A GROUP-THEORETIC CHARACTERIZATION

OF *M*-GROUPS

THESIS

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ABSTRACT

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I. Martin Isaacs, among others, has posed the problem of finding a purely group-theoretic characterization of M-groups which traditionally are defined via character theory. This thesis seeks to understand and answer Isaacs' question by finding a characterization of M-groups in purely group-theoretic terms. Such a characterization based on cyclic sections of the group G and the irreducible, monomial characters that proceed naturally from them does exist, and was first described by Alan Parks. He describes M-groups based on the notion of good pairs and an equivalence relation on them. If m_G is the number of classes of good pairs and n_G is the number of rational conjugacy classes for a group G, then $m_G = n_G$ if and only if G is an M-group.

In order to fully understand this group-theoretic characterization of M-groups, the traditional way of defining M-groups by induced characters had to be explored. First, fundamentals of representation and character theory (including irreducible and induced representations and characters, inner products of characters, the Mackey Theorems and the calculation of character tables) were researched. Next, M-groups themselves were studied, including many specific examples of well-known groups. At this point it was

possible to understand the traditional definition of an M-group as a group in which every irreducible character is induced from a linear character of some subgroup of the group. Before attempting to study Parks' group theoretic characterization, some background preparation in field theory and Galois theory was done. Finally Parks' article was studied and his characterization was validated.

The conclusion is that there is a verifiable group-theoretic characterization of M-groups. This alternative definition of M-groups serves to add to the body of knowledge about M-groups and how they behave.

CHAPTER 1

INTRODUCTION

This thesis is introductory research on M-groups and the two different ways of defining them. Traditionally, M-groups have been defined in terms of characters, but a new characterization using only concepts from group theory has been developed by Alan Parks [7]. This paper seeks to understand the group-theoretic definition of M-groups and to verify that it is indeed a valid characterization.

There are two parts to this thesis. The first part includes Chapters 2, 3 and 4 which discuss background information on representations, characters, M-groups, field theory and Galois theory. Part two, which consists of Chapter 5, develops the proof of Alan Parks' definition of an M-group using purely group-theoretic concepts.

Specifically, Chapter 2 is devoted to the basics of representation theory and includes the topics of equivalent, irreducible and induced representations. Further, Maschke's theorem is included along with a proof describing a method for counting the number of linear representations of any finite group.

Chapter 3 is a summary of the essentials of character theory whose highlights include inner products of characters, the Orthogonality Relations and their corollaries, calculations for the character tables of various groups, induced and conjugate characters, and the Mackey theorems

The fourth chapter introduces M-groups from the point of view of characters and gives several examples of them. Then basic facts from field and Galois theory such as the cyclotomic polynomial, primitive roots of unity, Galois groups, and irreducible polynomials are introduced. Finally Chapter 4 closes with an introduction to and a discussion of the Schur index.

We conclude with Chapter 5 and the group-theoretic definition of M-groups. First, the idea of good pairs is introduced, and then an equivalence relation on good pairs is given (and in fact proven to be a valid equivalence relation). Denoting the number of equivalency classes of good pairs as m_G and the number of rational conjugacy classes of the given group as n_G, we have $m_G = n_G$ if and only if G is an M-group. In order to prove this group-theoretic classification of M-groups is valid, Galois conjugacy classes of irreducible, complex-valued characters are studied and linked to the Schur index and the Berman-Witt theorem.

Presumably, those reading this thesis will have a basic understanding of group theory, linear algebra, and matrix theory as many results from these fields are not specifically cited but rather assumed. The reader will notice that some of the theorems in this thesis do not have a proof included (for example, Schur's theorem or Maschke's theorem). Whenever proofs are omitted, it is typically because the proof itself involves more advanced concepts of group theory, or that its proof would necessitate the proof of many lemmas and propositions which would serve to unnecessarily lengthen this thesis. Moreover, notation is introduced along the way and often references back to the original definitions are given in the text.

I have not attempted to trace the various theorems, propositions and definitions back to their original sources except where it was appropriate or necessary. For those that wish to explore in depth the background sources for this thesis, they are encouraged to review in detail the bibliography in conjunction with the following information. The foundation for Chapters 2 and 3 were Hill's book [3] and Grove's book [2]. The motivation for Chapter 4 included Isaacs' books on group theory [4] and character theory [5] and Grove's book [2]. Finally, Chapter 5 is based on Parks' article [7].

CHAPTER 2

REPRESENTATION THEORY

2.1 Introduction to Representations

An *F-representation of a group G* is defined as a homomorphism T from G to the general linear group over V for some finite dimensional vector space V and field F. A representation maps the given group into a set of invertible linear transformations over V. From linear algebra we know that a given linear transformation can be associated with a matrix. Given the basis $B = \{v_1, v_2, \ldots, v_n\}$ for V, if $v_i \cdot T(g) = \sum_{j=1}^{n} a_{ij} v_j$ for $a_{ij} \in F$, then the matrix associated with the linear transformation $T(g)$ is $A = [a_{ij}]$. In effect, a representation maps an arbitrary group to a set of matrices with entries from the field F where we can identify the group multiplication with matrix multiplication. The dimension of V is also called the *degree* of the representation and is denoted $\deg(T)$. A one-dimensional representation is called *linear*, and a representation is called *faithful* if it is one-to-one. Note that a representation is a homomorphism, so it preserves powers, inverses, and the identity. Because of this fact, it suffices to define a representation only on a set of generators of the group.

Consider a few examples of representations. Define the Klein-4 group as $K = \{1, a, b, ab\}$ (unless otherwise specified, the group K will always refer to the Klein-4 group). It has the following as a representation, $T : K \longrightarrow GL(2, \mathbb{C})$ defined as

$$T(a) = T(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Observe that $T(ab) = T(a)T(b) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so T is a representation, but T is not faithful.

As a second example consider the symmetric group on three elements, denoted $S_3 = \{1, r, r^2, c, rc, r^2c\}$ where $r = (123)$, $c = (23)$, and clearly $cr = r^2c$. We want to find all linear representations of S_3 over $\mathbb{C}$. Since S_3 is generated by r and c, then we simply define the representations on those elements. Because r has order 3 and c has order 2, then for T to be a representation the matrix $T(r)$ must have order 3, and the matrix $T(c)$ must have order 2. This forces $T(r)$ and $T(c)$ to be third and second roots of unity respectively. Thus $T(r) = [1], T(r) = [\frac{-1+i\sqrt{3}}{2}]$, or $T(r) = [\frac{-1-i\sqrt{3}}{2}]$, and $T(c) = [1]$ or $T(c) = [-1]$. There are 3 choices for $T(r)$ and two choices for $T(c)$, so there are at most 6 one-dimensional representations for S_3. Will all of the possibilities work?

Suppose $T(r) = [\frac{-1+i\sqrt{3}}{2}]$, then $T(r^2) = [\frac{-1-i\sqrt{3}}{2}]$. Since $cr = r^2c$ then $T(c)T(r) = T(r^2)T(c)$. Hence $[\frac{-1+i\sqrt{3}}{2}] = [\frac{-1-i\sqrt{3}}{2}]$ which is a contradiction. We conclude that $T(r) \neq [\frac{-1+i\sqrt{3}}{2}]$ and, similarly, $T(r) \neq [\frac{-1-i\sqrt{3}}{2}]$. The only choice is for $T(r) = [1]$. Thus there are only two linear representations of S_3; T_1 where $T_1(c) = T_1(r) = [1]$, and T_2 where $T_2(c) = [-1]$ and $T_2(r) = [1]$. Observe that $[T_2(c)]^2 = [1] = [T_2(r)]^3$, and $T_2(r^2c) = T_2(r^2)T_2(c) = (T_2(r))^2T_2(c) = [1]^2[-1] = [-1] = [-1][1] = T_2(c)T_2(r) = T_2(cr)$. Clearly T_2 is a valid representation of S_3. Note that T_1 maps all elements in S_3 to $[1]$ - this is called the trivial representation. In fact, the *trivial representation* of degree n is defined as the representation that maps all elements in the group to the $n \times n$ identity matrix.

2.2 Equivalent Representations

How can one determine if two representations are, for all intents and purposes, the same? This question introduces the concept of equivalent representations. Let T and U be n-dimensional representations of a group G. Then we say that T and U are *equivalent* if there exists an invertible matrix A such that

$$A \cdot T(g) = U(g) \cdot A \text{ for all } g \in G.$$

Equivalently, $T(g) = A^{-1}U(g)A$ for all $g \in G$. In other words, equivalent representations correspond to a change of basis for the vector space. When two representations T and U are equivalent we write $T \sim U$.

As an example, consider the cyclic group of order 4 generated by the element g and the following representations, $V(g) = [i], U(g) = [-i]$, and $T(g) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We will show that T is equivalent to the representation given by $W(g) = \begin{bmatrix} U(g) & 0 \\ 0 & V(g) \end{bmatrix}$. Pick $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ (note A is invertible) and observe that $A \cdot T(g) = \begin{bmatrix} -i & 1 \\ -1 & i \end{bmatrix} = W(g) \cdot A$. Since this equality holds for the generator, then it holds for all elements of G. We have shown that $T \sim W$.

2.3 The Right Regular Representation

A very important representation is the right regular representation. Let G be a finite group of order n and consider the action of a fixed element on the set G by right multiplication, which is simply a permutation of the elements of G. The *right regular representation* of a finite group is defined as follows. Let $G = \{x_1, x_2, \ldots, x_n\}$. For each x_i, where $1 \leq i \leq n$, define a bijection on G, $f_{x_i} : x_j \mapsto x_k$ when $x_j x_i = x_k$. Let

$$\begin{aligned} T(x_i) &= [a_{jk}]_{1 \leq j,k \leq n} \text{ where} \\ a_{jk} &= \begin{cases} 1 & \text{if } f_{x_i}(x_j) = x_k \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Because right multiplication is a permutation, it follows that $T(x_i)$ is a permutation matrix.

What is the right regular representation for the group S_3? Since S_3 is generated by r and c, then we only need to find matrices for these elements. First denote $1, r, r^2, c, rc, r^2c$ as $x_1, x_2, \ldots, x_6$ respectively. Now consider the following,

$$\begin{aligned} f_r(x_1) = f_r(1) &= 1 \cdot r = r = x_2 \\ f_r(x_2) = f_r(r) &= r \cdot r = r^2 = x_3 \end{aligned}$$

$$\begin{aligned} f_r(x_3) = f_r(r^2) &= r^2 \cdot r = 1 = x_1 \\ f_r(x_4) = f_r(c) &= c \cdot r = r^2 c = x_6 \\ f_r(x_5) = f_r(rc) &= rc \cdot r = c = x_4 \\ f_r(x_6) = f_r(r^2 c) &= r^2 c \cdot r = rc = x_5. \end{aligned}$$

The first entry above means that in row 1 and column 2 the entry is one. Likewise, the second entry means that in row 2, column 3, the entry is one. If T is the right regular representation of S_3, then we have

$$T(r) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Similarly, using right multiplication by the element c we can find that

$$T(c) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

2.4 Irreducible Representations

If T is a representation of a group G on the vector space V, then a subspace U of V is defined to be T-*invariant* if for all $g \in G$, $UT(g) \subseteq U$. Whenever a proper, non-zero, T-invariant subspace U exists we call the representation T, *reducible.* Since U is T-invariant then T can be restricted to U creating a representation of G on the subspace U. For a suitably chosen basis of V we can write T in the form,

$$\begin{bmatrix} A(g) & C(g) \\ 0 & B(g) \end{bmatrix}.$$

where A and B are representations of G of smaller degree than T, $C(g)$ is some matrix that depends on the element g, and 0 is a block of zeros of appropriate dimension so that the matrix is square. Observe that for $g, h \in G$,

$$\begin{aligned} T(g)T(h) &= \begin{bmatrix} A(g) & C(g) \\ 0 & B(g) \end{bmatrix} \cdot \begin{bmatrix} A(h) & C(h) \\ 0 & B(h) \end{bmatrix} \\ &= \begin{bmatrix} A(g)A(h) & A(g)C(h) + C(g)B(h) \\ 0 & B(g)B(h) \end{bmatrix} \\ &= \begin{bmatrix} A(gh) & A(g)C(h) + C(g)B(h) \\ 0 & B(gh) \end{bmatrix} \\ &= T(gh). \end{aligned}$$

Note that A is the representation of T restricted to U described above. If T cannot be written in this form, then we say that T is *irreducible.*

A representation T of a group G is *completely reducible* if for all $g \in G$, $T(g)$ is equivalent to a matrix of the form

$$\begin{bmatrix} A_1(g) & & & 0 \\ & A_2(g) & & \\ & & \ddots & \\ 0 & & & A_k(g) \end{bmatrix}$$

where each A_i is an irreducible representation of G for $i = 1, \ldots, k$ and 0 is a block of zeros of appropriate dimension to yield a square matrix. Note that a representation can be reducible but not completely reducible as evidenced by the following example.

Consider the infinite cyclic group generated by element g and the $\mathbb{R}$-representation $T(g) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It is easy to see that $T(g^n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ using induction. Let $U = \{(0, b) | b \in \mathbb{R}\}$ and observe that U is invariant under T. T is a reducible representation, but is it completely reducible? If it is, then because T has degree 2 there exist two one-dimensional representations $A(g)$ and $B(g)$ of the infinite, cyclic group G such that T is equivalent to $T^*(g) = \begin{bmatrix} A(g) & 0 \\ 0 & B(g) \end{bmatrix}$. This means there exists an invertible matrix

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and representations $A(g)$ and $B(g)$ such that

$$M \cdot T(g) = \begin{bmatrix} A(g) & 0 \\ 0 & B(g) \end{bmatrix} \cdot M.$$

Define $A(g) = [\gamma]$ and $B(g) = [\beta]$ where $\gamma, \beta \in \mathbb{R}$. So if T is equivalent to T^* then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & \beta \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} \gamma a & \gamma b \\ \beta c & \beta d \end{bmatrix}.$$

Now $a = \gamma \cdot a$ implies that $a = 0$ or $\gamma = 1$. If $\gamma = 1$, then $a+b = b$ and $a = 0$. So we have $a = 0$ in any case. Since $\det(M) \neq 0$, neither b nor c can be zero. However, $a + b = \gamma b$ which implies that $b = \gamma b$. Now either $\gamma = 1$ or $b = 0$, and since $b \neq 0$ then $\gamma = 1$. At this point we have $a = 0$ and $\gamma = 1$. But note that $c = \beta c$ so either $\beta = 1$ or $c = 0$. Since $c \neq 0$, then $\beta = 1$. The fact that $\beta = 1$ and $c + d = \beta \cdot d$ implies $c + d = d$. So $c = 0$; this is a contradiction. Hence no such matrix M or representations $A(g)$ and $B(g)$ exist. We conclude that the representation T is reducible but not completely reducible.

2.5 Maschke's Theorem

The field F of scalars is a determining factor for the question of complete reducibility for a representation. In fact, Maschke's Theorem deals with this idea.

Theorem 2.1 (Maschke's Theorem): *Suppose that G is a finite group, F is a field and $\text{char}(F) \nmid |G|$. Then every F-representation of G is completely reducible. In particular, a representation over the field $\mathbb{C}$ (or $\mathbb{R}$) is completely reducible.*

The proof of this theorem uses induction and is included in full in Grove's book [2, p.99], but would take us too far afield in this paper.

As an illustration of this theorem, consider the cyclic group of order 3 generated by the element x and the representation $T(x) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Since $(T(x))^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then this is indeed a valid representation of degree 2. We want to show that T is irreducible over $\mathbb{R}$ but reducible over $\mathbb{C}$. If we assume T is reducible over $\mathbb{R}$, then by Maschke's Theorem it is completely reducible. So there exist $\mu, \lambda \in \mathbb{R}$ such that $T(x)$ is equivalent to $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ where λ and μ are one-dimensional representations of the group G. This means for some nonsingular matrix M,

$$M \cdot T(x) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \cdot M.$$

Recall that for a nonsingular matrix M, $\mathrm{Tr}(M^{-1}NM) = \mathrm{Tr}(N)$ and $\det(M^{-1}NM) = \det(N)$. Since

$$T(x) = M^{-1} \cdot \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \cdot M,$$

then $-1 = \lambda + \mu$ and $\lambda\mu = 1$. The possible solutions for λ are $\frac{-1 \pm i\sqrt{3}}{2}$ which are complex numbers. So T is not reducible over $\mathbb{R}$.

Without loss of generality let $\lambda = \frac{-1+i\sqrt{3}}{2}$, and then $\mu = \frac{-1-i\sqrt{3}}{2}$. Now we simply need to find an appropriate invertible matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfies

$$M \cdot T(x) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \cdot M.$$

Pick $M = \begin{bmatrix} 1 & \lambda \\ 1 & \mu \end{bmatrix}$ and observe that

$$\begin{aligned} M \cdot T(x) &= \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \cdot M \text{ means that} \\ \begin{bmatrix} \lambda & -1-\lambda \\ \mu & -1-\mu \end{bmatrix} &= \begin{bmatrix} \lambda & \lambda^2 \\ \mu & \mu^2 \end{bmatrix}. \end{aligned}$$

Since $\lambda^2 + \lambda + 1 = 0$ and $\mu^2 + \mu + 1 = 0$, then $\lambda^2 = -\lambda - 1$ and $\mu^2 = -\mu - 1$ so that the above equation is true. Further since $\mu = \lambda^2$ then $\det(M) = -i\sqrt{3} \neq 0$. Therefore M

is invertible, and we have found values λ, μ and the matrix M to prove that the given representation T is completely reducible over the complex numbers.

2.6 Finite Abelian Groups

Now consider linear representations of finite, abelian groups.

Proposition 2.2: *If G is cyclic of order k and $\zeta \in \mathbb{C}$ is a primitive kth root of 1, then $T_j(x) = [\zeta^j]$ for j=1,2,...,k is a complete list of all one-dimensional $\mathbb{C}$-representations of G.*

Proof: We know $(T_j(x))^k = [\zeta^j]^k = [\zeta^{jk}] = [\zeta^{kj}] = [\zeta^k]^j = [1]^j = [1]$. So the T_j for $j = 1, \ldots, k$ are indeed representations of G. If T is a linear representation such that $T(x) = [\mu]$ where $\mu \in \mathbb{C}$, then $\mu^k = 1$. Hence μ is a kth root of unity but not necessarily a primitive root of unity. Regardless, it follows that μ can be written as a power of ζ and so $T = T_j$ for some j. $\diamondsuit$

By the Fundamental Theorem of Finite Abelian Groups if G is a finite, abelian group, then G is isomorphic to the direct product of a finite number of cyclic groups of prime power order. In other words, we have $G \cong \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \ldots \times \mathbb{Z}_{q_n}$ where each q_j is a power of a prime.

Proposition 2.3: *Let S be the set of all linear, irreducible characters of $\mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_n}$ and let $T = \{\prod_{i=1}^{n} \lambda_i | \lambda_i$ is a linear, irreducible character of $\mathbb{Z}_{q_i}$ for $i = 1, \ldots, n\}$. Then $T = S$ and $|T| = q_1 \ldots q_n = |S|$.*

Proof: First we want to show $S \subseteq T$. Let $\lambda \in S$ and then for all $(x_1, \ldots, x_n) \in \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_n}$ define

$$\left(\prod_{i=1}^{n} \lambda_i\right)(x_1, \ldots, x_n) = \prod_{i=1}^{n} \lambda_i(x_i)$$

where $\lambda_i : \mathbb{Z}_{q_i} \longrightarrow \mathbb{C}^{\times}$ such that $\lambda_i(x) = \lambda(1, \ldots, 1, x, 1, \ldots, 1)$ where x is in the ith position. Let $y_i = (1, \ldots, 1, x_i, 1, \ldots, 1)$ where x_i is in the ith spot of the n-tuple.

First, is λ_i an irreducible character of $\mathbb{Z}_{q_i}$? We need to show each λ_i is a homomorphism that preserves powers on the generator of $\mathbb{Z}_{q_i}$. Let $x_i, z_i \in \mathbb{Z}_{q_i}$ and note that $\lambda_i(x_i z_i) = \lambda(1, \ldots, 1, x_i z_i, 1, \ldots, 1) = \lambda(1, \ldots, 1, x_i, 1, \ldots, 1)\lambda(1, \ldots, 1, z_i, 1, \ldots, 1)$ $= \lambda_i(x_i)\lambda_i(z_i)$. If $\langle g_i \rangle = \mathbb{Z}_{q_i}$ then $\lambda_i(1) = \lambda_i((g_i)^n) = \lambda(1, \ldots, 1, (g_i)^n, 1, \ldots, 1) =$ $\lambda(1, \ldots, 1) = 1$ where $n = q_i$. So $(\lambda_i(g_i))^n = 1$ and $(\lambda_i(g_i))$ is an nth root of unity. Therefore λ_i is an irreducible, linear representation of $\mathbb{Z}_{q_i}$.

Now observe that

$$\begin{aligned}
\lambda(x_1, \ldots, x_n) &= \lambda((x_1, 1, \ldots,)(1, x_2, 1, \ldots, 1) \ldots (1, \ldots, x_n)) \\
&= \lambda(y_1)\lambda(y_2) \ldots \lambda(y_n) \\
&= \lambda_1(x_1)\lambda_2(x_2) \ldots \lambda_n(x_n) \\
&= \Pi_{i=1}^n \lambda_i(x_i) \\
&= (\Pi_{i=1}^n \lambda_i)(x_1, \ldots, x_n).
\end{aligned}$$

Since λ and $\prod_{i=1}^n \lambda_i$ agree on all domain values, then $\lambda = \prod_{i=1}^n \lambda_i$ and $\lambda \in T$.

Let $\prod_{i=1}^n \lambda_i \in T$ where each λ_i is a linear, irreducible character of $\mathbb{Z}_{q_i}$. Is $\prod_{i=1}^n \lambda_i$ an irreducible, linear representation of $\mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_n}$? Since each λ_i preserves multiplication, then obviously $\prod_{i=1}^n \lambda_i$ does as well. Further $\prod_{i=1}^n \lambda_i$ is a map into $\mathbb{C}^\times$. Thus $\prod_{i=1}^n \lambda_i$ is a linear representation of $\mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_n}$ and therefore irreducible as well. Hence $\prod_{i=1}^n \lambda_i \in S$. We have shown altogether that $T = S$.

Finally suppose $\lambda, \mu \in S$ such that $\lambda = \prod_{i=1}^n \lambda_i$, $\mu = \prod_{i=1}^n \mu_i$ and $\lambda \neq \mu$. This means there exists $j \in \{1, \ldots, n\}$ and $x_j \in \mathbb{Z}_{q_j}$ such that $\lambda_j(x_j) \neq \mu_j(x_j)$ Observe that $\lambda(1, \ldots, 1, x_j, 1, \ldots, 1) = (\prod_{i=1}^n \lambda_i)(1, \ldots, 1, x_j, 1, \ldots, 1) = \lambda_j(x_j)$ since $\lambda_i(1) = 1$ for all i. Also note that $\mu(1, \ldots, 1, x_j, 1, \ldots, 1) = (\prod_{i=1}^n \mu_i)(1, \ldots, 1, x_j, 1, \ldots, 1) = \mu_j(x_j)$ since $\mu_i(1) = 1$ for all i. Since $\mu_j(x_j) \neq \lambda_j(x_j)$ then $\lambda \neq \mu$ because there is at least one value in $\mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_n}$ on which μ and λ do not agree. We know $\mathbb{Z}_{q_i}$ is a cyclic group of order q_i, so there are exactly q_i one-dimensional $\mathbb{C}$-representations by Proposition 2.2. Using this fact and the rule of product, we have that $|T| = q_1 \cdot q_2 \cdot \ldots \cdot q_n$. Since $T = S$ then $|S| = q_1 \cdot q_2 \cdot \ldots \cdot q_n$. $\diamondsuit$

So when G is a finite, abelian group, then $G \cong \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_n}$, and $|G| = q_1 \cdot \ldots \cdot q_n$. Further, the linear, irreducible characters of G can be identified with the set S defined in Proposition 2.3 which has cardinality $q_1 \cdot \ldots \cdot q_n$. Thus the number of linear, irreducible characters of any finite, abelian group G is $q_1 \cdot \ldots \cdot q_n = |G|$.

Proposition 2.4: *Let $N \trianglelefteq G$ and let T be any representation of the factor group G/N with degree n. We can define $\widehat{T}$ on G such that $\widehat{T}(g) = T(Ng)$. Then $\widehat{T}$ is a representation of G. Moreover, if V is a representation of G/N then $\widehat{T} \sim \widehat{V}$ if and only if $T \sim V$.*

Proof: Let $g, h \in G$. Then $\widehat{T}(gh) = T(Ngh) = T(Ng \cdot Nh) = T(Ng)T(Nh) = \widehat{T}(g)\widehat{T}(h)$ so $\widehat{T}$ is a representation.

We will prove that $\widehat{T} \sim \widehat{V}$ if and only if $T \sim V$. Suppose that $\widehat{T}$ is equivalent to $\widehat{V}$. Then there exists an invertible matrix A such that $A \cdot \widehat{T}(g) = \widehat{V}(g) \cdot A$ for all $g \in G$, which implies $A \cdot T(Ng) = V(Ng) \cdot A$ for all $Ng \in G/N$. We conclude that T and V are equivalent representations of G/N. Since all of the preceding statements are if and only if statements, then $\widehat{T} \sim \widehat{V}$ if and only if $T \sim V$. $\diamondsuit$

The *derived group of G* (or the commutator subgroup), denoted as G', is the subgroup generated by all commutators of G, that is, $G' = \langle [x, y] = x^{-1}y^{-1}xy | x, y \in G \rangle$. The following theorems are stated here without proof, but can be found in any basic text on group theory ([4, p.37]).

Theorem 2.5: *Let $N \trianglelefteq G$. Then G/N is abelian if and only if $G' \leq N$.*

Corollary 2.6: *Let $\varphi : G \longrightarrow A$ be a homomorphism where A is abelian. Then $G' \leq \ker(\varphi)$.*

These theorems and the previous propositions lead to the following statement.

Proposition 2.7: *There are exactly $[G : G']$ one-dimensional $\mathbb{C}$-representations of a* *finite group G.*

Proof: Apply Theorem 2.5 by letting $N = G'$, and we conclude G/G' is abelian. Since we are only considering finite groups then G/G' is finite and abelian. From Proposition 2.3 and the remarks following, the number of linear $\mathbb{C}$-representations of G/G' is $[G : G']$. Define S to be the set of all linear $\mathbb{C}$-representations of G/G' and R to be the set of all linear $\mathbb{C}$-representations of G. Since the set S has cardinality $[G : G']$, then if we can find a bijection from S to R we can conclude that R also has cardinality $[G : G']$.

Let $f : S \longrightarrow R$, such that $f(T) = \widehat{T}$ where $\widehat{T}$ is defined as in Proposition 2.4. Suppose $T_1, T_2 \in S$ and $f(T_1) = f(T_2)$. This implies that $\widehat{T_1}(g) = \widehat{T_2}(g)$ for all $g \in G$, and thus $T_1(G'g) = T_2(G'g)$ for all $G'g \in G/G'$. We conclude that $T_1 = T_2$ and the function f is injective.

Is f surjective? Let $U \in R$. Observe that for any $x, y \in G$ we have

$$U(x^{-1}y^{-1}xy) = U(x^{-1})U(y^{-1})U(x)U(y) = U(x^{-1})U(x)U(y^{-1})U(y).$$

Note that the order of multiplication can be switched since U is one-dimensional and is therefore a homomorphism into the set of complex numbers (not including 0) which is an abelian group. So, $U(x^{-1}xy^{-1}y) = U(1) = [1]$, and hence for all $g \in G'$, $U(g) = [1]$.

Now for each $U \in R$ define a map T on G/G' into $\mathbb{C}^{\times}$ such that $T(G'g) = U(g)$ for each $G'g \in G/G'$. T is a homomorphism because it is a map into $\mathbb{C}^{\times}$. Is the map T well-defined on the group G/G'; that is, if $g_1 \in G'g_2$, then does $T(G'g_1) = T(G'g_2)$? Since $g_1 \in G'g_2$ then $g_1 \cdot g_2^{-1} \in G'$. Thus $T(G'g_1g_2^{-1}) = U(g_1g_2^{-1}) = [1]$, and $T(G'g_1) \cdot T(G'g_2^{-1}) = [1]$. We conclude that $T(G'g_1) = T(G'g_2)$ and the map is well-defined on the group G/G'. So T is a valid linear representation of G/G' and is in the set S. Observe that $f(T) = U$ and we have shown f to be an onto function.

The function f from the set of all linear, complex-valued representations of G/G' to the set of all linear, complex-valued representations of G is a bijection so we conclude that the sets R and S have the same cardinality. $\diamondsuit$

As an application of Proposition 2.7, we can find all one-dimensional $\mathbb{C}$-representations of D_4, the group of symmetries of the square. Denote D_4 as the set $\{1, r, r^2, r^3, v, rv, r^2v, r^3v\} =$

$\langle r, v\rangle$ where $r^4 = 1 = v^2$, $vr = r^3v$, the element r is equivalent to a 90 degree rotation or the cycle (1234), and the element v is equivalent to the product of transpositions $(12)(34)$ or a reflection over a vertical line. By Proposition 2.7, there are $[D_4 : D_4']$ one-dimensional representations of D_4. Since D_4' is normal in D_4 then there are four nontrivial possibilities of subgroups for D_4', and it is easy to see that $D_4' = \langle r^2\rangle$. Obviously $|D_4'| = 2$ and $[D_4 : D_4'] = 4$. Further $D_4/D_4' = \{\{1, r^2\}, \{r, r^3\}, \{v, r^2v\}, \{rv, r^3v\}\}$. We know there are four linear representations of D_4, and to find those we simply map the elements in each element of the factor group of D_4/D_4' to the same matrix indicated by the representation of D_4/D_4'. Because D_4' is the identity of D_4/D_4' then all linear representations must map D_4' to $[1]$. Further, observe that every coset of D_4/D_4' is its own inverse so they can only be mapped to $[1]$ or $[-1]$. The following then are the one-dimensional representations defined on the generators of D_4/D_4'.

$$\begin{array}{cccc}
\underline{T_1} & \underline{T_2} & \underline{T_3} & \underline{T_4} \\
\{r, r^3\} \longrightarrow [1] & \{r, r^3\} \longrightarrow [1] & \{r, r^3\} \longrightarrow [-1] & \{r, r^3\} \longrightarrow [-1] \\
\{v, r^2v\} \longrightarrow [1] & \{v, r^2v\} \longrightarrow [-1] & \{v, r^2v\} \longrightarrow [-1] & \{v, r^2v\} \longrightarrow [1]
\end{array}$$

To find the one-dimensional representations of D_4 we simply extend the representations just defined. The four linear representations of D_4 are $\widehat{T_1}$ such that $\widehat{T_1}(r) = [1]$ and $\widehat{T_1}(v) = [1]; \widehat{T_2}$ such that $\widehat{T_2}(r) = [1]$ and $\widehat{T_2}(v) = [-1]; \widehat{T_3}$ such that $\widehat{T_3}(r) = [-1]$ and $\widehat{T_3}(v) = [-1]$; and $\widehat{T_4}$ such that $\widehat{T_4}(r) = [-1]$ and $\widehat{T_4}(v) = [1]$.

As an additional example, consider the group $G = \langle a, b\rangle$ where $a^5 = b^4 = 1$, $ba = a^2b$, and G has order 20. Its only nontrivial, normal subgroup is $\langle a\rangle$. We will determine all one-dimensional representations of G.

Obviously $ab \neq a^2b = ba$ so G is not abelian. If $G' = \{1\}$, then $G/G' = G$ and G would be abelian. So $G' \neq \{1\}$. Consider the factor group $G/\langle a\rangle$ which is of order 4. Hence $G/\langle a\rangle$ is abelian, and by Theorem 2.5, $G' \leq \langle a\rangle$. It is clear $G' \neq G$, so $G' = \langle a\rangle$. We can write the factor group G/G' as $\{G', G'b, G'b^2, G'b^3\}$ and G/G' is isomorphic to $\mathbb{Z}_4$. It follows that the four linear representations of G/G', and therefore for G as well, are the four linear representations of $\mathbb{Z}_4$. The four linear representations of G are listed below.

$\underline{T_1}$	$\underline{T_2}$	$\underline{T_3}$	$\underline{T_4}$
$T_1(a) = [1]$	$T_2(a) = [1]$	$T_3(a) = [1]$	$T_4(a) = [1]$
$T_1(b) = [1]$	$T_2(b) = [-1]$	$T_3(b) = [i]$	$T_4(b) = [-i]$

2.7 Induced Representations

Let H be a subgroup of a finite group G, and let T be a $\mathbb{C}$-representation of degree m of the subgroup H. Define $\dot{T}$ on G such that

$$\dot{T}(g) = \begin{cases} T(g) & \text{if } g \in H \\ 0_m & \text{if } g \notin H \end{cases}$$

where 0_m is an $m \times m$ block of 0's.

Suppose also that $[G : H] = n$ and $g_1, g_2, \ldots, g_n$ are left coset representatives of H where $g_1 = 1$. Define $T^G(g) = [\dot{T}(g_i^{-1} g g_j)]_{i,j=1,\ldots,n}$. Observe that the dimension of the induced matrix T^G is nm since it is constructed of n rows (and columns) of $m \times m$ submatrices. Is T^G a representation? Before considering this question we present the following lemma.

Lemma 2.8: *If $g_1, \ldots, g_n$ are left coset representatives of H then $g_1^{-1}, \ldots, g_n^{-1}$ are right coset representatives of H.*

Proof: We want to show $G = \bigcup_{i=1}^{n} Hg_i^{-1}$, and if $i \neq j$ then $Hg_i^{-1} \cap Hg_j^{-1} = \emptyset$. Let $g \in G$, so there exists g_l (a left coset representative) such that $g^{-1} \in g_l H$. This implies the existence of $h_1 \in H$ such that $g^{-1} = g_l \cdot h_1$. Therefore, $g = (g^{-1})^{-1} = h_1^{-1} g_l^{-1}$, and $g \in Hg_l^{-1}$. So $G \subseteq \bigcup_{i=1}^{n} Hg_i^{-1}$ Further $\bigcup_{i=1}^{n} Hg_i^{-1} \subseteq G$ trivially.

Now suppose $i \neq j$ and $Hg_i^{-1} \cap Hg_j^{-1} \neq \emptyset$. Then there exists x such that $x = h_1 g_i^{-1}$ and $x = h_2 g_j^{-1}$. Thus $h_1 g_i^{-1} = h_2 g_j^{-1}$ which implies that $g_j = g_i \cdot h_1^{-1} \cdot h_2$. We conclude that $g_j \in g_i H$ which is a contradiction because $g_i H \cap g_j H = \emptyset$ when $i \neq j$. Thus our assumption is wrong, and when $i \neq j$ then $Hg_i^{-1} \cap Hg_j^{-1} = \emptyset$. $\diamondsuit$

Theorem 2.9: *The map T^G is a $\mathbb{C}$-representation of the group G.*

Proof: To show that T^G preserves multiplication let $g, h \in G$ and we must prove that $T^G(g) \cdot T^G(h) = T^G(gh)$. First,

$$T^G(g) \cdot T^G(h) = \left[\sum_{k=1}^{n}(\dot{T}(g_i^{-1}gg_k) \cdot \dot{T}(g_k^{-1}hg_j))\right]_{i,j}$$

Since $T^G(gh) = [\dot{T}(g_i^{-1}ghg_j)]_{i,j}$, what we really want to show is

$$\dot{T}(g_i^{-1}ghg_j) = \sum_{k=1}^{n}(\dot{T}(g_i^{-1}gg_k) \cdot \dot{T}(g_k^{-1}hg_j))$$

where i and j are fixed. There are two cases to consider, either the i,jth block of $T^G(gh)$ is a block of zeros or not.

Case I: Suppose the i,jth block of the matrix $T^G(gh)$ is not the matrix 0_{mxm}. This means $\dot{T}(g_i^{-1}ghg_j) = T(g_i^{-1}ghg_j)$ which implies $g_i^{-1}ghg_j \in H$. Let g_i be a fixed left coset representative. By Lemma 2.8 there exists a unique k such that $g_i^{-1}g \in Hg_k^{-1}$. This implies $g_i^{-1}gg_k \in H$. Since $(g_i^{-1}gg_k)(g_k^{-1}hg_j) = g_i^{-1}ghg_j \in H$, then $g_k^{-1}hg_j = (g_i^{-1}gg_k)^{-1}(g_i^{-1}ghg_j) \in H$. Therefore

$$\begin{aligned}\dot{T}(g_i^{-1}gg_k) &= T(g_i^{-1}gg_k) \neq 0_{mxm} \text{ and}\\ \dot{T}(g_k^{-1}hg_j) &= T(g_k^{-1}hg_j) \neq 0_{mxm}.\end{aligned}$$

Hence, the i,jth block of the matrix product $T^G(g) \cdot T^G(h)$ equals

$$\sum_{k=1}^{n}(\dot{T}(g_i^{-1}gg_k) \cdot \dot{T}(g_k^{-1}hg_j)) = 0 + \ldots + 0 + T(g_i^{-1}gg_k) \cdot T(g_k^{-1}hg_j) + 0 + \ldots + 0$$

(Because of the uniqueness of k all other summands will be 0). Since T is a representation and preserves multiplication, then the i,jth block of $T^G(g) \cdot T^G(h)$ is $T(g_i^{-1}ghg_j)$. But this is exactly the i,jth block of $T^G(gh)$.

Case II: Suppose the i,jth block of the matrix $T^G(gh)$ is 0_{mxm}
This means $\dot{T}(g_i^{-1}ghg_j) = 0_{m\times m}$ which implies $g_i^{-1}ghg_j \notin H$. We will show that the i,jth block of the matrix product $T^G(g) \cdot T^G(h) = [\sum_{k=1}^{n}(\dot{T}(g_i^{-1}gg_k) \cdot \dot{T}(g_k^{-1}hg_j))]_{i,j}$ equals 0_{mxm}.

To see this, for all $k = 1, 2, \ldots, n$ we will show $\dot{T}(g_i^{-1}gg_k)\cdot\dot{T}(g_k^{-1}hg_j) = 0_{m\times m}$. So suppose not. Suppose there exists k such that $\dot{T}(g_i^{-1}gg_k) \cdot \dot{T}(g_k^{-1}hg_j) \neq 0_{m\times m}$. It follows that $\dot{T}(g_i^{-1}gg_k) \neq 0_{m\times m}$ and $\dot{T}(g_k^{-1}hg_j) \neq 0_{m\times m}$; by the definition of $\dot{T}$, $g_i^{-1}gg_k \in H$ and $g_k^{-1}hg_j \in H$. We conclude that $(g_i^{-1}gg_k)(g_k^{-1}hg_j) = g_i^{-1}ghg_j \in H$. Thus the i, jth block of $T^G(gh)$ is not zero, and this contradicts our assumption. We conclude that for each $k = 1, 2, \ldots, n$, $\dot{T}(g_i^{-1}gg_k)\cdot\dot{T}(g_k^{-1}hg_j) = 0_{m\times m}$. Therefore, the i, jth block of $T^G(g)\cdot T^G(h)$ equals $0_{m\times m}$.

Altogether we have shown that the induced representation T^G does preserve multiplication.

Additionally we must show that $T^G(g)$ is invertible for each $g \in G$ in order to complete the proof that $T^G(g)$ is a representation.

Recall that the induced representation $T^G(g)$ has n rows and n columns each containing a matrix "block" that has dimensions $m \times m$. The i, jth block of $T^G(g)$ is $\dot{T}(g_i^{-1}gg_j)$ where g_i, g_j are left coset representatives of H with $1 \leq i, j \leq n$. We have also defined

$$\dot{T}(x) = \begin{cases} T(x) & \text{if } x \in H \\ 0_{mxm} & \text{if } x \notin H \end{cases}$$

By Lemma 2.8, for each g_i there exists a unique k such that $g_i^{-1}g \in Hg_k^{-1}$ which implies $g_i^{-1}gg_k \in H$. This means there is only one k such that $\dot{T}(g_i^{-1}gg_k) = T(g_i^{-1}gg_k)$ where i is fixed. So in the ith row of the induced representation as we move across the different columns there is only one nonzero column entry. In other words, $g_i^{-1}gg_j \in H$ if and only if $g_j = g_k$. Moreover, we know for each g_j there exists a unique l such that $g_l^{-1}gg_j \in H$. When j is fixed there exists one g_l such that $\dot{T}(g_l^{-1}gg_j) = T(g_l^{-1}gg_j)$. In other words, in the jth column of the induced representation as we move down the rows there is only one nonzero row entry. So $T^G(g)$ looks somewhat like a permutation matrix, but instead of having ones as entries it has the $m \times m$ matrix $T(g_i^{-1}gg_j)$.

Define a matrix $C(g)$ blockwise such that $[C(g)]_{i,j}$ is the i, jth block where

$$[C(g)]_{i,j} \quad = \quad \begin{cases} (T(g_j^{-1}gg_i))^{-1} & \text{if } g_j^{-1}gg_i \in H \\ 0_{m\times m} & \text{if } g_j^{-1}gg_i \notin H \end{cases}$$

We are assured the existence of $(T(g_j^{-1}gg_i))^{-1}$ since T was initially defined as a representation. All that remains is to show $T^G(g)C(g) = I_{mn} = C(g)T^G(g)$, and it follows that

$T^G(g)$ is invertible. Only the first half of this equation will be verified since a similar proof follows for the second half of the equation.

Denote $T^G C = A = [B_{i,j}]_{1 \leq i,j \leq n}$ where $B_{i,j}$ is the $m \times m$ matrix located in the i,jth block of the product matrix A. So the i,jth block of $T^G(g) \cdot C(g)$ equals $B_{i,j} = \sum_{k=1}^{n} \dot{T}(g_i^{-1} g g_k)(C(g))_{k,j}$. (Recall that $(C(g))_{k,j} = (\dot{T}(g_j^{-1} g g_k))^{-1}$ or $0_{m \times m}$ depending on whether $g_j^{-1} g g_k \in H$ or not). Now A is the identity matrix only if $B_{i,j} = I_{m \times m}$ when $i = j$ and $B_{i,j} = 0_{m \times m}$ when $i \neq j$.

Suppose $i = j$ and then $B_{i,i} = \sum_{k=1}^{n} \dot{T}(g_i^{-1} g g_k)(C(g))_{k,i}$. By Lemma 2.8 we know there exists exactly one left coset representative g_p such that $g_i^{-1} g g_p \in H$. Hence $B_{i,i} = T(g_i^{-1} g g_p) \cdot (T(g_i^{-1} g g_p))^{-1} = I_{m \times m}$.

Now suppose $i \neq j$ and we want to show $B_{i,j} = 0_{m \times m}$. Well suppose not, suppose $B_{i,j} = \sum_{k=1}^{n} \dot{T}(g_i^{-1} g g_k)(C(g))_{k,j} \neq 0_{m \times m}$. Then there exists k_1 such that $g_i^{-1} g g_{k_1} \in H$ and $g_j^{-1} g g_{k_1} \in H$. This implies $g g_{k_1} \in g_i H$ and $g g_{k_1} \in g_j H$. Since $i \neq j$ by assumption, then g_i and g_j are distinct left coset representatives and the element $g g_{k_1}$ cannot be in both $g_i H$ and $g_j H$ at the same time. We have the desired contradiction and conclude that $B_{i,j} = 0_{m \times m}$ when $i \neq j$.

Therefore, $T^G(g)$ is an invertible homomorphism for all $g \in G$, and we have shown that the induced representation is, indeed, a representation. $\diamondsuit$

As an example consider the group S_3 and its normal subgroup $H = \langle r \rangle$. Since r has order 3, then the three one-dimensional representations of the subgroup H are simply the 3rd roots of unity; $T(r) = [1], U(r) = [\frac{-1-i\sqrt{3}}{2}]$, and $V(r) = [\frac{-1+i\sqrt{3}}{2}]$. Find the induced representations T^G and U^G (since calculation of the induced representation V^G is similar to the representation U^G, we will simply state that representation without proof). Recall that the elements r and c generate the entire group S_3, so we must find $T^G(r), T^G(c), U^G(r)$ and $U^G(c)$. The set of right coset representatives of H is $\{1, c\}$, so define g_1 and g_2 as 1 and c respectively The degree of T^G is $[G : H] \cdot \deg(T) = 2$ as are the degrees of U^G and V^G. The i,jth block of $T^G(r)$ is defined as $\dot{T}(g_i^{-1} r g_j)$ where $1 \leq i \leq 2$ and $1 \leq j \leq 2$. Consider the following,

$$[T^G(r)]_{1,1} = \dot{T}(g_1^{-1} r g_1) = \dot{T}(1 \cdot r \cdot 1) = \dot{T}(r) = T(r) = [1]$$

$$
\begin{aligned}
[T^G(r)]_{1,2} = \dot{T}(g_1^{-1} r g_2) &= \dot{T}(1 \cdot r \cdot c) = \dot{T}(rc) = [0] \\
[T^G(r)]_{2,1} = \dot{T}(g_2^{-1} r g_1) &= \dot{T}(c \cdot r \cdot 1) = \dot{T}(cr) = [0] \\
[T^G(r)]_{2,2} = \dot{T}(g_2^{-1} r g_2) &= \dot{T}(c \cdot r \cdot c) = \dot{T}(r^2) = T(r^2) = [1]
\end{aligned}
$$

In the first row of calculations, $[T^G(r)]_{1,1}$ is the submatrix in the first row and first column for the induced representation on the element r. So we have,

$$
T^G(r) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Similarly one can find that $[T^G(c)]_{1,1} = [0]$, $[T^G(c)]_{1,2} = [1]$, $[T^G(c)]_{2,1} = [1]$, and $[T^G(c)]_{2,2} = [0]$. It follows that

$$
T^G(c) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$

Now to find the induced representation U^G consider the following,

$$
\begin{aligned}
[U^G(r)]_{1,1} &= \dot{U}(r) = U(r) = \left[\frac{-1 - i\sqrt{3}}{2}\right] \\
[U^G(r)]_{1,2} &= \dot{U}(rc) = [0] \\
[U^G(r)]_{2,1} &= \dot{U}(cr) = [0] \\
[U^G(r)]_{2,2} &= \dot{U}(r^2) = U(r^2) = \left[\frac{-1 + i\sqrt{3}}{2}\right]
\end{aligned}
$$

After performing similar calculations for the element c, we find that the induced representation U^G is defined as

$$
U^G(r) = \begin{bmatrix} \frac{-1-i\sqrt{3}}{2} & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} \end{bmatrix} \quad \text{and} \quad U^G(c) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$

Additionally the representation induced from V follows,

$$
V^G(r) = \begin{bmatrix} \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix} \quad \text{and} \quad V^G(c) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
$$

The reader might note that the induced representation T^G is completely reducible, since the representation for the element c, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is equivalent to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

using the invertible matrix $M = \begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix}$. Also since $T^G(r) = I_{2\times 2}$ then obviously $M \cdot T^G(r) = T^G(r) \cdot M$. The representations V^G and U^G are both irreducible, which will become clear on page 35 of this paper.

CHAPTER 3

CHARACTER THEORY

3.1 Introduction to Characters

The *character* χ of an F-representation T is defined as a function from G to a field F such that $\chi(x) = (\mathrm{Tr} \circ T)(x) = \mathrm{Tr}(T(x))$ for all $x \in G$ where $\mathrm{Tr}(T(x))$ is the trace of the square matrix $T(x)$. An F-character is a character afforded by an F-representation, meaning that all entries in the matrix are from the field F. Observe that $\chi(1) = \mathrm{Tr}(I) = \deg(T)$. We call this integer the *degree* of χ and write $\deg(\chi)$.

As a simple example of a character consider the right regular representation T of a finite group G. Recall that for $x \in G$ the entries on the diagonal of $T(x)$, a_{ii}, are 1 only if $f_x(x_i) = x_i$, or if $(x_i) \cdot x = x_i$. This only occurs when x is the identity of G. So the character associated with this representation is $\chi(x) = 0$ when $x \neq 1$ and $\chi(x) = \deg(T) = |G|$ when $x = 1$.

Before considering some basic facts of character theory remember from linear algebra that if A is an invertible matrix of degree n and B is a matrix of degree n, then $\mathrm{Tr}(A^{-1}BA) = \mathrm{Tr}(B)$. In the following propositions the field F is an arbitrary field.

Proposition 3.1: *If χ is an F-character of a group G, then χ is a class function; that is, χ is constant on conjugacy classes.*

Proof: Let T be a representation of G, let $x, y \in G$, and suppose y is a conjugate of x. This means there exists $g \in G$ such that $y = g^{-1}xg$. So $\chi(y) = \chi(g^{-1}xg) = \mathrm{Tr}(T(g^{-1}xg)) = \mathrm{Tr}(T(g)^{-1}T(x)T(g)) = \mathrm{Tr}(T(x)) = \chi(x)$. $\diamondsuit$

Proposition 3.2: *If S and T are equivalent F-representations of G with characters χ and ψ respectively, then $\chi = \psi$.*

Proof: Since S and T are equivalent representations then there exists an invertible matrix M such that $S(g) = M^{-1}T(g)M$ for all $g \in G$, and it follows that $\chi(g) = \mathrm{Tr}(S(g)) = \mathrm{Tr}(M^{-1}T(g)M) = \mathrm{Tr}(T(g)) = \psi(g)$. $\diamondsuit$

The converse of Proposition 3.2 is also true, but its proof requires the use of theorems we have not yet discussed. Therefore its formal statement and proof are postponed until a later section (see Proposition 3.12).

We know a representation is reducible over a suitable basis if for all $g \in G$, $T(g)$ can be written in the following form,

$$\begin{bmatrix} A(g) & C(g) \\ 0 & B(g) \end{bmatrix}$$

where $A(g)$ and $B(g)$ are representations of the group and $C(g)$ is a matrix depending on the element g. Let θ and φ be the characters associated with representations A and B respectively. The character χ of the representation T is equivalent to the sum of the characters of representations A and B; that is, $\chi = \theta + \varphi$. It is important to note that the sum of two characters is still a character, but unlike representations, characters are not necessarily homomorphisms from the group G to the field F. As an example consider the cyclic group on two elements with g as the generator and the representation $U(g) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. If χ is the character afforded by U then $\chi(g^2) = 2 \neq 0 = (\chi(g))^2$. However, all linear characters are homomorphisms because they are essentially the same map as their associated one-dimensional representation. So every linear character is a homomorphism where each group element is mapped into the multiplicative group $F \setminus \{0\}$.

3.2 Characteristic Roots and Characters

One major result of linear algebra has to do with eigenvalues and eigenvectors and their relationship to characteristic polynomials. If P is a linear transformation from a $\mathbb{C}$-vector space V to itself, $v \in V$ is a nonzero vector, and there exists a scalar λ such that $P(v) = \lambda v$, then v is an *eigenvector* of P. The scalar λ is an *eigenvalue* of P associated with vector v. If A is the $n \times n$ matrix associated with the linear transformation P for a given basis, then we can refer to the eigenvalues and eigenvectors for A ([6, p.207]). Further, for any $n \times n$ matrix A we can define the *characteristic polynomial* of A to be the polynomial $\det(XI_n - A)$ ([6, p.211]).

From linear algebra we know since A is a $n \times n$ matrix then its characteristic polynomial is of degree n. Also we know that the matrix A has at most n distinct eigenvalues. A critical result of linear algebra shows that the eigenvalues of the matrix A associated with the linear transformation P are the roots of the characteristic polynomial of A ([6, p.212]). If we define $A = [a_{ij}]$, then finding the characteristic polynomial of A using expansion by minors yields the following polynomial, $\det(XI_n - A) = (x - a_{11})(x - a_{22}) \ldots (x - a_{nn}) +$ terms in x^{n-2} and below. Using the Binomial Theorem to expand the product we have $\det(XI_n - A) = x^n - \left(\sum_{i=1}^{n} a_{ii}\right) x^{n-1} +$ terms in x^{n-2} and below. However, we also know that the roots of $\det(XI_n - A)$ are the eigenvalues of A. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A so $\det(XI_n - A) = (x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_n)$. Once again if we use the Binomial Theorem to expand the right hand side, we have $\det(XI_n - A) = x^n - \left(\sum_{i=1}^{n} \lambda_i\right) x^{n-1} +$ terms in x^{n-2} and below. We conclude that $x^n - \left(\sum_{i=1}^{n} a_{ii}\right) x^{n-1} +$ terms in x^{n-2} $= \det(XI_n - A) = x^n - \left(\sum_{i=1}^{n} \lambda_i\right) x^{n-1} +$ terms in x^{n-2}, and $\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$. Hence the trace of a square matrix A equals the sum of its eigenvalues ([3, p.135-6]).

Before using this information to give us some results about characters we need to recall some facts about complex conjugates. If $x \in \mathbb{C}$ such that $x = a + bi$, then $\overline{x} \in \mathbb{C}$ is called the *complex conjugate* of x and is simply $a - bi$. The *absolute value* of x is denoted $|x|$ and equals $\sqrt{a^2 + b^2}$. Moreover we know $\overline{a+c} = \overline{a} + \overline{c}$, $\overline{ac} = \overline{ac}$, and $\overline{\overline{a}} = a$. The following easy lemmas are presented without proof.

Lemma 3.3: *If ζ is a kth root of unity, then $|\zeta| = 1$ and $\zeta^{-1} = \overline{\zeta}$.*

Lemma 3.4: *If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and k is a positive integer, then the eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$, and the eigenvalues of A^{-1} are $\lambda_1^{-1}, \dots, \lambda_n^{-1}$.*

Proposition 3.5: *Let T be a $\mathbb{C}$-representation of a finite group G affording the character χ with degree n. Then $|\chi(g)| \leq \chi(1)$ for all $g \in G$.*

Proof: Since T is a $\mathbb{C}$-representation then by Maschke it is completely reducible and $T \sim U$ where

$$U(g) = \begin{bmatrix} T_1(g) & & & 0 \\ & T_2(g) & & \\ & & \ddots & \\ 0 & & & T_k(g) \end{bmatrix}$$

and each $T_i(g)$ is irreducible. Suppose $\mathrm{o}(g) = m$, then $(U(g))^m = I_{n \times n}$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the eigenvalues for the matrix $U(g)$. We know that the trace of a square matrix is the sum of its eigenvalues, so define $\psi(g) = \mathrm{Tr}(U(g))$, and $\psi(g) = \sum_{i=1}^{n} \lambda_i$. If we take the absolute value of both sides we have

$$|\psi(g)| = \left| \sum_{i=1}^{n} \lambda_i \right| \leq \sum_{i=1}^{n} |\lambda_i| \text{ by the Triangle Inequality.}$$

Since $(U(g))^m = I_{n \times n}$ and $\{\lambda_1^m, \dots, \lambda_n^m\}$ are the eigenvalues of $(U(g))^m$, then the eigenvalues of $U(g)$ are mth roots of 1. Thus $|\lambda_i| = 1$ for all $i = 1, \dots, n$ by Lemma 3.3. By Proposition 3.2 since $T \sim U$ then $\psi = \chi$ and

$$|\chi(g)| = |\psi(g)| \leq \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} 1 = n = \deg(\chi) = \chi(1).$$

$\diamond$

Proposition 3.6: *Let χ be a $\mathbb{C}$-character of a group G. Then $\chi(x^{-1}) = \overline{\chi(x)}$ for all $x \in G$.*

Proof: Let T be a $\mathbb{C}$-representation of the group G of degree n and χ its associated character. Let k be the order of x so the matrix $T(x)$ has order k as well. Let

$\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $T(x)$, then by Lemma 3.4 $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$ are the eigenvalues of $(T(x))^{-1}$. But $T(x)^k = I_{n \times n}$ so all eigenvalues of $T(x)$ are kth roots of one. By Lemma 3.3, $\lambda_i^{-1} = \overline{\lambda_i}$ for all $i = 1, \ldots, n$. Observe that $(T(x))^{-1} = T(x^{-1})$ since T is a homomorphism. Thus $\chi(x^{-1}) = \mathrm{Tr}(T(x^{-1})) = \mathrm{Tr}((T(x))^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\sum_{i=1}^{n} \lambda_i} = \overline{\mathrm{Tr}(T(x))} = \overline{\chi(x)}$. $\diamondsuit$

3.3 Orthogonality Relations

The *inner product* of two class functions χ and φ from G to a field F is

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \varphi(g^{-1}).$$

The inner product is symmetric and bilinear, meaning, $\langle \chi, \varphi \rangle = \overline{\langle \varphi, \chi \rangle}$ (symmetry), $\langle a\chi + b\psi, \varphi \rangle = a\langle \chi, \varphi \rangle + b\langle \psi, \varphi \rangle$, and $\langle \chi, a\psi + b\varphi \rangle = \bar{a}\langle \chi, \psi \rangle + \bar{b}\langle \chi, \varphi \rangle$(bilinearity). These statements are easily verified. One can also calculate the inner product of two characters using the sizes of the conjugacy classes of G. If a finite group G has the distinct conjugacy classes $C_1, \ldots, C_n$ and each class, C_j, contains h_j elements then by Proposition 3.1, $\sum_{x \in C_i} \chi(x)\varphi(x^{-1}) = h_i \chi(g_i)\varphi(g_i^{-1})$ where g_i is a representative element of the conjugacy class C_i. When calculating the inner product using the above definition we can sum over the conjugacy classes of G instead of summing over all of G. Hence,

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{j=1}^{n} h_j \chi(g_j) \varphi(g_j^{-1})$$

and by Proposition 3.6

$$\langle \chi, \varphi \rangle = \frac{1}{|G|} \sum_{j=1}^{n} h_j \chi(g_j) \overline{\varphi(g_j)}$$

Another important number that often occurs in character theory is the *Kronecker delta* which is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Theorem 3.7 (Schur's Theorem): *Suppose S and T are inequivalent, irreducible representations. Suppose that $S(x) = [s_{ij}(x)]$ and $T(x) = [t_{ij}(x)]$ for all $x \in G$ then,*

1. $\sum_{x \in G} s_{ij}(x) t_{kl}(x^{-1}) = 0$ for all i, j, k, l

2. If T is irreducible and $\deg(T) = n$ then $n \cdot \sum_{x \in G} t_{ij}(x) t_{kl}(x^{-1}) = \delta_{il}\delta_{jk} \cdot |G|$ for all i, j, k, l.

The proof of this theorem will not be included in this text, but the interested reader can refer to [2, p.103].

Proposition 3.8: *Suppose G is a finite group having irreducible $\mathbb{C}$-characters, $\chi_1, \chi_2, \ldots, \chi_k$. Then for an arbitrary $\mathbb{C}$-character χ of G afforded by the representation T,*

$$\chi = \sum_{i=1}^{k} a_i \chi_i$$

where $a_i \in \mathbb{N} \cup \{0\}$.

Proof: Since we are working over the field $\mathbb{C}$, then by Maschke's Theorem the representation T is completely reducible. We can decompose T into irreducible representations, $T_1, T_2, \ldots, T_n$. Since every irreducible representation gives rise to an irreducible character, then let ψ_i be the irreducible character of T_i. Thus, $\chi(g) = \mathrm{Tr}(T(g)) = \sum_{i=1}^{n} \mathrm{Tr}(T_i(g)) = \sum_{i=1}^{n} \psi_i(g)$. Since each $\psi_i \in \{\chi_1, \ldots, \chi_k\}$ for each i, $\chi(g) = \sum_{i=1}^{n} \psi_i(g) = \sum_{i=1}^{k} a_i \chi_i(g)$ where $a_i \in \mathbb{N} \cup \{0\}$. $\diamondsuit$

Theorem 3.9 (First Orthogonality Relation): *Let $\chi_1, \ldots, \chi_k$ be all of the irreducible $\mathbb{C}$-characters of G. Then $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ for all i, j.*

Proof: Let χ_i and χ_j be distinct irreducible characters of G ($i \neq j$), let χ_i be afforded by the representation T, and let χ_j be afforded by the representation S where $T = [t_{ij}]$ and $S = [s_{ij}]$. Note that by the contrapositive of Proposition 3.2 since $\chi_i \neq \chi_j$, then T

and S are not equivalent representations. We have,

$$\begin{aligned}
\langle \chi_i, \chi_j \rangle &= \frac{1}{|G|}\sum_{x\in G}\chi_i(x)\chi_j(x^{-1}) \\
&= \frac{1}{|G|}\sum_{x\in G}\operatorname{Tr}(T(x))\operatorname{Tr}(S(x^{-1})) \\
&= \frac{1}{|G|}\sum_{x\in G}\left[\left(\sum_i t_{ii}(x)\right)\left(\sum_j s_{jj}(x^{-1})\right)\right] \\
&= \frac{1}{|G|}\sum_{x\in G}\left(\sum_i\sum_j (t_{ii}(x)s_{jj}(x^{-1}))\right) \\
&= \frac{1}{|G|}\sum_i\sum_j\left(\sum_{x\in G} t_{ii}(x)s_{jj}(x^{-1})\right).
\end{aligned}$$

By Schur's Theorem $\sum_{x\in G} t_{ii}(x)s_{jj}(x^{-1}) = 0$. This is true only because S and T are not equivalent representations. So $\langle \chi_i, \chi_j \rangle = \frac{1}{|G|}\sum_i\sum_j 0 = 0$.

Now suppose $i = j$ and consider $\langle \chi_i, \chi_i \rangle$.

$$\begin{aligned}
\langle \chi_i, \chi_i \rangle &= \frac{1}{|G|}\sum_i\sum_j\sum_{x\in G} t_{ii}(x)t_{jj}(x^{-1}) \\
&= \frac{1}{|G|}\sum_i\sum_j\left(\frac{\delta_{ij}\cdot\delta_{ij}\cdot|G|}{\deg(T)}\right) \text{ by Schur's Theorem,} \\
&= \frac{1}{\deg(T)}\sum_i\sum_j(\delta_{ij})^2 \text{ and since } (\delta_{ij})^2 = \delta_{ij} \\
&= \frac{1}{\deg(T)}\sum_i\sum_j \delta_{ij}
\end{aligned}$$

For a fixed i, $\sum_j \delta_{ij} = \delta_{jj} = 1$, thus

$$\begin{aligned}
\langle \chi_i, \chi_i \rangle &= \frac{1}{\deg(T)}\sum_i 1 \\
&= \frac{1}{\deg(T)}\cdot\deg(T) \\
&= 1.
\end{aligned}$$

We have shown that $\langle \chi_i, \chi_j \rangle = 0$ when $i \neq j$ and $\langle \chi_i, \chi_j \rangle = 1$ when $i = j$. Hence $\langle \chi_i, \chi_j \rangle = \delta_{ij}$. $\Diamond$

Corollary 3.10: *If χ is a $\mathbb{C}$-character of G then χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

Proof: If χ is an irreducible $\mathbb{C}$-character of G then by Theorem 3.9 $\langle \chi, \chi \rangle = 1$. Now suppose that χ is a $\mathbb{C}$-character of G and $\langle \chi, \chi \rangle = 1$. We know that χ can be written as a linear combination of the irreducible characters of G by Proposition 3.8, so $\chi = \sum_{i=1}^{n} a_i \chi_i$ where $\{\chi_1, \ldots, \chi_n\}$ are the irreducible characters of G. We then have

$$\begin{aligned}
1 &= \langle \chi, \chi \rangle \\
&= \left\langle \sum_{i=1}^{n} a_i \chi_i, \sum_{i=1}^{n} a_i \chi_i \right\rangle \\
&= \sum_{i=1}^{n} a_i^2 \langle \chi_i, \chi_i \rangle \text{ by the bilinearity property of inner products,} \\
&= \sum_{i=1}^{n} a_i^2 \cdot 1 \\
&= \sum_{i=1}^{n} a_i^2.
\end{aligned}$$

Since each a_i is a nonnegative integer then if $1 = \sum_{i=1}^{n} a_i^2$, exactly one $a_i = 1$ and all others are 0. Thus $\chi = \sum_{i=1}^{n} a_i \chi_i = 1 \cdot \chi_i$ and χ is an irreducible character of G. $\diamondsuit$

Corollary 3.11: *Let G be a finite group having irreducible $\mathbb{C}$-characters $\chi_1, \ldots, \chi_k$. Then for any $\mathbb{C}$-character χ of G, $\chi = \sum_{i=1}^{k} \langle \chi, \chi_i \rangle \chi_i$.*

Proof: From Proposition 3.8, $\chi = \sum_{i=1}^{k} a_i \chi_i$ where $a_i \in \mathbb{N} \cup \{0\}$. It follows that for a fixed irreducible $\mathbb{C}$-character χ_j,

$$\langle \chi, \chi_j \rangle = \left\langle \sum_{i=1}^{k} a_i \chi_i, \chi_j \right\rangle = \sum_{i=1}^{k} a_i \langle \chi_i, \chi_j \rangle = \sum_{i=1}^{k} a_i \delta_{ij} = a_j \in \mathbb{N} \cup \{0\}.$$

Thus for each irreducible $\mathbb{C}$-character χ_j we have $\langle \chi, \chi_j \rangle = a_j$, and we conclude that $\chi = \sum_{i=1}^{k} a_i \chi_i = \sum_{i=1}^{k} \langle \chi, \chi_i \rangle \chi_i$. $\diamondsuit$

This result along with Proposition 3.8 are nice because they allows us to *reduce* a character to the sum of irreducible characters that compose the given character. The irreducible characters χ_i where $\langle \chi, \chi_i \rangle > 0$ are called the *constituents* of χ The integers $a_i = \langle \chi, \chi_i \rangle$ are the *multiplicities* of the constituents. At this point, we need to introduce some new notation. The set of all irreducible $\mathbb{C}$-characters of a group G will be written as $\mathrm{Irr}(G)$. To identify a different field F, we write $\mathrm{Irr}_F(G)$ to indicate the set of all irreducible F-characters (Remember that an F-character is a character whose associated representation has entries from the field F).

Proposition 3.12: *Suppose S and T are $\mathbb{C}$-representations of G with μ and ψ as their respective characters. If $\mu(g) = \psi(g)$ for all $g \in G$, then $S \sim T$.*

Proof: Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ and let R_i be a representation with character χ_i for $i = 1, \ldots, k$. Since we are working over $\mathbb{C}$ then by Maschke's Theorem both S and T are completely reducible. This means that $T(g)$ is equivalent to

$$T^*(g) = \begin{bmatrix} T_1(g) & & & 0 \\ & T_2(g) & & \\ & & \ddots & \\ 0 & & & T_l(g) \end{bmatrix}$$

and $S(g)$ is equivalent to

$$S^*(g) = \begin{bmatrix} S_1(g) & & & 0 \\ & S_2(g) & & \\ & & \ddots & \\ 0 & & & S_m(g) \end{bmatrix}$$

for suitable $l, m \in \mathbb{N}$ and $T_i, S_j \in \{R_1, \ldots, R_k\}$ for all i, j. Thus $\mathrm{Tr}(T(g)) = \mathrm{Tr}(T^*(g)) = \sum_{i=1}^k m_i \mathrm{Tr}(R_i(g))$ where $m_i \in \mathbb{N} \cup \{0\}$ and $m_i R_i(g)$ denotes m_i copies of the representation $R_i(g)$. Likewise $\mathrm{Tr}(S(g)) = \mathrm{Tr}(S^*(g)) = \sum_{i=1}^k n_i \mathrm{Tr}(R_i(g))$ where $n_i \in \mathbb{N} \cup \{0\}$. Observe that $\mu(g) = \psi(g)$ implies $\mu(g) - \psi(g) = 0$. So

$$\begin{aligned} 0 &= \mathrm{Tr}(S(g)) - \mathrm{Tr}(T(g)) \\ &= \sum_{i=1}^k n_i \mathrm{Tr}(R_i(g)) - \sum_{i=1}^k m_i \mathrm{Tr}(R_i(g)) \\ &= \sum_{i=1}^k n_i \chi_i(g) - \sum_{i=1}^k m_i \chi_i(g) \\ &= \sum_{i=1}^k (n_i - m_i) \chi_i(g). \end{aligned}$$

Now consider the following inner product,

$$\begin{aligned} 0 &= \left\langle \sum_{i=1}^{k}(n_i - m_i)\chi_i, \chi_j \right\rangle \\ &= \sum_{i=1}^{k}(n_i - m_i)\langle \chi_i, \chi_j \rangle \text{ by bilinearity of inner products} \\ &= (n_j - m_j)\langle \chi_j, \chi_j \rangle \\ &= (n_j - m_j) \cdot 1. \end{aligned}$$

Therefore, $n_j = m_j$ for all $j = 1, \dots, k$ and thus $l = m$. So $\sum_{i=1}^{k} n_i R_i(g) = \sum_{i=1}^{k} m_i R_i(g)$, and both representations have the same number of copies of the irreducible representations R_i. Therefore we can find an appropriate permutation matrix M such that

$$\begin{bmatrix} T_1(g) & & & 0 \\ & T_2(g) & & \\ & & \ddots & \\ 0 & & & T_l(g) \end{bmatrix} = M^{-1} \cdot \begin{bmatrix} S_1(g) & & & 0 \\ & S_2(g) & & \\ & & \ddots & \\ 0 & & & S_m(g) \end{bmatrix} \cdot M$$

We conclude that S and T are equivalent representations since their decomposition into irreducible representations is exactly the same. $\diamondsuit$

Theorem 3.13: *If* $\{\chi_1, \dots, \chi_k\} = \mathrm{Irr}(G)$ *then*

$$\sum_{i=1}^{k} \chi_i(1)^2 = |G|.$$

Proof: Denote T as the right regular representation and ρ as the character afforded by T. In Section 3.1 we showed that ρ has a value of 0 for all non-identity elements and that $\rho(1) = \deg(T) = |G|$. By Corollary 3.11, $\rho = \sum_{i=1}^{k}\langle \rho, \chi_i \rangle \chi_i$. By the definition of inner product, $\langle \rho, \chi_i \rangle = \frac{1}{|G|}\sum_{g \in G} \rho(g)\chi_i(g^{-1}) = \frac{1}{|G|}\rho(1)\chi_i(1) = \frac{1}{|G|} \cdot |G| \cdot \chi_i(1) = \chi_i(1)$. Hence $\rho = \sum_{i=1}^{k}\langle \rho, \chi_i \rangle \chi_i = \sum_{i=1}^{k} \chi_i(1)\chi_i$. Since $|G| = \rho(1)$, then $|G| = \sum_{i=1}^{k} \chi_i(1)^2$. $\diamondsuit$

Theorem 3.14: *The number of irreducible representations of a finite group G over $\mathbb{C}$ is equal to the number of conjugacy classes of G.*

The proof of this theorem requires the use of more advanced group theory results and is therefore not included in this paper; however, Issacs provides a full proof of this theorem in his book [5, p.16]. This theorem allows us to make a nice observation about the characters of abelian groups. For any finite abelian group G we know that $G' = 1$ so the number of linear characters is $[G : G'] = |G|$ (by Proposition 2.7). From the previous theorem $|\mathrm{Irr}(G)|$ is the number of conjugacy classes of G. Since G is abelian, then each element is in its own conjugacy class and $|\mathrm{Irr}(G)| = |G|$. Thus for any abelian group every irreducible character is linear.

Proposition 3.15: *If χ is a $\mathbb{C}$-character of a finite group G then $g \in \ker(\chi)$ if and only if $\chi(1) = \chi(g)$.*

Proof: Let T be the representation of G affording χ and suppose $\deg(T) = n$. If $g \in \ker(\chi)$ then $\chi(g) = \mathrm{Tr}(I_{n\times n}) = \deg(T(g)) = \chi(1)$.

Now suppose $g \in G$ such that $\chi(1) = \chi(g)$. As argued in the proof of Proposition 3.5, if $\mathrm{o}(g) = m$ then the n eigenvalues of $T(g)$ are all mth roots of unity; that is, $|\lambda_i| = 1$ for all $i = 1, \ldots, n$. Since the trace of $T(g)$ is the sum of its eigenvalues, $\chi(g) = \mathrm{Tr}(T(g)) = \sum_{i=1}^{n} \lambda_i$. But $n = \mathrm{Tr}(I_{n\times n}) = \chi(1)$. So we have $n = \sum_{i=1}^{n} \lambda_i$ and taking the absolute value of both sides we have $n = |n| = |\sum_{i=1}^{n} \lambda_i| \leq \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} 1 = n$. This implies that $\lambda_i = 1$ for all $i = 1, \ldots, n$. We want to show that $T(g)$ is diagonalizable. Restrict the representation T to the cyclic group $\langle g \rangle$. By Maschke's Theorem, this is a completely reducible representation so the matrices of $T|_{\langle g \rangle}$ (over a suitable basis) have the form

$$\begin{bmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & \ddots & \\ 0 & & & D_k \end{bmatrix}$$

where each D_i is an irreducible representation of $\langle g \rangle$. From the remarks following Proposition 3.14 we know that every irreducible representation D_i of the cyclic (and therefore abelian) group $\langle g \rangle$ is linear. Since $T|_{\langle g \rangle}(g) = T(g)$, then we conclude $T(g)$ is diagonalizable. So there exists an invertible matrix $A \in \mathrm{GL}(n, \mathbb{C})$ such that $T(g)$ is similar to

$A^{-1}T(g)A$. Further when diagonalizing $T(g)$ its eigenvalues remain the same. So,

$$A^{-1}T(g)A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Since $\lambda_i = 1$ for all $i = 1, \ldots, n$ then $A^{-1}T(g)A = I_{n\times n}$. Thus $T(g) = AI_{n\times n}A^{-1} = I_{n\times n}$. Since $T(g) = I_{n\times n}$ then $g \in \ker(\chi)$. $\diamondsuit$

Proposition 3.16: *Let G be a finite group with $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$, let $1 \leq i, j \leq k$, let h_i be the size of the conjugacy class C_i, and let $g_i \in C_i$. Then*

$$\sum_{t=1}^{k} h_t \chi_i(g_t)\overline{\chi_j(g_t)} = |G|\delta_{ij}.$$

Proof: From Theorem 3.14 we know since there are k irreducible characters of G, then G has k conjugacy classes. By the definition of inner product we have $\frac{1}{|G|}\sum_{t=1}^{k} h_t\chi_i(g_t)\overline{\chi_j(g_t)} = \langle\chi_i, \chi_j\rangle$. The First Orthogonality Relation gives us that $\langle\chi_i, \chi_j\rangle = \delta_{ij}$. So we have $\frac{1}{|G|}\sum_{t=1}^{k} h_t\chi_i(g_t)\overline{\chi_j(g_t)} = \delta_{ij}$. It follows that $\sum_{t=1}^{k} h_t\chi_i(g_t)\overline{\chi_j(g_t)} = |G|\delta_{ij}$. $\diamondsuit$

Theorem 3.17 (Second Orthogonality Relation): *Let G be a finite group with irreducible characters $\chi_1, \ldots, \chi_k$, let $1 \leq i, j \leq k$, let h_i be the size of the conjugacy class C_i, and let $g_i \in C_i$. Then*

$$\sum_{t=1}^{k} \chi_t(g_i)\overline{\chi_t(g_j)} = \frac{\delta_{ij}|G|}{h_i}.$$

Proof: Since $|\mathrm{Irr}(G)| = k$ then the number of conjugacy classes of G is k (Theorem 3 14) Proposition 3.16 gives us

$$\frac{1}{|G|}\sum_{t=1}^{k} h_t\chi_i(g_t)\overline{\chi_j(g_t)} = \delta_{ij}.$$

Since i and j are fixed then we can switch them and obtain

$$\frac{1}{|G|}\sum_{t=1}^{k} h_t\overline{\chi_i(g_t)}\chi_j(g_t) = \delta_{ji}$$

which by Proposition 3.6 can be rewritten as $\frac{1}{|G|}\sum_{t=1}^{k} h_t\chi_i(g_t^{-1})\chi_j(g_t) = \delta_{ji}$. Define the following matrices $A = [a_{it}]$ where $a_{it} = \chi_t(g_i)$ and $B = [b_{tj}]$ where $b_{tj} = \frac{h_j}{|G|}\chi_t(g_j^{-1})$. Then we have that

$$\begin{aligned} BA &= \left[\sum_{t=1}^{k} b_{it}a_{tj}\right]_{ij} \\ &= \left[\sum_{t=1}^{k} \frac{h_t}{|G|}\chi_i(g_t^{-1})\chi_j(g_t)\right]_{ij} \\ &= [\delta_{ji}]_{ij} \\ &= I_n. \end{aligned}$$

Since B and A are square matrices such that $BA = I_n$ then $AB = I_n$. So

$$\begin{aligned} I_n &= [\delta_{ij}]_{ij} \\ &= AB \\ &= \left[\sum_{t=1}^{k} a_{it}b_{tj}\right]_{ij} \\ &= \left[\sum_{t=1}^{k} \chi_t(g_i)\frac{h_j}{|G|}\chi_t(g_j^{-1})\right]_{ij}, \text{ thus} \\ \delta_{ij} &= \sum_{t=1}^{k} \chi_t(g_i)\frac{h_j}{|G|}\chi_t(g_j^{-1}). \end{aligned}$$

Taking the conjugate of both sides of the equation yields

$$\overline{\delta_{ij}} = \overline{\sum_{t=1}^{k} \chi_t(g_i)\frac{h_j}{|G|}\chi_t(g_j^{-1})}$$

which equals $\delta_{ij} = \frac{1}{|G|}\sum_{t=1}^{k} h_j\overline{\chi_t(g_i)}\ \overline{\overline{\chi_t(g_j)}}$. So $\delta_{ij} = \frac{1}{|G|}\sum_{t=1}^{k} h_j\chi_t(g_j)\overline{\chi_t(g_i)}$. Once again we can exchange i and j and $\delta_{ij} = \delta_{ji} = \frac{1}{|G|}\sum_{t=1}^{k} h_i\chi_t(g_i)\overline{\chi_t(g_j)}$, and we have $\frac{\delta_{ij}|G|}{h_i} = \sum_{t=1}^{k}\chi_t(g_i)\overline{\chi_t(g_j)}$. $\diamondsuit$

The previous theorems and propositions will be useful in calculating character tables for groups (in particular the last three will prove to be quite useful). A *character table* of G is an $m\times n$ matrix whose columns are indexed by the conjugacy classes of the group and whose rows are indexed by the irreducible $\mathbb{C}$-characters of G. By Theorem 3.14 we know that character tables are square matrices The i, jth entry in the table is the value of the

ith character on the jth conjugacy class. Proposition 3.16 and Theorem 3.17 describe the results when multiplying rows and columns in the character tables. Proposition 3.16 fixes the two characters and then multiplies the values over the different conjugacy classes. In Theorem 3.17 we find that the product of a column with the conjugate of another column is simply 0 whereas the product of a column with the conjugate of itself is $|G|/h_i$ where i is the ith column and therefore indexes the ith conjugacy classes.

3.4 Character Tables

In order to show the utility of the theorems and propositions in the previous section, we will consider the character tables of a few well-known groups.

First consider the group S_3. There are three conjugacy classes of S_3 (namely $\mathrm{cl}\{1\} = \{1\}$ where $\mathrm{cl}\{1\}$ is the conjugacy class with the element 1 as its representative, $\mathrm{cl}\{r\} = \{r, r^2\}$, and $\mathrm{cl}\{c\} = \{c, rc, r^2c\}$), so by Theorem 3.14 there are three irreducible characters of S_3. Define χ_1, χ_2, and χ_3 as the irreducible characters of S_3, and by Theorem 3.13 we find $|S_3| = 6 = \chi_1(1)^2 + \chi_2(1)^2 + \chi_3(1)^2$. Since $1^2 + 1^2 + 2^2 = 6$ then $\chi_1(1) = 1$, $\chi_2(1) = 1$ and $\chi_3(1) = 2$. Let χ_1 be the trivial character and since $\deg(\chi_1) = 1$ then the first row of the character table is all ones. For any symmetric group S_n, we can define the alternating representation,

$$T(g) = \begin{cases} [-1] & \text{if } g \text{ is odd} \\ [1] & \text{if } g \text{ is even} \end{cases} .$$

The corresponding character $\chi(g)$ is 1 when g is even and -1 when g is odd. Note that this is a linear character, so let χ_2 be the alternating character. Since $\mathrm{cl}\{r\}$ is comprised of three-cycles (which are even since they can be written as the product of two transpositions), the value for χ_2 in that column is 1. Similarly in $\mathrm{cl}\{c\}$ all elements are transpositions and are therefore odd; the value for χ_2 in that column is -1. At this point we can fill in the character table for S_3 as follows,

S_3	cl{1}	cl{r}	cl{c}
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2		

Obviously χ_3 is an irreducible character of degree 2. To find its values on the conjugacy classes of S_3 we will use the Second Orthogonality Relation. We know the product of the first column and the conjugate of the second is 0. Thus we have $0 = 1 \cdot \overline{1} + 1 \cdot \overline{1} + 2 \cdot \overline{x}$ where x is the entry in the 3rd row, 2nd column. So we have that $\overline{x} = -1$ and since the conjugate of a real number is just itself then $x = -1$. Similarly we can find the product of the first column and the conjugate of the third, and we have $0 = 1 \cdot \overline{1} + 1 \cdot \overline{-1} + 2 \cdot \overline{y}$ where y is the entry in the 3rd row, 3rd column. We have that $\overline{y} = 0$, and thus $y = 0$. So the complete character table for S_3 is,

S_3	cl{1}	cl{r}	cl{c}
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

As another example consider the group D_4, the group of symmetries of the square. Define D_4 as $\{1, r, r^2, r^3, v, rv, r^2v, r^3v\} = \langle r, v \rangle$ where $r^4 = 1 = v^2$, $vr = r^3v$, the element r is equivalent to a 90 degree rotation or the cycle (1234), and the element v is equivalent to the product of transpositions (12)(34) or a reflection over a vertical line. There are five conjugacy classes of D_4, cl{1}, cl{r^2}, cl{r}, cl{v}, and cl{rv}. This means there are five irreducible characters of D_4 (by Theorem 3.14). In Chapter 2 we found all four of the linear representations of D_4 (Section 2.6, p.14). So we can fill in the first four rows of the character table and we have,

D_4	cl$\{1\}$	cl$\{r^2\}$	cl$\{r\}$	cl$\{v\}$	cl$\{rv\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	-1	1	-1
χ_5					

To find the degree of the last irreducible character apply Theorem 3.13, and $|D_4| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + \chi_5(1)^2$. The only possible value for $\chi_5(1)$ is 2. Using the Second Orthogonality Relation four times, we can obtain the entire last row of the character table.

D_4	cl$\{1\}$	cl$\{r^2\}$	cl$\{r\}$	cl$\{v\}$	cl$\{rv\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	-1	1	-1
χ_5	2	-2	0	0	0

The two following character tables for A_4 and S_4 involve lengthy (yet easy) calculations so the explanations for both tables are not included. However, since the tables will be used in examples later in this chapter and in the following chapter on M-groups, they are presented below

A_4	cl$\{1\}$	cl$\{(12)(34)\}$	cl$\{(123)\}$	cl$\{(321)\}$
χ_1	1	1	1	1
χ_2	1	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$
χ_3	1	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$
χ_4	3	-1	0	0

S_4	cl{1}	cl{(12)(34)}	cl{(123)}	cl{(12)}	cl{(1234)}
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

3.5 Induced Characters

Let T be a $\mathbb{C}$-representation with character χ. If T^G is the induced representation as defined in Chapter 2, then the trace of T^G is the induced character χ^G. Before attempting to calculate the value of an induced character, we need to introduce the concept of extending a character and consider the following lemma. Suppose χ is a character of H where $H \leq G$. We can *extend* χ to a function on the entire group G in the following fashion,

$$\dot{\chi}(x) = \begin{cases} 0 & \text{if } x \notin H \\ \chi(x) & \text{if } x \in H \end{cases}.$$

In general, a function ζ defined on G such that $\zeta|_H = \chi$ is called an *extension* of χ or we say χ *extends* to ζ.

Lemma 3.18: *Let G be a finite group, $H \leq G$, and χ a character of H having degree n. If $g \in G$ and $h \in H$ then $\dot{\chi}(h^{-1}gh) = \dot{\chi}(g)$.*

Proof: There are two cases to consider, either $h^{-1}gh \in H$ or not. If $h^{-1}gh \in H$ then $g = hh_1h^{-1}$ for some $h_1 \in H$, and it follows that $g \in H$. Thus $\dot{\chi}(g) = \chi(g)$ and $\dot{\chi}(h^{-1}gh) = \chi(h^{-1}gh)$. We are given that χ is a character on H so by Proposition 3.1, $\dot{\chi}(g) = \chi(g) = \chi(h^{-1}gh) = \chi(h^{-1}gh)$. If $h^{-1}gh \notin H$, then $g \notin H$. So $\dot{\chi}(g) = 0_{n\times n}$ and $\dot{\chi}(h^{-1}gh) = 0_{n\times n}$ In both cases $\chi(h^{-1}gh) = \dot{\chi}(g)$ $\diamondsuit$

Proposition 3.19: *If $H \leq G, \chi$ is a $\mathbb{C}$-character of H afforded by the representation*

T, and $\{x_1, \ldots, x_m\}$ *is a left transversal for* H *in* G*, then*

$$\chi^G(x) = \sum_{i=1}^{m} \dot{\chi}(x_i^{-1}xx_i) = \frac{1}{|H|}\sum_{t\in G} \dot{\chi}(t^{-1}xt)$$

for all $x \in G$*.*

Proof: First, recall that a left transversal for H in G is a set of left coset representatives of H in G. Then $\chi^G(x) = \mathrm{Tr}(T^G(x)) = \sum_{i=1}^{m} \mathrm{Tr}(\dot{T}(x_i^{-1}xx_i)) = \sum_{i=1}^{m} \dot{\chi}(x_i^{-1}xx_i)$. Now consider $\sum_{t\in G} \dot{\chi}(t^{-1}xt)$ and since for all $t \in G$, t is in some left coset, then there exists exactly one x_i and $h \in H$ such that $t = x_i h$. We have that

$$\begin{aligned}
\sum_{t\in G} \dot{\chi}(t^{-1}xt) &= \sum_{i=1}^{m}\sum_{h\in H} \dot{\chi}((x_ih)^{-1}x(x_ih)) \\
&= \sum_{i=1}^{m}\sum_{h\in H} \dot{\chi}(h^{-1}(x_i^{-1}xx_i)h) \\
&= \sum_{i=1}^{m}\sum_{h\in H} \dot{\chi}(x_i^{-1}xx_i) \text{ (by Lemma 3.18)} \\
&= \sum_{i=1}^{m} \left(\dot{\chi}(x_i^{-1}xx_i)\sum_{h\in H}(1)\right) \\
&= \sum_{i=1}^{m} \dot{\chi}(x_i^{-1}xx_i)|H| \\
&= |H|\sum_{i=1}^{m} \dot{\chi}(x_i^{-1}xx_i).
\end{aligned}$$

Thus,

$$\frac{1}{|H|}\sum_{t\in G} \dot{\chi}(t^{-1}xt) = \sum_{i=1}^{m} \dot{\chi}(x_i^{-1}xx_i) = \chi^G(x).$$

◇

The preceding proposition and proof describe how to calculate the induced character. As an example, consider the group S_4 with the normal subgroup H generated by $(12)(34)$ and $(13)(24)$. Let χ be a character of degree 1 on H given by $\chi((12)(34)) = [1]$ and $\chi((13)(24)) = [-1]$. Let ψ be the character on H associated with the representation given by,

$$T((12)(34)) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; T((13)(24)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We will determine χ^G and ψ^G and reduce each to its irreducible components.

Since $H \trianglelefteq G$, then both χ^G and ψ^G are 0 when evaluated at $g \notin H$. Observe that G/H is a group of order 6 such that $G/H = \{H, (1234)H, (123)H, (12)H, (234)H, (1243)H\}$. To find the values of the induced character we need to calculate $\chi^G(g) = \sum_{i=1}^{6} \dot{\chi}(g_i^{-1} g g_i)$ where 1, (1234), (123), (12), (234), (1243) are denoted $g_1, \ldots, g_6$ respectively. First we calculate $\chi^G(1) = \deg(\chi^G) = [G : H] = 6$. Since χ^G is a class function, then for all $h \in \mathrm{cl}\{(12)(34)\}$, $\chi^G(h) = \chi^G((12)(34))$. Observe that

$$\begin{aligned}
\chi^G((12)(34)) &= \dot{\chi}(1^{-1}(12)(34)1) + \dot{\chi}((1234)^{-1}(12)(34)(1234)) + \\
&\quad \dot{\chi}((123)^{-1}(12)(34)(123)) + \dot{\chi}((12)^{-1}(12)(34)(12)) + \\
&\quad \dot{\chi}((234)^{-1}(12)(34)(234)) + \dot{\chi}((1243)^{-1}(12)(34)(1243)) \\
&= \dot{\chi}((12)(34)) + \dot{\chi}((14)(23)) + \dot{\chi}((14)(23)) + \\
&\quad \dot{\chi}((12)(34)) + \dot{\chi}((13)(24)) + \dot{\chi}((13)(24)) \\
&= 1 - 1 - 1 + 1 - 1 - 1 = -2.
\end{aligned}$$

And finally, for $g \notin H$, $\chi^G(g) = 0$. Similarly

$$\begin{aligned}
\psi^G((12)(34)) &= 2 \cdot \dot{\psi}((12)(34)) + 2 \cdot \dot{\psi}((14)(23)) + 2 \cdot \dot{\psi}((13)(24)) \\
&= 0 + 2 \cdot \dot{\psi}((14)(23)).
\end{aligned}$$

Using the representation T defined above we find that

$$T((14)(23)) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $\dot{\psi}((14)(23)) = -2$. So $\psi^G((12)(34)) = -4$. The character ψ^G is defined by

$$\begin{aligned}
\psi^G(g) &= 0 \text{ when } g \notin H, \\
\psi^G(g) &= -4 \text{ when } g \in \mathrm{cl}\{(12)(34)\}, \text{ and} \\
\psi^G(1) &= \deg(\psi^G) = [G : H] \cdot \deg(\psi) = 12.
\end{aligned}$$

To reduce ψ^G and χ^G it would be advantageous to compare the values of ψ^G and χ^G

over the conjugacy classes to the character table of S_4.

S_4	cl{1}	cl{(12)(34)}	cl{(123)}	cl{(12)}	cl{(1234)}
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

	cl{1}	cl{(12)(34)}	cl{(123)}	cl{(12)}	cl{(1234)}
ψ^G	12	-4	0	0	0

	cl{1}	cl{(12)(34)}	cl{(123)}	cl{(12)}	cl{(1234)}
χ^G	6	-2	0	0	0

It's obvious that $\chi^G = \chi_4 + \chi_5$ and that $\psi^G = 2\chi_4 + 2\chi_5$.

Now consider the following proposition about the kernel of an induced character.

Proposition 3.20: *Let θ be a character of $H \leq G$. Then* $\ker(\theta^G) = \bigcap_{x\in G}(\ker(\theta))^x$.

Proof: First, we clarify the notation $(\ker(\theta))^x$ which is conjugation of the kernel of θ by x. Recall that we can compute $\theta^G(g)$ using Proposition 3.19 and

$$\theta^G(g) = \frac{1}{|H|}\sum_{x\in G}\dot{\theta}(x^{-1}gx)$$

Now by Proposition 3.15 $g \in \ker(\theta^G)$ if and only if

$$\theta^G(g) = \theta^G(1) = [G:H]\cdot\theta(1) = \frac{1}{|H|}\sum_{x\in G}\theta(1).$$

So $g \in \ker(\theta^G)$ if and only if $\sum_{x\in G}\theta(1) = \sum_{x\in G}\dot{\theta}(x^{-1}gx)$. Since $\theta(1) \in \mathbb{N}$, then $\sum_{x\in G}\theta(1) \in \mathbb{N}$ which implies $|\sum_{x\in G}\theta(1)| = \sum_{x\in G}\theta(1)$. Now taking the absolute value of both sides of the equation we have

$$\sum_{x\in G}\theta(1) = \left|\sum_{x\in G}\theta(1)\right| = \left|\sum_{x\in G}\theta(x^{-1}gx)\right| \leq \sum_{x\in G}|\dot{\theta}(x^{-1}gx)| \leq \sum_{x\in G}\theta(1).$$

(We obtained the last part of the inequality by Proposition 3.5 and observing if $x^{-1}gx \notin H$, then $|\dot{\theta}(x^{-1}gx)| = 0 \leq \theta(1)$). Therefore, $g \in \ker(\theta^G)$ if and only if

$$\sum_{x\in G} |\theta(x^{-1}gx)| = \sum_{x\in G} \theta(1).$$

We know $|\theta(x^{-1}gx)| \leq \theta(1)$ for all $x \in G$ (see Proposition 3.5) so each complex number $\theta(x^{-1}gx)$ is either on the unit circle in the complex plane with radius $\theta(1)$ or inside of that circle. Since

$$\sum_{x\in G} |\theta(x^{-1}gx)| = \sum_{x\in G} \theta(1) = |G| \cdot \theta(1)$$

then for each $x \in G$, $|\theta(x^{-1}gx)| = \theta(1)$ or the previous equation cannot be true.

We have $|\theta(x^{-1}gx)| = \theta(1) \in \mathbb{N}$, and $\sum_{x\in G} \theta(x^{-1}gx) = \theta(1) \cdot |G|$, so $\theta(x^{-1}gx) = \theta(1)$ for all $x \in G$. This is true if and only if $x^{-1}gx \in \ker(\theta)$ for all $x \in G$ by Proposition 3.15. But this is true if and only if for all $x \in G$, $g \in \ker(\theta)^{x^{-1}}$ if and only if $g \in \bigcap_{x\in G}(\ker(\theta))^x$. Since all of the statements are if and only if statements then we have shown $\ker(\theta^G) = \bigcap_{x\in G}(\ker(\theta))^x$. $\diamondsuit$

3.6 Inner Products of Induced Characters

If χ is a character of a subgroup H, then the *restriction of* χ *to* H is denoted $\chi|_H$ and is defined to be $\chi(h)$ when $h \in H$ and undefined when $h \notin H$.

Theorem 3.21 (Frobenius Reciprocity): *If* $H \leq G$, χ *is a character of* G, *and* μ *is a character of* H, *then* $\langle \mu, \chi|_H \rangle = \langle \mu^G, \chi \rangle$.

Proof:

$$\begin{aligned} \langle \mu^G, \chi \rangle &= \frac{1}{|G|} \sum_{g\in G} \mu^G(g)\overline{\chi(g)} \\ &= \frac{1}{|G|} \sum_{g\in G} \left(\frac{1}{|H|} \sum_{t\in G} \dot{\mu}(t^{-1}gt) \right) \overline{\chi(g)} \\ &= \frac{1}{|G||H|} \sum_{g\in G} \sum_{t\in G} \mu(t^{-1}gt)\overline{\chi(g)} \end{aligned}$$

$$= \frac{1}{|G||H|}\sum_{t\in G}\sum_{g\in G}\dot{\mu}(t^{-1}gt)\overline{\chi(t^{-1}gt)} \text{ by Proposition 3.1}$$

Since for a fixed $t \in G$, as g runs through all elements of G, so does $t^{-1}gt$, it follows that $\frac{1}{|G||H|}\sum_{t\in G}\sum_{g\in G}\dot{\mu}(t^{-1}gt)\overline{\chi(t^{-1}gt)} = \frac{1}{|G||H|}\sum_{t\in G}\sum_{x\in G}\dot{\mu}(x)\overline{\chi(x)}$. Thus,

$$\begin{aligned}\langle \mu^G, \chi\rangle &= \frac{1}{|G||H|}\sum_{x\in G}\dot{\mu}(x)\overline{\chi(x)}\sum_{t\in G}1\\ &= \frac{1}{|H|}\sum_{x\in G}\dot{\mu}(x)\overline{\chi(x)}.\end{aligned}$$

Since $\dot{\mu}(x) = 0$ when $x \notin H$ then

$$\begin{aligned}\frac{1}{|H|}\sum_{x\in G}\dot{\mu}(x)\overline{\chi(x)} &= \frac{1}{|H|}\sum_{x\in H}\mu(x)\overline{\chi(x)}\\ &= \langle \mu, \chi|_H\rangle.\end{aligned}$$

We have shown $\langle \mu, \chi|_H\rangle = \langle \mu^G, \chi\rangle$. ◇

Corollary 3.22: *Let $H \leq G$, let χ be a character of G, and let μ be a character of H. Then the multiplicity of χ in μ^G is equal to the multiplicity of μ in $\chi|_H$.*

Proof: The multiplicity of χ in μ^G is $\langle \mu^G, \chi\rangle$, and the multiplicity of μ in $\chi|_H$ is $\langle \chi|_H, \mu\rangle$. Since the multiplicity is simply a natural number, $\langle \chi|_H, \mu\rangle = \overline{\langle \chi|_H, \mu\rangle}$. In Section 3.3 we discussed that the inner product is symmetric so $\overline{\langle \chi|_H, \mu\rangle} = \langle \mu, \chi|_H\rangle$. By the Frobenius Reciprocity, $\langle \mu^G, \chi\rangle = \langle \mu, \chi|_H\rangle = \langle \chi|_H, \mu\rangle$. ◇

Theorem 3.23 (Transitivity of Induction): *If $K \leq H \leq G$ and χ is a character of K. Then $(\chi^H)^G = \chi^G$.*

Proof: Define $\varphi = \chi^H$ and so $(\chi^H)^G = \varphi^G$. Observe that $\chi^G(x) = \frac{1}{|K|}\sum_{g\in G}\dot{\chi}(g^{-1}xg)$ Now consider

$$\begin{aligned}(\chi^H)^G(x) &= \varphi^G(x)\\ &= \frac{1}{|H|}\sum_{t\in G}\dot{\varphi}(t^{-1}xt)\\ &= \frac{1}{|H|}\sum_{t\in G}\dot{\chi}^H(t^{-1}xt).\end{aligned}$$

If $t^{-1}xt \in H$ then $\dot{\chi}^H(t^{-1}xt) = \chi^H(t^{-1}xt)$ and 0 otherwise. Now we will use the definition for the induced character χ^H and will have two dots in the summation.

$$\frac{1}{|H|}\sum_{t\in G}\chi^H(t^{-1}xt) = \frac{1}{|H|}\sum_{t\in G}\frac{1}{|K|}\left(\sum_{v\in H}\dot{\chi}(v^{-1}(t^{-1}xt)v)\right)$$

Recall that the inside dot means $\dot{\chi}(v^{-1}(t^{-1}xt)v)$ has a nonzero value only if $v^{-1}(t^{-1}xt)v \in K$. Since $K \leq H$ when $v^{-1}(t^{-1}xt)v \in K$, then $t^{-1}xt \in H$, and so the outside dot is unnecessary. Thus,

$$\begin{aligned}
\frac{1}{|H|}\sum_{t\in G}\frac{1}{|K|}\left(\sum_{v\in H}\dot{\chi}(v^{-1}(t^{-1}xt)v)\right) &= \frac{1}{|H|}\sum_{t\in G}\frac{1}{|K|}\sum_{v\in H}\dot{\chi}(v^{-1}(t^{-1}xt)v) \\
&= \frac{1}{|H||K|}\sum_{v\in H}\sum_{t\in G}\dot{\chi}((tv)^{-1}x(tv))
\end{aligned}$$

Since right multiplication is a permutation of a group, then for each $v \in H$, $\{t \cdot v | t \in G\} = G$. So it follows that

$$\begin{aligned}
\frac{1}{|H||K|}\sum_{v\in H}\sum_{t\in G}\dot{\chi}((tv)^{-1}x(tv)) &= \frac{1}{|H||K|}\sum_{v\in H}\sum_{g\in G}\dot{\chi}(g^{-1}xg) \\
&= \frac{1}{|H||K|}\sum_{g\in G}\dot{\chi}(g^{-1}xg)\sum_{v\in H}1 \\
&= \frac{1}{|K|}\sum_{g\in G}\dot{\chi}(g^{-1}xg) \\
&= \chi^G(x).
\end{aligned}$$

Therefore we have shown that $(\chi^H)^G = \chi^G$ and induction is transitive. $\diamondsuit$

As an application of the Frobenius Reciprocity, consider the following corollary.

Corollary 3.24: *If $A \leq G$, A is abelian and $\chi \in \mathrm{Irr}(G)$ then $\chi(1) \leq [G : A]$.*

Proof: If we restrict χ to A then we can write $\chi|_A$ as a linear combination of irreducible, linear characters of A by Proposition 3.8 and the remarks following Theorem 3.14. Let φ be an irreducible character of A such that the multiplicity of φ in $\chi|_A$ is positive By Corollary 3.22, $\langle \chi|_A, \varphi\rangle = \langle \varphi^G, \chi\rangle > 0$. We can also write φ^G as a linear combination of irreducible characters of G. So if $\mathrm{Irr}(G) = \{\psi_i | i = 1, \ldots, m\}$ then $\varphi^G = \sum_{i=1}^m \langle \varphi^G, \psi_i\rangle\psi_i$ by Corollary 3.11. Specifically, $\chi = \psi_j$ for some $j = 1, \ldots, m$. Hence,

$\varphi^G(1) = \sum_{i=1}^m \langle \varphi^G, \psi_i \rangle \psi_i(1) \geq \langle \varphi^G, \chi \rangle \chi(1)$. Since $\langle \varphi^G, \chi \rangle \chi(1) \leq \varphi^G(1)$ and $\langle \varphi^G, \chi \rangle > 0$ then $\chi(1) \leq \varphi^G(1)$. Hence $\chi(1) \leq \varphi^G(1) = [G : A] \cdot \deg(\varphi) = [G : A] \cdot 1 = [G : A]$ (this is because A is abelian and thus all irreducible characters are linear). $\diamondsuit$

3.7 Mackey's Theorem

Let $H \trianglelefteq G$ and suppose T is a $\mathbb{C}$-representation of H with character χ. The nonstandard notation ${}^x h$ for $x \in G$ and $h \in H$ is defined to equal xhx^{-1} for the remainder of the text. For $x \in G$ define the map T^x on the set H such that $T^x(h) = T({}^x h) = T(xhx^{-1})$ for all $h \in H$. This map is defined since $H \trianglelefteq G$ and thus $xhx^{-1} \in H$. Since $T^x(hj) = T({}^x(hj)) = T(xhjx^{-1}) = T(xhx^{-1}xjx^{-1}) = T(xhx^{-1})T(xjx^{-1}) = T({}^x h)T({}^x j) = T^x(h)T^x(j)$ then T^x preserves multiplication. Also $T^x(h)$ is invertible since $T^x(h) = T(xhx^{-1})$ and $T(xhx^{-1})$ is invertible. Therefore T^x is a representation called the *conjugate representation* of T. The character of $T^x(h)$ is denoted $\chi^x(h)$ and is called the *conjugate character.* Observe that $\chi^x(h) = \mathrm{Tr}(T^x(h)) = \mathrm{Tr}(T(xhx^{-1})) = \chi(xhx^{-1}) = \chi({}^x h)$ so we can now calculate χ^x using what we know about the character χ. Further, conjugation of characters behaves in a "multiplicative" fashion just as we would desire as evidenced by the following calculation, $\chi^{xz}(h) = \chi(xzhz^{-1}x^{-1}) = \chi^x(zhz^{-1}) = (\chi^x)^z(h)$ for all $x, z \in G$.

Proposition 3.25: *If χ, φ are characters of $H \trianglelefteq G$ and $x \in G$ then,*

$$\langle \chi^x, \varphi^x \rangle = \langle \chi, \varphi \rangle.$$

Proof:

$$\begin{aligned}
\langle \chi^x, \varphi^x \rangle &= \frac{1}{|H|} \sum_{h \in H} \chi^x(h) \varphi^x(h^{-1}) \\
&= \frac{1}{|H|} \sum_{h \in H} \chi(xhx^{-1}) \varphi(xh^{-1}x^{-1}) \\
&= \frac{1}{|H|} \sum_{h \in H} \chi(h) \varphi(h^{-1}) \\
&= \langle \chi, \varphi \rangle.
\end{aligned}$$

$\diamondsuit$

One important inference we draw from this proposition is if χ is an irreducible character of a normal subgroup H, then χ^x is also irreducible on H.

We can broaden this definition of conjugate characters with some slight modifications so that the subgroup H does not have to be normal. If $H \leq G$ and T is a $\mathbb{C}$-representation of G, then define T^x on H^x such that $T^x(h^x) = T(h)$ for all $h \in H$. Since T^x is defined in terms of T, then the given matrix is clearly invertible. Also, $T^x((hj)^x) = T(hj) = T(h)T(j) = T^x(h^x)T(j^x)$ for all $h, j \in H$. T^x is, in fact, a representation of G. If χ is the character associated with T, then χ^x is the character associated with T^x where $\chi^x(h^x) = \mathrm{Tr}(T^x(h^x)) = \mathrm{Tr}(T(h)) = \chi(h)$.

The following are called the Mackey Theorems and deal with calculating the inner product of induced characters using conjugate characters. First recall that the *double coset* of K and H containing g is defined as the subset $KgH = \{kgh | k \in K,\ h \in H\}$. Note that $a \in KgH$ if and only if $a = kgh$ for some $k \in K$ and $h \in H$. Further let $\{g_1, \ldots, g_s\}$ be a set of double coset representatives of K and H. Then G is the disjoint union of double cosets; that is, $G = \bigcup_{i=1}^{s} Kg_iH$. The set $\{g_1, \ldots, g_s\}$ is also called a *K,H transversal.*

Throughout the remainder of this section we will use the following notation. Suppose $K, H \leq G$ and let $x_1, \ldots, x_s$ be a set of K and H double coset representatives in G with $x_1 = 1$. For each i set $K^{(i)} = K^{x_i} = x_i^{-1}Kx_i$ and $H_i = K^{(i)} \cap H$. If χ is a character of K then define χ_i as a character of $K^{(i)}$ such that $\chi_i = \chi^{x_i}$.

Proposition 3.26: *Let $\{h_{i1}, h_{i2}, \ldots, h_{is_i}\}$ be a set of right coset representatives for H_i in H. Then Kx_iH is the disjoint union $Kx_ih_{i1} \cup Kx_ih_{i2} \cup \cdots \cup Kx_ih_{is_i}$.*

Proof: First let $g \in Kx_iH$, and we will show $g \in \bigcup_{j=1}^{s_i} Kx_ih_{ij}$. Let $g = k_1x_ih_1$ where $k_1 \in K$ and $h_1 \in H$. Since $h_1 \in H$, then there exists h_{ij} where $1 \leq j \leq s_i$ such that $h_1 \in H_ih_{ij}$ (since $H = \bigcup_{k=1}^{s_i} H_ih_{ik}$). This means $h_1(h_{ij})^{-1} \in H_i$, and since $H_i = K^{x_i} \cap H$ then $h_1(h_{ij})^{-1} \in K^{x_i}$. We have that $h_1 \in (x_i^{-1}Kx_i)h_{ij}$, and $g = k_1x_ih_1 =$

$k_1x_i(x_i^{-1}k_2x_ih_{ij})$ where k_2 is an element in K. So $g = k_1k_2x_ih_{ij} \in Kx_ih_{ij}$. This proves that $Kx_iH \subseteq Kx_ih_{i1} \cup \cdots \cup Kx_ih_{is_i}$.

Since the reverse inclusion is trivial, altogether $Kx_iH = \bigcup_{j=1}^{s_i} Kx_ih_{ij}$.

Finally we will prove the union is disjoint. Suppose $Kx_ih_{im} \cap Kx_ih_{in} \neq \emptyset$ for some $m, n \in \{1, \ldots, s_i\}$. So let $g = k_1x_ih_{im}$ and $g = k_2x_ih_{in}$ where h_{in} and h_{im} are right coset representatives of H_i in H and $k_1, k_2 \in K$. This means that $x_i^{-1}g = x_i^{-1}k_1x_ih_{im} = x_i^{-1}k_2x_ih_{in}$. Thus $x_i^{-1}g \in K^{x_i}h_{im}$ and $x_i^{-1}g \in K^{x_i}h_{in}$. Since $K^{x_i}h_{im}$ and $K^{x_i}h_{in}$ are two right cosets of K^{x_i} with one element in common, then $K^{x_i}h_{im} = K^{x_i}h_{in}$. This means $n = m$ and we are done. ◇

A direct consequence of the proposition above is that since $\{x_1, \ldots, x_s\}$ is a set of double coset representatives for K and H in G and $\{h_{i1}, \ldots, h_{is_i}\}$ is a set of coset representatives for H_i in H, then $\{x_ih_{ij} | 1 \leq i \leq s \text{ and } 1 \leq j \leq s_i\}$ is a set of right coset representatives for K in G.

Theorem 3.27: *Let $\{x_1, \ldots, x_s\}$ be a set of K and H double coset representatives in G. If χ is a character of K then $(\chi^G)|_H = \sum_{i=1}^{s}(\chi_i|_{H_i})^H$. Recall that χ_i is the character χ^{x_i} defined on the set K^{x_i}.*

Proof: Let $y \in H$ and consider $\chi^G|_H(y)$. Since $\{x_ih_{ij} | 1 \leq i \leq s \text{ and } 1 \leq j \leq s_i\}$ is a set of right coset representatives for K in G then $\chi^G|_H(y) = \sum_{i=1}^{s}\sum_{j=1}^{s_i} \dot{\chi}(x_ih_{ij}yh_{ij}^{-1}x_i^{-1})$ where $\dot{\chi}(a) = \chi(a)$ if $a \in K$ and 0 otherwise. Observe that

$$\sum_{i=1}^{s}\sum_{j=1}^{s_i} \dot{\chi}(x_ih_{ij}yh_{ij}^{-1}x_i^{-1}) = \sum_{i=1}^{s}\sum_{j=1}^{s_i} \dot{\chi}({}^{x_i}(h_{ij}yh_{ij}^{-1}))$$

since ${}^{x}h = xhx^{-1}$. If ${}^{x_i}(h_{ij}yh_{ij}^{-1}) \in K$ then $h_{ij}yh_{ij}^{-1} \in K^{x_i}$. Moreover, if ${}^{x_i}(h_{ij}yh_{ij}^{-1}) \notin K$ then $h_{ij}yh_{ij}^{-1} \notin K^{x_i}$. It follows that $\dot{\chi}({}^{x_i}(h_{ij}yh_{ij}^{-1})) = (\dot{\chi^{x_i}})(h_{ij}yh_{ij}^{-1})$ (Remember that χ^{x_i} is defined on K^{x_i} and χ is defined on K). Thus,

$$\begin{aligned} \sum_{i=1}^{s}\sum_{j=1}^{s_i} \dot{\chi}({}^{x_i}(h_{ij}yh_{ij}^{-1})) &= \sum_{i=1}^{s}\sum_{j=1}^{s_i} (\dot{\chi^{x_i}})(h_{ij}yh_{ij}^{-1}) \\ &= \sum_{i=1}^{s}\sum_{j=1}^{s_i} (\dot{\chi_i})(h_{ij}yh_{ij}^{-1}) \end{aligned}$$

using the notation in the statement of the theorem. Note that $(\dot{\chi_i})(a) = \chi_i(a)$ if $a \in K^{x_i}$ and 0 otherwise. Now if $h_{ij}yh_{ij}^{-1} \in K^{x_i}$ then since $y \in H$ we know that $h_{ij}yh_{ij}^{-1} \in H$. Hence $h_{ij}yh_{ij}^{-1} \in H \cap K^{x_i} = H_i$. Further if $h_{ij}yh_{ij}^{-1} \notin K^{x_i}$ then $h_{ij}yh_{ij}^{-1} \notin H_i$. So we have $(\dot{\chi_i})(h_{ij}yh_{ij}^{-1}) = (\chi_i|_H)\dot{}\,(h_{ij}yh_{ij}^{-1})$. The dot in the right hand side of the previous equation is to the right of the expression $(\chi_i|_H)$ because of the awkwardness of the notation of having a dot above. Thus

$$(\chi^G|_H)(y) = \sum_{i=1}^{s}\sum_{j=1}^{s_i}(\dot{\chi_i})(h_{ij}yh_{ij}^{-1}) = \sum_{i=1}^{s}\sum_{j=1}^{s_i}(\chi_i|_{H_i})\dot{}\,(h_{ij}yh_{ij}^{-1}).$$

Note that by the definition of induced character,

$$(\chi_i|_{H_i})^H(y) = \sum_{j=1}^{s_i}(\chi_i|_{H_i})\dot{}\,(h_{ij}yh_{ij}^{-1})$$

and substituting this into

$$(\chi^G|_H)(y) = \sum_{i=1}^{s}\sum_{j=1}^{s_i}(\chi_i|_{H_i})\dot{}\,(h_{ij}yh_{ij}^{-1})$$

yields

$$(\chi^G|_H)(y) = \sum_{i=1}^{s}(\chi_i|_{H_i})^H(y).$$

$\diamond$

Theorem 3.28 (Mackey): *If χ is a character of H and ψ is a character of K, then*

$$\langle \chi^G, \psi^G\rangle = \sum_{i=1}^{s}\langle \chi|_{H_i}, \psi_i|_{H_i}\rangle.$$

Proof: By the Frobenius Reciprocity, $\langle \chi^G, \psi^G\rangle = \langle \chi, (\psi^G)|_H\rangle$. By the previous theorem, $(\psi^G)|_H = \sum_{i=1}^{s}(\psi_i|_{H_i})^H$ where $\{x_1, \ldots, x_s\}$ are a set of double coset representatives of K and H. It follows that

$$\langle \chi, (\psi^G)|_H\rangle = \left\langle \chi, \sum_{i=1}^{s}(\psi_i|_{H_i})^H \right\rangle = \sum_{i=1}^{s}\left\langle \chi, (\psi_i|_{H_i})^H\right\rangle.$$

Restrict χ to H_i and apply the Frobenius Reciprocity again to obtain $\langle \chi, (\psi_i|_{H_i})^H\rangle = \langle \chi|_{H_i}, \psi_i|_{H_i}\rangle$. Hence

$$\sum_{i=1}^{s}\langle \chi, (\psi_i|_{H_i})^H\rangle = \sum_{i=1}^{s}\langle \chi|_{H_i}, \psi_i|_{H_i}\rangle$$

Therefore we have shown $\langle \chi^G, \psi^G \rangle = \sum_{i=1}^{s} \langle \chi|_{H_i}, \psi_i|_{H_i} \rangle$. $\diamondsuit$

Proposition 3.29: *Let $H \leq G$ with $\lambda \in \mathrm{Irr}(H)$ such that λ is linear and* $\ker(\lambda) = M$. *Then*

$$\langle \lambda^G, \lambda^G \rangle = \langle (\lambda^x)^G, (\lambda^x)^G \rangle.$$

Proof: Using the definition of inner product we have

$$\begin{aligned}
\langle (\lambda^x)^G, (\lambda^x)^G \rangle &= \frac{1}{|G|} \sum_{g \in G} (\lambda^x)^G(g) \overline{(\lambda^x)^G(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H^x|} \sum_{t \in G} (\dot{\lambda^x})(t^{-1}gt) \overline{\frac{1}{|H^x|} \sum_{t \in G} (\dot{\lambda^x})(t^{-1}gt)} \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H|} \sum_{t \in G} (\dot{\lambda^x})(t^{-1}gt) \overline{\frac{1}{|H|} \sum_{t \in G} (\dot{\lambda^x})(t^{-1}gt)} \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H|} \sum_{t \in G} \dot{\lambda}(xt^{-1}gtx^{-1}) \overline{\frac{1}{|H|} \sum_{t \in G} \dot{\lambda}(xt^{-1}gtx^{-1})} \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H|} \sum_{t \in G} \dot{\lambda}((tx^{-1})^{-1}g(tx^{-1})) \overline{\frac{1}{|H|} \sum_{t \in G} \dot{\lambda}((tx^{-1})^{-1}g(tx^{-1}))} \right).
\end{aligned}$$

Since $x \in G$ is fixed then as t runs over all of G, tx^{-1} does as well. Therefore,

$$\begin{aligned}
\langle (\lambda^x)^G, (\lambda^x)^G \rangle &= \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H|} \sum_{v \in G} \dot{\lambda}(v^{-1}gv) \overline{\frac{1}{|H|} \sum_{v \in G} \dot{\lambda}(v^{-1}gv)} \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \lambda^G(g) \overline{\lambda^G(g)} \\
&= \langle \lambda^G, \lambda^G \rangle.
\end{aligned}$$

$\diamondsuit$

This proposition shows us if $\lambda^G \in \mathrm{Irr}(G)$, then $(\lambda^x)^G \in \mathrm{Irr}(G)$ as well.

CHAPTER 4

M-GROUPS AND MISCELLANEOUS BACKGROUND MATERIAL

4.1 Introduction to M-groups

An M-group is defined as follows. Let χ be a character of a group G. Then χ is *monomial* if $\chi = \lambda^G$, where λ is a linear character of some subgroup (not necessarily proper) of G. The group G is an *M-group* if every $\chi \in \mathrm{Irr}(G)$ is monomial, that is, every irreducible character of G is induced from a linear character of some subgroup of G.

From Isaacs' book on group characters, we learn that every nilpotent group is an M-group [5, p.83], and also that every M-group is solvable (this proof will follow later). We also know that every finite p-group is nilpotent so it follows that every finite p-group is also an M-group [4, p.105]. Another well-known example of an M-group is abelian groups since every abelian group is nilpotent.

Before proving some of these results about M-groups, we will consider a few examples of M-groups. Since M-groups have irreducible characters that are induced from subgroups of the group, it will be helpful to have the character tables for some well-known groups at hand. The character tables for S_3 and D_4 are originally found on pages 35 and 36 of this thesis. Since $\mathbb{Z}_3$ and $\mathbb{Z}_4$ are cyclic, their character tables are easy to calculate and were found using Proposition 2.2. The Klein-4 group's character table was motivated by the discussion in the middle of page 14. Additionally, the reader can refer to [3, p.150, 152] to find the character tables for A_4 and S_4.

S_3	cl{1}	cl{(123)}	cl{(23)}
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

D_4	cl{1}	cl{(13)(24)}	cl{(1234)}	cl{(12)(34)}	cl{(24)}
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	-1	1	-1
χ_5	2	-2	0	0	0

A_4	cl{1}	cl{(12)(34)}	cl{(123)}	cl{(321)}
χ_1	1	1	1	1
χ_2	1	1	$\frac{-1+\imath\sqrt{3}}{2}$	$\frac{-1-\imath\sqrt{3}}{2}$
χ_3	1	1	$\frac{-1-\imath\sqrt{3}}{2}$	$\frac{-1+\imath\sqrt{3}}{2}$
χ_4	3	-1	0	0

S_4	cl{1}	cl{(12)(34)}	cl{(123)}	cl{(12)}	cl{(1234)}
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

$\mathbb{Z}_4$	cl{1}	cl{(1234)}	cl{(13)(24)}	cl{(4321)}
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	$-i$
χ_4	1	$-i$	-1	i

K	cl{1}	cl{(12)(34)}	cl{(13)(24)}	cl{(14)(23)}
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1
χ_4	1	1	-1	-1

$\mathbb{Z}_3$	cl{1}	cl{(123)}	cl{(321)}
χ_1	1	1	1
χ_2	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$
χ_3	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$

Proposition 4.1: *The group S_4 is an M-group.*

Proof: To prove S_4 is an M-group we must show that each of the irreducible characters is monomial. Write $G = S_4$ and since χ_1 and χ_2 are linear then they are monomial. Is χ_3 monomial? We know $A_4 \leq S_4$ such that $[S_4 : A_4] = 2$ and from the character table of A_4 we know $\chi_2 \in \mathrm{Irr}(A_4)$ is linear. Define $\psi = \chi_2 \in \mathrm{Irr}(A_4)$ and we will calculate the induced character ψ^G. First we see $S_4/A_4 = \{A_4, (1234)A_4\}$ so $T = \{1, (1234)\}$ is a left transversal for A_4 in S_4. Using Proposition 3 19 we find $\psi^G(x) = \psi(1^{-1}x1) + \psi((4321)x(1234))$ for all $x \in G$. Calculating ψ^G on the conjugacy classes of S_4 yields

$$\psi^G(1) = 2\psi(1) = 2,$$

$$
\begin{aligned}
\psi^G((12)(34)) &= 2\dot{\psi}((12)(34)) = 2, \\
\psi^G((123)) &= \dot{\psi}((123)) + \dot{\psi}((234)) = \frac{-1+i\sqrt{3}}{2} + \frac{-1-i\sqrt{3}}{2} = -1, \\
\psi^G((12)) &= \dot{\psi}((12)) + \dot{\psi}((23)) = 0, \text{ and} \\
\psi^G((1234)) &= 2\dot{\psi}((1234)) = 0.
\end{aligned}
$$

Writing the values of ψ^G in a table allows us to compare the values of ψ^G over the conjugacy classes.

	cl{1}	cl{(12)(34)}	cl{(123)}	cl{(12)}	cl{(1234)}
ψ^G	2	2	-1	0	0

Thus $\psi^G = \chi_3$ and χ_3 is a monomial character of S_4.

In the same fashion we need to show that $\chi_4 \in \mathrm{Irr}(S_4)$ is also monomial. Consider the dihedral group D_4, and referring to its character table define φ as the linear character $\chi_3 \in \mathrm{Irr}(D_4)$. Note that $S_4/D_4 = \{D_4, (123)D_4, (321)D_4\}$ so $T = \{1, (123), (321)\}$ is a left transversal for D_4 in S_4. By Proposition 3.19 we can calculate $\varphi^G(x) = \dot{\varphi}(1^{-1}x1) + \dot{\varphi}((321)x(123)) + \dot{\varphi}((123)x(321))$. Since characters are constant on conjugacy classes we only need five calculations.

$$
\begin{aligned}
\varphi^G(1) &= 3\dot{\varphi}(1) = 3, \\
\varphi^G((12)(34)) &= \dot{\varphi}((12)(34)) + \dot{\varphi}((14)(23)) + \dot{\varphi}((13)(24)) = -1 - 1 + 1 = -1, \\
\varphi^G((123)) &= 3\dot{\varphi}((123)) = 0, \\
\varphi^G((12)) &= \dot{\varphi}((12)) + \dot{\varphi}((23)) + \dot{\varphi}((13)) = 0 + 0 + 1 = 1, \text{ and} \\
\varphi^G((1234)) &= \dot{\varphi}((1234)) + \dot{\varphi}((1423)) + \dot{\varphi}((1243)) = -1 + 0 + 0 = -1.
\end{aligned}
$$

Hence the values of φ^G over the conjugacy classes are

	cl{1}	cl{(12)(34)}	cl{(123)}	cl{(12)}	cl{(1234)}
φ^G	3	-1	0	1	-1

Comparing this to χ_4 in the character table of S_4 shows that $\varphi^G = \chi_4$, and since φ is linear then χ_4 is monomial. Similarly if we define θ as the linear character χ_2 in the character table of D_4, then we find that $\theta^G = \chi_5 \in \mathrm{Irr}(S_4)$. Therefore, every irreducible character of S_4 is monomial so S_4 is an M-group. $\diamondsuit$

Using the same idea, we can show that A_4 is an M-group since its only non-linear irreducible character is induced from the linear character χ_3 in the character table for K. Additionally, D_4 is an M-group because its irreducible character of degree two is induced from the linear character χ_3 in the character table for $\mathbb{Z}_4$. Finally, S_3 is also an M-group with its irreducible character of degree two induced from either of the nontrivial characters on the group $\mathbb{Z}_3$.

4.2 Solvable Groups and M-Groups

After looking at some familiar examples of M-groups, now we will look at some general results on M-groups. Earlier in this chapter we stated that every M-group was solvable. A group is *solvable* if there exists a finite collection of normal subgroups $1 = G_0 \leq G_1 \leq \ldots \leq G_n = G$ such that G_{i+1}/G_i is abelian for $0 \leq i < n$. We can also describe solvability in terms of the derived subgroups where we have $G = G^{(0)} \geq G^{(1)} \geq \ldots \geq G^{(n-1)} \geq G^{(n)}$. This series of groups is called the *derived series* of G. If G is solvable then the smallest integer such that $G^{(n)} = 1$ is called the *derived length* of G, $\mathrm{dl}(G)$.

As an example, consider the group S_4. We will show that S_4 is solvable and find its the derived length. Since A_4 is of index two in S_4 it is normal, but is $A_4 = S_4'$? Note that S_4/A_4 is isomorphic to $\mathbb{Z}_2$ so it is abelian and $S_4' \leq A_4$ by Theorem 2.5. Suppose $S_4' = H \neq A_4$ where $H \triangleleft S_4$ and $H < A_4$ Considering only the group A_4, we see that it has four subgroups of order 3 generated by the three-cycles (for example, $\langle(123)\rangle$) and one subgroup of order 4 generated by 2 two-cycles ($\langle(12)(34),(13)(24)\rangle$). If H is a subgroup generated by one of the three-cycles then H is not normal in S_4 since $(123)^{(14)(23)} = (243) \notin \langle(123)\rangle$. Thus H cannot possibly be one of the subgroups of order 3. If $H = 1$ then S_4/H is abelian which implies that S_4 is abelian. Since this is not true, $H \neq 1$. Now if $H \leq \langle(12)(34),(13)(24)\rangle$ then consider a four-cycle and a three-cycle $(abcd)$ and (abc) and their commutator $[(abcd),(abc)] = (abcd)(abc)(dcba)(cba) = (a)(bdc) \notin H$ (since no three-cycles are in H). We have found an element that is obviously in the commutator subgroup of S_4 which is not in H. We conclude then that $A_4 = S_4'$.

Now we need to find $A_4' = S_4^{(2)}$. Let $H = A_4'$. We have already explored some of the subgroup structure of A_4. By Lagrange, $|H| = 1, 2, 3, 4$ or 6. Observe that no subgroup generated by a three-cycle is normal in A_4 since $(abc)^{(ab)(cd)} = (adb) \notin \langle (abc) \rangle$. Thus $A_4' \neq \langle (abc) \rangle$. If $H = \{1\}$ then $A_4/H = A_4$ and A_4 is abelian. Since this is not true then $A_4' \neq \{1\}$. We now know that A_4' has order 2 or 4 or 6.

Note that $K \leq A_4$ since each element in the Klein-4 group has a decomposition into two transpositions. Moreover, $|A_4/K| = 3$ so A_4/K is abelian and $A_4' \leq K$. So $|A_4'| = 2$ or 4. Since no subgroup of A_4 of order two is normal in A_4 then A_4' cannot possibly be of order two. Let $H = \langle (ab)(cd), (ac)(bd) \rangle$, and because there is only one subgroup of order 4 in A_4, then H is normal in A_4. Thus $A_4' = H = \langle (ab)(cd), (ac)(bd) \rangle$.

Now we must find $S_4^{(3)}$. Since $S_4^{(2)} = A_4'$ is of order 4 with two generators, then $S_4^{(2)} \cong K$ (the Klein-4 group). We know that K is abelian; therefore, $S_4^{(3)} = \{1\}$, and $1 = S_4^{(3)} \leq S_4^{(2)} \leq S_4' \leq S_4$ where $S_4' = A_4$, and $S_4^{(2)} = \langle (ab)(cd), (ac)(bd) \rangle$. It follows that S_4 is solvable and $\mathrm{dl}(S_4) = 3$.

Theorem 4.2: *Let G be an M-group and let $1 = f_1 < f_2 < \ldots < f_k$ be the distinct degrees of the irreducible characters of G. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = f_i$. Then $G^{(i)} \leq \ker(\chi)$.*

Proof by Induction: For the base case let $i = 1$ so $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = f_1 = 1$. We want to show that $G' \leq \ker(\chi)$. Since $\chi(1) = 1$ then χ is linear. For $g, h \in G$ we have $\chi([g,h]) = \chi(g^{-1}h^{-1}gh) = \chi(g^{-1})\chi(h^{-1})\chi(g)\chi(h) = \chi(g^{-1})\chi(g)\chi(h^{-1})\chi(h) = 1$ which shows that $G' \leq \ker(\chi)$ as wanted

Let $i > 1$. By the inductive hypothesis for $j < i$, if $\chi(1) = f_j$ then $G^{(i-1)} \leq G^{(j)} \leq \ker(\chi)$.

Let $\chi \in \mathrm{Irr}(G)$ and since χ is monomial and $i > 1$ then there exists $H < G$ (a proper subgroup) and a linear character $\lambda \in \mathrm{Irr}(H)$ such that $\lambda^G = \chi$. Denote 1_G as the trivial character on G and 1_H as the trivial character on H. By the Frobenius Reciprocity

(Theorem 3.21)

$$\langle 1_H, 1_G|_H\rangle = \langle (1_H)^G, 1_G\rangle.$$

Since $1_G|_H = 1_H$ and $1_H \in \mathrm{Irr}(G)$ then

$$1 = \langle 1_H, 1_H\rangle = \langle (1_H)^G, 1_G\rangle.$$

If $(1_H)^G \in \mathrm{Irr}(G)$ then $1_G = (1_H)^G$ which means $1_G(1) = (1_H)^G(1)$, or $1 = [G : H]$. Since H is a proper subgroup of G this is impossible. Thus $(1_H)^G \notin \mathrm{Irr}(G)$. By Corollaries 3.8 and 3.11

$$(1_H)^G = \sum_{j=1}^{n} a_j\psi_j \text{ where } \mathrm{Irr}(G) = \{\psi_1, \ldots, \psi_n\} \text{ and } a_j = \langle (1_H)^G, \psi_j\rangle.$$

Since $(1_H)^G$ is not irreducible, we have $a_j \geq 2$ for some j or at least two of the a_j are greater than 0. This implies that $\deg(\psi_j) < \deg((1_H)^G)$ for all $j = 1, \ldots, n$ with $a_j \neq 0$. By relabeling the ψ_j we may assume that $\psi_1, \ldots, \psi_m$ for $m \leq n$ are the constituents of $(1_H)^G$. Then we know that $(1_H)^G = \sum_{j=1}^{m} a_j\psi_j$ where $a_j > 0$. So for all $j = 1, \ldots, m$

$$\psi_j(1) < (1_H)^G(1) = [G : H] = \lambda^G(1) = \chi(1) = f_j.$$

By the inductive hypothesis $G^{(i-1)} \leq \ker(\psi_j)$ for $j = 1, \ldots, m$. For all $g \in G^{(i-1)}$ we have $\psi_j(g) = \psi_j(1)$ for all $j = 1, \ldots, m$ by Proposition 3.15. But for all $g \in G^{(i-1)}$, we have $(1_H)^G(g) = \sum_{j=1}^{m} a_j\psi_j(g) = \sum_{j=1}^{m} a_j\psi_j(1) = (1_H)^G(1)$. Thus $g \in \ker((1_H)^G)$, again by Proposition 3.15. It follows that $G^{(i-1)} \leq \ker((1_H)^G)$. By Proposition 3.20, $\ker((1_H)^G) = \bigcap_{x\in G}(\ker(1_H))^x$. But $\ker(1_H) = H$ so $G^{(i-1)} \leq \bigcap_{x\in G} H^x \leq H$ (by taking $x = 1$). It follows that $\left(G^{(i-1)}\right)' \leq H'$ and because $G^{(i)} = \left(G^{(i-1)}\right)'$, then $G^{(i)} \leq H'$. Since $\lambda \in \mathrm{Irr}(H)$ and λ is linear, then λ is a homomorphism into the abelian group $\mathbb{C}^\times$. It follows that $H' \leq \ker(\lambda)$ by Corollary 2.6. Thus $G^{(i)} \leq \ker(\lambda)$. Observe that $G^{(i)}$ is normal in G so $(G^{(i)})^x \leq (\ker(\lambda))^x$ for all $x \in G$ implies $G^{(i)} \leq \bigcap_{x\in G}(\ker(\lambda))^x = \ker(\lambda^G)$ by Proposition 3.20. Recall that $\lambda^G = \chi$ so $G^{(i)} \leq \ker(\chi)$. $\diamondsuit$

Corollary 4.3: *If G is an M-group, then G is solvable.*

Proof: Let G be an M-group and $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = f_i$. Then by Theorem 4.2, $G^{(i)} \leq \ker(\chi)$. Since this is true for all irreducible characters,

$$G^{(k)} \leq G^{(i)} \leq \ker(\chi) \text{ for all } i = 1, 2, \ \ldots, k$$

where $k = |\mathrm{Irr}(G)|$. Hence $G^{(k)} \leq \bigcap_{\psi \in \mathrm{Irr}(G)} \ker(\psi)$. Let $g \in \ker(\psi)$ such that $g \neq 1$ and $\psi \in \mathrm{Irr}(G)$. In the proof of Theorem 3.13 we show for the character ρ afforded by the right regular representation, that the multiplicity of each irreducible character of G in ρ is simply the degree of that character. In other words, $\langle \rho, \psi \rangle = \psi(1)$. So by Corollary 3.11 we can express ρ as

$$\rho(g) = \sum_{\psi \in \mathrm{Irr}(G)} \langle \rho, \psi \rangle \psi(g) = \sum_{\psi \in \mathrm{Irr}(G)} \psi(1)\psi(g).$$

Recall that in Section 3.1 we showed that ρ has a value of 0 for all non-identity elements so if $g \neq 1$ then

$$0 = \rho(g) = \sum_{\psi \in \mathrm{Irr}(G)} \psi(1)\psi(g).$$

But since $g \in \ker(\psi)$ for all $\psi \in \mathrm{Irr}(G)$ then $\psi(g) = \psi(1)$ (see Proposition 3.15). Hence

$$0 = \sum_{\psi \in \mathrm{Irr}(G)} (\psi(1))^2$$

which implies $\psi(1) = 0$ for all $\psi \in \mathrm{Irr}(G)$. This cannot happen so the assumption that $g \neq 1$ is incorrect. Thus for $g \in \ker(\psi)$, $g = 1$ and $G^{(k)} \leq \bigcap_{\psi \in \mathrm{Irr}(G)} \ker(\psi) \leq 1$. Hence $G^{(k)} = 1$, and G is solvable with $\mathrm{dl}(G) \leq k$. $\diamondsuit$

One might think that all solvable groups are M-groups, but that is simply not true. A counterexample is the group $\mathrm{SL}(2,3)$ which is solvable but not an M-group [5, p.67].

4.3 Miscellaneous Background Material

The goal of this thesis is to explain Alan Parks' characterization of M-groups in purely group-theoretic terms, not in terms of characters as was done above. In order to discuss this new definition of M-groups, we need to review some basic facts of group theory, Galois theory, and field theory. The following theorems and propositions are results we will need in the next chapter when explaining Parks' article.

Recall the following basic results from field and Galois theory presented without formal proof

<u>Result 1:</u> Let $\mathcal{G} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) = \{\sigma \in \mathrm{Aut}(\mathbb{C}) | \sigma(x) = x \text{ for all } x \in \mathbb{Q}\}$, $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, $f(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial in $\mathbb{Q}[X]$ (meaning the coefficients $a_i \in \mathbb{Q}$), and α be a root of $f(x)$. Then $0 = [f(\alpha)]^\sigma = \sigma(\sum_{i=0}^{n} a_i \alpha^i) = \sum_{i=0}^{n} \sigma(a_i) \cdot \sigma(\alpha^i) = \sum_{i=0}^{n} a_i \cdot \sigma(\alpha^i) = \sum_{i=0}^{n} a_i \cdot (\sigma(\alpha))^i = f(\alpha^\sigma)$. This means that σ permutes the roots of $f(x)$ [4, p.277], or more generally each automorphism $\sigma \in \mathcal{G} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ permutes all the roots of a polynomial in $\mathbb{Q}[X]$.

<u>Result 2:</u> If we define $\Omega = \{\alpha \in \mathbb{C} \mid f(\alpha) = 0\}$ and assume $f \neq 0$ is an irreducible polynomial in $\mathbb{Q}[X]$, then the action of the Galois group over $\mathbb{C}$ is transitive on Ω [4, p.277]. Recall that a nonconstant polynomial in $\mathbb{Q}[X]$ is *irreducible* over $\mathbb{Q}$ if it cannot be expressed as the product of two polynomials in $\mathbb{Q}[X]$ both of lower degree than the given polynomial. Now because the action is transitive we know if $\omega_1, \omega_2 \in \Omega$ then there exists $\sigma \in \mathcal{G}$ such that $\sigma(\omega_1) = \omega_2$.

<u>Result 3:</u> Define the nth cyclotomic polynomial as $\Phi_n(x) = \Pi_\epsilon(x - \epsilon)$ where ϵ runs over the primitive complex roots of unity. Further, the set of the nth roots of unity is $\{e^{i\frac{2\pi k}{n}} \mid k = 0, \ldots, n-1\}$. Since $e^{i\theta} = \sin\theta + i\cos\theta$ then for every value of θ, $e^{i\theta}$ is a complex number on the unit circle in the complex plane. The values of $e^{i\theta}$ that when raised to some power yield one, are the roots of unity. For ϵ to be a primitive nth root of unity, $\epsilon^m \neq 1$ for $1 \leq m \leq n$. Isaacs book on group theory states several important results about the cyclotomic polynomial in chapter 20 [4]:

1. The nth cyclotomic polynomial has integer coefficients [4, p.309],
2. The set of nth roots of unity form a cyclic group under multiplication [4, p.307],
3. For a prime p, the pth cyclotomic polynomial $\Phi_p(x) = \Pi_\epsilon(x - \epsilon) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$ [4, p.309],
4. The number of primitive nth roots of unity is $\phi(n)$ which is Euler's function [4, p.308], and
5. The nth cyclotomic polynomial is irreducible in $\mathbb{Q}[X]$ for all integers $n \geq 1 [4, p.311]$.

This last statement is the most important because when the nth cyclotomic polynomial is irreducible in $\mathbb{Q}[X]$, then for every pair of roots of the polynomial $\Phi_n(x)$ there is an automorphism in $\mathcal{G} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ that carries one root to the other root (because of the transitivity discussed in Result 2).

Before using this information to prove the following proposition some new notation needs to be introduced and explained. If f is a function, $f : G \longrightarrow F$ where G is a group, F is a field, and $\sigma \in \mathrm{Aut}(F)$, then we write $f^\sigma = \sigma \circ f$. Equivalently, $f^\sigma(g) = (\sigma \circ f)(g) = \sigma(f(g))$.

Proposition 4.4: *Let f and h be injective functions defined on a finite, cyclic group G into $\mathbb{C}^\times$; that is,*

$$f : G \longrightarrow \mathbb{C}^\times \qquad \textit{and} \qquad h : G \longrightarrow \mathbb{C}^\times.$$

Then there exists $\sigma \in \mathcal{G} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ such that $f^\sigma = h$.

Proof: G is cyclic so suppose $\langle g \rangle = G$ and let $|G| = a$. In order to find the automorphism σ that will map f to h we need to know how the functions behave. Since both functions are defined on a cyclic group, then the assignment of the generator, g, completely determines all other values of the function. Let $f(g) = e^{\imath \frac{2\pi k}{a}} = \beta_1$ for some $k \in \{0, 1, \ldots, a-1\}$ where $\gcd(k, a) = 1$. Likewise $h(g) = e^{\imath \frac{2\pi k'}{a}} = \beta_2$ for some $k' \in \{0, 1, \ldots, a-1\}$ where $\gcd(k', a) = 1$. Note that both β_1 and β_2 are primitive roots of unity, and therefore roots of the ath cyclotomic polynomial which is irreducible in $\mathbb{Q}[X]$. From Galois theory we know that since the action of $\mathcal{G}$ on the primitive ath roots of unity is transitive, then there exists $\sigma \in \mathcal{G}$ such that $\sigma(\beta_1) = \beta_2$. So $f^\sigma(g) = \sigma(f(g)) = \sigma(\beta_1) = \beta_2 = h(g)$. Since f^σ and h agree on their generator g, then $f^\sigma(g^n) = h(g^n)$ where $n \in \{0, 1, \ldots, a-1\}$. We have shown there exists $\sigma \in \mathcal{G}$ such that $f^\sigma = h$. $\diamondsuit$

Proposition 4.5: *If $\xi \in \mathcal{G}$ and φ is a character of $H \leq G$, then $(\varphi^G)^\xi = (\varphi^\xi)^G$.*

Proof: We will simply apply the definition of an induced character and use the fact that automorphisms of a field preserve multiplication and addition.

$$\begin{aligned} (\varphi^G)^\xi(g) &= \left(\xi \circ \varphi^G\right)(g) \\ &= \xi(\varphi^G(g)) \end{aligned}$$

$$\begin{aligned}
&= \xi\left(\frac{1}{|H|}\sum_{t\in G}\dot{\varphi}(t^{-1}gt)\right)\\
&= \frac{1}{|H|}\sum_{t\in G}\xi(\dot{\varphi}(t^{-1}gt))\\
&= \frac{1}{|H|}\sum_{t\in G}(\xi\circ\dot{\varphi})(t^{-1}gt))\\
&= \frac{1}{|H|}\sum_{t\in G}(\dot{\varphi})^{\xi}(t^{-1}gt)\\
&= \frac{1}{|H|}\sum_{t\in G}(\dot{\varphi^{\xi}})(t^{-1}gt)\\
&= (\varphi^{\xi})^{G}(g) \text{ for all } g\in G.
\end{aligned}$$

$\diamondsuit$

Proposition 4.6: *If λ is a linear character of $H \leq G$ such that* $\ker(\lambda) = M$, *then* $\ker(\lambda^x) = M^x$.

Proof: Let $g \in \ker(\lambda^x)$, then $\lambda^x(g) = 1$ which implies $\lambda(xgx^{-1}) = 1$. Hence $xgx^{-1} \in \ker(\lambda) = M$. It follows that $g \in x^{-1}Mx$, or $g \in M^x$. Similarly if we let $g \in M^x$, then $g = x^{-1}mx$ where $m \in M$. Then $\lambda^x(g) = \lambda^x(x^{-1}mx) = \lambda(x(x^{-1}mx)x^{-1})) = \lambda(m) = 1$ since M is the kernel of λ. Thus $\lambda^x(g) = 1$ and $g \in \ker(\lambda^x)$; therefore, $M^x = \ker(\lambda^x)$. $\diamondsuit$

4.4 The Schur Index

The last critical piece of background information is the Schur index. Before we can begin to discuss the Schur index, we must understand some of the ideas behind the concept. The following discussion and two propositions will provide the framework for understanding the Schur index.

Suppose F and K are fields such that $F \subseteq K$ and K is a finite extension of F. So let $[K : F] = n$ and this implies the basis for K as a vector space over F has n vectors. Define $\{b_1, \ldots, b_n\}$ as that basis. Let $a \in K$ and $a = \sum_{i=1}^{n} f_i b_i$ where $f_i \in F$ since every element in K can be written as a linear combination of vectors in the basis Let $\tilde{a}$ be an F-linear transformation of K into K such that $\tilde{a}(b) = ab$ for all $b \in K$. Observe that this is, in fact, an F-linear transformation since $\tilde{a}(f \cdot b) = f \cdot \tilde{a}(b)$ and $\tilde{a}(b+c) = \tilde{a}(b)+\tilde{a}(c)$ for every $f \in F$ and $b, c \in K$. Every linear transformation of K is completely determined by where it sends the basis vectors. We know $\tilde{a}(b) \in K$ so suppose $\tilde{a}(b_1) = \sum_{i=1}^{n} f_{i1} b_i$ where $f_{i1} \in F$ for all i. Similarly $\tilde{a}(b_2) = \sum_{i=1}^{n} f_{i2} b_i$, and we continue in this fashion so that $\tilde{a}(b_n) = \sum_{i=1}^{n} f_{in} b_i$. We can form the matrix associated with the linear transformation $\tilde{a}$ and denote it as $\breve{a}$ where

$$\breve{a} = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{bmatrix}.$$

Now define the function $\mathrm{Tr}_{K/F} : K \longrightarrow F$ such that $\mathrm{Tr}_{K/F}(a) = \mathrm{Tr}(\tilde{a}) = \mathrm{Tr}(\breve{a}) = \sum_{i=1}^{n} f_{ii} \in F$. Let $x \in F$, so $\tilde{x}(b) = xb$ for all $b \in K$, and $\tilde{x}(b_1) = x \cdot b_1 = \sum_{i=1}^{n} f_{i1} b_i$. Observe that $0 = (f_{11} - x)b_1 + f_{21}b_2 + \ldots + f_{n1}b_n$. Since $\{b_1, \ldots b_n\}$ is a basis, then the vectors are linearly independent. Hence $f_{i1} = 0$ for all $i \geq 2$ and $(f_{11} - x) = 0$. This implies $f_{11} = x$. Similarly $\tilde{x}(b_2) = x \cdot b_2 = \sum_{i=1}^{n} f_{i2} b_i$, and $0 = f_{12} + (f_{22} - x)b_2 + \ldots f_{n2} b_n$. Once again $f_{12}, f_{32}, \ldots, f_{n2} = 0$ and $(f_{22} - x) = 0$; it follows that $f_{22} = x$. In the same manner we find that $f_{ij} = x$ when $i = j$ and $f_{ij} = 0$ when $i \neq j$. The matrix representation of this linear transformation is,

$$\breve{x} = \begin{bmatrix} x & 0 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x \end{bmatrix}.$$

Since there are n vectors in the basis, $\breve{x}$ is an $n \times n$ matrix . Hence $\mathrm{Tr}(\breve{x}) = n \cdot x = [K : F] \cdot x$ since $[K : F] = n$. So if $x \in F$ then $\mathrm{Tr}_{K/F}(x) = \mathrm{Tr}(\breve{x}) = [K : F] \cdot x$.

Proposition 4.7: *Suppose F and K are fields such that $F \subseteq K \subseteq \mathbb{C}$, $[K : F]$ is finite and that T is a K-representation of G with character χ and degree k. Then there exists an F-representation of G, denoted T' with associated character $\chi' = Tr_{K/F} \circ \chi$.*

Proof: Choose a basis of K over F, say $\{b_1, \ldots, b_n\}$ where $[K \cdot F] = n$. Define $T(g) = [t_{ij}(g)]$ where $t_{ij}(g) \in K$. Now define $T'(g)$ as the matrix which has a submatrix inserted into each position $[t_{ij}(g)]$. Replace each entry $t_{ij}(g)$ with the matrix associated with the linear transformation $\tilde{t_{ij}}(g)$. Using $\tilde{t_{ij}}(g)$ we can determine where the basis vectors of K are mapped, and thereby find $\check{t_{ij}}(g)$. Clearly T' is an F-representation because each entry in $\check{t_{ij}}$ is an element of F. Further $\chi'(g) = \mathrm{Tr}(T'(g)) = \sum_{i=1}^{k} \mathrm{Tr}([\check{t_{ii}}(g)]) = \sum_{i=1}^{k} \mathrm{Tr}_{K/F}(t_{ii}(g)) = \mathrm{Tr}_{K/F} \sum_{i=1}^{k} t_{ii}(g) = \mathrm{Tr}_{K/F}(\chi(g)) = (\mathrm{Tr}_{K/F} \circ \chi)(g)$. $\diamondsuit$

We can now state and sketch the proof of the key proposition which gives rise to the Schur index.

Proposition 4.8: *If $F \subseteq \mathbb{C}$ and $\chi \in \mathrm{Irr}(G)$ then there exists $m > 0$, $m \in \mathbb{N}$ such that $m\chi$ is an $F(\chi)$-character.*

Proof: First note that $F(\chi)$ is an extension of the field F where all values of χ are adjoined to F. By basic results from field theory it can be shown that there exists a finite extension K of the field $F(\chi)$ of which χ is a K-character. So the representation $T = [t_{ij}]$ affording the irreducible character χ might only have entries from $F(\chi)$ but not necessarily - it could have entries from K. By Proposition 4.7 there exists an $F(\chi)$-representation of G with associated character $\chi' = \mathrm{Tr}_{K/F(\chi)} \circ \chi$. Take the K-representation and each entry in the matrix is replaced with the matrix $\check{t_{ij}}(x)$. Pick $m = [K : F(\chi)]$ and since $\chi(x) \in F(\chi)$ for all $x \in G$ (meaning the character has values all in $F(\chi)$) then

$$\chi'(x) = (\mathrm{Tr}_{K/F(\chi)} \circ \chi)(x) = \mathrm{Tr}_{K/F(\chi)}(\chi(x)) = [K : F(\chi)] \cdot \chi(x) = m\chi(x)$$

for all $x \in G$. So there exists $m \in \mathbb{N}$ such that $m\chi = \chi'$ is indeed an $F(\chi)$-character. $\diamondsuit$

We call the smallest such m the *Schur index* of χ relative to F and we write $m_F(\chi)$. So $m_F(\chi)$ is the least positive integer m for which $m\chi$ is an $F(\chi)$-character. Observe that if $\sigma \in \mathrm{Gal}(F(\chi)/F)$ that the Schur index for χ^σ is simply the Schur index for χ as can easily be checked.

CHAPTER 5

A GROUP-THEORETIC CHARACTERIZATION OF M-GROUPS

5.1 Good Pairs and Induced Characters

Recall from Chapter 4 that an M-group is a group where every irreducible character is induced from a linear character of some subgroup of G. I. Martin Isaacs posed the problem of finding a characterization of M-groups in purely group theoretic terms [5, p.67], and Alan Parks has found such a characterization based on cyclic subgroups and the notion of good pairs as described in his article [7].

Let $M, H \leq G$ and $M \trianglelefteq H$ such that H/M is cyclic. Then we define (H, M) to be a *pair*. For $H \leq G$ and $g \in G$, define $F_H(g)$ to be the set of commutators $[g, H \cap H^{g^{-1}}]$, that is, $F_H(g) = \{[g, x] \mid x \in H \cap H^{g^{-1}}\}$. We define (H, M) as a *good pair* if, for all $g \in G \setminus H$, $F_H(g) \nsubseteq M$. Note that $F_H(g) \subseteq H$ because if $g^{-1}x^{-1}gx \in F_H(g)$ for some $x \in H \cap H^{g^{-1}}$ then in particular $x \in H^{g^{-1}}$. Thus $x \in gHg^{-1}$, so $x = ghg^{-1}$ for some $h \in H$ which implies $x^{-1} = gh^{-1}g^{-1}$. Then $g^{-1}x^{-1}gx = g^{-1}(gh^{-1}g^{-1})gx = h^{-1}x \in H$. The distinction for a good pair is that although $F_H(g) \subseteq H$ for all $g \in G$, $F_H(g) \nsubseteq M$ if $g \notin H$.

For any pair (H, M) there exists a linear, irreducible character of H whose kernel is

M. This character is said to *proceed* from (H, M). Observe that we are guaranteed the existence of this character since H/M is cyclic. If the order of H/M is k, and hM is a generator of H/M, define λ to be a character of H as follows: $\lambda(g) = \varepsilon^i$ if $(hM)^i = gM$ and ε is a primitive kth root of unity. We can see $\ker(\lambda) = M$ since for $m \in M$, $(hM)^k = mM$ which implies $\lambda(m) = \varepsilon^k = 1$. Further, if $\lambda(x) = 1 = \varepsilon^k$ then $xM = (hM)^k = M$, and $x \in M$. Thus λ is a linear, irreducible character of H with kernel M, and therefore λ proceeds from (H, M).

Now for Parks' characterization of M-groups. We define a relation on the set of good pairs as follows. Suppose (H, M) and (K, L) are good pairs. Then $(H, M) \sim (K, L)$ if there exists $g \in G$ such that

$$H^g \cap L = K \cap M^g.$$

Further, we call this relation on good pairs S_G. Below we will show that S_G is indeed an equivalence relation (see Corollary 5.5), so let m_G be the number of equivalence classes of S_G.

We can identify another relation on elements of G defined as follows: $x \sim y$ if and only if there exists $g \in G$ such that $\langle x \rangle^g = \langle y \rangle$. The equivalence classes in this relation are often referred to as the *rational conjugacy classes* of G, and we will denote the number of rational conjugacy classes of G to be n_G. Observe that this is an equivalence relation since the reflexive, symmetric and transitive properties hold as demonstrated below.

<u>Reflexive:</u> We must show that $\langle x \rangle$ is conjugate to itself, so simply pick $1 \in G$ and observe that $\langle x \rangle^1 = \langle x \rangle$.

<u>Symmetric:</u> We must show if $\langle x \rangle^g = \langle y \rangle$ for some $g \in G$, then there exists $k \in G$ where $\langle y \rangle^k = \langle x \rangle$. Since $\langle x \rangle^g = \langle y \rangle$, pick $k = g^{-1}$. We know $\langle y \rangle = g^{-1}\langle x \rangle g$ which implies $g\langle y \rangle g^{-1} = \langle x \rangle$. Hence $\langle y \rangle^{g^{-1}} = \langle x \rangle$, or $\langle y \rangle^k = \langle x \rangle$.

<u>Transitive:</u> We must show if $\langle x \rangle^g = \langle y \rangle$ and $\langle y \rangle^h = \langle z \rangle$ for some $g, h \in G$, then there exists $j \in g$ such that $\langle x \rangle^j = \langle z \rangle$. Pick $j = gh$ and $\langle x \rangle^j = \langle x \rangle^{gh} = (gh)^{-1}\langle x \rangle(gh) = h^{-1}(g^{-1}\langle x \rangle g)h = h^{-1}\langle y \rangle h = \langle z \rangle$.

Using the equivalence relations we just described, we can state Parks' theorem.

Theorem 5.1: *Let G be a finite group. Then $m_G \leq n_G$ with equality if and only if G is an M-group.*

This theorem provides the desired characterization of M-groups without any dependence on characters. Since its proof comprises the remainder of this thesis, it will be presented later.

Lemma 5.2: *Let λ proceed from the pair (H, M) and let $x \in G$. Then $\langle \lambda_{H^x \cap H}, (\lambda^x)_{H^x \cap H} \rangle = 1$ if and only if $F_H(x^{-1}) \subseteq M$.*

Proof: First recall that $F_H(x^{-1})$ is the set of all commutators $[x^{-1}, H \cap H^x]$. Let $K = H \cap H^x$, and since λ proceeds from (H, M) then λ is linear. Certainly a restriction of λ is still a linear character. Hence λ_K and $(\lambda^x)_K$ are linear. Since linear characters are irreducible then either $\langle \lambda_K, (\lambda^x)_K \rangle = 0$ or $\langle \lambda_K, (\lambda^x)_K \rangle = 1$.

Now suppose that $\langle \lambda_K, (\lambda^x)_K \rangle = 1$. By Theorem 3.9 $(\lambda^x)_K(g) = \lambda_K(g)$ for all $g \in K$. So $\lambda_K(xgx^{-1}) = \lambda_K(g)$, and since λ is linear then we can view the character as a homomorphism from the group G into $\mathbb{C}^\times$ under the group operation of multiplication. Therefore, $\lambda_K(xgx^{-1}) \cdot (\lambda_K(g))^{-1} = 1$, and $\lambda_K(xgx^{-1}g^{-1}) = 1$. So for all $g \in K$, $xgx^{-1}g^{-1} \in \ker(\lambda_K)$, or equivalently, $[x^{-1}, g^{-1}] \in \ker(\lambda_K)$ for all $g \in K$. Recall that λ proceeds from (H, M) so $\ker(\lambda_K) \subseteq M$. Hence $[x^{-1}, g^{-1}] \in M$ for all $g \in K$. It follows that $[x^{-1}, K] \subseteq M$ and thus $F_H(x^{-1}) \subseteq M$.

Next suppose $F_H(x^{-1}) \subseteq M$, then $[x^{-1}, g^{-1}] \in M$ for all $g \in K$ where $K = H^x \cap H$ That is, $[x^{-1}, g^{-1}] \in \ker(\lambda_K)$, or, $\lambda_K(xgx^{-1}g^{-1}) = 1$ Since λ is linear then $1 = \lambda_K(xgx^{-1}) \cdot \lambda_K(g^{-1})$. This yields, $1 = \lambda_K(xgx^{-1}) \cdot (\lambda_K(g))^{-1}$ and it follows that $\lambda_K(g) = \lambda_K(xgx^{-1})$. But this is equivalent to $\lambda_K(g) = (\lambda^x)_K(g)$ for all $g \in K$. Thus by Theorem 3.9 we have $\langle \lambda_K, (\lambda^x)_K \rangle = 1$. $\diamondsuit$

Proposition 5.3: *Let (H, M) be a pair with $H \leq G$ and let λ proceed from (H, M). Then (H, M) is a good pair if and only if λ^G is irreducible.*

Proof: Suppose that $\lambda^G \in \mathrm{Irr}(G)$ and then $\langle \lambda^G, \lambda^G \rangle = 1$ by the First Orthogonality Relation. Pick some $x \in G \setminus H$ and there is an H, H transversal T such that $1, x \in T$ (Recall that an H, H transversal is simply a set of double coset representatives for the subgroups H and H - refer to Section 3.7, p.45). By Mackey's Theorem $\langle \lambda^G, \lambda^G \rangle = \sum_{t\in T} \langle \lambda_{H^t \cap H}, (\lambda^t)_{H^t\cap H} \rangle$. So

$$\begin{aligned} 1 &= \langle \lambda^G, \lambda^G \rangle \\ &\geq \langle \lambda_{H^x \cap H}, (\lambda^x)_{H^x \cap H} \rangle + \langle \lambda_{H^1 \cap H}, (\lambda^1)_{H^1 \cap H} \rangle \\ &= \langle \lambda_{H^x \cap H}, (\lambda^x)_{H^x \cap H} \rangle + \langle \lambda_H, \lambda_H \rangle. \end{aligned}$$

Obviously $\langle \lambda_H, \lambda_H \rangle = 1$ which implies that $\langle \lambda_{H^x \cap H}, (\lambda^x)_{H^x \cap H} \rangle = 0$. By Lemma 5.2 since $\langle \lambda_{H^x \cap H}, (\lambda^x)_{H^x \cap H} \rangle \neq 1$, then $F_H(x^{-1}) \nsubseteq M$. Hence (H, M) is a good pair since for all $x \in G \setminus H$, $F_H(x) \nsubseteq M$.

Now suppose that (H, M) is a good pair and we must show that λ^G is irreducible. Since (H, M) is a good pair then for all $x \in G \setminus H$, $F_H(x) \nsubseteq M$. Then by Lemma 5.2 we know that $\langle \lambda_{H^x \cap H}, (\lambda^x)_{H^x \cap H} \rangle \neq 1$. Since λ is linear and therefore irreducible, then by the First Orthogonality Relation $\langle \lambda_{H^x \cap H}, (\lambda^x)_{H^x \cap H} \rangle = 0$ for all $x \in G \setminus H$. By Mackey's Theorem, $\langle \lambda^G, \lambda^G \rangle = \sum_{t\in T} \langle \lambda_{H^t \cap H}, (\lambda^t)_{H^t \cap H} \rangle = \langle \lambda_H, \lambda_H \rangle + \sum_{t \in T\setminus 1} \langle \lambda_{H^t \cap H}, (\lambda^t)_{H^t \cap H} \rangle$. Since $\langle \lambda_{H^t \cap H}, (\lambda^t)_{H^t \cap H} \rangle = 0$ for all $t \in T \setminus 1$, then $\langle \lambda^G, \lambda^G \rangle = \langle \lambda_H, \lambda_H \rangle = 1$. Therefore λ^G is irreducible by Corollary 3.10. $\diamondsuit$

The following proposition will help to clarify the relationship between characters and related good pairs.

Proposition 5.4: *If (H, M) and (K, L) are good pairs then they are related if and only if there are characters λ and μ proceeding from (H, M) and (K, L), respectively, such that $\lambda^G = \mu^G$.*

Proof: Suppose λ and μ proceed from good pairs (H, M) and (K, L) respectively and $\lambda^G = \mu^G$. We want to show that (H, M) and (K, L) are related, that is, we need to show there exists $x \in G$ such that $H^x \cap L = K \cap M^x$.

Let T be an H, K transversal in G such that $T = \{x_\imath | 1 = 1, \ldots, t\}$ and $\{Hx_\imath K | \imath =$

$1,\ldots,t\}$ is a set of double cosets whose union is all of G. By Mackey's Theorem, $\langle\lambda^G,\mu^G\rangle = \sum_{i=1}^{t}\langle\mu_{H^{x_i}\cap K},(\lambda^{x_i})_{H^{x_i}\cap K}\rangle$. By assumption, $\lambda^G=\mu^G$ and because of Proposition 5.3 we know that λ^G and μ^G are irreducible. So $\langle\lambda^G,\mu^G\rangle = 1$. Thus, $1 = \sum_{i=1}^{t}\langle\mu_{H^{x_i}\cap K},(\lambda^{x_i})_{H^{x_i}\cap K}\rangle$ and there exists $x\in T$ such that $\langle\mu_{H^x\cap K},(\lambda^x)_{H^x\cap K}\rangle \neq 0$. Since μ and λ proceed from good pairs by the assumption, then they are both linear characters on K and H respectively. Thus $\lambda_{H^x\cap K}$ and $\mu_{H^x\cap K}$ are also linear and so is $(\lambda^x)_{H^x\cap K}$. Linear characters are irreducible so $\langle\mu_{H^x\cap K},(\lambda^x)_{H^x\cap K}\rangle = 0$ or 1 for all $x\in T$. We conclude there exists $x\in T$ such that $\langle\mu_{H^x\cap K},(\lambda^x)_{H^x\cap K}\rangle = 1$ which implies $(\lambda^x)_{H^x\cap K}=\mu_{H^x\cap K}$. Recall that characters are functions so $\ker(\mu_{H^x\cap K}) = \ker((\lambda^x)_{H^x\cap K})$.

By the definition of kernel, $\ker(\mu_{H^x\cap K}) = \{y\in H^x\cap K|\mu(y)=1\}$. Since μ proceeds from (K,L) then $\ker(\mu)=L$ by definition. μ is defined on K and since $H^x\cap K\leq K$ then $\ker(\mu_{H^x\cap K}) = \ker(\mu)\cap H^x\cap K = L\cap H^x\cap K$. Now λ is defined on H and proceeds from (H,M) so $\ker(\lambda)=M$. We know that λ^x is defined on H^x and $H^x\cap K\leq H^x$. Moreover, $\ker(\lambda^x)=M^x$ by Proposition 4.6. Thus $\ker((\lambda^x)_{H^x\cap K}) = \ker(\lambda^x)\cap H^x\cap K = M^x\cap H^x\cap K$.

We have $\ker(\mu_{H^x\cap K}) = \ker((\lambda^x)_{H^x\cap K})$ so we conclude that $L\cap H^x\cap K = M^x\cap H^x\cap K$. Since $M\leq H$, then $M^x\leq H^x$ and $M^x\cap H^x = M^x$. Similarly $L\leq K$ so $L\cap K = L$. It follows that $L\cap H^x\cap K = L\cap H^x$ and $M^x\cap H^x\cap K = M^x\cap K$. Thus, $H^x\cap L = K\cap M^x$ which is precisely the definition of two good pairs being related. We have shown that $(H,M)\sim(K,L)$.

Now suppose that good pairs (H,M) and (K,L) are related, and we want to show there exist λ and μ proceeding from (H,M) and (K,L) respectively, such that $\lambda^G=\mu^G$.

Because (H,M) and (K,L) are related then $H^x\cap L = K\cap M^x$ for some $x\in G$ and it follows that $(H^x\cap L)\cap K = K\cap M^x$, and $H^x\cap L\cap K = H^x\cap(K\cap M^x)$ Define $N = H^x\cap L\cap K$ and obviously $N\leq H^x\cap K$. Now observe $L\trianglelefteq K$ and $H^x\cap K\leq G$ so $N = L\cap(H^x\cap K)\trianglelefteq K\cap(H^x\cap K) = K\cap H^x$. Thus $N\trianglelefteq H^x\cap K$ so we can consider the factor group $(H^x\cap K)/N$.

By the Diamond Theorem [4, p.33], since $(H^x\cap K)\leq G$ and $L\trianglelefteq K$, then $(H^x\cap K)/((H^x\cap K)\cap L)\cong((H^x\cap K)L)/L$. Also $(H^x\cap K)\leq K$ and $L\trianglelefteq K$, so $(H^x\cap K)L\leq K$ and $(H^x\cap K)/N$ is isomorphic to some subgroup U of K/L Because (K,L) is a pair then

K/L is cyclic. Since all irreducible characters of cyclic subgroups are linear then there exists a linear character, $\nu \in \mathrm{Irr}((H^x \cap K)/N)$ such that ν is faithful. (Simply define the value of ν on the generator to be a kth primitive root when the order of the cyclic group $(H^x \cap K)/N$ is k). Since U and $(H^x \cap K)/N$ are isomorphic we can identify the values of ν on $(H^x \cap K)/N$ with the corresponding elements in U. Then we can extend ν to an irreducible character of K denoted μ such that $\ker(\mu) = L$.

Now $N = H^x \cap K \cap M^x \trianglelefteq H^x \cap K$. Because $M \trianglelefteq H$ then certainly $M^x \trianglelefteq H^x$ and by the Diamond Theorem [4, p.33], $(H^x \cap K)/N \cong (H^x \cap K)M^x/M^x$. Observe that $H^x \cap K \leq H^x$ and $M^x \trianglelefteq H^x$. So $(H^x \cap K)M^x/M^x$ is isomorphic to some subgroup of H^x/M^x call it V. We know that (H, M) is a good pair, so there exists ψ proceeding from (H, M) such that ψ^G is irreducible. Define ψ^x on H^x as the conjugate character of ψ. Since $\ker(\psi) = M$ then $\ker(\psi^x) = M^x$. Moreover, since ψ is linear then so is ψ^x. Finally by Proposition 3.29 we have that $1 = \langle \psi^G, \psi^G \rangle = \langle (\psi^x)^G, (\psi^x)^G \rangle$. Thus $(\psi^x)^G$ is irreducible, (H^x, M^x) is a good pair by Proposition 5.3, and H^x/M^x is cyclic. Hence $(H^x \cap K)/N = (H^x \cap K)/(H^x \cap K \cap M^x) \cong V$ which is some cyclic subgroup of H^x/M^x. Once again consider the faithful character ν defined on $(H^x \cap K)/N$, and we will identify its values with the corresponding elements of V. Then we can extend ν to an irreducible character of H^x denoted λ^x, where $M^x = \ker(\lambda^x)$ and λ is an irreducible character of H such that $M = \ker(\lambda)$.

We now have $\lambda^x \in \mathrm{Irr}(H^x)$ and $\mu \in \mathrm{Irr}(K)$ where $L = \ker(\mu)$ and $M^x = \ker(\lambda^x)$. We know there exists an H, K transversal T including x (recall that x is the element guaranteed to exist such that $H^x \cap L = K \cap M^x$). By Mackey's Theorem, $\langle \lambda^G, \mu^G \rangle = \sum_{g \in T} \langle \mu_{H^g \cap K}, (\lambda^g)_{H^g \cap K} \rangle \geq \langle \mu_{H^x \cap K}, (\lambda^x)_{H^x \cap K} \rangle$. Since μ and λ^x are extensions of ν into K and H^x, then $\mu_{H^x \cap K} = \nu$ and $(\lambda^x)_{H^x \cap K} = \nu$. Therefore $\langle \mu_{H^x \cap K}, (\lambda^x)_{H^x \cap K} \rangle = \langle \nu, \nu \rangle = 1$ because ν is a linear, irreducible character for $H^x \cap K$ Since (H, M) and (K, L) are good pairs, then by Proposition 5.3 we know λ^G and μ^G are both irreducible. The inner product of two irreducible characters is 0 or 1 and $\langle \lambda^G, \mu^G \rangle \geq 1$ Obviously $\langle \lambda^G, \mu^G \rangle \neq 0$, and we conclude that $\langle \lambda^G, \mu^G \rangle = 1$. By the First Orthogonality Relation it follows that $\mu^G = \lambda^G$. $\diamondsuit$

Corollary 5.5: *S_G is an equivalence relation on the good pairs of G.*

Proof: Let (H, M), (K, L) and (J, P) be good pairs. We will show the reflexive, symmetric and transitive properties are true for S_G.

First we need to show that $(H, M) \sim (H, M)$. This is trivial as $H^g \cap M = H \cap M^g$ for $g = 1$.

Second suppose that $(H, M) \sim (K, L)$, and we need to show that $(K, L) \sim (H, M)$. As $(H, M) \sim (K, L)$ there exists $g \in G$ such that $H^g \cap L = K \cap M^g$. Conjugating this equation by g^{-1} yields $K^{g^{-1}} \cap M = H \cap L^{g^{-1}}$. Thus $(K, L) \sim (H, M)$.

Finally suppose $(H, M) \sim (K, L)$ and $(K, L) \sim (J, P)$, and we need to show that $(H, M) \sim (J, P)$. We know by Proposition 5.4 that there exist λ and μ proceeding from (H, M) and (K, L) respectively such that $\mu^G = \lambda^G$. Additionally there exist φ and ψ proceeding from (K, L) and (J, P) respectively such that $\varphi^G = \psi^G$. As described in Proposition 2.4, we can identify a character of K with a corresponding character of K/L. Thus one can view λ and φ as faithfully representing the same cyclic group K/L. Then by Proposition 4.4, there exists an automorphism $\sigma \in \mathcal{G}$ such that $\varphi^\sigma = \lambda$ (Recall that $\mathcal{G}$ is the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and that $\varphi^\sigma = \sigma \circ \varphi$, or $\varphi^\sigma(g) = \sigma(\varphi(g))$ for all $g \in G$). Hence

$$\mu^G = \lambda^G = (\varphi^\sigma)^G = (\varphi^G)^\sigma = (\psi^G)^\sigma = (\psi^\sigma)^G.$$

Note that $(\varphi^\sigma)^G = (\varphi^G)^\sigma$ by Proposition 4.5. Because σ is an automorphism that fixes $\mathbb{Q}$, then ψ^σ is in fact a character proceeding from (J, P) as well (since $\ker(\psi^\sigma) = P$). Therefore $(H, M) \sim (J, P)$

Since all three properties are true, then S_G is an equivalence relation. $\diamondsuit$

5.2 Galois Conjugate Characters

To continue the characterization of M-groups using the concept of good pairs we need to introduce the idea of Galois conjugate characters. This will entail a review of some basic facts of field theory. We know that the Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ is defined as

$\mathcal{G} = \{\sigma \in \mathrm{Aut}(\mathbb{C}) \mid \sigma(x) = x \text{ for all } x \in \mathbb{Q}\}$. Let T be a $\mathbb{C}$-representation of G with associated character χ such that $T(x) = [t_{ij}(x)]$ where $t_{ij}(x) \in \mathbb{C}$ for all i, j. Then for all $\sigma \in \mathcal{G}$ define $T^\sigma(x)$ as the representation $[\sigma(t_{i,j}(x))]$, and it follows that the character of $T^\sigma(x)$ is $\chi^\sigma(x) = \mathrm{Tr}(T^\sigma(x)) = \mathrm{Tr}[\sigma(t_{i,j}(x))] = \sum_i \sigma(t_{i,i}(x)) = \sigma(\sum_i t_{i,i}(x)) = \sigma(\chi(x)) = (\sigma \circ \chi)(x)$ for all $x \in G$. If $\psi, \chi \in \mathrm{Irr}(G)$, where both are $\mathbb{C}$-characters, then ψ is a *Galois conjugate* of χ if there exists $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ such that $\chi^\sigma = \psi$. In other words, there must be some automorphism of $\mathbb{C}$, namely σ, that fixes $\mathbb{Q}$ element-wise and $\sigma(\chi(x)) = \psi(x)$ for all $x \in G$.

Proposition 5.6: *χ is irreducible if and only if χ^σ is irreducible.*

Proof: Suppose $\chi \in \mathrm{Irr}(G)$ and we want to show $\langle \chi^\sigma, \chi^\sigma \rangle = 1$. Since $\chi \in \mathrm{Irr}(G)$ then $\langle \chi, \chi \rangle = 1$. Recall that σ is an automorphism and therefore σ preserves addition and multiplication. By the definition of inner product,

$$\begin{aligned}
\langle \chi^\sigma, \chi^\sigma \rangle &= \frac{1}{|G|} \sum_{x \in G} \chi^\sigma(x)\chi^\sigma(x^{-1}) \\
&= \frac{1}{|G|} \sum_{x \in G} \sigma(\chi(x))\sigma(\chi(x^{-1})) \\
&= \frac{1}{|G|} \sum_{x \in G} \sigma(\chi(x)\chi(x^{-1})) \\
&= \sigma\left(\frac{1}{|G|} \sum_{x \in G} \chi(x)\chi(x^{-1})\right) \\
&= \sigma(\langle \chi, \chi \rangle) \\
&= \sigma(1) \\
&= 1
\end{aligned}$$

Hence χ^σ is irreducible by Corollary 3.10. A similar proof exists to show that $\chi^\sigma \in \mathrm{Irr}(G)$ implies $\chi \in \mathrm{Irr}(G)$. $\diamondsuit$

5.3 Galois Conjugacy Classes and Irreducible $\mathbb{Q}$-Characters

To prove Parks' theorem we must find a connection between the Galois conjugacy classes of irreducible, monomial characters and the classes of related good pairs. This relationship is not at all obvious at this point as more background needs to be introduced to motivate the connection.

For convenience once again denote the Galois group, $\mathrm{Gal}(\mathbb{C}/\mathbb{Q}) = \mathcal{G}$. As usual we will use $\mathrm{Irr}(G)$ to represent the set of all irreducible characters of G over the field $\mathbb{C}$, and $\mathrm{Irr}_{\mathbb{Q}}(G)$ as the set of all irreducible characters of G over the field $\mathbb{Q}$ or the set of all $\mathbb{Q}$-characters of G. In other words, the representation affording each character in $\mathrm{Irr}_{\mathbb{Q}}(G)$ has entries only from $\mathbb{Q}$. Note that when we are working over $\mathbb{C}$ no subscript to indicate the field is necessary. Define the set $\{\chi^{\mathcal{G}} | \sigma \in \mathcal{G}\}$ as the Galois conjugacy class with representative χ. Let $\chi \in \mathrm{Irr}(G)$ and consider $\sum_{\sigma\in\mathcal{G}} \chi^\sigma$. Because each χ^σ is a character and the sum of characters is a character, then $\sum_{\sigma\in\mathcal{G}} \chi^\sigma$ is also a character of the group G. Further observe $(\sum_{\sigma\in\mathcal{G}} \chi^\sigma)(g) \in \mathbb{Q}$ by the following lemma.

Lemma 5.7: *If χ is an irreducible $\mathbb{C}$-character of a group G and $\mathcal{G} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, then $(\sum_{\sigma\in\mathcal{G}} \chi^\sigma)(g) \in \mathbb{Q}$.*

Proof: Let $\alpha \in \mathcal{G}$ and consider $(\sum_{\sigma\in\mathcal{G}} \chi^\sigma)^\alpha$. We have

$$\left(\sum_{\sigma\in\mathcal{G}} \chi^\sigma\right)^\alpha = \alpha \circ \sum_{\sigma\in\mathcal{G}} \chi^\sigma = \sum_{\sigma\in\mathcal{G}} \alpha \circ \chi^\sigma$$

since α is a homomorphism and preserves addition in $\mathbb{C}$. Further, by the definition of Galois conjugate we can rewrite $\sum_{\sigma\in\mathcal{G}} \alpha \circ \chi^\sigma$ as $\sum_{\sigma\in\mathcal{G}} (\chi^\sigma)^\alpha$ which is equal to $\sum_{\sigma\in\mathcal{G}} (\chi)^{\alpha\,\sigma}$. Since $\sigma, \alpha \in \mathcal{G}$ and $\mathcal{G}$ is a group then for a fixed α, $\{\alpha \cdot \sigma | \sigma \in \mathcal{G}\}$ is simply a permutation or reordering of the group elements in $\mathcal{G}$. Thus $\sum_{\sigma\in\mathcal{G}} \chi^{\alpha\,\sigma} = \sum_{\sigma\in\mathcal{G}} \chi^\sigma$ for all $g \in G$. Hence $(\sum_{\sigma\in\mathcal{G}} \chi^\sigma)^\alpha = \sum_{\sigma\in\mathcal{G}} \chi^\sigma$, and α fixes $\sum_{\sigma\in\mathcal{G}} \chi^\sigma$. Since our choice of $\alpha \in \mathcal{G}$ was arbitrary, then $\sum_{\sigma\in\mathcal{G}} \chi^\sigma(g)$ is fixed by all $\alpha \in \mathcal{G}$ for all $g \in G$. Basic Galois theory tells us that $\sum_{\sigma\in\mathcal{G}} \chi^\sigma(g) \in \mathbb{Q}$ for all $g \in G$. $\diamondsuit$

Proposition 5.8: *For a finite group G, $|\mathrm{Irr}_{\mathbb{Q}}(G)|$ equals the number of Galois conjugacy classes of irreducible characters over $\mathbb{C}$.*

Proof: First recall that a $\mathbb{Q}$-character is a character afforded by a $\mathbb{Q}$-representation. Let $\theta \in \mathrm{Irr}(G)$. We will show that the character $m_{\mathbb{Q}}(\theta)\sum_{\sigma\in\mathcal{G}}\theta^{\sigma}$ is an irreducible character afforded by a $\mathbb{Q}$-representation where $m_{\mathbb{Q}}(\theta)$ is the Schur index of θ in $\mathbb{Q}$ (Refer to Section 4.4 for information on the Schur index). Using results from Grove (which we do not prove since they would take us too far afield from the problem at hand), if $\theta \in \mathrm{Irr}(G)$, then $m_{\mathbb{Q}}(\theta)$ is the multiplicity of θ as a constituent of some irreducible $\mathbb{Q}$-character ψ of the group G [2, p.189]. So $\langle\psi,\theta\rangle = m_{\mathbb{Q}}(\theta)$ and $\psi = m_{\mathbb{Q}}(\theta)\cdot\theta + \sum_{\chi\in Irr_{\mathbb{Q}}(G)\setminus\{\theta\}} a_{\chi}\chi$ for suitable $a_{\chi} \in \mathbb{N}\cup\{0\}$. By Proposition 10.2.7 in Grove's book [2, p.188], since $\psi \in \mathrm{Irr}_{\mathbb{Q}}(G)$ there is a unique, irreducible $\chi \in \mathrm{Irr}(G)$ such that $\psi = m_{\mathbb{Q}}(\chi)\sum_{\sigma\in\mathcal{G}}\chi^{\sigma}$ where $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. In addition, this proposition tells us that ψ is the only character in $\mathrm{Irr}_{\mathbb{Q}}(G)$ such that $\langle\psi,\chi\rangle > 0$.

At this point we know there exists $\chi \in \mathrm{Irr}(G)$ such that $\psi = m_{\mathbb{Q}}(\chi)\sum_{\sigma\in\mathcal{G}}\chi^{\sigma}$ where the irreducible constituents for ψ are χ^{σ}. Since θ is a constituent of ψ too, then $\theta = \chi^{\tau}$ for some $\tau \in \mathcal{G}$. So $\theta^{\tau^{-1}} = \chi$ and $\psi = m_{\mathbb{Q}}(\chi)\sum_{\sigma\in\mathcal{G}}\chi^{\sigma}$ becomes $m_{\mathbb{Q}}(\theta^{\tau^{-1}})\sum_{\sigma\in\mathcal{G}}(\theta^{\tau^{-1}})^{\sigma} = m_{\mathbb{Q}}(\theta^{\tau^{-1}})\sum_{\sigma\in\mathcal{G}}\theta^{\sigma}$. Since the Schur index of any Galois conjugate of a character equals the Schur index of the character, then for a given $\theta \in \mathrm{Irr}(G)$ we have that $\psi = m_{\mathbb{Q}}(\theta)\sum_{\sigma\in\mathcal{G}}\theta^{\sigma}$ is the unique, irreducible character afforded by a $\mathbb{Q}$-representation that has θ as a consituent.

In the same manner if ψ is an irreducible $\mathbb{Q}$-character of G, then we will show there is a unique Galois conjugacy class of irreducible $\mathbb{C}$-characters corresponding to ψ. Then we would have a one-to-one correspondence between the Galois conjugacy classes of irreducible characters of G over $\mathbb{C}$ and the irreducible $\mathbb{Q}$-characters of G.

Grove [2, p.188] states if $\psi \in \mathrm{Irr}_{\mathbb{Q}}(G)$ then there exists $\chi \in \mathrm{Irr}(G)$ where $\psi = m_{\mathbb{Q}}(\chi)\sum_{\sigma\in\mathcal{G}}\chi^{\sigma}$. We want to show the conjugacy class $\chi^{\mathcal{G}}$ is unique. Suppose there exists $\gamma \notin \{\chi^{\sigma}|\sigma\in\mathcal{G}\}$ such that $\psi = m_{\mathbb{Q}}(\gamma)\sum_{\sigma\in\mathcal{G}}\gamma^{\sigma}$ as well. Since conjugacy classes are either equivalent or disjoint, then $\gamma^{\mathcal{G}} \cap \chi^{\mathcal{G}} = \varnothing$ So

$$0 = \psi - \psi = \sum_{\sigma\in\mathcal{G}} m_{\mathbb{Q}}(\chi)\chi^{\sigma} - \sum_{\sigma\in\mathcal{G}} m_{\mathbb{Q}}(\gamma)\gamma^{\sigma}.$$

These summations cannot be combined because for all $\sigma \in \mathcal{G}$, $\gamma^\sigma \neq \chi$ for all Galois conjugacy class representatives γ. The set of irreducible $\mathbb{C}$-characters forms a basis for all $\mathbb{C}$-characters on G [2, p.108]. Thus $\mathrm{Irr}(G)$ is a linearly independent set that spans the set of class functions on G. Since

$$0 = \sum_{\sigma \in \mathcal{G}} m_{\mathbb{Q}}(\chi)\chi^\sigma - \sum_{\sigma \in \mathcal{G}} m_{\mathbb{Q}}(\gamma)\gamma^\sigma,$$

then all coefficients must be 0. This is a contradiction since the Schur index must be a positive integer. Thus $\gamma \in \{\chi^\sigma | \sigma \in \mathcal{G}\}$, and there is indeed a unique Galois conjugacy class for each $\psi \in \mathrm{Irr}_{\mathbb{Q}}(G)$. $\diamondsuit$

5.4 Irreducible $\mathbb{Q}$-Characters and Rational Conjugacy Classes

We have now shown in Proposition 5.8 that the number of Galois conjugacy classes of irreducible $\mathbb{C}$-characters of the group G is equal to the number of irreducible characters of the group G over $\mathbb{Q}$. One might wonder how this relates to Parks' classification of M-groups using good pairs.

Proposition 5.9: *For a finite group G, $|\mathrm{Irr}_{\mathbb{Q}}(G)| = n_G$.*

This proposition is attributed to the Berman-Witt Theorem [1, p.282], and is not included in this thesis because the methods used in its proof are beyond the scope of this text.

Corollary 5.10: *The number of rational conjugacy classes of G, n_G, equals the number of Galois conjugacy classes of irreducible $\mathbb{C}$-characters.*

Proof: This is immediate from Propositions 5 8 and 5.9 $\diamondsuit$

5.5 A Group-Theoretic Characterization of M-Groups

To complete Parks' classification we must show $m_G \leq n_G$ with equality if and only if G is an M-group (Remember that m_G is the number of classes of related good pairs). Let $(H_i, M_i), 1 \leq i \leq m_G$, be a set of representatives of each equivalence class of related good pairs. For all i, there exists λ_i proceeding from (H_i, M_i). Since (H_i, M_i) is a good pair, then by Proposition 5.3, λ_i^G is irreducible. Before proceeding with the proof that the number of Galois conjugacy classes of monomial, irreducible characters over $\mathbb{C}$ is m_G, we need the following lemma.

Lemma 5.11: *Let G be a finite group and $\lambda \in \mathrm{Irr}(G)$ with $\lambda(1) = 1$. Then $G/\ker(\lambda)$ is cyclic.*

Proof: Since $\lambda(1) = 1$ then $\lambda : G \longrightarrow \mathbb{C}^\times$ is a homomorphism into an abelian group under multiplication. By the Isomorphism Theorem [4, p.31] $G/\ker(\lambda) \cong \lambda(G)$. Suppose that $|G| = n$ and let ζ be a primitive nth root of unity. Certainly $g^n = 1$ for all $g \in G$ so $\lambda(g^n) = 1$ which implies $(\lambda(g))^n = 1$ for all $g \in G$. Thus $\lambda(g)$ is an nth root of unity. So $\lambda(G) \subseteq \{\zeta^i | i = 0, \ldots, n-1\}$ and certainly $\langle\zeta\rangle = \{\zeta^i | i = 0, \ldots, n-1\}$ is a cyclic group. Therefore $\lambda(G)$ is cyclic and we conclude that $G/\ker(\lambda)$ is also cyclic. $\diamondsuit$

Proposition 5.12: *m_G equals the number of distinct Galois conjugacy classes of monomial, irreducible characters over $\mathbb{C}$.*

Proof: Let $\chi \in \mathrm{Irr}(G)$ such that χ is monomial. χ must be in some Galois conjugacy class of the irreducible, monomial characters of G. We will show there exists a unique i, $1 \leq i \leq m_G$, such that χ is Galois conjugate to λ_i^G where λ_i proceeds from the good pair (H_i, M_i). In other words, each representative of a Galois conjugacy class must be associated with one and only one character proceeding from its respective good pair (H_i, M_i).

Since $\chi \in \mathrm{Irr}(G)$ is monomial, then χ is induced from a linear character of some subgroup of G. So there exists $H \leq G$ and $\mu \in \mathrm{Irr}(H)$ such that μ is linear and $\mu^G = \chi$.

Further define $M = \ker(\mu)$. Since M is the kernel of μ then M is normal in H. H/M is cyclic by Lemma 5.11 and (H, M) is therefore a pair.

The goal is to show there is a unique i such that $\chi^{\sigma} = \lambda_i^G$ for some $\sigma \in \mathcal{G}$. Since χ is irreducible, then (H, M) is a good pair from which μ proceeds (Proposition 5.3). The set of all good pairs is partitioned into equivalency classes of related good pairs so there exists i, $1 \leq i \leq m_G$, such that $(H, M) \sim (H_i, M_i)$. By Proposition 5.4 there are characters μ' and λ' proceeding from (H, M) and (H_i, M_i), respectively, such that $(\mu')^G = (\lambda')^G$. Note that we know virtually nothing about the characters μ' and λ' except for the fact that they exist. Further, we cannot assume that $\mu' = \mu$ or that $\lambda' = \lambda_i$, but we do know that μ and μ' faithfully represent the same cyclic group H/M and that λ_i and λ' faithfully represent the same cyclic group H_i/M_i.

Because of the irreducibility of cyclotomic polynomials and by Proposition 4.4, there exist $\sigma, \tau \in \mathcal{G}$ such that $\mu^{\sigma} = \mu'$ and $(\lambda')^{\tau} = \lambda_i$.

We know

$$\chi^{\sigma\tau} = (\mu^G)^{\sigma\tau} = [(\mu^{\sigma})^G]^{\tau} = [(\mu')^G]^{\tau} = [(\lambda')^G]^{\tau} = [(\lambda')^{\tau}]^G = (\lambda_i)^G.$$

By Proposition 4.5 we can switch the application of automorphisms and induction of characters as done in the above equation.

At this point we have shown for every distinct representative, χ, of the Galois conjugacy classes of monomial, irreducible $\mathbb{C}$-characters of G there exists i, $1 \leq i \leq m_G$ such that $\chi^{\sigma\tau} = \lambda_i^G$ where $\sigma, \tau \in \mathcal{G}$. Since $\mathcal{G}$ is a group, then $\sigma \cdot \tau \in \mathcal{G}$ as well, and we conclude λ_i^G is indeed a Galois conjugate of χ. Further, we associate with each Galois conjugacy class representative, χ, the related good pairs representative (H_i, M_i) from which the character λ_i proceeds. However, the question if our choice of i is unique still remains.

Assume χ is also conjugate to λ_j^G. Then there exists $\theta \in \mathcal{G}$ such that $\chi^{\theta} = \lambda_j^G$. Thus $[([(\lambda_i)^G]^{\tau^{-1}})^{\sigma^{-1}}]^{\theta} = \lambda_j^G$. Denote $\psi \in \mathcal{G}$ such that $\psi = \tau^{-1} \cdot \sigma^{-1} \cdot \theta$ and $((\lambda_i)^G)^{\psi} = (\lambda_j)^G$, or $((\lambda_i)^{\psi})^G = \lambda_j^G$. Since $(\lambda_i)^{\psi}$ and λ_j proceed from good pairs (H_i, M_i) and (H_j, M_j) respectively, then $(H_i, M_i) \sim (H_j, M_j)$ which implies $i = j$ and λ_i is unique. At this point we have shown that there is a well-defined map,

$$f : \{\chi^{\mathcal{G}} | \chi \in \mathrm{Irr}(G)\} \longrightarrow \{1, \ldots, m_G\}.$$

We must show that f is a bijection.

Suppose $f(\chi) = i$ and $f(\chi) = j$. Since $f(\chi) = i$ then there exists $\sigma \in \mathcal{G}$ such that $\chi^\sigma = (\lambda_i)^G$. Also, $f(\chi) = j$ implies there exists $\theta \in \mathcal{G}$ such that $\chi^\theta = (\lambda_j)^G$, or, $\chi = ((\lambda_j)^G)^{\theta^{-1}}$. So $(\lambda_i)^G = (((\lambda_j)^G)^{\theta^{-1}})^\sigma$, and $(\lambda_i)^G = ((\lambda_j)^{\theta^{-1}\sigma})^G$. Since λ_i and $(\lambda_j)^{\theta^{-1}\sigma}$ both proceed from good pairs, then by Proposition 5.4 $(H_i, M_i) \sim (H_j, M_j)$ and $i = j$. Hence f is an injective function.

Let $i \in \{1, \ldots, m_G\}$ and we must show there exists a representative χ of the Galois conjugacy classes of monomial characters such that $f(\chi) = i$. Consider the character $(\lambda_i)^G$ where λ_i proceeds from the good pair (H_i, M_i). It is irreducible by Proposition 5.3 and also monomial. Thus $(\lambda_i)^G$ is in some conjugacy class of the monomial Galois-conjugate characters. Define θ as the monomial class representative such that $(\lambda_i)^G \in \{\theta^\sigma | \sigma \in \mathcal{G}$ and $\theta \in \mathrm{Irr}(G)\}$. Hence $f(\theta) = i$ and since f is one-to-one and onto then $|\{\chi^{\mathcal{G}} | \chi \in \mathrm{Irr}(G)| = m_G$. We have now shown that the number of classes of related good pairs, m_G, is equal to the number of Galois conjugacy classes of monomial, irreducible $\mathbb{C}$-characters. $\diamondsuit$

Proof of Theorem 5.1: Denote the number of Galois conjugacy classes of monomial, irreducible $\mathbb{C}$-characters as a. By the previous proposition we have

$$m_G = a \leq \text{the number of Galois conjugacy classes of irreducible } \mathbb{C}\text{-characters } = n_G.$$

Thus $m_G \leq n_G$. G is an M-group if and only if the conjugacy classes of irreducible $\mathbb{C}$-characters are exactly the conjugacy classes of monomial, irreducible $\mathbb{C}$-characters if and only if $m_G = n_G$. Thus Parks' group-theoretic characterization of M-groups is complete. $\diamondsuit$

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VITA

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