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CONCENTRATION AND DYNAMIC SYSTEM OF SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. In this article, we use the concentration of solutions of the semilinear elliptic equations in axially symmetric bounded domains to prove that the equation has three positive solutions. One solution is y-symmetric and the other are non-axially symmetric. We also study the dynamic system of these solutions.

1. Introduction

Consider the semilinear elliptic equation

$$\begin{gathered} -\Delta u + u = |u|^{p-2}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{gathered} \tag{1.1}$$

where $N \geq 2$, $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$, $2 < p < 2^*$, Ω is a domain in $\mathbb{R}^N$, and $H_0^1(\Omega)$ is the Sobolev space in Ω with dual space $H^{-1}(\Omega)$. Associated with equation (1.1), we consider the energy functionals a, b, and J defined for each $u \in H_0^1(\Omega)$ as follows:

$$\begin{gathered} a(u) = \int_\Omega (|\nabla u|^2 + u^2), \quad b(u) = \int_\Omega |u|^p, \\ J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u). \end{gathered}$$

By Rabinowitz [9, Proposition B. 10], a, b, and J are of class $C^{1,1}$. It is well-known that the solutions of equation (1.1) are the critical points of the energy functional J. Let $z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Denote the N-ball $B^N(z_0; s)$ in $\mathbb{R}^N$, the infinite strip $\mathbf{A}^r$, the upper half strip $\mathbf{A}_0^r$, and the finite strip $\mathbf{A}_{s,t}^r$ as follows:

$$\begin{gathered} B^N(z_0; s) = \{z \in \mathbb{R}^N : |z - z_0| < s\}, \\ \mathbf{A}^r = \{(x, y) \in \mathbb{R}^N : |x| < r\}, \\ \mathbf{A}_0^r = \{(x, y) \in \mathbf{A}^r : 0 < y\}, \\ \mathbf{A}_{s,t}^r = \{(x, y) \in \mathbf{A}^r : s < y < t\}. \end{gathered}$$

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We should point out here that the precise definition of the finite strip $\mathbf{A}^r_{s,t}$ is the domain which is symmetric in $y-$axis and has been smoothed out at the corners of $\{(x,y) \in \mathbf{A}^r : s < y < t\}$. By the Rellich compactness theorem, there is a positive solution of equation (1.1) in the finite strip $\mathbf{A}^r_{-t,t}$ for each $t > 0$. Moreover, $\mathbf{A}^r_{-t,t}$ is convex in x and in y. Thus, by Gidas-Ni-Nirenberg [6], every positive solution of equation (1.1) in $\mathbf{A}^r_{-t,t}$ for each $t > 0$ is radially symmetric in x and axially symmetric in y. Actually, Dancer [5] proved that the positive solution of equation (1.1) in $\mathbf{A}^r_{-t,t}$ for each $t > 0$ in $\mathbb{R}^2$ is unique. However, the axially symmetry and uniqueness of positive solution generally fails if Ω is not convex in the y-direction. First, we consider a perturbation of the finite strip $\mathbf{A}^r_{-t,t}$, that is dumbbell type domain

$$D = B^N((0;-t), r_0) \cup \mathbf{A}^r_{-t,t} \cup B^N((0;t), r_0) \qquad \text{for } B^{N-1}(0;r) \subset B^{N-1}(0;r_0).$$

Then the dumbbell domain D is symmetric in $y-$axis, but not convex in y-direction. Moreover, the Dancer [5] and Byeon [2], [3] proved that the equation (1.1) in D has at least three positive solutions, for $B^{N-1}(0;r)$ is sufficiently close to a point x_0 in $\mathbb{R}^{N-1}$. And Chen-Ni-Zhou [4] use computational showed that the equation (1.1) in some dumbbell-type domains has multiple positive solutions and describe the concentration of these solutions.

The main purpose of this paper is using the Palais-Smale theory to present another perturbation. Let ω be a y-symmetric bounded set such that $\mathbf{A}^r \backslash \overline{\omega} \subsetneqq \mathbf{A}^r$ is a domain in $\mathbb{R}^N$ for some $t > 0$, consider the finite strip with holes

$$\Theta_t = \mathbf{A}^r_{-t,t} \backslash \overline{\omega}.$$

Then there exists a $t' > 0$ such that Θ_t is also symmetric in $y-$axis, but not convex in $y-$direction for each $t > t'$. We prove that there exists a $t_0 > 0$ such that for $t \geq t_0$, the equation (1.1) in Θ_t has three positive solutions which one is y-symmetric and the other are non-axially symmetric. Moreover, we describe the concentration and dynamic system of these solutions. Although, Wang-Wu [10] used the symmetry of positive solutions showed the same multiple results in a finite strip with hole $\mathbf{A}^r_{-t,t} \backslash B^N(0;r')$ for t sufficiently large. However, they have not describe the concentration and dynamic system of solutions.

This article is organized as follow. In section 2, we describe various preliminaries. In section 3, we describe various compactness results. In section 4, we describe some properties of the large domains in $\mathbf{A}^r$. In section 5 and section 6, we present the concentration and dynamic system of the solutions.

2. Preliminary

In this article, we focus on the problems on two Hilbert spaces: the whole Sobolev space $H^1_0(\Omega)$ and its closed linear subspace $H_s(\Omega)$ defined as follows: Let $z = (x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and Ω be a domain in $\mathbb{R}^N$.

Definition 2.1. (i) Ω is y-symmetric provided $z = (x,y) \in \Omega$ if and only if $(x,-y) \in \Omega$;

(ii) Let Ω be a y-symmetric domain in $\mathbb{R}^N$. A function $u : \Omega \to \mathbb{R}$ is y-symmetric (axially symmetric) if $u(x,y) = u(x,-y)$ for $(x,y) \in \Omega$.

In this article, we let Ω be a y-symmetric domain in $\mathbb{R}^N$ and $H_s(\Omega)$ the H^1-closure of the space $\{u \in C^\infty_0(\Omega) : u \text{ is } y\text{-symmetric}\}$ and let $X(\Omega)$ be either the whole space $H_0^1(\Omega)$ or the y-symmetric Sobolev space $H_s(\Omega)$. Then $H_s(\Omega)$ is a closed linear subspace of $H_0^1(\Omega)$. Let $H_s^{-1}(\Omega)$ be the dual space of $H_s(\Omega)$.

We define the Palais-Smale (simply by (PS)) sequences, (PS)-values and (PS)-conditions in $X(\Omega)$ for J as follows.

Definition 2.2. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a (PS)$_\beta$-sequence in $X(\Omega)$ for J if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $X^{-1}(\Omega)$ as $n \to \infty$

(ii) $\beta \in \mathbb{R}$ is a (PS)-value in $X(\Omega)$ for J if there is a (PS)$_\beta$-sequence in $X(\Omega)$ for J

(iii) J satisfies the (PS)$_\beta-$condition in $X(\Omega)$ if every (PS)$_\beta$-sequence in $X(\Omega)$ for J contains a convergent subsequence.

Now, we consider the Nehari minimization problem

$$\alpha_X(\Omega) = \inf_{u \in \mathbf{M}(\Omega)} J(u),$$

where $\mathbf{M}(\Omega) = \{u \in X(\Omega)\backslash\{0\} : a(u) = b(u)\}$. Note that $\mathbf{M}(\Omega)$ contains every nonzero solution of equation (1.1) in Ω, $\alpha_X(\Omega) > 0$, and if $u_0 \in \mathbf{M}(\Omega)$ achieves $\alpha_X(\Omega)$, then u_0 is a positive (or negative) solution of equation (1.1) in Ω (see Wang-Wu [10] or Willem [11]). We have the following useful lemma, whose proof can be found in Wang-Wu [10, Lemma 7].

Lemma 2.3. *Let $\{u_n\}$ be in $X(\Omega)$. Then $\{u_n\}$ is a (PS)$_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J if and only if $J(u_n) = \alpha_X(\Omega) + \mathrm{o}(1)$ and $a(u_n) = b(u_n) + \mathrm{o}(1)$. In particular, every minimizing sequence $\{u_n\}$ in $\mathbf{M}(\Omega)$ for $\alpha_X(\Omega)$ is a (PS)$_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J.*

We denote $\alpha_X(\Omega)$ by $\alpha(\Omega)$ for $X(\Omega) = H_0^1(\Omega)$. We denote $\alpha_X(\Omega)$ by $\alpha_s(\Omega)$ for $X(\Omega) = H_s(\Omega)$. We denote $\mathbf{M}(\Omega)$ by $\mathbf{M}_0(\Omega)$ for $X(\Omega) = H_0^1(\Omega)$. We denote $\mathbf{M}(\Omega)$ by $\mathbf{M}_s(\Omega)$ for $X(\Omega) = H_s(\Omega)$.

Remark 2.4. By the Principle of symmetric criticality (see Palais [8]), we have a (PS)$_\beta$-sequence in $X(\Omega)$ for J is a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J.

3. Palais-Smale Conditions

In this section, we present several (PS)$_{\alpha_X(\Omega)}-$conditions in $X(\Omega)$ for J which are used to prove our main results in section 4 and section 5. Since for each (PS)$_{\alpha_X(\Omega)}$-sequence $\{u_n\}$ in $X(\Omega)$ for J, there exists a subsequence $\{u_n\}$ and u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then u is a solution of equation (1.1) in Ω. Moreover, we have the following result, whose proof can be found in Bahri-Lions [1] and in Wang-Wu [10].

Lemma 3.1. *For each (PS)$_{\alpha_X(\Omega)}$-sequence $\{u_n\}$ in $X(\Omega)$ for J, there exists a subsequence $\{u_n\}$and a nonzero u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$ if and only if the (PS)$_{\alpha_X(\Omega)}-$condition holds in $X(\Omega)$ for J.*

Let Ω be any unbounded domain and $\xi \in C^\infty([0,\infty))$ such that $0 \le \xi \le 1$ and

$$\xi(t) = \begin{cases} 0, & \text{for } t \in [0,1] \\ 1, & \text{for } t \in [2,\infty). \end{cases}$$

Let

$$\xi_n(z) = \xi(\frac{2|z|}{n}). \tag{3.1}$$

Then we have the following results.

Proposition 3.2. *The equation* (1.1) *in* Ω *does not admit any solution* u_0 *such that* $J(u_0) = \alpha_X(\Omega)$ *if and only if for each* $(PS)_{\alpha_X(\Omega)}$*-sequence* $\{u_n\}$ *in* $X(\Omega)$ *for* J*, there exists a subsequence* $\{u_n\}$ *such that* $\{\xi_n u_n\}$ *is also a* $(PS)_{\alpha_X(\Omega)}$*-sequence in* $X(\Omega)$ *for* J*.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J. Then there exist a subsequence $\{u_n\}$ and $u_0 \in X(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $X(\Omega)$. Since the equation (1.1) in Ω does not admit any solution u_0 such that $J(u_0) = \alpha_X(\Omega)$, by Lemma 3.1, we have $u_0 = 0$. Let $v_n = \xi_n u_n$. First, we need to show

$$a(u_n - v_n) = o(1). \tag{3.2}$$

Note that

$$a(u_n - v_n) = a(u_n) + a(v_n) - 2\left\langle u_n, v_n\right\rangle_{H^1}.$$

Thus, it suffices to show that $\langle u_n, v_n\rangle_{H^1} = a(u_n) + o(1) = a(v_n) + o(1)$. Since

$$\begin{aligned}\langle u_n, v_n\rangle_{H^1} &= \int_\Omega \nabla u_n \nabla v_n + u_n v_n\\ &= \int_\Omega \xi_n \left[|\nabla u_n|^2 + u_n^2\right] + \int_\Omega u_n \nabla u_n \nabla \xi_n.\end{aligned}$$

Note that $|\nabla \xi_n| \le \frac{c}{n}$ and $\{u_n\}$ is a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J, so

$$\int_\Omega \xi_n^q u_n \nabla u_n \nabla \xi_n = o(1) \quad \text{for } q > 0. \tag{3.3}$$

Hence,

$$\langle u_n, v_n\rangle_{H^1} = \int_\Omega \xi_n \left[|\nabla u_n|^2 + u_n^2\right] + o(1). \tag{3.4}$$

Similarly, we have

$$a(v_n) = \int_\Omega \xi_n^2 \left[|\nabla u_n|^2 + u_n^2\right] + o(1). \tag{3.5}$$

For $r \ge 1$. Since $\{\xi_n^r u_n\}$ is bounded in $X(\Omega)$, we have

$$\begin{aligned}o(1) &= \langle J'(u_n), \xi_n^r u_n\rangle\\ &= \int_\Omega (\xi_n^r |\nabla u_n|^2 + r\xi_n^{r-1} u_n \nabla \xi_n \nabla u_n + \xi_n^r u_n^2) - \int_\Omega \xi_n^r |u_n|^p.\end{aligned}$$

By (3.3), we conclude that

$$\int_\Omega \xi_n^r (|\nabla u_n|^2 + u_n^2) = \int_\Omega \xi_n^r |u_n|^p + o(1). \tag{3.6}$$

Since $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, there exists a subsequence $\{u_n\}$ such that $u_n \to 0$ strongly in $L^p_{loc}(\Omega)$, or there exists a subsequence $\{u_n\}$ such that

$$\int_{Q(n)} |u_n|^p = o(1),$$

where $Q(n) = \Omega \cap B^N(0; n)$. Clearly,

$$\int_\Omega \xi_n^r |u_n|^p = \int_\Omega |u_n|^p + o(1). \tag{3.7}$$

By (3.4), (3.5), (3.6) and (3.7), we have

$$\langle u_n, v_n\rangle_{H^1} = a(u_n) + o(1) = a(v_n) + o(1).$$

Moreover, by the compact imbedding theorem, we obtain

$$b(v_n) = b(u_n) + o(1). \tag{3.8}$$

Since $a(u_n) = b(u_n) + o(1)$. Thus, from (3.2) and (3.8), we obtain

$$a(v_n) = b(v_n) + (1)\,,\ J(v_n) = \alpha_X(\Omega) + o(1)\,.$$

By Lemma 2.3, we can conclude that $\{\xi_n u_n\}$ is a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J. Conversely, assume that the equation (1.1) in Ω admits a solution u_0 such that $J(u_0) = \alpha_X(\Omega)$. We may assume that u_0 is a positive solution. Let $u_n = u_0$ for each $n \in \mathbb{N}$, then $\{u_n\}$ is a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J. By hypothesis, we have $\{\xi_n u_0\}$ is also a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J. We obtain

$$\int_\Omega |\xi_n u_0|^p = \frac{2p}{p-2}\alpha_X(\Omega) + o(1).$$

Thus, there exist n_0 and $d > 0$ such that

$$\int_\Omega |\xi_n u_0|^p > d \quad \text{for each } n \ge n_0. \tag{3.9}$$

However, $u_0 \in L^p(\Omega)$. Hence

$$\int_\Omega |\xi_n u_0|^p \le \int_{\left[B^N(0;\frac{n}{2})\right]^c} |u_0|^p = o(1) \quad \text{as } n \to \infty,$$

this contradicts to (3.9). □

Proposition 3.3. *J does not satisfy the $(PS)_{\alpha_X(\Omega)}-$condition in $X(\Omega)$ for J if and only if there exists a $(PS)_{\alpha_X(\Omega)}$-sequence $\{u_n\}$ in $X(\Omega)$ for J such that $\{\xi_n u_n\}$ is also a $(PS)_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J.*

The proof of this proposition is similar to the proof of Proposition 3.2 and therefore, it is omitted.

Let $\Omega_1 \subsetneqq \Omega_2$, clearly $\alpha_X(\Omega_1) \ge \alpha_X(\Omega_2)$. Then we have the following useful results.

Lemma 3.4. *Let $\Omega_1 \subsetneqq \Omega_2$ and $J : X(\Omega_2) \to \mathbb{R}$ be the energy functional. Suppose that $\alpha_X(\Omega_1) = \alpha_X(\Omega_2)$. Then*

- (i) *The equation (1.1) in Ω_1 does not admit any solution u_0 such that $J(u_0) = \alpha_X(\Omega_1)$*
- (ii) *J does not satisfy the $(PS)_{\alpha_X(\Omega_2)}-$condition.*

The proof of this lemma can be found in Wang-Wu [10, Lemma 13]. By the Rellich compact theorem, J satisfies the $(\mathrm{PS})_{\alpha_X(\Omega)}-$condition in $X(\Omega)$ if Ω is a bounded domain.

Lemma 3.5. *Let Ω be a bounded domain in $\mathbb{R}^N$. Then the $(PS)_{\alpha_X(\Omega)}-$condition holds in $X(\Omega)$ for J. Furthermore, the equation (1.1) in Ω has a positive solution u_0 such that $J(u_0) = \alpha_X(\Omega)$.*

4. Large Domains in $\mathbf{A}^r$

Definition 4.1. A domain Ω in $\mathbf{A}^r$ is large if for any $m>0$, there exist $s<t$ such that $t-s=m$ and $\mathbf{A}^r_{s,t}\subset\Omega$.

Lemma 4.2. *If Ω is a large domain in $\mathbf{A}^r$, then $\alpha(\Omega)=\alpha(\mathbf{A}^r)$. Furthermore, if Ω is a proper large domain in $\mathbf{A}^r$, then the equation* (1.1) *in Ω does not admit any solution u_0 such that $J(u_0)=\alpha(\Omega)$.*

The proof of this lemma follows by Lien-Tzeng-Wang [7, Lemma 2.5] and Lemma 3.4.

We need the following symmetric result to assert our main result.

Lemma 4.3. *Suppose that Ω is a y-symmetric large domain in $\mathbf{A}^r$. Then $\alpha_s(\Omega)\le 2\alpha(\mathbf{A}^r)$.*

Proof. Since Ω is a y-symmetric large domain in $\mathbf{A}^r$. Thus, there exist $t_0>0$, Ω_1 and Ω_2 are large domains in $\mathbf{A}^r$ such that $\Omega\backslash\overline{\mathbf{A}^r_{-t_0,t_0}}=\Omega_1\cup\Omega_2$. Let $\{u_n^1\}$ be a $(\mathrm{PS})_{\alpha(\Omega_1)}$-sequence in $H_0^1(\Omega_1)$ for J and let $u_n^2(x,y)=u_n^1(x,-y)$. Clearly, $\{u_n^2\}$ is a $(\mathrm{PS})_{\alpha(\Omega_2)}$-sequence in $H_0^1(\Omega_2)$ for J. Take $v_n=u_n^1+u_n^2$, then $v_n\in H_s(\Omega)$, $a(v_n)=b(v_n)+o(1)$ and

$$J(v_n)=\alpha(\Omega_1)+\alpha(\Omega_2)+o(1).$$

Moreover, there exists $s_n>0$ such that $s_nv_n\in\mathbf{M}_s(\Omega)$ and

$$J(s_sv_n)=\alpha(\Omega_1)+\alpha(\Omega_2)+o(1).$$

From Lemma 4.2 and the definition of Nehari minimization problem, we can conclude $\alpha_s(\Omega)\le 2\alpha(\mathbf{A}^r)$. □

Then we have the following symmetric compactness.

Proposition 4.4. *Suppose that Ω is a y-symmetric large domain in $\mathbf{A}^r$. Then J satisfies the $(PS)_{\alpha_s(\Omega)}-$condition in $H_s(\Omega)$ if and only if $\alpha_s(\Omega)<2\alpha(\mathbf{A}^r)$.*

Proof. Suppose that J satisfies the $(\mathrm{PS})_{\alpha_s(\Omega)}-$condition in $H_s(\Omega)$. By Lemma 4.3, we have $\alpha_s(\Omega)\le 2\alpha(\mathbf{A}^r)$. Suppose that $\alpha_s(\Omega)=2\alpha(\mathbf{A}^r)$. By the definition of domain in $\mathbb{R}^N$, we may take a domain $\tilde\Omega=\Omega\backslash\overline{B^N(0;\tilde r)}$ for some $\tilde r>0$ such that $\tilde\Omega\subsetneqq\Omega$ and $\tilde\Omega$ is a proper y-symmetric large domain in $\mathbf{A}^r$. By Lemma 3.4, we have $2\alpha(\mathbf{A}^r)=\alpha_s(\Omega)<\alpha_s(\tilde\Omega)$. This contradicts to Lemma 4.3. Conversely, suppose that J does not satisfy the $(\mathrm{PS})_{\alpha_s(\Omega)}-$condition. By Proposition 3.3, there exists a $(\mathrm{PS})_{\alpha_s(\Omega)}$-sequence $\{u_n\}$ in $H_s(\Omega)$ for J such that $\{\xi_nu_n\}$ is also a $(\mathrm{PS})_{\alpha_s(\Omega)}$-sequence in $H_s(\Omega)$ for J, where ξ_n is as in (3.1). Let $v_n=\xi_nu_n$, we obtain

$$\begin{gathered}J(v_n)=\alpha_s(\Omega)+\mathrm{o}(1),\\ J'(v_n)=\mathrm{o}(1)\quad\text{in }H^{-1}(\Omega).\end{gathered}\tag{4.1}$$

Since Ω is a y-symmetric large domain in $\mathbf{A}^r$, there exists a $n_0\in\mathbb{N}$ such that $v_n=0$ in $\overline{\Omega_{n_0}}$ for $n>2n_0$, and two disjoint subdomains Ω_1 and Ω_2 such that

$$\begin{gathered}(x,y)\in\Omega_2\quad\text{if and only if}\quad(x,-y)\in\Omega_1,\\ \Omega\setminus\overline{\Omega_{n_0}}=\Omega_1\cup\Omega_2,\end{gathered}$$

where $\Omega_n = \{z \in \Omega : -n < y < n\}$. Note that Ω_1 and Ω_2 are also large domains in $\mathbf{A}^r$. Moreover, $v_n = v_n^1 + v_n^2$ and for $i = 1, 2$,

$$v_n^i(z) = \begin{cases} v_n(z), & \text{for } z \in \Omega_i \\ 0, & \text{for } z \notin \Omega_i, \end{cases}$$

this implies $v_n^i \in H_0^1(\Omega_i)$. By (4.1), we obtain

$$J'(v_n^i) = \mathrm{o}(1) \quad \text{strongly in } H^{-1}(\Omega_i) \quad \text{for } i = 1, 2.$$

We have $v_n^1(x, y) = v_n^2(x, -y)$, $J(v_n^1) = J(v_n^2)$ and

$$\alpha_s(\Omega) + o(1) = J(v_n) = J(v_n^1) + J(v_n^2) = 2J(v_n^i) \quad \text{for } i = 1, 2,$$

or

$$J(v_n^i) = \frac{1}{2}\alpha_s(\Omega) + o(1) \quad \text{for } i = 1, 2.$$

Therefore, $\frac{1}{2}\alpha_s(\Omega)$ is a positive (PS)-value in $H_0^1(\Omega_i)$ for J. By the definition of Nehari minimization problem and Lemma 4.2, we have

$$\frac{1}{2}\alpha_s(\Omega) \geq \alpha(\Omega_i) = \alpha(\mathbf{A}^r),$$

which is a contradiction. □

Corollary 4.5. *Suppose that Ω is a y-symmetric large domain in $\mathbf{A}^r$. Then $\alpha(\Omega) = \alpha_s(\Omega)$ if and only if the equation* (1.1) *in Ω has a y-symmetric solution u_0 such that $J(u_0) = \alpha(\Omega)$.*

Proof. By Lemma 4.2, we have

$$\alpha(\mathbf{A}^r) = \alpha(\Omega) = \alpha_s(\Omega) < 2\alpha(\mathbf{A}^r).$$

By Proposition 4.4, J satisfies the $(\mathrm{PS})_{\alpha_s(\Omega)}-$condition in $H_s(\Omega)$. Thus, there exists a y-symmetric positive solution u_0 such that

$$J(u_0) = \alpha_s(\Omega) = \alpha(\Omega).$$

Conversely, use the definition of the Nehari minimization problem. □

Proposition 4.6. *Suppose that Ω is a y-symmetric large domain in $\mathbf{A}^r$ such that $\alpha_s(\Omega) = 2\alpha(\mathbf{A}^r)$. If $\widetilde{\Omega} \subsetneqq \Omega$ is also y-symmetric large domain in $\mathbf{A}^r$, then $\alpha_s(\widetilde{\Omega}) = 2\alpha(\mathbf{A}^r)$ and the equation* (1.1) *in $\widetilde{\Omega}$ does not admit any solution u_0 such that $J(u_0) = \alpha_s(\widetilde{\Omega})$.*

The proof of this proposition follows from Lemma 3.4 and Lemma 4.3.

Remark 4.7. From Lemma 4.3, Proposition 4.4 and Proposition 4.6, the y-symmetric large domains in $\mathbf{A}^r$ can be classify into three kinds. If Ω is a y-symmetric large domain in $\mathbf{A}^r$, then it satisfies one of the following conditions:

(1) $\alpha_s(\Omega) < 2\alpha(\mathbf{A}^r)$
(2) $\alpha_s(\Omega) = 2\alpha(\mathbf{A}^r)$ and the equation (1.1) in Ω has a solution u_0 such that $J(u_0) = \alpha_s(\Omega)$
(3) $\alpha_s(\Omega) = 2\alpha(\mathbf{A}^r)$ and the equation (1.1) in Ω does not admit any solution u_0 such that $J(u_0) = \alpha_s(\Omega)$.

5. Concentration of Solutions

For the rest of this article, let ω be a y-symmetric bounded set such that $\mathbf{A}^r\backslash\overline{\omega}$ is a y-symmetric proper large domain in $\mathbf{A}^r$. We need the following notation:

$$\begin{gathered}\mathbf{S} = \mathbf{A}^r\backslash\overline{\omega};\\ \mathbf{S}_{k,l} = \{(x,y)\in\mathbf{S} : k<y<l\};\\ \mathbf{S}^+_{-l} = \{(x,y)\in\mathbf{S} : y\geq -l\};\\ \mathbf{S}^-_{l} = \{(x,y)\in\mathbf{S} : y\leq l\}.\end{gathered}$$

Note that $\mathbf{S},\mathbf{S}^+_{-l}$ and $\mathbf{S}^-_{l}$ are proper large domains in $\mathbf{A}^r$ for all $l\geq 0$. By Lemma 4.2, we have $\alpha(\mathbf{S})=\alpha(\mathbf{A}^r)$ and the equation (1.1) in $\mathbf{S}$ does not admit any solution u_0 such that $J(u_0)=\alpha(\mathbf{S})$. We need the following lemmas to show our main results.

Lemma 5.1. *For each positive number $\varepsilon(\frac{p}{p-2})\alpha(\mathbf{A}^r)$ and $l\geq 0$, there exists a $\delta(\varepsilon,l)>0$ such that if $u\in\mathbf{M}_0(\mathbf{S})$ and $J(u)\leq\alpha(\mathbf{A}^r)+\delta(\varepsilon,l)$, then either $\int_{\mathbf{S}^+_{-l}}|u|^p<\varepsilon$ or $\int_{\mathbf{S}^-_{l}}|u|^p<\varepsilon$.*

Proof. We divide the proof into the following steps:
Step 1: Suppose that there exist $c>0, l_0\geq 0$ and $\{u_n\}\subset\mathbf{M}_0(\mathbf{S})$ such that

$$J(u_n)=\alpha(\mathbf{A}^r)+o(1)\,, \tag{5.1}$$

$$\int_{\mathbf{S}^+_{-l_0}}|u_n|^p\geq c\,, \tag{5.2}$$

$$\int_{\mathbf{S}^-_{l_0}}|u_n|^p\geq c\,. \tag{5.3}$$

From Lemma 2.3, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha(\mathbf{A}^r)}$-sequence in $H^1_0(\mathbf{S})$ for J. Since $\mathbf{S}$ is a proper large domain in $\mathbf{A}^r$, by Proposition 3.2 and Lemma 4.2, there exists a subsequence $\{u_n\}$ such that $\{\xi_n u_n\}$ is also a $(\mathrm{PS})_{\alpha(\mathbf{S})}$-sequence in $H^1_0(\mathbf{S})$ for J, where ξ_n is as in (3.1). Let $v_n=\xi_n u_n$, we obtain

$$\begin{gathered}J(v_n)=\alpha(\mathbf{A}^r)+o(1),\\ J'(v_n)=o(1)\quad\text{in } H^{-1}(\mathbf{S}),\end{gathered}\tag{5.4}$$

and there exists a $n_0>l_0$ such that $v_n=0$ in $\overline{\mathbf{A}(n_0)}$ for $n>2n_0$, where $\mathbf{A}(n)=\mathbf{S}_{-n,n}$. Moreover, $v_n=v_n^+ + v_n^-$ and

$$v_n^{\pm}(z)=\begin{cases}v_n(z) & \text{for } z\in\mathbf{S}^{\pm}_{\mp l_0},\\ 0 & \text{for } z\notin\mathbf{S}^{\pm}_{\mp l_0}.\end{cases}$$

Then $v_n^{\pm}\in H^1_0(\mathbf{S}^{\pm}_{\mp l_0})$ and $a(v_n^{\pm})=b(v_n^{\pm})+o(1)$. By (5.4), we obtain

$$J'(v_n^{\pm})=o(1)\quad\text{strongly in } H^{-1}(\mathbf{S}^{\pm}_{\mp l_0}).$$

Thus,

$$\alpha(\mathbf{A}^r)+o(1)=J(v_n)=J(v_n^+)+J(v_n^-).$$

Assume that $J(v_n^{\pm})=c^{\pm}+o(1)$. Then

$$c^+ + c^- = \alpha(\mathbf{A}^r). \tag{5.5}$$

Since $c^{\pm}$ are (PS)-values in $H_0^1(\mathbf{S}^{\pm}_{\mp l_0})$ for J, they are nonnegative. Moreover, the half strips $\mathbf{S}^{+}_{-l_0}$ and $\mathbf{S}^{-}_{l_0}$ are proper large domains in $\mathbf{A}^r$, From Lemma 4.2, we have

$$\alpha(\mathbf{A}^r) = \alpha(\mathbf{S}^{+}_{-l_0}) = \alpha(\mathbf{S}^{-}_{l_0}). \tag{5.6}$$

Thus, by (5.5), (5.6) and the definition of Nehari minimization problem, we may assume that $c^+ = \alpha(\mathbf{S}^{+}_{-l_0}) = \alpha(\mathbf{A}^r)$ and $c^- = 0$. Next, for $n > 2n_0$,

$$\begin{aligned}\int_{\mathbf{S}} |u_n|^p &= \int_{\mathbf{S}} |v_n|^p + o(1)\\ &= \int_{\mathbf{S}^{+}_{-l_0}} |v_n^+|^p + \int_{\mathbf{S}^{-}_{l_0}} |u_n|^p + o(1).\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbf{S}^{-}_{l_0}} |u_n|^p &= \int_{\mathbf{S}} |u_n|^p - \int_{\mathbf{S}^{+}_{-l_0}} |v_n^+|^p + o(1)\\ &= (\frac{2p}{p-2})\alpha(\mathbf{A}^r) - (\frac{2p}{p-2})\alpha(\mathbf{A}^r) + o(1)\\ &= o(1),\end{aligned}$$

which contradicts to (5.3).
Step 2: Suppose that there exists a $u_0 \in \mathbf{M}_0(\mathbf{S})$ with $J(u_0) < \alpha(\mathbf{A}^r) + \delta(\varepsilon)$ such that

$$\int_{\mathbf{S}^{+}_{-l_0}} |u_0|^p < \varepsilon \quad\text{and}\quad \int_{\mathbf{S}^{-}_{l_0}} |u_0|^p < \varepsilon.$$

Then

$$\begin{aligned}\frac{2p}{(p-2)}\alpha(\mathbf{S}) \le \int_{\mathbf{S}} |u_0|^p &= \int_{\mathbf{S}^{+}_{l_0}} |u_0|^p + \int_{\mathbf{S}^{-}_{l_0}} |u_0|^p\\ &< \frac{p}{(p-2)}\alpha(\mathbf{A}^r) + \frac{p}{(p-2)}\alpha(\mathbf{A}^r)\\ &= \frac{2p}{(p-2)}\alpha(\mathbf{A}^r),\end{aligned}$$

which is also a contradiction. ∎

Lemma 5.2. *If $\alpha_s(\mathbf{S}) < 2\alpha(\mathbf{A}^r)$. Then for each $0 < \varepsilon \le (\frac{p}{p-2})\alpha(\mathbf{A}^r)$, there exist positive numbers $l(\varepsilon)$ and $\delta(\varepsilon)$ such that if $u \in \mathbf{M}_s(\mathbf{S})$ and $J(u) < \alpha_s(\mathbf{S}) + \delta(\varepsilon)$, then $\int_{(\mathbf{S}_{-l(\varepsilon),l(\varepsilon)})^c} |u|^p < \varepsilon$.*

Proof. If not, there exist a positive number $c \le (\frac{p}{p-2})\alpha(\mathbf{A}^r)$ and $\{u_n\} \subset \mathbf{M}_s(\mathbf{S})$ such that

$$\begin{gathered}J(u_n) = \alpha_s(\mathbf{S}) + \frac{1}{n}\,,\\ \int_{(\mathbf{S}_{-n,n})^c} |u_n|^p \ge c \quad \text{for all } n = 1, 2, \dots.\end{gathered} \tag{5.7}$$

By Lemma 2.3, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha_s(\mathbf{S})}$-sequence in $H_s(\mathbf{S})$ for J. Since $\alpha_s(\mathbf{S}) < 2\alpha(\mathbf{A}^r)$. By Proposition 4.4, J is satisfying $(\mathrm{PS})_{\alpha_s(\mathbf{S})}-$condition in $H_s(\mathbf{S})$. Thus, there exist a subsequence $\{u_n\}$ and $u_0 \in H_s(\mathbf{S})$ such that

$$u_n \to u_0 \quad \text{strongly in } H_s(\mathbf{S}).$$

By the Sobolev imbedding theorem and the Vitali convergence theorem, there exists a $l_0 > 0$ such that

$$\int_{(\mathbf{S}_{-l_0,l_0})^c} |u_n|^p < \frac{c}{2} \quad \text{for all } n,$$

which contradicts to (5.7). $\square$

Lemma 5.3. *Suppose that the equation* (1.1) *in* $\mathbf{S}$ *does not admit any solution* u_0 *such that* $J(u_0) = \alpha_s(\mathbf{S})$*. Then for each positive number* $\varepsilon \le (\frac{2p}{p-2})\alpha_s(\mathbf{S})$ *and* l*, there exists a* $\delta(\varepsilon, l) > 0$ *such that if* $u \in \mathbf{M}_s(\mathbf{S})$ *and* $J(u) < \alpha_s(\mathbf{S}) + \delta(\varepsilon, l)$*, then* $\int_{\mathbf{S}_{-l,l}} |u|^p < \varepsilon$*.*

The proof of this lemma is similar to the proof of Lemma 5.1, and is omitted here.

For $\Theta_t = \mathbf{A}^r_{-t,t} \backslash \overline{\omega}$, consider the filtration of J in $\mathbf{M}(\Theta_t)$,

$$F(\Theta_t) = \{u \in \mathbf{M}_0(\Theta_t) : J(u) \le \alpha_s(\mathbf{S})\}.$$

Note that if $F(\Theta_t)$ is a nonempty set, then

$$\alpha(\Theta_t) = \inf_{v \in F(\Theta_t)} J(v).$$

Note that $\Theta_{t_1} \subset \Theta_{t_2}$ for $t_1 < t_2$. Thus, $\alpha_X(\Theta_{t_1}) > \alpha_X(\Theta_{t_2})$ for $t_1 < t_2$. Then we have the following result.

Lemma 5.4. $\alpha_X(\Theta_t) \searrow \alpha_X(\mathbf{S})$ *as* $t \nearrow \infty$*.*

The proof of this lemma is similar to the proof of Lien-Tzeng-Wang [7, Lemma 2.5] and omitted here.

Theorem 5.5. *There exists a positive number* t_0 *such that* $F(\Theta_t)$ *is non-empty and* $F(\Theta_t) \cap \mathbf{M}_s(\Theta_t) = \phi$ *for* $t \ge t_0$*. Furthermore, the equation* (1.1) *in* Θ_t *has three positive solutions which one is* y*-symmetric and the other are non-axially symmetric for* $t \ge t_0$*.*

Proof. First, we need to show that $\alpha(\mathbf{S}) < \alpha_s(\mathbf{S})$. Assume the contrary, $\alpha(\mathbf{S}) = \alpha_s(\mathbf{S})$. By Corollary 4.5, the equation (1.1) in $\mathbf{S}$ admits a solution u_0 such that $J(u_0) = \alpha(\mathbf{S})$, this contradicts the fact of Lemma 4.2. Since $\mathbf{S}$ is a proper large domain in $\mathbf{A}^r$. From Lemma 4.2, we have

$$\alpha(\mathbf{A}^r) = \alpha(\mathbf{S}) < \alpha_s(\mathbf{S}). \tag{5.8}$$

By (5.8) and Lemma 5.4, there exists a $t_0 > 0$ such that

$$\alpha(\mathbf{S}) < \alpha(\Theta_t) \le \alpha_s(\mathbf{S}) \quad \text{for all } t \ge t_0. \tag{5.9}$$

Since Θ_t is a y-symmetric bounded domain, by Lemma 3.5, $F(\Theta_t)$ is nonempty for all $t \ge t_0$. Moreover,

$$\alpha_s(\Theta_t) = \inf_{v \in \mathbf{M}_s(\Theta_t)} J(v)$$

and

$$\alpha_s(\mathbf{S}) < \alpha_s(\Theta_t) \quad \text{for all } t > 0. \tag{5.10}$$

We can conclude that $F(\Theta_t) \cap \mathbf{M}_s(\Theta_t) = \phi$ for all $t \ge t_0$. By (5.9), (5.10) and Lemma 3.5, we have

$$\alpha(\Theta_t) \le \alpha_s(\mathbf{S}) < \alpha_s(\Theta_t) \quad \text{for all } t \ge t_0 \tag{5.11}$$

and the equation (1.1) in Θ_t admit two disjoint positive solutions u_1, u_2 such that $J(u_1) = \alpha_s(\Theta_t)$ and $J(u_2) = \alpha(\Theta_t)$. Take $u_3(x,y) = u_2(x,-y)$, then $J(u_3) = \alpha(\Theta_t)$, $u_3 \in \mathbf{M}_0(\Theta_t)$ and u_3 is third positive solution. $\square$

Remark 5.6. By Theorem 5.5, there exists a $t_0 > 0$ such that for $t \geq t_0$, the equation (1.1) in Θ_t has one $y-$symmetric positive solution u_1 and two non-axially symmetric positive solutions u_2 and u_3. Moreover,

$$\int_{\Theta_t} |u_1|^p = \frac{2p}{p-2}\alpha_s(\Theta_t) > \frac{2p}{p-2}\alpha_s(\mathbf{S})$$

and

$$\int_{\Theta_t} |u_i|^p = \frac{2p}{p-2}\alpha(\Theta_t) \leq \frac{2p}{p-2}\alpha_s(\mathbf{S}) \quad \text{for } i = 2, 3.$$

Thus, we can conclude that

$$\int_{\Theta_t^+} |u_1|^p = \int_{\Theta_t^-} |u_1|^p > \frac{p}{p-2}\alpha_s(\mathbf{S}),$$
$$\int_{\Theta_t^+} |u_2|^p \leq \frac{p}{p-2}\alpha_s(\mathbf{S})$$
$$\int_{\Theta_t^-} |u_3|^p \leq \frac{p}{p-2}\alpha_s(\mathbf{S}),$$

where $\Theta_t^+ = \{(x,y) \in \Theta_t : y \geq 0\}$ and $\Theta_t^- = \{(x,y) \in \Theta_t : y \leq 0\}$.

Next, we describe the concentration of solutions of equation (1.1) in Θ_t. We need the following notation:

$$\Theta_t(-l,l) = \{(x,y) \in \Theta_t : -l \leq y \leq l\}\,;$$
$$\Theta_t^+(l) = \{(x,y) \in \Theta_t : y \geq l\}\,;$$
$$\Theta_t^-(l) = \{(x,y) \in \Theta_t : y \leq l\}\,.$$

Then we have the following results.

Theorem 5.7. *Suppose that $\alpha_s(\mathbf{S}) < 2\alpha(\mathbf{A}^r)$. Then for each positive number $\varepsilon \leq (\frac{p}{p-2})\alpha(\mathbf{A}^r)$, there exist positive numbers $t_0 > l_0$ such that for $t > t_0$ the equation* (1.1) *in Θ_t has three positive solutions u_1, u_2 and u_3. Moreover,*

(i) $\int_{(\Theta_t(-l_0,l_0))^c} |u_1|^p < \varepsilon$

(ii) $\int_{\Theta_t^+(-l_0)} |u_2|^p < \varepsilon$ *and* $\int_{\Theta_t^-(l_0)} |u_3|^p < \varepsilon$.

Proof. Since $\alpha_s(\mathbf{S}) < 2\alpha(\mathbf{A}^r)$. By Lemma 5.2, for each positive number $\varepsilon \leq (\frac{p}{p-2})\alpha(\mathbf{A}^r)$, there exist positive numbers l_0 and $\delta(\varepsilon)$ such that if $u \in \mathbf{M}_s(\mathbf{S})$ and $J(u) < \alpha_s(\mathbf{S}) + \delta(\varepsilon)$, then $\int_{(\mathbf{S}_{-l_0,l_0})^c} |u|^p < \varepsilon$. Moreover, by Lemma 5.4, there exists a $t_1 > 0$ such that $\alpha_s(\Theta_t) < \alpha_s(\mathbf{S}) + \delta(\varepsilon)$ for all $t > t_1$. Since Θ_t is a bounded domain, by Lemma 3.5, the equation (1.1) in Θ_t admits a positive solution $u_1 \in H_0^1(\Theta_t)$ such that $J(u_1) = \alpha_s(\Theta_t)$. Thus, $u_1 \in \mathbf{M}_s(\mathbf{S})$,

$$J(u_1) < \alpha(\mathbf{A}^r) + \delta(\varepsilon)\,,$$
$$\int_{(\mathbf{S}_{-l_0,l_0})^c} |u_1|^p = \int_{(\Theta_t(-l_0,l_0))^c} |u_1|^p < \varepsilon.$$

Fixed the positive numbers ε, l_0. By Lemma 5.1, there exists a $\delta(\varepsilon, l_0) > 0$ such that if $u \in \mathbf{M}_0(\mathbf{S})$ and $J(u) < \alpha(\mathbf{A}^r) + \delta(\varepsilon, l_0)$, then $\int_{\mathbf{S}^+_{-l_0}} |u|^p < \varepsilon$ or $\int_{\mathbf{S}^-_{l_0}} |u|^p < \varepsilon$.

Moreover, by Lemma 5.4, there exists a $t_2 > 0$ such that $\alpha(\Theta_t) < \alpha(\mathbf{A}^r) + \delta(\varepsilon)$ for all $t > t_2$. Since Θ_t is a bounded domain, by Lemma 3.5, the equation (1.1) in Θ_t admits a positive solution u_2 such that $J(u_2) = \alpha(\Theta_t)$. Then $u_2 \in \mathbf{M}_0(\Theta_t) \subset \mathbf{M}_0(\mathbf{S})$, $J(u_2) < \alpha(\mathbf{A}^r) + \delta(\varepsilon)$ and either

$$\int_{\Theta_t^+(-l_0)} |u_2|^p < \varepsilon \text{ or } \int_{\Theta_t^-(l_0)} |u_2|^p < \varepsilon. \tag{5.12}$$

Without loss of generality, we may assume that

$$\int_{\Theta_t^+(-l_0)} |u_2|^p < \varepsilon.$$

Take $u_3(x, y) = u_2(x, -y)$, then u_3 is third positive solution and

$$\int_{\Theta_t^-(l_0)} |u_3|^p < \varepsilon.$$

Now, let $t_0 = \max\{t_1, t_2\}$. Since $\varepsilon \le (\frac{p}{p-2})\alpha(\mathbf{A}^r)$, u_i is disjoint for $i = 1, 2, 3$. $\square$

Theorem 5.8. *Suppose that the equation* (1.1) *in* $\mathbf{S}$ *does not admit any solution* u_0 *such that* $J(u_0) = \alpha_s(\mathbf{S})$. *Then for positive numbers* $\varepsilon \le (\frac{p}{p-2})\alpha(\mathbf{A}^r)$ *and* l, *there exists a positive number* t_0 *such that for* $t > t_0$, *the equation* (1.1) *in* Θ_t *has three positive solutions* u_1, u_2 *and* u_3. *Moreover,*

(i) $\int_{(\Theta_t(-l,l))^c} |u_1|^p < \varepsilon$

(ii) $\int_{\Theta_t^+(-l)} |u_2|^p < \varepsilon$ *and* $\int_{\Theta_t^-(l)} |u_3|^p < \varepsilon$.

The proof of this theorem is similar to the proof of Theorem 5.7 and therefore omitted here.

Note that if u_1, u_2 and u_3 are positive solutions as in Theorem 5.7 or Theorem 5.8, then u_1 is y-symmetric and u_2, u_3 are non-axially symmetric.

6. Dynamic System of Solutions

For $m = 1, 2, \cdots$, define $\Theta_m = \mathbf{A}^r_{-m,m} \backslash \overline{\omega}$, then $\{\Theta_m\}$ is an increasing sequence and

$$\mathbf{S} = \mathbf{A}^r \backslash \overline{\omega} = \cup_{m=1}^{\infty} \Theta_m.$$

By Theorem 5.5, there exists a $t_0 > 0$ such that for $m \ge t_0$, the equation (1.1) in Θ_m admit one $y-$symmetric positive solution u_m^1 and two non-axially symmetric positive solutions u_m^2 and u_m^3. Note that

$$J(u_m^2) = J(u_m^3) = \alpha(\Theta_m) < \alpha_s(\Theta_m) = J(u_m^1) \quad \text{for all } m \ge t_0.$$

Then we have the following results.

Theorem 6.1. (i) *The sequence* $\{u_m^1\}$ *is a* $(PS)_{\alpha_s(\mathbf{S})}$*-sequence in* $H_s(\mathbf{S})$ *for* J

(ii) *If* $\alpha_s(\Theta_{m_0}) < 2\alpha(\mathbf{A}^r)$ *for some* $m_0 > 0$, *then there exist a subsequence* u_m^1 *and* $u^1 \in H_s(\Omega)$ *such that* $u_m^1 \to u^1$ *strongly in* $L^p(\mathbf{S})$ *in* $H_s(\mathbf{S})$ *as* $m \to \infty$ *and* $J(u^1) = \alpha_s(\mathbf{S})$

(iii) *If the equation* (1.1) *in* $\mathbf{S}$ *does not admit any solution* u_0 *such that* $J(u_0) = \alpha_s(\mathbf{S})$, *then* $u_m^1 \rightharpoonup 0$ *weakly in* $L^p(\mathbf{S})$ *and in* $H_0^1(\mathbf{S})$ *as* $m \to \infty$.

Proof. (i) By Lemma 5.4, we have $J(u_m^1) = \alpha_s(\Theta_m) = \alpha_s(\mathbf{S}) + o(1)$. Since $u_m^1 \in \mathbf{M}_s(\Theta_m) \subset \mathbf{M}_s(\mathbf{S})$, from Lemma 2.3 we can conclude that $\{u_m^1\}$ is a $(\mathrm{PS})_{\alpha_s(\mathbf{S})}$-sequence in $H_s(\mathbf{S})$ for J.
(ii) Since $\alpha_s(\Theta_{m_0}) < 2\alpha(\mathbf{A}^r)$ for some $m_0 > 0$ and $\Theta_m \subset \Theta_{m+1} \subset \mathbf{S}$ for each m, we have $\alpha_s(\mathbf{S}) < 2\alpha(\mathbf{A}^r)$. By Proposition 4.4, J satisfies the $(\mathrm{PS})_{\alpha_s(\mathbf{S})}-$condition in $H_s(\mathbf{S})$. Then there exist a subsequence $\{u_m^1\}$ and a y-symmetric positive solution u^1 of equation (1.1) in $\mathbf{S}$ such that $u_m^1 \to u^1$ strongly in $L^p(\mathbf{S})$ and in $H_s(\mathbf{S})$ and $J(u^1) = \alpha_s(\mathbf{S})$.
(iii) Let $v \in L^q(\mathbf{S})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then for each $\varepsilon > 0$, there exists a $l > 0$ such that
$$\int_{(\mathbf{S}_{-l,l})^c} |v|^q < \varepsilon^q.$$
Moreover, by Theorem 5.8, there exists a m_0 such that
$$\int_{\mathbf{S}_{-l,l}} |u_m^1|^q < \varepsilon^p \quad \text{for all } m > m_0.$$
Thus, for each $\varepsilon > 0$, there exists a m_0 such that
$$\begin{aligned}
\int_{\mathbf{S}} u_m^1 v &= \int_{(\mathbf{S}_{-l,l})^c} u_m^1 v + \int_{\mathbf{S}_{-l,l}} u_m^1 v \\
&\le \Big(\int_{(\mathbf{S}_{-l,l})^c} |u_m^1|^p \Big)^{1/p} \Big(\int_{(\mathbf{S}_{-l,l})^c} |v|^q \Big)^{\frac{1}{q}} + \Big(\int_{\mathbf{S}_{-l,l}} |u_m^1|^p \Big)^{1/p} \Big(\int_{\mathbf{S}_{-l,l}} |v|^q \Big)^{\frac{1}{q}} \\
&\le (c_1 + c_2)\varepsilon \quad \text{for all } m > m_0,
\end{aligned}$$
where $c_1 = (\frac{2p}{p-2}\alpha_s(\Theta_1))$ and $c_2 = \|v\|_{L^q}$. This implies $u_m^1 \rightharpoonup 0$ weakly in $L^p(\mathbf{S})$ as $m \to \infty$. Since u_m^1 is a solution of equation (1.1) in Θ_m, we have
$$\int_{\Theta_m} \nabla u_m^1 \nabla \varphi + u_m^1 \varphi = \int_{\Theta_m} |u_m^1|^{p-2} u_m^1 \varphi \quad \text{for all } \varphi \in H_0^1(\Theta_m).$$
First, we need to show for each $\varepsilon > 0$ and $\varphi \in C_c^1(\mathbf{S})$, there exists m_0 such that
$$\int_{\Theta_m} \nabla u_m^1 \nabla \varphi + u_m^1 \varphi < \varepsilon \quad \text{for all } m > m_0.$$
For $\varphi \in C_c^1(\mathbf{S})$. Let $K = \operatorname{supp} \varphi$, then $K \subset \mathbf{S}$ is compact and there exists a m_1 such that $K \subset \Theta_m$ for all $m \ge m_1$. Thus, by Theorem 5.8 for each $\varepsilon > 0$, there exist $l_0 > 0$ and m_0 such that $\varphi \in H_0^1(\Theta_m)$,
$$\int_{(\mathbf{S}_{-l_0,l_0})^c} |\varphi|^p = 0\,,$$
$$\int_{\mathbf{S}_{-l_0,l_0}} |u_m^1|^p < \varepsilon^{\frac{p-1}{p}} \quad \text{for all } m > m_0.$$
We obtain
$$\begin{aligned}
\int_{\Theta_m} |u_m^1|^{p-2} u_m^1 \varphi &= \int_{(\mathbf{S}_{-l_0,l_0})^c} |u_m^1|^{p-2} u_m^1 \varphi + \int_{\mathbf{S}_{-l,l_0}} |u_m^1|^{p-2} u_m^1 \varphi \\
&\le \Big(\int_{(\mathbf{S}_{-l_0,l_0})^c} |u_m^1|^p \Big)^{\frac{p-1}{p}} \Big(\int_{(\mathbf{S}_{-l_0,l_0})^c} |\varphi|^p \Big)^{1/p} \\
&\quad + \Big(\int_{\mathbf{S}_{-l_0,l_0}} |u_m^1|^p \Big)^{\frac{p-1}{p}} \Big(\int_{\mathbf{S}_{-l_0,l_0}} |\varphi|^p \Big)^{1/p} \le c\varepsilon
\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{S}} \nabla u_m^1 \nabla \varphi + \int_{\mathbf{S}} u_m^1 \varphi &= \int_{\Theta_m} \nabla u_m^1 \nabla \varphi + \int_{\Theta_m} u_m^1 \varphi \\ &= \int_{\Theta_m} |u_m^1|^{p-2} u_m^1 \varphi \quad \text{for all } m > m_0.\end{aligned}$$

This follows that

$$\int_{\mathbf{S}} \nabla u_m^1 \nabla \varphi + \int_{\mathbf{S}} u_m^1 \varphi \le c\varepsilon \quad \text{for all } m > m_0. \tag{6.1}$$

Since $\alpha_s(\Theta_{m+1}) < \alpha_s(\Theta)$, there exists a $C > 0$ such that $\|u_m^1\|_{H^1} \le C$. Thus, for each $\varepsilon > 0$ and $\psi \in H_0^1(\mathbf{S})$, there exists a $\varphi \in C_c^1(\mathbf{S})$ such that

$$\|\psi - \varphi\|_{H^1} < \frac{\varepsilon}{C}. \tag{6.2}$$

From (6.1) and (6.2), we can conclude that for each $\varepsilon > 0$ and $\psi \in H_0^1(\mathbf{S})$, there exists a $m_0 > 0$ such that

$$\begin{aligned}\langle u_m^1, \psi \rangle_{H^1} &= \langle u_m^1, \psi - \varphi \rangle_{H^1} + \langle u_m^1, \varphi \rangle_{H^1} \\ &\le C\|\psi - \varphi\|_{H^1} + \langle u_m^1, \varphi \rangle_{H^1} \\ &< \varepsilon + c\varepsilon \quad \text{for } m > m_0.\end{aligned}$$

This implies $u_m^1 \rightharpoonup 0$ weakly in $H_0^1(\mathbf{S})$. □

Theorem 6.2. (i) *The sequence $\{u_n^i\}$ is a (PS)$_{\alpha(\mathbf{S})}$-sequence in $H_0^1(\mathbf{S})$ for J, for $i = 2, 3$*

(ii) *$u_n^i \rightharpoonup 0$ weakly in $L^p(\mathbf{S})$ and in $H_0^1(\mathbf{S})$ as $n \to \infty$, for $i = 2, 3$.*

The proof of this theorem is similar to the proof of Theorem 6.1 (i) and (iii).

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