely continuous (a.c. for short) if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\sum \|x(t_i) - x(s_i)\| \leq \varepsilon$ whenever the finitely many intervals $[s_i, t_i] \subset J$ do not overlap and $\sum |t_i - s_i| \leq \delta$. In particular every Lipschitzean map is a.c. When X is of finite dimension it is known that x is a.c. if and only if x is differentiable almost everywhere (a.e. for short) and $x' \in L^1(J, X)$, but if X is of infinite dimension and X is not reflexive, then an a.c. function need not be differentiable at any point (see e.g Deimling [6] p.138).

By a solution of the Cauchy problem

$$x' \in F(t,x)\,; \quad x(t_0) = x_0 \tag{1}$$

in some interval I (with $t_0 \in I$), we mean a continuous function on I, a.c. in every compact subset of I, differentiable a.e., and that satisfies (1) a.e. on I.

The collection non-empty compact convex subsets of X will be denoted by $CV(X)$.

2 Boundedness and periodicity of solutions

We begin by giving a result concerning the existence of a global solutions. Let $(X, \|.\|)$ be a real reflexive Banach space. Consider the multi-valued Cauchy problem

$$x'(t) \in F(t, x(t)) \tag{2}$$

$$x(0) = x_0\,, \tag{3}$$

where $F : \mathbb{R}^+ \times X \to CV(X)$ is u.s.c. and bounding.

Theorem 1 *If for all $(t,x,y) \in \mathbb{R}^+ \times X \times X$, $\langle F(t,x) - F(t,y), x-y\rangle_- \leq 0$, then the Cauchy problem (2)-(3) has a unique solution defined on $\mathbb{R}^+$.*

Remark. This theorem is well known for the inclusion of type

$$x' \in -Ax + f(t)$$

where A is a multi-valued maximal monotone operator on a Hilbert space and f is a uni-valued map (see Brezis [4]).

Proof of Theorem 1. Since F is u.s.c. with convex values, by the approximate selection theorem (see Cellina [1]) for each $n \geq 0$ there exists a locally lipschitzean map $f_n : \mathbb{R}^+ \times X \to X$ such that

$$f_n(t,x) \in F(\mathbb{R}^+ \times X \cap B_{1/n}(t,x)) + B_{\frac{1}{n}}(0) \quad \forall (t,x) \in \mathbb{R}^+ \times X\,,$$

where $B_{1/n}(t,x)$ is a ball in $\mathbb{R}^+ \times X$ and $B_{1/n}(0)$ is a ball in X. Since F is u.s.c. at (t,x), for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(\mathbb{R}^+ \times X \cap B_\delta(t,x)) \subset F(t,x) + B_\varepsilon(0)\,.$$

Then for n large we can choose δ such that $B_{1/n}(t,x) \subset B_\delta(t,x)$ and

$$F(\mathbb{R}^+ \times X \cap B_{1/n}(t,x)) \subset F(t,x) + B_\varepsilon(0)\,.$$

Consequently, for $\varepsilon = 1/m$ with $m \geq n$ we obtain

$$f_n(t,x) \in F(t,x) + B_{1/n}(0) + B_{1/n}(0) \subset F(t,x) + B_{2/n}(0)\,.$$

Now we show that for any $a > 0$ the uni-valued Cauchy problem

$$x'(t) = f_n(t,x(t)) \tag{4}$$

$$x(0) = x_0 \tag{5}$$

has a unique solution x_n on $[0,a]$ and the sequence x_n converges uniformly to the solution of the Cauchy problem (2)-(3).

Consider f_n from $[0,a] \times X$ to X, then f_n satisfies
i) f_n is continuous and locally lipschitzean with respect to x.
ii) $\langle f_n(t,x) - f_n(t,y), x-y\rangle_- \leq \frac{4}{n}\|x-y\|$.
For proving ii), we take $f_n(t,x) \in F(t,x) + B_{2/n}(0)$ and $f_n(t,y) \in F(t,y) + B_{2/n}(0)$, so that $f_n(t,x) = a + \alpha_n$ and $f_n(t,y) = b + \beta_n$ with $a \in F(t,x), b \in F(t,y)$ and $\alpha_n, \beta_n \in B_{2/n}(0)$. Then

$$\begin{aligned}
\langle f_n(t,x) - f_n(t,y), x-y\rangle_- &= \langle a + \alpha_n - b - \beta_n, x-y\rangle_- \\
&\leq \langle a-b, x-y\rangle_- + \langle \alpha_n - \beta_n, x-y\rangle_+ \\
&\leq \langle \alpha_n - \beta_n, x-y\rangle_- \\
&\leq \|\alpha_n - \beta_n\|\|x-y\| \\
&\leq \frac{4}{n}\|x-y\|.
\end{aligned}$$

It is well known that by i) the uni-valued Cauchy problem (4)-(5) has a unique local solution x_n, and that by ii) this solution can be extended on $[0,a]$. This statement is proven by the standard procedure of bounding the derivative of x_n.

Taking $y = 0$ in ii) , we obtain

$$\langle f_n(t,x) - f_n(t,0), x\rangle_- \leq \frac{4}{n}\|x\|.$$

Therefore,

$$\begin{aligned}\langle x_n'(t), x_n(t)\rangle_- &= \langle f_n(t,x_n(t)) - f_n(t,0) + f_n(t,0), x_n(t)\rangle_- \\ &\leq \langle f_n(t,x_n(t)) - f_n(t,0), x_n(t)\rangle_- + \langle f_n(t,0), x_n(t)\rangle_+ \\ &\leq \frac{4}{n}\|x_n(t)\| + \|f_n(t,0)\|\|x_n(t)\| \\ &\leq (1 + \sup_{t\in[0,a]} \|f_n(t,0)\|)\|x_n(t)\|.\end{aligned}$$

We deduce that (see appendix II)

$$D^-\|x_n(t)\| \leq 1 + \sup_{t\in[0,a]} \|f_n(t,0)\| = k_n$$

with k_n a constant which does not depend on t. This follows because there is $t_0^n \in [0,a]$ such that

$$\sup_{t\in[0,a]} \|f_n(t,0)\| = \|f_n(t_0^n,0)\|.$$

consequently, we have a sequence $x_n \in C([0,a],X)$ that satisfies

$$x_n'(t) \in F(t,x_n(t)) + B_{2/n}(0)\,. \tag{6}$$

Next we show that x_n is a Cauchy sequence. Let $\Phi_{n,m}(t) = \|x_n(t) - x_m(t)\|$. Then $\Phi_{n,m}(0) = 0$ and using the same technique as for proving ii) we deduce that

$$\begin{aligned}\Phi_{n,m}(t)D^-\Phi_{n,m}(t) &= \langle x_n'(t) - x_m'(t), x_n(t) - x_m(t)\rangle_- \\ &\leq (\frac{2}{n} + \frac{2}{m})\Phi_{n,m}(t)\,.\end{aligned}$$

Therefore, $\Phi_{n,m}(t) \leq (\frac{2}{n} + \frac{2}{m})a$ and then

$$\sup_{t\in[0,a]} \|x_n(t) - x_m(t)\| \to 0 \quad \text{as } n,m \to +\infty$$

Let x be the limit of x_n. Then we have in particular $x(0) = x_0$, now we have to show that x is a.e. differentiable and satisfies

$$x'(t) \in F(t,x(t)) \quad \text{a.e. in } [0,a].$$

Since F is u.s.c. and $x_n \to x$ uniformly on $[0,a]$, we deduce that for n large,

$$F(t,x_n(t)) \subset F(t,x(t)) + B_1(0)\,.$$

Since F is bounding, by (6) we have $\|x_n'(t)\| \leq c$ uniformly on $[0,a]$ for some $c > 0$.

Put $J = [0,a]$, then we have $x_n' \in L^\infty(J,X) \subset L^2(J,X)$. Since $L^2(J,X)$ is reflexive (because X is reflexive), there is a subsequence (which we denote by the same symbol) such that $x_n' \rightharpoonup y \in L^2(J,X)$ so

$$\begin{aligned} x_n(t) &= x_0 + \int_0^t x_n'(s)\,ds = x_0 + \int_J \chi_{[0,t]}(s)x_n'(s)\,ds \\ &\rightharpoonup x_0 + \int_J \chi_{[0,t]}(s)y(s)\,ds = x_0 + \int_0^t y(s)ds\,. \end{aligned}$$

Since $x_n(t) \to x(t)$, it follows that $x_n(t) \rightharpoonup x(t)$. Consequently

$$x(t) = x_0 + \int_0^t y(s)ds \quad \text{and} \quad x'(t) = y(t) \quad \text{a.e. in} J.$$

We deduce that $x_n' \rightharpoonup x'$ in $L^2(J,X)$ for the weak topology $\sigma(L^2(J,X), L^2(J,X^*))$. Let $\varepsilon > 0$ and put

$$A_\varepsilon = \left\{ z \in L^2(J,X) : z(t) \in F(t,x(t)) + \overline{B}_\varepsilon(0) \text{ a.e. } \right\}$$

Then A_ε is nonempty (because $x_n(t) \to x(t)$ and F is u.s.c., so $x_n' \in A_\varepsilon$ for n large), A_ε is closed and convex, hence A_ε is weakly closed. Since $x_n' \in A_\varepsilon$ and $x_n' \rightharpoonup x'$ we deduce that

$$x'(t) \in \overline{F(t,x(t))} = F(t,x(t)) \text{ a.e.}$$

So x is a solution of the Cauchy problem (2)-(3). Since $a > 0$ is arbitrary we deduce that the sequence x_n converges in the Banach space $C(\mathbb{R}^+, X)$ equipped with the topology of uniform convergence in compact subsets of $\mathbb{R}^+$.

That x is unique follows from the dissipativeness of F. Indeed let x and y be two solutions of the Cauchy problem (2)-(3), then we have

$$\langle x'(t) - y'(t), x(t) - y(t)\rangle_- \le 0 \quad \text{and} \quad \frac{1}{2}D^-\|x(t) - y(t)\|^2 \le 0\,.$$

Hence the map $t \mapsto \|x(t) - y(t)\|^2$ is non increasing, and consequently

$$\|x(t) - y(t)\| \le \|x(0) - y(0)\|. \tag{7}$$

Now we present a result that gives us the relationship between the existence of bounded solution and the existence of an ω-periodic solution of (2) when F is ω-periodic. Observe that under the hypothesis of Theorem 1 the condition: There exists a positive R such that

$$< F(t,x), x >_- \le 0 \quad \text{for } \|x\| > R$$

ensures the existence of a bounded solution on $[0, +\infty[$ (see Browder [3] and Hanebaly [8]).

Theorem 2 *Under the hypothesis of Theorem 1, assuming that X is uniformly convex, and $F(t + \omega, x) = F(t,x)$ ($\omega > 0$), the equation (2) has an ω-periodic solution if and only if it has a bounded solution on $[0, +\infty[$.*

Proof. The necessity condition is obvious because a continuous periodic map is bounded. Conversely we consider the Poincaré map $P : X \to X$ defined by $Px_0 = x(\omega)$ where x_0 is given in X and x is a solution of (2) which satisfies $x(0) = x_0$. The map P is well defined because of the uniqueness of solutions for the Cauchy problem (2)-(3). Now let x be the solution of (2) which is bounded on $[0, +\infty[$ and put

$$\begin{aligned} x_1 &= Px_0 = x(\omega) \\ x_2 &= Px_1 = x(2\omega) \\ &\vdots \\ x_n &= Px_{n-1} = x(n\omega) \end{aligned}$$

Note that the solution x is bounded, so the sequence x_n is bounded, and that P is non-expansive. Indeed, let y and z be two solutions of (2) such that $y(0) = y_0$ and $z(0) = z_0$ so by dissipativeness of F and the inequality (7) we have

$$\|y(t) - z(t)\| \le \|y(0) - z(0)\| = \|y_0 - z_0\|.$$

Taking $t = \omega$ we deduce that

$$\|Py_0 - Pz_0\| \le \|y_0 - z_0\|.$$

So by the Browder-Petryshyn's fixed point theorem (see Petryshyn [2]), P has a fixed point. So there is a solution $\widetilde{x}$ of (2) which satisfies $\widetilde{x}(0) = \widetilde{x}(\omega)$ and $\widetilde{x}$ is ω-periodic. Indeed, put $\widetilde{y}(t) = \widetilde{x}(t + \omega)$ then

$$\widetilde{y}'(t) = \widetilde{x}'(t + \omega) \in F(t + \omega, \widetilde{x}(t + \omega)) = F(t, \widetilde{y}(t))\,.$$

Now since $\widetilde{y}(0) = \widetilde{x}(\omega) = \widetilde{x}(0)$, by (7) we deduce that

$$\widetilde{x}(t) = \widetilde{y}(t) = \widetilde{x}(t + \omega)$$

hence $\widetilde{x}$ is ω-periodic. $\diamondsuit$

Remark. Let x be an ω-periodic solution of (2), if y is another ω-periodic solution (respectively an T-periodic solution with $\frac{\omega}{T} \notin \mathbb{Q}$) then $\|x(t) - y(t)\|$ is constant for all $t \in \mathbb{R}^+$. From the dissipativeness of F it follows that the map $t \mapsto \|x(t) - y(t)\|$ is decreasing. Since it is continuous and periodic (respectively almost-periodic) we conclude that it is constant.

Example. Consider $(\mathbb{R}^n, \|.\|)$ with $\|.\|$ the Euclidean norm and $\langle ., . \rangle$ the associated inner product. We consider the differential equation

$$x' + x\|x\|^\alpha + \beta\,\mathrm{sgn}(x) = f(t)$$

where $\alpha \ge 0$, $\beta \ge 0$, $f : \mathbb{R}^+ \to \mathbb{R}^n$ is continuous and ω-periodic, and

$$\mathrm{sgn}(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } x \ne 0 \\ \overline{B}(0,1) & \text{if } x = 0 \end{cases}$$

Then the above equation becomes $x' \in F(t,x)$ where $F(t,x) = f(t) - x\|x\|^\alpha - \beta\,\mathrm{sgn}(x)$ is a bounding multi-valued map with compact and convex values. To conclude that the inclusion has an ω-periodic solution, we have to prove the following lemma.

Lemma 1 *1) F is upper semi-continuous on $\mathbb{R}^+ \times \mathbb{R}^n$.*
2) There exist a positive c_α and $r_\alpha \geq 2$ such that for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

$$\langle F(t,x) - F(t,y), x - y\rangle \leq -c_\alpha \|x-y\|^{r_\alpha} .$$

In particular F is dissipative with respect to x
3) Every solution of the inclusion $x' \in F(t,x)$ is bounded.

Proof of 1) We have to show that for every closed $A \subset \mathbb{R}^n$, the set

$$F^{-1}(A) = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : F(t,x) \cap A \neq \emptyset\}$$

is closed in $\mathbb{R}^+ \times \mathbb{R}^n$. Let $(t_n, x_n) \in \mathbb{R}^+ \times \mathbb{R}^n$ be such that $(t_n, x_n) \to (t,x)$ and $F(t_n,x_n) \cap A \neq \emptyset$. We have to show that $F(t,x) \cap A \neq \emptyset$. Let $y_n \in F(t_n,x_n) \cap A$, then $y_n = f(t_n) - x_n\|x_n\|^\alpha - \beta\gamma_n$ with $\|\gamma_n\| \leq 1$, γ_n has a subsequence (which we denote by the same) such that $\gamma_n \to \gamma$ with $(\|\gamma\| \leq 1)$, so

$$y_n = f(t_n) - x_n\|x_n\|^\alpha - \beta\gamma_n \to y := f(t) - x\|x\|^\alpha - \beta\gamma \in F(t,x) \cap A.$$

Hence $F(t,x) \cap A \neq \emptyset$ and F is upper semi-continuous on $\mathbb{R}^+ \times \mathbb{R}^n$.

Proof of 2) It is easy to see that for all $x, y \in \mathbb{R}^n$, $\langle \mathrm{sgn}(x) - \mathrm{sgn}(y), x-y\rangle \geq 0$. Now let $x, y \in \mathbb{R}^n$, then

$$\begin{aligned}
&\langle x\|x\|^\alpha - y\|y\|^\alpha, x-y\rangle \\
&\quad= \langle x\|x\|^\alpha - y\|x\|^\alpha + x\|y\|^\alpha - y\|y\|^\alpha + y\|x\|^\alpha - x\|y\|^\alpha, x-y\rangle \\
&\quad= \|x-y\|^2(\|x\|^\alpha + \|y\|^\alpha) + \langle y\|x\|^\alpha - x\|y\|^\alpha, x-y\rangle \\
&\quad= \frac{1}{2}\|x-y\|^2(\|x\|^\alpha + \|y\|^\alpha) + \frac{1}{2}\langle (x+y)\|x\|^\alpha - (x+y)\|y\|^\alpha, x-y\rangle \\
&\quad= \frac{1}{2}\|x-y\|^2(\|x\|^\alpha + \|y\|^\alpha) + \frac{1}{2}(\|x\|^\alpha - \|y\|^\alpha)(\|x\|^2 - \|y\|^2) \\
&\quad\geq \frac{1}{2}\|x-y\|^2(\|x\|^\alpha + \|y\|^\alpha)
\end{aligned}$$

The last inequality comes from the fact that the map $\varphi(t) = t^\alpha$ is increasing on $\mathbb{R}^+$, so $(\|x\|^\alpha - \|y\|^\alpha)(\|x\| - \|y\|) \geq 0$. Hence for $\alpha = 0$,

$$\langle x\|x\|^\alpha - y\|y\|^\alpha, x-y\rangle \geq \|x-y\|^2 .$$

If $0 < \alpha \leq 1$ then $\|x\|^\alpha + \|y\|^\alpha \geq (\|x\| + \|y\|)^\alpha \geq \|x-y\|^\alpha$, (because the map $\varphi(t) = 1 + t^\alpha - (1+t)^\alpha$ is positive on $\mathbb{R}^+$), so

$$\langle x\|x\|^\alpha - y\|y\|^\alpha, x-y\rangle \geq \frac{1}{2}\|x-y\|^{\alpha+2}$$

If $\alpha \geq 1$ then the map $\varphi(t) = t^\alpha$ is convex on $\mathbb{R}^+$, so

$$\|x\|^\alpha + \|y\|^\alpha \geq \frac{1}{2^{\alpha-1}}(\|x\| + \|y\|)^\alpha \geq \frac{1}{2^{\alpha-1}}\|x-y\|^\alpha \,.$$

Hence

$$\langle x\|x\|^\alpha - y\|y\|^\alpha, x-y\rangle \geq \frac{1}{2^\alpha}\|x-y\|^{\alpha+2}$$

Proof of 3) From 2) we deduce that

$$\langle F(t,x) - F(t,0), x\rangle \leq -c_\alpha\|x\|^{r_\alpha} \,,$$

where

$$\begin{array}{ll} c_\alpha = 1 \text{ and } r_\alpha = 2 & \text{if } \alpha = 0 \\ c_\alpha = 1/2 \text{ and } r_\alpha = \alpha+2 & \text{if } 0 < \alpha \leq 1 \\ c_\alpha = 1/2^\alpha \text{ and } r_\alpha = \alpha+2 & \text{if } \alpha \geq 1 \end{array}$$

Let x be a solution of $x' \in F(t,x)$, and let $a \in F(t,0)$. Then $a = f(t) - \beta\gamma$, ($\|\gamma\| \leq 1$), and we have

$$\begin{aligned} \langle x'(t), x(t)\rangle &= \langle x'(t) - a + a, x(t)\rangle \\ &= \langle x'(t) - a, x(t)\rangle + \langle a, x(t)\rangle \\ &\leq -c_\alpha\|x(t)\|^{r_\alpha} + (M+\beta)\|x(t)\| \,, \end{aligned}$$

where $M = \sup_{t\in\mathbb{R}} \|f(t)\|$. Therefore,

$$\frac{d}{2dt}\|x(t)\|^2 \leq 0 \quad \text{for } \|x(t)\| \geq \big(\frac{M+\beta}{c_\alpha}\big)^{1/(r_\alpha-1)} .$$

Consequently

$$\sup_{t\in\mathbb{R}} \|x(t)\| \leq \max\big[\|x(0)\|, (\frac{M+\beta}{c_\alpha})^{1/(r_\alpha-1)}\big]$$

because the map $t \mapsto \|x(t)\|^2$ is decreasing outside $B\big(0, (\frac{M+\beta}{c_\alpha})^{1/(r_\alpha-1)}\big)$.

3 Almost periodic solutions

Let $(E, \|.\|)$ be a uniformly convex Banach space with E^* uniformly convex. We consider the problem

$$x'(t) \in -Ax(t) + f(t) \,, \tag{8}$$

where $f : \mathbb{R} \to E$ is a continuous almost periodic function (see appendix I for the definition of almost periodicity) and $A : E \to 2^E \setminus \emptyset$ is a hyper-accretive multi-valued map which means that for all $\lambda > 0$, $\mathrm{Im}(I + \lambda A) = E$ and $\langle Ax - Ay, x-y\rangle_+ \geq 0$ for all $x, y \in E$.

Theorem 3 *Problem (8) has a solution on* $[t_0, +\infty[$ *(*$t_0 \in \mathbb{R}$*), which is uniformly continuous with precompact range if and only if it has a weak almost periodic solution.*

Remark. Since a continuous almost periodic map is uniformly continuous with precompact range, it is convenient to relate the existence of a solution to that of uniformly continuous with the precompact range.

Proof of Theorem 3. The proof will be divided into four steps.

Step 1. The Cauchy problem

$$x'(t) \in -Ax(t) + f(t) \tag{9}$$

$$x(t_0) = x_0 \tag{10}$$

has a unique weak solution on $[t_0, +\infty[$. (weak solution means that there are sequences x_n and f_n where x_n is a strong solution and $x_n \to x$ uniformly in every compact subset J of $[t_0, +\infty[$ and $f_n \to f$ in $L^1(J, E)$). Indeed, Since E and E^* are uniformly convex, the Cauchy problem

$$\begin{gathered} x'(t) \in -Ax(t) \\ x(t_0) = x_0 \end{gathered}$$

has a unique strong solution on $[t_0, +\infty[$ (see Deimling [6]). Since f is almost periodic, $f \in L^1(J, E)$ for every compact $J \subset [t_0, +\infty[$, with $t_0 \in J$, so there is a sequence f_n of stairs functions which converges uniformly to f, hence $f_n \to f$ in $L^1(J, E)$. On the other hand for every f_n there is x_n such that

$$\begin{gathered} x_n'(t) \in -Ax_n(t) + f_n(t) \\ x_n(t_0) = x_0 \,. \end{gathered}$$

Because if g is a stair function defined on $a = b_0 < b_1 < ... < b_p = T$ ($T > a$) by $g(t) = y_i$ on $[b_{i-1}, b_i[$ the Cauchy problem

$$\begin{gathered} x'(t) \in -Ax(t) + g(t) \\ x(t_0) = x_0 \end{gathered}$$

has also a unique strong solution x defined by $x(t) = S_i(t - b_{i-1}).x(b_{i-1})$ for $t \in [b_{i-1}, b_i]$ and $x(t_0) = x_0$ where $S_i(t)$ is the semigroup generated by the hyper-accretive operator $-(A - y_i)$.

Let us show that (x_n) is a Cauchy sequence in the Banach space $C([t_0, +\infty[, E)$ equipped with the topology of uniformly convergence in compact subsets. Since $-A$ is dissipative, we have

$$\langle x_n'(t) - f_n(t) - x_p'(t) + f_p(t), x_n(t) - x_p(t)\rangle_- \leq 0 \,,$$

E^* is uniformly convex, $\langle ., . \rangle_- = \langle ., . \rangle_+$, and $\langle ., . \rangle_-$ is linear on the first argument. Then

$$\frac{d^-}{2dt}\|x_n(t) - x_p(t)\|^2$$

$$\begin{aligned}
&= \langle x_n'(t) - x_p'(t), x_n(t) - x_p(t)\rangle_- \\
&= \langle x_n'(t) - f_n(t) - x_p'(t) + f_p(t) + f_n(t) - f_p(t), x_n(t) - x_p(t)\rangle_- \\
&= \langle x_n'(t) - f_n(t) - x_p'(t) + f_p(t), x_n(t) - x_p(t)\rangle_- \\
&\quad + \langle f_n(t) - f_p(t), x_n(t) - x_p(t)\rangle_- \\
&\le \langle f_n(t) - f_p(t), x_n(t) - x_p(t)\rangle_- \\
&\le \|f_n(t) - f_p(t)\| \|x_n(t) - x_p(t)\|
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_n(t) - x_p(t)\| &\le \|x_n(t_0) - x_p(t_0)\| + \int_{t_0}^{t} \|f_n(s) - f_p(s)\| ds \\
&= \int_{t_0}^{t} \|f_n(s) - f_p(s)\| ds \to 0 \quad \text{as } n, p \to +\infty\,.
\end{aligned}$$

Without loss of generality, we can assume that the Cauchy problem (9)-(10) has a strong solution. Let $x : [t_0, +\infty[\to E$ be the uniformly continuous solution of the Cauchy problem (9)-(10) with $x([t_0, +\infty[)$ precompact. Since f is almost periodic, there is $t_n \to +\infty$ such that $f(t + t_n) \to f(t)$ uniformly on $\mathbb{R}$ (see appendix I). Consider the sequences of translated functions

$$x_n(t) = x(t + t_n) \quad \text{and} \quad f_n(t) = f(t + t_n)$$

which are defined on the real interval $[a, +\infty[$ when $n \ge n(a)$. Since $x([t_0, +\infty[)$ is precompact, we deduce that $\{x_n(t), t \ge a, n \ge n(a)\}$ is also precompact. On the other hand that $\{x_n, n \ge n(a)\}$ is equi-continuous follows from the following lemma which is easy to proof.

Lemma 2 *Let E be a Banach space, $J \subset \mathbb{R}$ be an interval and $\mathcal{M}$ a bounded subset of the Banach space $C_b(J, E)$ of continuous bounded functions. Then $\mathcal{M}$ is uniformly equi-continuous if and only if the mapping $(\psi, t) \mapsto \psi(t)$ of $\mathcal{M} \times J \subset C_b(J, E) \times \mathbb{R}$ into E is uniformly continuous on $\mathcal{M} \times J$.*

Now applying Ascoli's theorem in the intervals $[-N, N]$, $N = 1, 2, \dots$ and using the diagonal procedure (see Zaidman [13]) it is possible to find a subsequence which converges uniformly in every compact subset J of $\mathbb{R}$. But f_n is almost periodic, so $f_n \to f$ in $L^1(J, E)$. Therefore, we obtain a weak solution x^* of (8) defined on $\mathbb{R}$ which is uniformly continuous with range contained in the closure of $x([t_0, +\infty[)$, hence with precompact range.

Step 2. Put $K_0 = \overline{Co}(x^*(\mathbb{R}))$, so that K_0 is a compact convex subset of E. Let

$$\Omega = \big\{x : \mathbb{R} \to E | x(\mathbb{R}) \subset K_0\big\}$$

with x a uniformly continuous solution of (8) and $J : \Omega \to \mathbb{R}^+$ defined by $Jx = \sup_{t\in\mathbb{R}} \|x(t)\|$. Put $\mu = \inf_{x\in\Omega} Jx$, so there is $x_n \in \Omega$ such that $J(x_n) \to \mu$. By Lemma 2 and Ascoli's theorem there is a subsequence of x_n which converges uniformly in every compact subset of $\mathbb{R}$, let $\widetilde{x}$ be this limit, then $\widetilde{x} \in \Omega$ and $J\widetilde{x} = \mu$.

Step 3. We show that $\widetilde{x}$ is unique. Assume that there are x_1 and x_2 in Ω such that $Jx_1 = Jx_2 = \mu$. Since f is almost periodic, there is $t_n \to -\infty$ such that $f(t+t_n) \to f(t)$ uniformly on $\mathbb{R}$. By Ascoli's theorem, we can extract from t_n a subsequence (which we denote by the same symbol) such that $x_1(t+t_n)$ and $x_2(t+t_n)$ converge uniformly in every compact subset of $\mathbb{R}$. Let

$$y_1 = \lim x_1(t+t_n) \quad \text{and} \quad y_2 = \lim x_2(t+t_n\,.)$$

Then y_1 and y_2 are weak solutions of (8), and $y_1, y_2 \in \Omega$ with $J(y_1) = J(y_2) = \mu$. Now since

$$x_1'(t+t_n) \in -Ax_1(t+t_n) + f(t+t_n)$$

and

$$x_2'(t+t_n) \in -Ax_2(t+t_n) + f(t+t_n)$$

and $-A$ is dissipative, we deduce that

$$\langle x_1'(t+t_n) - x_2'(t+t_n), x_1(t+t_n) - x_2(t+t_n)\rangle_- \le 0$$

So

$$\frac{d^-}{2dt}\|x_1(t+t_n) - x_2(t+t_n)\|^2 \le 0$$

Consequently the map $t \longmapsto \|x_1(t+t_n) - x_2(t+t_n)\|$ is non increasing. Since $x_i(\mathbb{R}) \subset K_0$ for i=1,2, we deduce that

$$\begin{aligned} \|y_1(t) - y_2(t)\| &= \lim_{n\to+\infty} \|x_1(t+t_n) - x_2(t+t_n)\| \\ &= \lim_{\tau\to-\infty} \|x_1(\tau) - x_2(\tau)\| \\ &= \sup_{t\in\mathbb{R}} \|x_1(t) - x_2(t)\| \\ &= \text{a constant} \end{aligned} \tag{11}$$

To continue, we need the following lemma.

Lemma 3 *Let E be a strictly convex Banach space, C a closed convex subset of E. Let $T : C \to C$ be a non expansive map and x_0, y_0 in C such that*

$$\|Tx_0 - Ty_0\| = \|x_0 - y_0\|\,.$$

Then

$$T(\frac{x_0+y_0}{2}) = \frac{Tx_0 + Ty_0}{2}.$$

Let the operator $T_t : E \to E$ be defined by by $T_t x(0) = x(t)$ where $x(.)$ is a weak solution of (8). Then $T_t y_1(0) = y_1(t)$ and $T_t y_2(0) = y_2(t)$ where y_1 and y_2 are in Ω. By (11),

$$\|T_t y_1(0) - T_t y_2(0)\| = \|y_1(t) - y_2(t)\| = \|y_1(0) - y_2(0)\|\,.$$

So that by Lemma 3,

$$T_t(\frac{y_1(0)+y_2(0)}{2}) = \frac{T_t y_1(0)+y_2(0)}{2} = \frac{y_1(t)+y_2(t)}{2}$$

and $y(t) := \frac{y_1(t)+y_2(t)}{2}$ is also a solution of (8) satisfying $y(0) = \frac{y_1(0)+y_2(0)}{2}$. Since K_0 is convex, $y(\mathbb{R}) \subset K_0$ and $y \in \Omega$. We have

$$Jy_1 = Jy_2 = \mu$$

So, $\mu = \inf\limits_{x\in\Omega} Jx$ and $\frac{y_1+y_2}{2} \in \Omega$. We deduce that

$$\mu \leq J(\frac{y_1+y_2)}{2}) \leq \frac{Jy_1}{2} + \frac{Jy_2}{2} = \mu$$

and consequently $Jy = \mu$. Since $J(\frac{y_1+y_2}{2}) = \frac{Jy_1}{2} + \frac{Jy_2}{2}$ we have

$$\sup_{t\in\mathbb{R}} \|\frac{y_1(t)+y_2(t)}{2}\| = \frac{1}{2}\sup_{t\in\mathbb{R}}\|y_1(t)\| + \frac{1}{2}\sup_{t\in\mathbb{R}}\|y_2(t)\|$$

So there is $s_n \in \mathbb{R}$ such that

$$\begin{aligned} \mu - \frac{1}{n} &< \|\frac{y_1(s_n)+y_2(s_n)}{2}\| \\ &\leq \frac{\|y_1(s_n)\|}{2} + \frac{\|y_2(s_n)\|}{2} \\ &\leq \mu \end{aligned}$$

and since $y_1(s_n) \in K_0; y_2(s_n) \in K_0$ there is a subsequence (which we denote by the same symbol) such that $y_1(s_n) \to l_1$ and $y_2(s_n) \to l_2$. Then

$$\|\frac{l_1+l_2}{2}\| = \frac{\|l_1\|}{2} + \frac{\|l_2\|}{2} = \mu\,.$$

On the other hand $\|y_i(s_n)\| \leq \mu$ implies $\|l_i\| \leq \mu$ and $\frac{\|l_1\|}{2} + \frac{\|l_2\|}{2} = \mu$ implies $\|l_i\| \geq \mu$ for $i = 1, 2$. Hence $\|l_1\| = \|l_2\| = \mu$. Since the norm of E is strictly convex, we deduce that $l_1 = l_2$ and consequently

$$\begin{aligned} \|l_1 - l_2\| &= \|y_1(t) - y_2(t)\| \\ &= \lim_{\tau\to-\infty}\|x_1(\tau) - x_2(\tau)\| \\ &= \|x_1(-\infty) - x_2(-\infty)\| \\ &= \sup_{t\in\mathbb{R}}\|x_1(t) - x_2(t)\| \end{aligned}$$

So $x_1(t) = x_2(t)$ for every $t \in \mathbb{R}$.

Remark. In the case of a Hilbert space, by the parallelogram formula and by (11), we deduce directly that $x_1(t) = x_2(t)$ for all $t \in \mathbb{R}$.

Step 4. Finally we show that $\widetilde{x}$ the unique element of Ω which satisfies $J\widetilde{x} = \inf_{x\in\Omega} Jx$ is almost periodic. For this purpose, we use the 2^{nd} Bochner's characterization of almost periodicity (see appendix I). Let t_n and s_n be two real sequences, then by Ascoli's theorem there is a subsequence of t_n (which we denote by the same symbol) such that $\widetilde{x}(t+t_n) \to y(t)$ uniformly in every compact subset of $\mathbb{R}$. Then $y(.)$ is a weak solution of

$$x' \in -Ax + g(t)\,, \tag{12}$$

where $g(t) = \lim f(t+t_n)$. Now consider $\widetilde{x}(t+t_n+s_n)$ and $y(t+s_n)$, then by Ascoli's theorem we can extract from t_n and s_n sub-sequences such that $\widetilde{x}(t+t_n+s_n) \to z_1(t)$ and $y(t+s_n) \to z_2(y)$, but $f(t+t_n+s_n)$ and $g(t+s_n)$ have the same limit which we denote by $h(t)$. Then $z_1(.)$ and $z_2(.)$ are weak solutions of

$$x' \in -Ax + h(t) \tag{13}$$

so $\mu = J_f(K_0) = J_h(K_0) \le Jz_i \quad i = 1,2$ where

$$J_f(K_0) = \inf\big\{Jx : x \text{is a weak solution of } (8), x(\mathbb{R}) \subset K_0\big\}$$

and

$$J_h(K_0) = \inf\big\{Jx : x \text{is a weak solution of } (13), x(\mathbb{R}) \subset K_0\big\}\,.$$

We have $\mu = Jz_1 = Jz_2$, but the equation (13) has the same property as the equation (8) because the map $h(.)$ is almost periodic. Therefore, there is a unique solution which satisfies

$$J_h(K_0) = \inf\big\{Ju : u \text{ is a weak solution of } (13), u(\mathbb{R}) \subset K_0\big\}.$$

Consequently $z_1 = z_2$. Also $\widetilde{x}(t+t_n+s_n)$ and $y(t+s_n)$ have the same limit, hence $\widetilde{x}$ is almost periodic. $\diamondsuit$

Example. Let $E = (\mathbb{R}^n, \|.\|)$ with the Euclidean norm $\|.\|$, and let $\varphi(x) = \|x\|$. Consider

$$Ax = \partial\varphi(x) + kx$$

where $k > 0$ and $\partial\varphi$ is the sub-differential of φ. Since φ is continuous and convex,

$$\overbrace{\mathrm{Dom}(\varphi)}^{\circ} \subset \mathrm{Dom}(\partial\varphi) \quad \text{so} \quad \mathrm{Dom}(A) = \mathbb{R}^n.$$

The problem

$$x' \in -Ax + f(t)\,,$$

with $f : \mathbb{R} \to \mathbb{R}^n$ continuous and almost periodic, has a strong solution defined on $[t_0, +\infty[$ ($t_0 \in \mathbb{R}$) (see Brezis [4]). Now since $0 \in \partial\varphi(0)$ we have

$$\langle f(t) - kx - x'(t), x(t)\rangle \ge 0\,.$$

Therefore,

$$\begin{aligned} \langle x'(t), x(t)\rangle &\leq \langle f(t), x(t)\rangle - k\|x(t)\|^2 \\ &\leq (M - k\|x(t)\|)\|x(t)\| \end{aligned}$$

where $M = \sup_{t\in\mathbb{R}} \|f(t)\|$. We deduce that

$$D^-\|x(t)\| \leq M$$

and

$$\frac{d}{2dt}\|x(t)\|^2 \leq 0 \quad \text{for } \|x(t)\| \geq \frac{M}{k}.$$

The first inequality shows that x is lipschitzean, hence uniformly continuous and the second one shows that the map $t \longmapsto \|x(t)\|$ is non increasing outside of the ball $B(0, \frac{M}{k})$. Consequently

$$\|x(t)\| \leq \sup(\|x(t_0)\|, \frac{M}{k}) \quad \forall t \geq t_0.$$

So that the problem $x' \in -Ax + f(t)$ has a uniformly continuous solution which is bounded, hence with precompact range, so it has an almost periodic solution. $\diamondsuit$

Appendix I

Let E be a real Banach space, a map $f : \mathbb{R} \to E$ is said to be almost periodic if for each $\varepsilon > 0$ there exists l_ε such that for all $a \in \mathbb{R}$ there exists $\tau \in [a, a + l_\varepsilon]$ such that

$$\|f(t + \tau) - f(t)\| \leq \varepsilon \quad \forall t \in \mathbb{R}.$$

If f is almost periodic then there exist $t_n \to +\infty$ and $s_n \to -\infty$ such that $f(t + t_n) \to f(t)$ and $f(t + s_n) \to f(t)$ uniformly on $\mathbb{R}$. In practice, we use the following Bochner's characterizations of almost periodicity (Yoshisawa [12]).

First characterization. $f \in C(\mathbb{R}, E)$ is almost periodic if and only if from every real sequence t'_n one can extract a subsequence t_n such that $\lim f(t + t_n)$ exists uniformly on the real line, furthermore the limit is also almost periodic.

Second characterization. $f \in C(\mathbb{R}, E)$ is almost periodic if and only if for every pair of real sequences h'_n and k'_n there are sub-sequences h_n and k_n such that $f(t+h_n)$ has a pointwise limit $g(t)$ on $\mathbb{R}$, and $f(t+h_n+k_n)$ and $g(t+k_n)$ have a same limit $h(t)$ on $\mathbb{R}$, and h is also almost periodic.

Appendix II

Let E be a real Banach space and $x : [a, b] \subset \mathbb{R} \to E$ differentiable, and put $\Phi(t) = \|x(t)\|$. Then

$$\Phi(t)D^{-}\Phi(t) =< x'(t), x(t) >_{-}$$

where

$$D^{-}\Phi(t) = \limsup_{h \to 0^{-}} \frac{\Phi(t+h) - \Phi(t)}{h}$$

is the upper Dini's derivative of Φ (see e.g Deimling [6]).

References

[1] J. Aubin, A. Cellina, *Differential inclusions multivalued maps and viability theory*, Springer-Verlag (1984).

[2] F. Browder, W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bul Amer Math Soc, 72 (1966), 571–575.

[3] F. Browder, *Periodic solutions of nonlinear equations of evolution in infinite dimensional spaces*, "Lecture in differential equations" A.K Aziz. ed, V1 Van Norstrand, Newyork,(1969), 71–96.

[4] H. Brezis, *Opérateurs maximaux monotones et semigroupes de contraction dans les espaces de Hilbert*, North-Holand Amsterdam (1973).

[5] J. Baillon A. Haraux, *Comportement à l'infinie pour les équations d'évolution avec forcing périodique*, Arch. Rat. Mec. Anal, 67 (1977), 101–109.

[6] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag (1985).

[7] E. Hanebaly, *Solutions périodiques d'équations différentielles non linéaires en dimension infinite*, C.R.A.S, Serie A (1979), 623–626.

[8] E. Hanebaly, *Solutions presque-périodiques d'équations différentielles monotones*, C.R.A.S, 296 Serie I (1983), 263.

[9] E. Hanebaly, *Contribution à l'étude des solutions périodiques et presque-périodiques des équations différentielles non linéaires sur les espaces de Banach*, Thèse d'état Université de Pau I.U.R.S (1988).

[10] E. Hanebaly, *Un théorème du point fixe et solutions périodiques d'équations différentielles V-dissipatives*, πP.R.N.M.S (252) Fixed point theory and applications, (1991), 221–230. Edit=82 par J.B.Baillon M.A. thera.

[11] G. Lumer, *Semi-inner product spaces*, Trans. Math. Math .Soc, (1961), 29–43.

[12] T. Yoshisawa, *stability theory and the existence of periodic solutions and almost periodic solutions*, Spriger-Verlag (1975).

[13] S. Zaidman, *Solution of almost periodic abstract differential equations with relatively compact range*, J. Nonlinear analysis, Vol. 8, No. 9 (1984), 1091–1093.

E. Hanebaly
Université Mohammed V Faculté des Sciences
Département de Mathématiques, Rabat, Maroc.
e-mail: hanebaly@fsr.ac.ma

Brahim Marzouki
Université Mohammed I Faculté des Sciences
Département de Mathématiques, Oujda, Maroc.
e-mail: marzouki@sciences.univ-oujda.ac.ma