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EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO SUPERLINEAR PERIODIC PARABOLIC PROBLEMS

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and let a, b, c be three (possibly discontinuous and unbounded) T-periodic functions with $c \geq 0$. We study existence and nonexistence of positive solutions for periodic parabolic problems $Lu = \lambda(a(x,t)u^p - b(x,t)u^q + c(x,t))$ in $\Omega \times \mathbb{R}$ with Dirichlet boundary condition, where $\lambda > 0$ is a real parameter and $p > q \geq 1$. If a and b satisfy some additional conditions and $p < (N+2)/(N+1)$ multiplicity results are also given. Qualitative properties of the solutions are discussed as well. Our approach relies on the sub and supersolution method (both to find the stable solution as well as the unstable one) combined with some facts about linear problems with indefinite weight. All results remain true for the corresponding elliptic problems. Moreover, in this case the growth restriction becomes $p < N/(N-1)$.

1. Introduction

Let Ω be a $C^{2+\theta}$ bounded domain in $\mathbb{R}^N$, $\theta \in (0,1)$, $N \geq 2$. For $T > 0$ and $1 \leq p \leq \infty$, let L_T^p be the Banach space of T-periodic functions h on $\Omega \times \mathbb{R}$ (i.e. satisfying $h(x,t) = h(x,t+T)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$) such that $h_{|\Omega \times (0,T)} \in L^p(\Omega \times (0,T))$, equipped with the norm $\|h\|_{L_T^p} := \|h_{|\Omega \times (0,T)}\|_{L^p(\Omega \times (0,T))}$. Let $C_T^{1+\theta,(1+\theta)/2}$, $C_T^{1,0}$ be the spaces of T-periodic functions on $\overline{\Omega} \times \mathbb{R}$ belonging to $C^{1+\theta,(1+\theta)/2}(\overline{\Omega} \times \mathbb{R})$ and $C^{1,0}(\overline{\Omega} \times \mathbb{R})$ respectively, and denote by

$$P^\circ := \text{the interior of the positive cone of } C_T^{1+\theta,(1+\theta)/2}.$$

Let $\{a_{ij}\}$, $\{b_j\}$, $1 \leq i,j \leq N$, be two families of T-periodic functions satisfying $a_{ij} \in C^{0,1}(\overline{\Omega} \times \mathbb{R})$, $b_j \in L_T^\infty$, $a_{ij} = a_{ji}$ and

$$\sum a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2$$

for some $\alpha > 0$ and all $(x,t) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^N$. Let A be the $N \times N$ matrix whose i,j entry is a_{ij}, let $\overline{b} = (b_1, \dots, b_N)$, let $0 \leq c_0 \in L_T^\infty$ and let L be the parabolic operator given by

$$Lu = u_t - div(A\nabla u) + \langle \overline{b}, \nabla u \rangle + c_0 u.$$

For $1 \leq r \leq \infty$ let $W_r^{2,1}(\Omega \times (t_0,t_1))$ be the Sobolev space of the functions $u \in L^r(\Omega \times (t_0,t_1))$, $u = u(x,t)$, $x = (x_1, \dots, x_N)$ such that u_t, u_{x_j} and $u_{x_ix_j}$

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belong to $L^r(\Omega \times (t_0, t_1))$ for $1 \leq i, j \leq N$, and let $W^{2,1}_{r,T}$ be the space of T-periodic functions such that $u_{|\Omega\times(0,T)} \in W^{2,1}_r(\Omega \times (0, T))$. For $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $r > 1$ we say that $u \in W^{2,1}_{r,T}$ is a (strong) solution of the periodic problem

$$\begin{gathered} Lu = g \quad \text{in } \Omega \times \mathbb{R} \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \\ u \quad T\text{-periodic} \end{gathered} \tag{1.1}$$

if the equation holds a.e. in the pointwise sense. It is known that for $g \in L^r_T$ with $1 < r < \infty$ there exists a unique solution $u \in W^{2,1}_{r,T}$ of (1.1) and that the associated solution operator $L^{-1} : L^r_T \to W^{2,1}_{r,T}$ is continuous (see e.g. [20, Section 4]). Moreover, if $r > N + 2$ then $W^{2,1}_{r,T} \subset C^{1+\theta,(1+\theta)/2}_T$ for some $\theta \in (0, 1)$ and so $u \in C^{1+\theta,(1+\theta)/2}_T$ (e.g. [19, Lemma 3.3, p. 80]), and in particular the boundary and periodicity conditions are satisfied pointwise.

Our aim in this paper is to study existence, nonexistence and multiplicity of (strictly) positive solutions for periodic parabolic problems of the form

$$\begin{gathered} Lu = \lambda(a(x, t)u^p - b(x, t)u^q + c(x, t)) \quad \text{in } \Omega \times \mathbb{R} \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \\ u \quad T\text{-periodic} \end{gathered} \tag{1.2}$$

where $a, b, c \in L^r_T$ for some $r > N + 2$, $c \geq 0$, $\lambda > 0$ is a real parameter and $p > q \geq 1$.

To avoid unnecessary complexity we restrict ourselves to (1.2), but one can see that most of the results are still valid for increasing nonlinearities that behave like u^p and u^q near the origin and infinity. Let us also mention that as a consequence of our proofs all results remain true for the corresponding elliptic problems. Let

$$\Lambda := \sup\{\lambda > 0 : \text{there exists a solution } u_\lambda > 0 \text{ of } (1.2)\}. \tag{1.3}$$

If $c \not\equiv 0$, constructing (well ordered) sub and supersolutions we shall prove that there exists some $\overline{\Lambda} > 0$ such that for all $\lambda \in (0, \overline{\Lambda}]$ there exists $u_\lambda \in P^\circ$ solution of (1.2). Moreover, we shall see that there exist $k_1, k_2 > 0$ not depending on λ such that $k_1\lambda \leq \|u_\lambda\|_\infty \leq k_2\lambda$ for such λ's. Also, if in addition $a \geq 0$ and $b^+/c \in L^\infty_T$, by means of the implicit function theorem we shall show that u_λ can be chosen such that $\lambda \to u_\lambda$ is differentiable and increasing for all $\lambda \in (0, \beta)$ for some $\beta > 0$ (see Theorem 3.1 (i) and (ii) respectively). Under an additional condition (which is fulfilled if for instance $b \leq \min\{a, c\}$) we shall see that (1.2) has a solution for every $\lambda \in (0, \Lambda)$. Furthermore, when $a \not\equiv 0$ we will prove that $\Lambda < \infty$ and we will provide some upper estimates for Λ (see Theorem 3.1 (iii)). Let us note that if $a \equiv 0 \leq b$ then (1.2) becomes "sublinear" and it is known in this case that $\Lambda = \infty$ (see e.g. [12]).

On the other hand, suppose $a, b, c \in L^\infty_T$ with $0 \leq a \not\equiv 0$ with $\inf_{\Omega\times\mathbb{R}}(a/b^+) > 0$. Then for $p < (N + 2)/(N + 1)$ we shall prove employing (non-well-ordered) sub and supersolutions that there exists a solution $v_\lambda \in P^\circ$ for all $\lambda \in (0, \alpha)$ for some $\alpha > 0$, and that v_λ satisfies that $\|v_\lambda\|_\infty \geq k\lambda^{-1/(p-1)}$ for all $\lambda > 0$ small enough and $k > 0$ not depending on λ. If additionally either the aforementioned condition in Theorem 3.1 (iii) holds or $c \equiv 0$, then we shall prove existence of a positive solution for all $\lambda \in (0, \Lambda)$ (see Theorem 3.3 (i) and (ii) respectively). Moreover, in many situations in which $c \equiv 0$ we shall show that $\Lambda = \infty$ (see Theorem 3.3 (iii)).

Also, as a consequence of the above results we shall obtain the existence of at least two positive solutions of (1.2), and in the case $c \equiv 0$ and $q = 1$ we shall prove similar results even without any relation between b and a or c (see Corollaries 3.4 and 3.5). Let us point out that for the analogous elliptic problem, Theorem 3.3 and Corollaries 3.4 and 3.5 are still valid for $p < N/(N-1)$ (see Remark 2.5 (ii) below).

Problems of the form (1.2) have been studied by several authors. If $b = c \equiv 0$, Esteban [8, Theorem 4] proved existence of a positive solution assuming that L has θ-Hölder continuous coefficients, $p < (N+2)/N$ and that $a = a(t) \in C_T^{\theta/2}(\mathbb{R})$ with $\min_{\mathbb{R}} a > 0$. If in addition $a \in W_T^{1,\infty}(\mathbb{R})$ and satisfies some technical conditions, she gave the same result in [8, Theorem 7] for $L = \partial/\partial_t - \Delta$ and $p < (3N+8)/(3N-4)$, and later on in [9] she improved this last theorem to the case $p < N/(N-2)$. Also, Quittner in [22] obtained a positive solution (also for the heat operator and $a \in W_T^{1,\infty}$ as above) for $p < (N+2)/(N-2)$, and an extension of this result under some additional hypothesis for $a = a_1(x)a_2(t)$ with $a_1 \in C^1(\overline{\Omega})$, $a_2 \in W_T^{1,\infty}$, $\inf_{\Omega\times\mathbb{R}}\{a_1, a_2\} > 0$ and Ω convex can be found in [18]. In all these works the main tools used are topological degree arguments together with several a priori estimates. We would like to point out that while our approach poses a stronger restriction on p, the assumptions on $a(x,t)$ and L are considerably weakened and the proofs given here are completely different and (in our opinion) quite more simple. We mention also that in the elliptic case existence of a positive solution of (1.2) with $b = c \equiv 0$ is well known (even if a changes sign) but to our knowledge it is always asked that either $a \in C(\overline{\Omega})$ or $a \in L^\infty(\Omega)$ but with several additional assumptions (see e.g. [2, 1] and the references therein).

On the other hand, when $b \equiv 0 \not\equiv c$ Esteban [8, Section V] showed the existence of at least two positive solutions for all $0 < \lambda < \Lambda$ under the aforementioned hypothesis in [8, Theorem 7] and assuming that $0 \le c \in C(\overline{\Omega} \times \mathbb{R})$. If $a \equiv 1$, $0 \le c \in L_T^\infty$ and $p < (N+2)/(N-2)$, Hirano and Mizoguchi found also in the case of the heat operator two positive solutions for $\lambda > 0$ small enough and studied their stability/instability (see [17]), and an extension for a similar problem and sign changing c's with $c \in C(\mathbb{R}, L^\infty(\Omega))$ was later established in [5]. We observe that again all these results mainly rely on topological degree arguments and a priori bounds which require restrictions on p, while we do not impose any condition on p in order to prove the existence of one of the solutions (namely, the stable one). Furthermore, we allow in this case a, b and c to be unbounded and a, b may have indefinite sign.

Finally, as far as we know no results are available specifically for (1.2) neither when $b \not\equiv 0 \equiv c$ nor if $b \not\equiv 0 \not\equiv c$. There are, however, some bifurcation results available for convex nonlinearities (e.g. [16, Chapter 3]) or increasing nonlinearities (e.g. [8, Section V]), but under strong regularity conditions on the coefficients of L and the nonlinearity. Let us note that for example when $a \ge 0$, the right member of (1.2) is convex either if $q = 1$ or $b \le 0$, and if $q > 1$ and $b \ge 0$ then it becomes "concave-convex". As far as the elliptic problem is concerned, (1.2) with a, b, c positive constants and $p \le N/(N-2)$ is included in some of the many types of nonlinearities covered in the nice paper [21, Theorem 6.21]. When Ω is a ball and $L = -\Delta$, it is proved there that $\Lambda < \infty$ and that there exist exactly two positive solutions for $\lambda \in (0, \Lambda)$ and exactly one for $\lambda = \Lambda$. Let us also mention that the nonlinearities that arise in Corollary 3.5 are included (for Ω, L and p as above, and

$a, \pm b$ positive constants) in [21, Theorems 6.5 and 6.11], and it is also proved there that in this cases the solution is unique for every $\lambda \in (0, \Lambda)$. We remark that all these last results are obtained applying variational and symmetry arguments which of course are not eligible in our case.

2. Preliminaries

We start by collecting some necessary facts about periodic parabolic problems with indefinite weight.

Remark 2.1. (i) Let $m \in L_T^r$ with $r > (N+2)/2$, and let

$$P_\Omega(m) := \int_0^T \operatorname{esssup}_{x\in\Omega} m(x,t)dt. \tag{2.1}$$

Then $P_\Omega(m) > 0$ is necessary and sufficient for the existence of a (unique and simple) positive principal eigenvalue $\lambda_1(L,m)$ (or $\lambda_1(m)$ if no confusion arises) for the problem

$$\begin{gathered} Lu = \lambda m u \quad \text{in } \Omega\times\mathbb{R} \\ u = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \\ u \quad \text{-periodic} \end{gathered} \tag{2.2}$$

(cf. [10, Theorem 3.6]). We note that $P_\Omega(m) = +\infty$ is allowed (cf. [10, p. 218]) and that no regularity on $\partial\Omega$ is needed. It also holds that $m \to \lambda_1(m)$ is continuous (cf. [10, Theorem 3.9]). If $\lambda_1(m)$ exists, we will denote (from now on) by Φ the positive principal eigenfunction normalized by $\|\Phi\|_\infty = 1$. If in addition Ω has $C^{2+\theta}$ boundary and $r > N+2$, then $\Phi \in P^\circ$.

(ii) The following comparison principle holds: if $m_1, m_2 \in L_T^r$ with $r > (N+2)/2$, $P_\Omega(m_1) > 0$ and $m_1 \le m_2$ in $\Omega\times\mathbb{R}$, then $\lambda_1(m_1) \ge \lambda_1(m_2)$ and, if in addition $m_1 < m_2$ in a set of positive measure, then $\lambda_1(m_1) > \lambda_1(m_2)$ (cf. [10, Remark 3.7]).

Remark 2.2. (i) Let $m \in L_T^r$ with $r > (N+2)/2$. For $\lambda \in \mathbb{R}$, let $\mu_{L,m}(\lambda)$ (or simply $\mu_m(\lambda)$ if no confusion arises) be defined as the unique $\mu \in \mathbb{R}$ such that the Dirichlet periodic problem $Lu = \lambda m u + \mu_m(\lambda)u$ in $\Omega\times\mathbb{R}$ has a positive solution u. Then $\mu_m(\lambda)$ is well defined, $\mu_m(0) > 0$, μ_m is concave and continuous, and a given $\lambda \in \mathbb{R}$ is a principal eigenvalue for (2.2) if and only if $\mu_m(\lambda) = 0$ (cf. [10, Lemmas 3.2 and 3.5]). In particular, for $\lambda > 0$, if $P_\Omega(m) > 0$ then $\mu_m(\lambda) > 0$ if and only if $\lambda < \lambda_1(m)$, and $\mu_m(\lambda) > 0$ for all $\lambda > 0$ if $P_\Omega(m) \le 0$.

(ii) Let $m, h \in L_T^r$ with $r > (N+2)$. Then, if $\mu_m(\lambda) > 0$, the problem

$$\begin{gathered} Lu = \lambda m u + h \quad \text{in } \Omega\times\mathbb{R} \\ u = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \\ u \quad T\text{-periodic} \end{gathered} \tag{2.3}$$

has a unique solution $u \in W_{r,T}^{2,1}$ which is positive if $h \ge 0$, and the solution operator $h \to u$ is continuous (cf. [11, Lemma 2.9]). Conversely, if $\lambda_1(m)$ exists and $Lu \gneqq \lambda m u$ (respectively $\lneqq$) for some $\lambda > 0$ and $u > 0$ in $\Omega\times\mathbb{R}$ with $u = 0$ on $\partial\Omega\times\mathbb{R}$, then $\lambda < \lambda_1(m)$ (respectively $\lambda > \lambda_1(m)$) (cf. [14, Remark 2.1 (e)]).

We will need the following elementary lemma to provide one of the upper estimates for Λ.

Lemma 2.3. *Let $p > q \geq 1$ and let $h(\xi) := \xi^p - \xi^q - c_{p,q}\xi + 1$, where $c_{p,q} > 0$ is defined by*

$$c_{p,q} := \begin{cases} \frac{p}{(p-1)^{(p-1)/p}} - 1 & \text{if } q = 1 \\ (\frac{q}{p-1})^{(p-1)/(p-q)} & \text{if } q > 1 \text{ and } p - q \geq 1 \\ p - q & \text{if } q > 1 \text{ and } p - q \leq 1 \end{cases} \tag{2.4}$$

Then $h(\xi) \geq 0$ for all $\xi \geq 0$.

Proof. Suppose $q = 1$. Then h attains its unique minimum at $\xi_0 := ((1+c_{p,q})/p)^{1/(p-1)}$. Moreover, after some computations we get

$$h(\xi_0) = \Big(\frac{1 + c_{p,q}}{p}\Big)^{1/(p-1)} \Big(\frac{1 + c_{p,q}}{p} - (1 + c_{p,q})\Big) + 1 = 0.$$

Suppose now $q > 1$ and $p - q \geq 1$. Define $\xi_0 := (q/(p-1))^{1/(p-q)}$. Then $p\xi_0^{p-1} - q\xi_0^{q-1} = c_{p,q}$ and hence $h'(\xi_0) = 0$. Furthermore, taking into account this we find that

$$\begin{aligned} h(\xi_0) &= (1 - p)\xi_0^p + (q - 1)\xi_0^q + 1 \\ &= \Big(\frac{q}{p-1}\Big)^{\frac{q}{p-q}} \Big(\frac{(1-p)q}{(p-1)} + (q - 1)\Big) + 1 \geq 0 \end{aligned}$$

because $p - q \geq 1$.

Finally, suppose $q > 1$ and $p-q \leq 1$. Since in this case $c_{p,q} = p-q$ it follows that $h'(1) = 0$. Moreover, $h(1) \geq 0$ because $p - q \leq 1$ and this concludes the proof. $\square$

We say that $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an L^r_T-Carathéodory function if $f(x, t, \xi)$ is T-periodic in T, $(x, t) \to f(x, t, \xi)$ is measurable for all $\xi \in \mathbb{R}$, $\xi \to f(x, t, \xi)$ is continuous on $\mathbb{R}$ a.e. $(x, t) \in \Omega \times \mathbb{R}$; and, for each $\rho > 0$, there exists $h \in L^r_T$ such that $|f(x, t, \xi)| \leq h(x, t)$ for a.e. $(x, t) \in \Omega \times \mathbb{R}$ and every $\xi \in [-\rho, \rho]$. Also, if $r > N+2$, we will say that $v \in W^{2,1}_{r,T}$ is a subsolution (respectively a supersolution) of

$$\begin{gathered} Lu = f(x, t, u) \quad \text{in } \Omega \times \mathbb{R} \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \\ u \quad T\text{-periodic} \end{gathered} \tag{2.5}$$

if $Lv \leq f(x, t, v)$ (resp. $Lv \geq f(x, t, v)$) in $\Omega \times \mathbb{R}$ and $v \leq 0$ (resp. $v \geq 0$) on $\partial\Omega \times \mathbb{R}$. Finally, we say that a subsolution v of (2.5) is strict if for every solution u of (2.5) with $v \leq u$ one has $v < u$ in $\Omega \times \mathbb{R}$ and either $v < u$ or $v = u$ and $\partial_\nu u > \partial_\nu v$ on $\partial\Omega \times \mathbb{R}$, ν being the unit outer normal to $\partial\Omega$. A strict supersolution is defined analogously.

We state for the reader's convenience the following existence result in the presence of non-well-ordered sub and supersolutions (for the proof, see [15, Lemma 2.3]). Let us mention that for $m \equiv 1$ this lemma can be found in [4, Theorem 3.2].

Lemma 2.4. *Let $m \in L^\infty_T$ such that $P_\Omega(m) > 0$ and let $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying*

- (H1) *f is an L^r_T-Carathéodory function for some $r > N + 2$.*
- (H2) *There exist $\gamma \in (0, 1)$, $\delta \in (1, (N + 2 - \gamma)/(N + 1))$ and $\sigma_0 > 0$ such that*

$$\frac{f(x, t, \xi) - \lambda_1(m)m\xi}{|\xi|^\gamma} \geq -1 \quad \text{and} \quad \frac{f(x, t, \xi) - \lambda_1(m)m\xi}{|\xi|^\delta} \leq 1$$

a.e. $(x, t) \in \Omega \times \mathbb{R}$ for all ξ such that $|\xi| > \sigma_0$.

Suppose that there exist v, w sub and supersolutions respectively of (2.5) *such that* $v \not\leq w$. *Then* (2.5) *has a solution* $u \in \overline{\mathcal{O}}$ *where*

$$\mathcal{O} := \{u \in C_T^{1,0} : v \not\leq u \text{ and } u \not\leq w\}.$$

Remark 2.5. (i) In the same way as in [4, Theorem 3.2 and Remark 2.2], if v and w are strict sub and supersolutions, every solution $u \in \overline{\mathcal{O}}$ actually satisfies $u \in \mathcal{O}$.

(ii) The restrictions $r > N + 2$ and $\delta \in (1, (N + 2 - \gamma)/(N + 1))$ come from the use of the strong maximum principle in the proof of Lemma 2.4. Thus, in the elliptic case one can take $r > N$ and $\delta \in (1, (N - \gamma)/(N - 1))$ and obtain exactly the same conclusions (in fact, for $m \equiv 1$ this is done for instance in [3, Section 4]).

3. Main results

As usual, we write $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. For $a, b, c \in L_T^r$ with $r > N + 2$ and $p > q \geq 1$ we set

$$\overline{\Lambda} := 1/\|L^{-1}(a^+ + b^- + c)\|_{L_T^\infty}, \quad \underline{\Lambda} := \|(L + \overline{\Lambda}(a^- + b^+))^{-1}c\|_{L_T^\infty}, \tag{3.1}$$

$$\beta_0 := \min\{\overline{\Lambda}, (\overline{\Lambda}^{q-1}\lambda_1(pa + qb^-))^{1/q}\}. \tag{3.2}$$

Recall that Λ is given by (1.3).

Theorem 3.1. *Let $a, b, c \in L_T^r$ for some $r > N + 2$ such that $0 \leq c \not\equiv 0$, and let $p > q \geq 1$. Then*

(i) *Equation* (1.2) *has a solution $u_\lambda \in P^\circ$ for all $\lambda \in (0, \overline{\Lambda}]$ and*

$$\underline{\Lambda}\lambda \leq \|u_\lambda\|_{L_T^\infty} \leq \overline{\Lambda}^{-1}\lambda \tag{3.3}$$

for such λ (in particular, $\Lambda \geq \overline{\Lambda}$).

(ii) *Assume in addition that $a \geq 0$ and $b^+/c \in L_T^\infty$. Then there exists $\beta > \beta_0$ such that $\lambda \to u_\lambda$ is a C^1 increasing map from $(0, \beta)$ into P°.*

(iii) *Assume in addition that*

$$K_1 := \|b^+/c\|_{L_T^\infty}^{1/q} \leq \inf_{\Omega \times \mathbb{R}} (a/b^+)^{1/(p-q)} := K_2. \tag{3.4}$$

Let $m(x, t) := \min\{a(x, t), c(x, t)\}$, let $c_{p,q}$ be given by (2.4), *and let $w \in P^\circ$ be the solution of* (1.1) *with c in place of g. Then* (1.2) *has a solution $u_\lambda \in P^\circ$ for all $\lambda \in (0, \Lambda)$ and*

$$\Lambda \leq \begin{cases} \lambda_1(m)/c_{p,q} & \text{if } m \not\equiv 0 \\ \max\{1, \lambda_1(aw^{p-1})\} & \text{if } m \equiv 0 \text{ and } a \not\equiv 0 \end{cases} \tag{3.5}$$

Proof. Let $\lambda > 0$, and let $\Phi_\lambda \in P^\circ$ be the unique positive principal eigenfunction of (2.2) with c and $L + \lambda(a^- + b^+)$ in place of m and L respectively, normalized by $\|\Phi_\lambda\|_\infty = 1$ (since $c \not\equiv 0$ such Φ_λ exists). Let $\lambda^* := \lambda_1(L + \lambda(a^- + b^+), c)$ and let $k_0 = k_0(\lambda) := \lambda/\lambda^*$. We first claim that $k\Phi_\lambda$ is a subsolution of (1.2) for every $0 < k \leq k_0$. Indeed, taking into account that $p, q \geq 1$ we find that

$$\lambda(a^+(k\Phi_\lambda)^p + b^-(k\Phi_\lambda)^q + c) \geq \lambda c \geq \lambda^* ck\Phi_\lambda = (L + \lambda(a^- + b^+))k\Phi_\lambda$$

and since $k\Phi_\lambda \leq 1$ the claim follows.

On the other hand, let $\overline{\Lambda}$ be given by (3.1), let $0 < \lambda \leq \overline{\Lambda}$ and let $z_\lambda \in P^\circ$ be the unique solution of the periodic problem $Lz_\lambda = \lambda(a^+ + b^- + c)$ in $\Omega \times \mathbb{R}$, $z_\lambda = 0$

on $\partial\Omega \times \mathbb{R}$. It is easy to check that z_λ is a supersolution for (1.2). Indeed, clearly $\|z_\lambda\|_\infty = \overline{\Lambda}^{-1}\lambda \le 1$ and then

$$Lz_\lambda \ge \lambda(a^+\|z_\lambda\|_\infty^p + b^-\|z_\lambda\|_\infty^q + c) \ge \lambda(az_\lambda^p - bz_\lambda^q + c).$$

Hence, if $k = k(\lambda)$ is small enough, [7, Theorem 1] gives some $u_\lambda \in L_T^\infty$ solution of (1.2) satisfying $k\Phi_\lambda \le u_\lambda \le z_\lambda$ for all $\lambda \in (0, \overline{\Lambda}]$. Moreover, $u_\lambda \in P^\circ$ and $\|u_\lambda\|_\infty \le \overline{\Lambda}^{-1}\lambda$ for such λ's (i.e., the second inequality in (3.3) holds). Also, taking into account this last fact, from (1.2) we obtain

$$Lu_\lambda \ge \lambda(-a^-u_\lambda - b^+u_\lambda + c) \tag{3.6}$$

and thus $(L + \overline{\Lambda}(a^- + b^+))u_\lambda \ge \lambda c$ (because $\lambda \le \overline{\Lambda}$) and the first inequality in (3.3) follows. So, (i) is proved.

To prove (ii) we first note that we may assume without loss of generality that $b \le c$. Indeed, if $\|b^+/c\|_\infty = 0$ then $b \le 0$ and so $b \le c$. If not, take $k := \|b^+/c\|_\infty^{1/q}$ and define $a_k := ak^{1-p}$, $b_k := bk^{1-q}$ and $c_k := ck$. It follows that $b_k \le c_k$. Furthermore, u is a solution of (1.2) if and only if ku is a solution of (1.2) with a_k, b_k and c_k in place of a, b and c respectively. Henceforth we assume that $b \le c$.

Let $\lambda > 0$, $u_\lambda > 0$ be the solution of (1.2) found in (i), and let $m_\lambda := pau_\lambda^{p-1} - qbu_\lambda^{q-1}$. We claim that the implicit function theorem can be applied in a point (λ, u_λ) for any $\lambda > 0$ sufficiently small. Indeed, a direct computation shows that in order to see this it suffices to prove that for a given $h \in L_T^r$ there is a unique solution $u \in W_{r,T}^{2,1}$ of problem (2.3) with m_λ in place of m and that the solution operator for this problem is continuous. Thus, recalling Remark 2.2 (ii) the claim will follow if $\lambda_1(m_\lambda) > \lambda$ (if such $\lambda_1(m_\lambda)$ exists; if $\lambda_1(m_\lambda)$ does not exist we have nothing to prove). Now, let β_0 be given by (3.2) and let $0 < \lambda \le \beta_0$. Since $p > q$ and $\lambda \le \overline{\Lambda}$, by the second inequality in (3.3) we have

$$m_\lambda \le u_\lambda^{q-1}(pau_\lambda^{p-q} + qb^-) < (\overline{\Lambda}^{-1}\lambda)^{q-1}(pa + qb^-)$$

and therefore the comparison principle in Remark 2.1 (ii) yields

$$\lambda_1(m_\lambda) > (\overline{\Lambda}\lambda^{-1})^{q-1}\lambda_1(pa + qb^-) \ge \lambda$$

(if $\lambda_1(pa + qb^-)$ does not exist then $m_\lambda \le 0$ and we are done). Hence, the claim is proved.

Let $I := (\alpha_1, \alpha_2)$ be a maximal interval centered at β_0 provided by the implicit function theorem in which $\lambda \to u_\lambda$ is a C^1 map into P°. Differentiating (1.2) with respect to λ and taking into account that $a \ge 0$ and $b \le c$ we obtain

$$(L - \lambda m_\lambda)\frac{\partial u_\lambda}{\partial \lambda} = au_\lambda^p - bu_\lambda^q + c \ge c(1 - u_\lambda^q) \tag{3.7}$$

for all $\lambda \in I$. So, since u_{β_0} satisfies (3.3), it follows from (3.7) and Remark 2.2 (ii) that $\partial u_\lambda/\partial\lambda > 0$ for some $(\alpha, \beta) \subset I$ with $\beta_0 \in (\alpha, \beta)$. We next observe that $\alpha = \alpha_1 = 0$. Indeed, suppose first $\alpha > \alpha_1$. In this case $\partial u_\lambda/\partial\lambda_{|\lambda=\alpha} = 0$, but since $\lambda \to u_\lambda$ is increasing in (α, β) and $\|u_{\beta_0}\|_\infty \le 1$, again (3.7) and Remark 2.2 (ii) yield $\partial u_\lambda/\partial\lambda_{|\lambda=\alpha} > 0$. Assume now $\alpha > 0$, and let $u_j \in P^\circ$ be the solutions of (1.2) corresponding to some sequence $\lambda_j \searrow \alpha$. Then u_j is decreasing and so the continuity of the solution operator L^{-1} supplies some $u_\alpha \ge 0$ solution of (1.2) for $\lambda = \alpha$. Furthermore, $\alpha > 0$ implies $\|u_\alpha\|_\infty > 0$ (because $c \not\equiv 0$) and hence we can apply the implicit function theorem in the point (α, u_α), contradicting the maximality of (α, β). Consequently, $\alpha = 0$ and (ii) is proved.

Let us prove (iii). By (3.4), as in the beginning of the proof of (ii) we may now assume that $b \le \min\{a, c\}$. Indeed, take $0 < k \in [K_1, K_2]$ (where K_1 and K_2 are given by (3.4)) and define a_k, b_k and c_k as in (ii). Then $b_k \le \min\{a_k, c_k\}$ and as before u is a solution of (1.2) if and only if ku is a solution of (1.2) with a_k, b_k and c_k in place of a, b and c.

Now, let $\lambda \in (0, \Lambda)$ and let $\overline{\lambda} \in (\lambda, \Lambda)$ such that there exists $u_{\overline{\lambda}} > 0$ solution of (1.2) with $\overline{\lambda}$ in place of λ. Since $b \le \min\{a, c\}$ and $a, c \ge 0$ it is easy to check that $a\xi^p - b\xi^q + c \ge 0$ for all $\xi \ge 0$ a.e. $(x,t) \in \Omega \times \mathbb{R}$. Thus $u_{\overline{\lambda}}$ is a supersolution for (1.2). Therefore, since for all $\lambda > 0$ the first paragraph of the proof provides subsolutions for (1.2) of the form $k\Phi_\lambda$, making $k > 0$ sufficiently small we can again apply [7, cite 1], and obtain a solution of (1.2).

We prove (3.5). Suppose first $0 \not\equiv m(x,t) := \min\{a(x,t), c(x,t)\}$. We observe that $\lambda_1(m)$ exists because $m \ge 0$. Let $u > 0$ be a solution of (1.2). Taking into account Lemma 2.3 we get

$$Lu = \lambda(au^p - bu^q + c) \ge \lambda m(u^p - u^q + 1) \ge \lambda m c_{p,q} u$$

and then Remark 2.2 (ii) says that $\lambda \le \lambda_1(mc_{p,q}) = \lambda_1(m)/c_{p,q}$.

On the other hand, if $m \equiv 0$, from $b \le \min\{a, c\}$ and $a, c \ge 0$ we have that $b \le 0$. Suppose now the last inequality in (3.5) is not valid. Let $w \in P^\circ$ be the unique solution of (1.1) with c in place of g. Since $c \not\equiv 0$ it holds that $w > 0$. Moreover, $0 \le aw^{p-1} \not\equiv 0$ because $a \not\equiv 0$. Choose $\lambda > \max\{1, \lambda_1(aw^{p-1})\}$ such that there exists $u > 0$ solution of (1.2). We observe that since $\lambda > 1$ the maximum principle yields $u \ge w$. Also,

$$Lu = \lambda(au^p - bu^q + c) \ge \lambda au^p \ge \lambda aw^p$$

and so again employing Remark 2.2 (ii) we obtain $\lambda \le \lambda_1(aw^{p-1})$. Contradiction. $\square$

Let us note that if in Theorem 3.1 (iii) it holds that $m = a \equiv 0$ then (1.2) becomes $Lu = \lambda(b^- u^q + c)$ and hence upper bounds for Λ can be obtained in the same way as there.

Lemma 3.2. *Let $a, b, c \in L^\infty_T$ such that $a, c \ge 0$ and $\inf_{\Omega\times\mathbb{R}}(a/b^+) > 0$, and let $1 \le q < p < (N+2)/(N+1)$. Assume there exist $v, w \ge 0$ sub and supersolutions respectively of* (1.2) *such that neither of them is a solution and $v \not\le w$. Then there exists $u \in P^\circ$ solution of* (1.2) *satisfying $v \not\le u \not\le w$.*

Proof. We note first that v is a strict subsolution. Indeed, if z is a solution of (1.2) with $v \le z$ then

$$L(z - v) > -b^+(z^q - v^q) \ge -q\|z\|_\infty^{q-1} b^+(z - v)$$

and hence the assertion follows from the strong maximum principle (as stated e.g. in [6, Theorem 13.5]). In the same way w is a strict supersolution.

Let $\widetilde{f}$ be defined by $\widetilde{f}(x,t,\xi) = \lambda(a(x,t)\xi^p - b(x,t)\xi^q + c(x,t))$ for $\xi \ge 0$ and $\widetilde{f}(x,t,\xi) = \lambda c(x,t)$ for $\xi < 0$. Let $\mu := (N+2)/(N+1) - p > 0$, and choose $\alpha, \gamma > 0$ small enough such that $\alpha + \gamma/(N+1) < \mu$. Since $\inf_{\Omega\times\mathbb{R}}(a/b^+) > 0$, reasoning as in Theorem 3.1 (iii) we may assume that $b \le a$. Taking into account this, it is easy to check that the function $\widetilde{f}$ satisfies the assumptions of Lemma 2.4 with γ as above, $\delta := p + \alpha$, $m := a$ and any $r > N + 2$. Therefore, Lemma 2.4 provides a solution $u \in C^{1,0}_T$ of (2.5) with $\widetilde{f}$ in place of f satisfying $v \not< u \not< w$.

Moreover, since v and w are strict sub and supersolutions, from Remark 2.5 (i) we get $v \nleq u \nleq w$. In particular $u \not\equiv 0$ because $w \not\equiv 0$ (observe that if $w \equiv 0$ then $c \equiv 0$ and therefore w is a solution of (1.2), contradicting the hypothesis). Let $U := \{(x,t) \in \Omega \times \mathbb{R} : u(x,t) < 0\}$. If $U \neq \varnothing$ we have $Lu = \lambda c \geq 0$ in U and $u = 0$ on ∂U and so the maximum principle (as stated e.g. in [13, Lemma 2.3]) implies $u \geq 0$ in U which is not possible. Thus $u \geq 0$ in $\Omega \times \mathbb{R}$ and hence by the aforementioned strong maximum principle in [6] $u \in P^\circ$. It follows that u is a solution of (1.2) and this ends the lemma. $\square$

We focus now on what happens when $a \not\equiv 0$. The special case $c \equiv 0$ and $q = 1$ will be considered separately in Corollary 3.5 below. For $\overline{\Lambda}$ as in (3.1) and $\varepsilon > 0$ we set

$$\begin{gathered} \delta_0 := \min\{\lambda_1(a), \overline{\Lambda}\} \\ \Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}, \quad \Omega_\varepsilon^c := \Omega - \Omega_\varepsilon. \end{gathered} \tag{3.8}$$

Theorem 3.3. *Let $a, b, c \in L^\infty_T$ with $a, c \geq 0$, $a \not\equiv 0$ and $\inf_{\Omega\times\mathbb{R}}(a/b^+) > 0$. Let $1 \leq q < p < (N+2)/(N+1)$. Then*

- (i) *Equation (1.2) has a solution $v_\lambda \in P^\circ$ for all $\lambda \in (0, \delta_0)$ and there exists $k > 0$ not depending on λ such that for all $\lambda > 0$ small enough*
$$\|v_\lambda\|_{L^\infty_T} \geq k\lambda^{\frac{-1}{p-1}}. \tag{3.9}$$
- (ii) *Assume that either $c \equiv 0$ or (3.4) holds. Then (1.2) has a positive solution for all $\lambda \in (0, \Lambda)$.*
- (iii) *If $c \equiv 0$, $q > 1$ and $b \leq 0$ in $\Omega_\sigma \times \mathbb{R}$ for some $\sigma > 0$, then $\Lambda = \infty$.*

Proof. As in the above lemma we assume that $b \leq a$. Let $\lambda \in (0, \lambda_1(a))$ ($a \not\equiv 0$ and so such $\lambda_1(a)$ exists) and $m_\varepsilon := a\chi_{\Omega_\varepsilon^c \times\mathbb{R}} - b^+\chi_{\Omega_\varepsilon\times\mathbb{R}}$. Since $0 \leq a \not\equiv 0$ and the positive principal eigenvalue is continuous with respect to the weight (cf. Remark 2.1), we can choose $\varepsilon > 0$ small enough such that $\lambda_1(m_\varepsilon)$ exists and $\lambda_1(m_\varepsilon) \geq \lambda$. Let Φ be the unique positive principal eigenfunction of (2.2) with m_ε in place of m, normalized by $\|\Phi\|_\infty = 1$. Let $0 < \delta := \min_{\Omega_\varepsilon^c\times\mathbb{R}} \Phi$ and let $K_0 = K_0(\lambda) := ((1 + \lambda_1(m_\varepsilon)/\lambda)^{1/(p-q)})/\delta$. We claim that $k\Phi$ is a subsolution of (1.2) for every $k \geq K_0$. Indeed, let $k \geq K_0$ and let us first write $f(x,t,\xi) := \lambda(a\xi^p - b\xi^q + c)$ and

$$\begin{gathered} A_k := \{(x,t) \in \Omega \times \mathbb{R} : k\Phi \leq 1\}, \quad A_k^c := (\Omega \times \mathbb{R}) - A_k, \\ B_k := (\Omega_\varepsilon \times \mathbb{R}) \cap A_k, \quad B_k^c := (\Omega_\varepsilon \times \mathbb{R}) \cap A_k^c, \\ C_k := (\Omega_\varepsilon^c \times \mathbb{R}) \cap A_k, \quad C_k^c := (\Omega_\varepsilon^c \times \mathbb{R}) \cap A_k^c. \end{gathered} \tag{3.10}$$

We observe that $C_k = \varnothing$ because $k\Phi \geq K_0\Phi > \delta^{-1}\Phi \geq 1$ in $\Omega_\varepsilon^c \times \mathbb{R}$. Now, taking into account that $b \leq a$ and $1 \leq q < p$ we get that

$$\begin{aligned} f(x,t,k\Phi) &\geq -\lambda b^+(k\Phi)^q\chi_{B_k} + \lambda a((k\Phi)^p - (k\Phi)^q)\chi_{B_k^c \cup C_k^c} \\ &\geq -\lambda b^+ k\Phi\chi_{B_k} + \lambda a(k\Phi)^q((k\delta)^{p-q} - 1)\chi_{C_k^c} \\ &\geq -\lambda_1(m_\varepsilon)b^+k\Phi\chi_{B_k} + \lambda_1(m_\varepsilon)ak\Phi\chi_{C_k^c} \\ &\geq \lambda_1(m_\varepsilon)m_\varepsilon k\Phi = L(k\Phi) \end{aligned} \tag{3.11}$$

and this proves the claim.

On the other side, let $0 < \lambda < \overline{\Lambda}$ and let $z_\lambda \geq 0$ be defined as in the second paragraph of the proof of Theorem 3.1. Since $\|z_\lambda\|_\infty = \overline{\Lambda}^{-1}\lambda$, there exists $\alpha_\lambda > 0$ such that if $\overline{z}_\lambda := \alpha_\lambda + z_\lambda$ then $\|\overline{z}_\lambda\|_\infty \leq 1$. Furthermore, in a similar way as there one can see that $\overline{z}_\lambda$ is a supersolution of (1.2) for all $\lambda \in (0, \overline{\Lambda})$. Therefore, by Lemma 3.2 there exists some $v_\lambda \in P^\circ$ solution of (1.2) for every $\lambda \in (0, \min\{\lambda_1(a), \overline{\Lambda}\})$ and satisfying $k\Phi \not\leq v_\lambda \not\leq \overline{z}_\lambda$. In particular, $\|v_\lambda\|_\infty \geq \alpha_\lambda$ for all such λ's.

To prove (3.9) we proceed by contradiction. Let λ_j be a sequence with $\lambda_j \searrow 0$, let v_j be the corresponding solutions of (1.2) found above, and suppose that $\lambda_j \|v_j\|_\infty^{p-1} \to 0$. Without loss of generality we can assume that $\|v_j\|_\infty \geq \alpha$ for all j large enough and $\alpha > 0$ not depending on λ. Let $w_j := v_j/\|v_j\|_\infty$. Dividing (1.2) by $\|v_j\|_\infty$ we get

$$Lw_j = \lambda_j \|v_j\|_\infty^{p-1} \big(a w_j^p - b w_j^q / \|v_j\|_\infty^{p-q} + c/\|v_j\|_\infty \big). \tag{3.12}$$

Now, going to the limit in (3.12), the continuity of the solution operator L^{-1} yields that $w_j \to 0$ when $j \to 0$, which is not possible.

Let us prove (ii). Assume first that (3.4) holds. In this case we start arguing as in the first part of the proof but defining now $m_\varepsilon := a\chi_{\Omega_\varepsilon^c \times \mathbb{R}}$. Then $m_\varepsilon \geq 0$ in $\Omega \times \mathbb{R}$ and $m_\varepsilon = 0$ in the sets B_k and B_k^c given by (3.10). Moreover, (since by (3.4) we may suppose that $b \leq c$) we have

$$f(x, t, k\Phi) \geq \lambda(-b(k\Phi)^q + c) \geq 0 \quad \text{in } B_k$$

and hence we do not need to impose the restriction $\lambda \leq \lambda_1(m_\varepsilon)$ in (3.11). Furthermore, a quick look at (3.11) shows that the other bounds remain the same, and thus as there we obtain a positive subsolution of the form $k\Phi$ but now for all $\lambda > 0$ (with $k \geq K_0(\lambda)$ as in (3.11)).

Next, we claim that reasoning as in the proof of Theorem 3.1 (iii) we obtain a solution to (1.2) for all $\lambda \in (0, \Lambda)$. Indeed, let $\lambda \in (0, \Lambda)$ and take $\overline{\lambda} \in (\lambda, \Lambda)$ such that there exists $u_{\overline{\lambda}} > 0$ solution of (1.2) with $\overline{\lambda}$ in place of λ. As before, since we are assuming that $b \leq \min\{a, c\}$, $f(., \xi) \geq 0$ for all $\xi > 0$. So, $u_{\overline{\lambda}}$ is a supersolution for (1.2) and therefore the claim follows from the above paragraph and Lemma 3.2.

Suppose now that $c \equiv 0$ and $q > 1$ (the case $q = 1$ is included in Corollary 3.5 below). In this case multiplying (1.2) by $\lambda^{1/(q-1)}$ and writing $v := \lambda^{1/(q-1)} u$ we transform (1.2) into the equivalent problem

$$Lv = \lambda^{-(p-q)/(q-1)} a v^p - b v^q \quad \text{in } \Omega \times \mathbb{R}. \tag{3.13}$$

From (i) (3.13) has a positive solution for all $\lambda > 0$ small enough. Moreover, readily (3.13) has a positive solution for every $\lambda \in (0, \Lambda)$. Indeed, let $\lambda \in (0, \Lambda)$ and take $\underline{\lambda} > 0$ small enough and $\overline{\lambda} \in (\lambda, \Lambda)$ such that there exist $u_{\underline{\lambda}}, u_{\overline{\lambda}} > 0$ solutions of (3.13) with $\underline{\lambda}$ and $\overline{\lambda}$ in place of λ respectively. Then $u_{\underline{\lambda}}$ and $u_{\overline{\lambda}}$ are super and subsolutions respectively of (3.13) and therefore (either if they are well-ordered or not) we obtain a solution for (3.13), and hence for (1.2).

To prove (iii) we shall supply a solution of (1.2) for every $\lambda > 0$. We note first that since $b \leq 0$ in $\Omega_\sigma \times \mathbb{R}$ for some $\sigma > 0$ (Ω_σ given by (3.8)), for any $\lambda > 0$, the subsolution constructed in the first paragraph of the proof of (ii) can still be used in this situation choosing there $\varepsilon \leq \sigma$. Indeed, as in (ii) $m_\varepsilon = 0$ in the sets B_k and B_k^c, $f(., k\Phi) \geq c \geq 0$ in B_k and the rest also stays the same. On the other hand, let $\underline{\lambda} < \lambda$ small enough such that there exists $\underline{v}$ solution of (3.13) with $\underline{\lambda}$ in place of λ. Clearly $\lambda^{-1/(q-1)} \underline{v}$ is a supersolution of (1.2) and then again Lemma 3.2 gives a solution of (1.2) and this concludes the proof. □

Corollary 3.4. *Let $a, b, c \in L_T^\infty$ such that $a, c \geq 0$, $a \not\equiv 0 \not\equiv c$, and let $1 \leq q < p < (N+2)/(N+1)$.*

(i) *If* $\inf_{\Omega\times\mathbb{R}}(a/b^+) > 0$*, then for all* $\lambda > 0$ *small enough there exist two positive solutions of* (1.2)*.*
(ii) *If in addition* (3.4) *holds, then (i) is true for all* $\lambda \in (0, \Lambda)$*.*

Proof. (i) is an immediate consequence of (3.3) and (3.9). Let us prove (ii). We assume $b \le \min\{a, c\}$ and argue as before. Let $\lambda \in (0, \Lambda)$, let $\overline{\lambda} \in (\lambda, \Lambda)$ and let $v_{\overline{\lambda}}$ be the solution of (1.2) with $\overline{\lambda}$ in place of λ given by Theorem 3.3 (ii). We have that $v_{\overline{\lambda}}$ is a supersolution for (1.2). Also, the first paragraph of the proof of Theorem 3.1 provides some positive subsolution u_λ of (1.2) such that $u_\lambda \le v_{\overline{\lambda}}$ and hence [7, Theorem 1] gives some $\underline{w}_\lambda$ solution of (1.2) satisfying $u_\lambda \le \underline{w}_\lambda \le v_{\overline{\lambda}}$. On the other hand, as in the first part of the proof of Theorem 3.3 (ii) we can construct another subsolution $\widetilde{u}_\lambda$ such that $\widetilde{u}_\lambda \not\le v_{\overline{\lambda}}$ and thus recalling Lemma 3.2 we obtain a solution $\overline{w}_\lambda \in P^\circ$ of (1.2) satisfying $\widetilde{u}_\lambda \not\le \overline{w}_\lambda \not\le v_{\overline{\lambda}}$. In particular $\underline{w}_\lambda \ne \overline{w}_\lambda$ and this proves (ii). $\square$

For the case $c \equiv 0$ and $q = 1$ no relation between b and a or c is needed. Let us rewrite (1.2) as

$$\begin{gathered} Lu = \lambda(a(x,t)u^p + b(x,t)u) \quad \text{in } \Omega\times\mathbb{R} \\ u = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \\ u \quad T\text{-periodic} \end{gathered} \tag{3.14}$$

We recall that P_Ω and Λ are given by (2.1) and (1.3) respectively. We have

Corollary 3.5. *Let* $a, b \in L^\infty_T$ *such that* $0 \le a \not\equiv 0$*, and let* $1 < p < (N+2)/(N+1)$*. Then* (3.14) *has a solution* $v_\lambda \in P^\circ$ *for all* $\lambda \in (0, \Lambda)$*. Moreover,* $\Lambda = \lambda_1(b)$ *if* $P_\Omega(b) > 0$ *and* $\Lambda = \infty$ *if* $P_\Omega(b) \le 0$*.*

Proof. Let us note first that since (3.14) can be written as $(L + \lambda b^-)u = \lambda(au^p + b^+u)$, arguing as in the first part of the proof of Theorem 3.3 (ii) we get some positive subsolution $k\Phi$ of (3.14) for any $\lambda > 0$.

On the other hand, let μ_b be defined as in Remark 2.2. Then there exists some $u \in P^\circ$ satisfying $Lu = \lambda bu + \mu_b(\lambda)u$ in $\Omega\times\mathbb{R}$, $u = 0$ on $\partial\Omega\times\mathbb{R}$. Furthermore, by the results listed in Remarks 2.1 and 2.2 it holds that $\mu_b(\lambda) > 0$ for all $\lambda \in (0, \lambda_1(b))$ if $P_\Omega(b) > 0$ and $\mu_b(\lambda) > 0$ for every $\lambda > 0$ if $P_\Omega(b) \le 0$. Taking this into account, it is easy to check that for such λ's u is a supersolution of (3.14) if one takes $\|u\|_\infty$ sufficiently small (in fact, it suffices $\|u\| \le (\mu_b(\lambda)/(\lambda\|a\|))^{1/(p-1)}$). Hence Lemma 3.2 applies and gives a solution for all $\lambda < \lambda_1(b)$ if $P_\Omega(b) > 0$ and for every $\lambda > 0$ if $P_\Omega(b) \le 0$. That is, $\Lambda \ge \lambda_1(b)$ in the first case and $\Lambda = \infty$ in the second one. To end the proof we observe that (3.14) implies $Lu \ge \lambda bu$ and so Remark 2.2 (ii) says that $\Lambda \le \lambda_1(b)$ when $P_\Omega(b) > 0$. $\square$

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