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BIFURCATION OF POSITIVE SOLUTIONS FOR A SEMILINEAR EQUATION WITH CRITICAL SOBOLEV EXPONENT

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Abstract. In this note we consider bifurcation of positive solutions to the semilinear elliptic boundary-value problem with critical Sobolev exponent

$$\begin{gathered}-\Delta u = \lambda u - \alpha u^p + u^{2^*-1}, \quad u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega.\end{gathered}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded C^2-domain $\lambda > \lambda_1$, $1 < p < 2^* - 1 = \frac{n+2}{n-2}$ and $\alpha > 0$ is a bifurcation parameter. Brezis and Nirenberg [2] showed that a lower order (non-negative) perturbation can contribute to regain the compactness and whence yields existence of solutions. We study the equation with an indefinite perturbation and prove a bifurcation result of two solutions for this equation.

1. Introduction and main result

It is well known that the following equation with a critical exponent has no solution on the star-shaped domains, [12],

$$\begin{gathered}-\Delta u = u^{\frac{n+2}{n-2}}, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

due to the lack of compactness in the embedding $H_0^1(\Omega) \hookrightarrow L_{\frac{2n}{n-2}}(\Omega)$. In their seminal work [2], Brezis and Nirenberg show that perturbation by a lower order term suffices to regain the compactness and hence existence of a solution. Consider particularly for the following equation

$$\begin{gathered}-\Delta u = \lambda u + u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.2}$$

where λ is considered as a bifurcation parameter, let $\lambda_1 > 0$ be the first eigenvalue of Laplacian with a Dirichlet boundary, then they show the following result.

Theorem 1.1 ([2]). *There is a constant $\lambda^* \in [0, \lambda_1)$, such that* (1.2) *has a solution if $\lambda \in (\lambda^*, \lambda_1)$ and has no solution, if $\lambda \geq \lambda_1$.*

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Thereafter, there are many papers devoted to study of problems with critical Sobolev exponent (see [9, 14] and references therein). Effects of concave and convex combination on bifurcation have been studied in [1, 4, 5, 6, 15]. In this paper we consider the equation with an indefinite lower order perturbation. For simplicity consider the prototype equation

$$\begin{gathered} -\Delta u = \lambda u - \alpha u^p + u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where λ is a fixed positive constant, and $\alpha > 0$ is considered as a bifurcation parameter. The main result of this note is the the following theorem showing the existence of two solutions.

Theorem 1.2. *If $\lambda > \lambda_1$ and $3 \le n \le 5$, $1 < p < 4/(n-2)$, then there is a constant $\alpha_0 > 0$ such that* (1.3) *has at least two solutions for $\alpha > \alpha_0$ and has no solution if $\alpha < \alpha_0$*

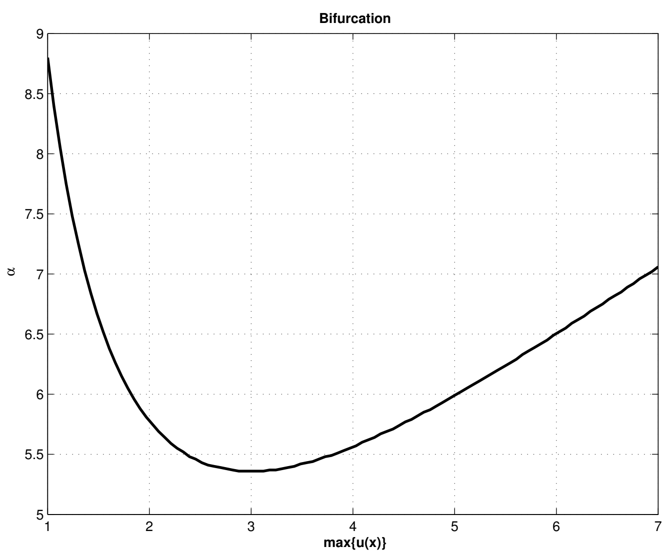

FIGURE 1. Bifurcation diagram of (1.3)

2. Auxiliary lemmas

In this section we establish some estimates which are needed in the proof of Theorem 1.2. Without loss of generality, we assume that the domain Ω contains the origin and choose $R > 0$ small enough so that $\{x : |x| \le 2R\} \subset \Omega$. Let $\psi(x)$ be a cut-off function such that

$$\psi(x) \equiv \begin{cases} 1, & |x| \le R, \\ 0, & |x| \ge 2R, \end{cases}$$

and $N = \sqrt{n(n-2)}$. Also let

$$u_\varepsilon(x) = \psi(x) u_{0\varepsilon}(x), \quad u_{0\varepsilon}(x) = \big(\frac{N\varepsilon}{\varepsilon^2 + |x|^2}\big)^{(n-2)/2}.$$

Then $\|\nabla u_{0\varepsilon}\|_2^2 = S^{n/2} = \|u_{0\varepsilon}\|_{2^*}^{2^*}$ for all $\varepsilon > 0$. The following estimates will be needed in the proof of Theorem 1.2.

Lemma 2.1. *The following estimates hold for some constant $K = K(q) > 0$*

(a) $\|\nabla u_\varepsilon\|_2^2 = S^{n/2} + O(\varepsilon^{n-2})$

(b) $\|u_\varepsilon\|_{2^*}^{2^*} = S^{n/2} + O(\varepsilon^n)$

(c) $1 \le q < 2^*$,

$$\|u_\varepsilon\|_q^q \begin{cases} = K\varepsilon^{\frac{2n-(n-2)q}{2}} + O(\varepsilon^{\frac{(n-2)q}{2}}), & q > n/(n-2) \\ = \varepsilon^{n/2}(K|\ln\varepsilon| + O(1)), & q = n/(n-2) \\ \approx \varepsilon^{(n-2)q/2}, & q < n/(n-2). \end{cases}$$

Proof. The estimate in (a) and (b) are known. Estimate (c) can be shown similarly as in [9, 14]. $\square$

Lemma 2.2. *There are constants $\beta, \beta_1, \beta_2 > 0$ such that the following inequalities hold for all $a, b \ge 0$*

(1) $p \ge 2$, $\beta_1(a^{p-1}b + ab^{p-1}) \ge (a+b)^p - a^p - b^p \ge \beta_2(a^{p-1}b + ab^{p-1})$.

(2) $p \in (1,2)$, $(a+b)^p - a^p - b^p \le \beta a^{p-1}b$.

Proof. The inequalities follow from the facts that $h(t) = \frac{(1+t)^p - 1 - t^p}{t + t^{p-1}} \to p$ as either $t \to 0+$ or $t \to +\infty$; $h_0(t) = \frac{(1+t)^p - 1 - t^p}{t} \to p$ as $t \to 0+$ and $h_0(t) \to 0$ as $t \to +\infty$. $\square$

We would like to point out here that if $1 < p < 2$ then there is no constant $\beta > 0$ such that the following estimate holds for all $a, b \ge 0$,

$$(a+b)^p \ge a^p + b^p + \beta a^{p-1}b.$$

3. Proof of Theorem 1.2

Now we consider

$$\begin{gathered} -\Delta u = \lambda u - \alpha u^p + u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

We first observe that for small $\alpha > 0$ there is no solution for (3.1) by comparison, because $f(u) := \lambda u - \alpha u^p + u^{\frac{n+2}{n-2}}$ satisfies the inequality $f(u) > \lambda_1 u$ on $(0,\infty)$. On the other hand, if α is big enough, then $f(u)$ vanishes somewhere on $(0,\infty)$ and whence a constant $u_+(x) = M$ suffices for a super-solution. To find a sub-solution, we can take $u_-(x) = t\phi_1(x) > 0$, where $\phi_1(x) > 0$ is the normalized eigenfunction associated to λ_1, because

$$-\Delta(t\phi_1) - \lambda(t\phi_1) + \alpha(t\phi_1)^p - (t\phi_1)^{2^*-1} = t(\lambda_1 - \lambda)\phi_1 + \alpha(t\phi_1)^p - (t\phi_1)^{2^*-1} < 0. \tag{3.2}$$

Thus by the sub- and super-solution method, there is a solution for (3.1). Furthermore for given $\alpha_0 > 0$ if the problem (3.1) has a solution u_{α_0}, we shall show then for any $\alpha > \alpha_0$ the problem (3.1) has also a solution. Clearly u_{α_0} is a super-solution for (3.1), because

$$-\Delta u_{\alpha_0} - \lambda u_{\alpha_0} + \alpha u_{\alpha_0}^p - u_{\alpha_0}^{2^*-1} = (\alpha - \alpha_0)u_{\alpha_0}^p > 0, \tag{3.3}$$

and moreover $t\phi_1(x)$ still suffices as a sub-solution. Further, by the Hopf's lemma $\frac{\partial u_{\alpha_0}}{\partial \nu} > 0$ on $\partial\Omega$, we deduce that $t\phi_1(x) < u_{\alpha_0}(x)$ on the whole domain Ω and thus again via sub- and super-solution method we obtain a solution $u_\alpha(x)$ for (3.1), where $u_\alpha(x)$ is a minimizer of

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{\lambda}{2}u^2 + \frac{\alpha}{p+1}|u|^{p+1} + \frac{1}{2^*}|u|^{2^*}\,dx$$

over the convex set $K = \{u \in H_0^1(\Omega) : t\phi_1(x) \le u(x) \le u_{\alpha_0}(x) \text{ a. e. in } \Omega\}$. Furthermore, since $t\phi_1, u_{\alpha_0}$ are not solutions of $(3)_\alpha$, we deduce that $t\phi_1(x) < u(x) < u_{\alpha_0}(x)$ on Ω. If we choose $k > 0$ large then $(\lambda + k)u - \alpha u^p + u^{\frac{n+2}{n-2}}$ will be increasing on $(0, \infty)$ and whence we deduce from [3, Theorem 2] that u_α is a local minimizer for J in $H_0^1(\Omega)$-topology.

We now define α_0 to be the infimum of all $\alpha > 0$ such that (3.1) has a solution, then we infer that $\alpha_0 > 0$ is an finite number, and it remains to show that for all $\alpha > \alpha_0$ there are two solutions for (3.1).

Let $\alpha > \alpha_0$ be given, and u_α be the solution of (3.1) obtained by the sub- and super-solution method. To establish the second solution we exploit the truncation and translation technique and define $v = u - u_\alpha$ and

$$g(x,v) = \begin{cases} \lambda v - \alpha((v+u_\alpha)^p - u_\alpha^p) + (v+u_\alpha)^{2^*-1} - u_\alpha^{2^*-1} & v \ge 0 \\ 0 & v < 0. \end{cases}$$

In the sequel we shall study the boundary-value problem

$$\begin{gathered} -\Delta v = g(x,v) \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.4}$$

First we notice that any nontrivial solution v of (3.4) must be non-negative and then by the strong maximal principle it should be strictly positive on Ω. Whence if $v \ne 0$ is a solution of (3.4), then $u = v + u_\alpha$ will be a positive solution to the problem (3.1), which is bigger than u_α.

We will exploit the critical point method and whence will study the associated functional to the problem (3.4),

$$E(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - G(x,v), \quad G(x,v) = \int_0^v g(x,t)\,dx.$$

Given any $v \in H$, decomposed into positive part v_+, and negative part v_-, then we test the equation (3.1) for the solution u_α by v_+ and obtain

$$\int_\Omega \nabla u_\alpha \cdot \nabla v_+ = \int_\Omega (\lambda u_\alpha - \alpha u_\alpha^p + u_\alpha^{\frac{n+2}{n-2}})v_+\,.$$

Furthermore we obtain the relation

$$E(v) = J(v_+ + u_\alpha) - J(u_\alpha) + \frac{1}{2}\|v_-\|^2, \tag{3.5}$$

which shows that zero is even a local minimizer for E.

Lemma 3.1. *The equation* (3.4) *satisfies the Palais-Smale condition (P.S.)$_c$ for any $c \in (0, \frac{1}{n}S^{n/2})$.*

Proof. Arguments in [14, Lemma 2.3] works also here. □

By the min-max principle, if we can find $v > 0$ such that

$$c = \inf_{\phi \in \Gamma} \max\{E(\phi(t)) : t \in [0,1]\}$$

is finite and E satisfies the local Palais-Smale condition $(\mathrm{P.S.})_c$, where

$$\Gamma = \{\phi \in C([0,1], H) : \phi(0) = 0, \phi(1) = v\} \tag{3.6}$$

then there is a critical point u of E at level c. It follows from (3.5) that $c \geq 0$. If $c > 0$, then we will have a nontrivial solution u. If $c = 0$, then by [7, Theorem 5.10], see also [8, 10], we deduce that there is a continua of minimizers $u^\varepsilon(x), \varepsilon \in (0, \varepsilon_0)$ such that $E(u^\varepsilon) = E(u_\alpha)$. So we are also done even in this case.

To find the function v in (3.6), we shall test $v = tu_\varepsilon$. For $n = 3$, we may assume $p \in (2, 3)$ then we have $2^* = 6$, $\frac{n}{n-2} = 3$ and by Lemma 2.2 we obtain

$$(v + u_\alpha)^{2^*-1} - u_\alpha^{2^*-1} \geq v^5 + 4v^4 u_\alpha,$$
$$(v + u_\alpha)^p - u_\alpha^p \leq v^p + \beta_1(v^{p-1}u_\alpha + vu_\alpha^{p-1})$$

and consequently

$$G(x, v) \geq \frac{\lambda}{2}v^2 - \alpha(\frac{1}{p+1}v^{p+1} + \beta(\frac{1}{2}v^2 u_\alpha^{p-1} + \frac{1}{p}v^p u_\alpha)) + \frac{1}{6}v^6 + \frac{\beta_2}{5}v^5 u_\alpha.$$

Since u_α is strictly positive on Ω, so there are constants $C_1 \geq C_2 > 0$ such that $C_1 \geq u_\alpha(x) \geq C_2$, for all $x \in \Omega, |x| \leq 2R$. We deduce that for some constants $C_3, C_4 > 0$,

$$E(tu_\varepsilon) \leq \int_\Omega \frac{t^2}{2}|\nabla u_\varepsilon|^2 + C_4(t^2 u_\varepsilon^2 + t^p u_\varepsilon^p + t^{p+1}u_\varepsilon^{p+1}) - C_3 t^5 u_\varepsilon^5 - \frac{t^6}{6}u_\varepsilon^6.$$

In view of lemma 2.1, we obtain

$$\|u_\varepsilon\|_2^2 \leq A\varepsilon, \quad \|u_\varepsilon\|_p^p \leq A\varepsilon^{p/2}, \quad \|u_\varepsilon\|_{p+1}^{p+1} = K(p+1)\varepsilon^{(5-p)/2} + O(\varepsilon^{(p+1)/2}),$$
$$\|u_\varepsilon\|_5^5 = K(3.3)\sqrt{\varepsilon} + O(\varepsilon^{5/2}), \quad \|u_\varepsilon\|_6^6 = S^{3/2} + O(\varepsilon^3)$$

thus

$$E(tu_\varepsilon) \leq \frac{t^2}{2}(S^{3/2} + O(\varepsilon)) + C_4(t^2 A\varepsilon + t^p A\varepsilon^{p/2} + t^{p+1}(K(p+1)\varepsilon^{\frac{5-p}{2}} + O(\varepsilon^{\frac{p+1}{2}})))$$

$$- t^5 C_3(K(5)\sqrt{\varepsilon} + O(\varepsilon^{5/2})) - \frac{t^6}{6}(S^{3/2} + O(\varepsilon^3)) := h_3(t).$$

The function $h_3(t)$ attains its maximum on $(0, \infty)$ at $t_{max3} := 1 - \frac{5K(3.3)C_3}{4S^{3/2}}\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$. Moreover $h_3(t_{max3}) = \frac{1}{3}S^{3/2} - C_3 K(3.3)\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$. Therefore, we deduce that for $\varepsilon > 0$ enough small

$$c = \inf_{\phi \in \Gamma} \max\{E(\phi(t)) : t \in [0, 1]\} \leq h_3(t_{max3}) < \frac{1}{3}S^{3/2}$$

and obtain via the mountain pass theorem that (3.4) admits a positive solution u. The proof is complete for the case of dimension 3.

If $n = 4$ or 5, then by the assumption $p < 4/(n-2) \leq 2$ and thus it follows from the lemma 2.2 that

$$(v + u_\alpha)^p - u_\alpha^p \leq v^p + \beta v u_\alpha; \quad (v + u_\alpha)^{2^*-1} - u_\alpha^{2^*-1} \geq v^{2^*-1} + \beta_2 v^{2^*-2}u_\alpha,$$
$$g(x, v) \geq \lambda v - \alpha(v^p + \beta v u_\alpha^{p-1}) + v^{2^*-1} + \beta_2 v^{2^*-2}u_\alpha$$

and consequently

$$G(x,v) \geq \frac{\lambda}{2}v^2 - \alpha(\frac{1}{p+1}v^{p+1} + \frac{\beta}{2}v^2u_\alpha^{p-1}) + \frac{1}{2^*}v^{2^*} + \frac{\beta_2}{2^*-1}v^{2^*-1}u_\alpha,$$

$$E(v) \leq \int_\Omega \frac{1}{2}|\nabla v|^2 - (\frac{\lambda}{2}v^2 - \alpha(\frac{1}{p+1}v^{p+1} + \frac{\beta}{2}v^2u_\alpha^{p-1}) + \frac{1}{2^*}v^{2^*} + \frac{\beta_2}{2^*-1}v^{2^*-1}u_\alpha).$$

In analogy as the case $n = 3$, we deduce that for some constants $C_3, C_4 > 0$.

$$E(tu_\varepsilon) \leq \int_\Omega \frac{t^2}{2}|\nabla u_\varepsilon|^2 + C_4(t^2u_\varepsilon^2 + t^{p+1}u_\varepsilon^{p+1}) - C_3t^{2^*-1}u_\varepsilon^{2^*-1} - \frac{t^{2^*}}{2^*}u_\varepsilon^{2^*}.$$

For $n = 4$, we have

$$\begin{aligned} E(tu_\varepsilon) \leq &\frac{t^2}{2}(S^2 + 0(\varepsilon^2)) + C_4(t^2(\varepsilon^2(K(2)|\ln\varepsilon| + O(1)) + t^{p+1}(K(p+1)\varepsilon^{3-p} \\ &+ O(\varepsilon^{p+1}))) - t^3C_3(K(3)\varepsilon + O(\varepsilon^3)) - \frac{t^4}{4}(S^2 + O(\varepsilon^4)) := h_4(t)\,. \end{aligned}$$

Then $h_4(t)$ attains its maximum on $(0,\infty)$ at $t_{max4} := 1 - \frac{3K(3)C_3}{2S^2}\varepsilon + o(\varepsilon)$, which satisfies

$$\begin{aligned} &S^2 + O(\varepsilon^2) + C_4(2\varepsilon^2(K(2)|\ln\varepsilon| + O(1)) + t^{p-1}(p+1)(K(p+1)\varepsilon^{3-p} + O(\varepsilon^{p+1}))) \\ &= t3C_3(K(3)\varepsilon + O(\varepsilon^3)) + t^2(S^2 + O(\varepsilon^4)) \end{aligned}$$

and moreover $h_4(t_{max4}) = \frac{1}{4}S^2 - C_3K(3)\varepsilon + o(\varepsilon) < \frac{1}{4}S^2$, for sufficient small $\varepsilon > 0$. So we are done in this case.

If $n = 5$, we obtain in a similar way that

$$\begin{aligned} E(tu_\varepsilon) \leq &\frac{t^2}{2}(S^{5/2} + O(\varepsilon^3)) + C_4(t^2(\varepsilon^2K(2) + O(\varepsilon^3)) + t^{p+1}(K(p+1)\varepsilon^{(7-3p)/2} \\ &+ O(\varepsilon^{\frac{3(p+1)}{2}}))) - t^{\frac{7}{3}}C_3(K(\frac{7}{3})\varepsilon^{\frac{3}{2}} + O(\varepsilon^{\frac{7}{2}})) - \frac{3t^{\frac{10}{3}}}{10}(S^{\frac{5}{2}} + O(\varepsilon^5)) := h_5(t). \end{aligned}$$

Because $p < 4/3$, we see that $(7-3p)/2 > 3/2$ and whence $h_5(t)$ attends its maximum on $(0,\infty)$ at $t_{max5} := 1 - \frac{7K(7/3)C_3}{4S^{5/2}}\varepsilon^{3/2} + o(\varepsilon^{3/2})$, which satisfies

$$\begin{aligned} &S^{5/2} + C_4(2\varepsilon^2K(2) + O(\varepsilon^3) + (p+1)t^{p-1}(K(p+1)\varepsilon^{3-p} + O(\varepsilon^{p+1}))) \\ &= \frac{7}{3}C_3t^{1/3}(K(7/3)\varepsilon^{3/2} + O(\varepsilon^{7/2})) + t^{4/3}(S^{5/2} + O(\varepsilon^5))\,. \end{aligned}$$

Moreover $h_5(t_{max5}) = \frac{1}{5}S^{5/2} - C_3K(7/3)\varepsilon^{3/2} + o(\varepsilon^{3/2}) < \frac{1}{5}S^{5/2}$, for sufficient small $\varepsilon > 0$. So the proof is complete in this case.

4. An example

In this part we show a numerical result of solutions for an equation on the the unite ball in $\mathbb{R}^3$. we consider an equation with a critical exponent $\Omega = \{x \in \mathbb{R}^3 : \|x\| < 1\}$,

$$\begin{gathered} -\Delta u(x) = 4\pi u(x) - \alpha u^2(x) + u^5(x), \quad \|x\| < 1, \\ u(x) = 0, \quad \|x\| = 1. \end{gathered}$$

By Gidas, Ni and Nirenberg [11], any positive solution must be radial symmetric, i.e. $u(x) = u(r)$, $r = \|x\|$ and thus satisfies ordinary differential equation

$$-(r^2u'(r))' = r^2(4\pi u(r) - \alpha u^2(r) + u^5(r)), \quad r \in (0, 1),$$
$$u'(0) = 0, \quad u(1) = 0.$$

By a numerical simulation for $\alpha = 7.5$, we find two positive solutions, where their maxima of the solutions are $u_1(0) = 0.575$ and $u_2(0) = 3.44$.

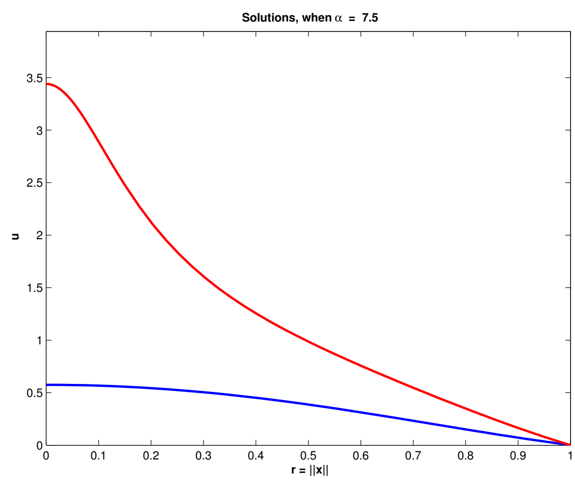

Figure 2. Numerical simulation of solutions on unit ball in $\mathbb{R}^3$

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