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SEQUENCES OF SMALL HOMOCLINIC SOLUTIONS FOR DIFFERENCE EQUATIONS ON INTEGERS

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Abstract. In this article, we determine a concrete interval of positive parameters λ, for which we prove the existence of infinitely many homoclinic solutions for a discrete problem

$$-\Delta\big(a(k)\phi_p(\Delta u(k-1))\big)+b(k)\phi_p(u(k))=\lambda f(k,u(k)),\quad k\in\mathbb{Z},$$

where the nonlinear term $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ has an appropriate oscillatory behavior at zero. We use both the general variational principle of Ricceri and the direct method introduced by Faraci and Kristály [11].

1. Introduction

In this article we study the nonlinear second-order difference equation

$$\begin{gathered}-\Delta\left(a(k)\phi_p(\Delta u(k-1))\right)+b(k)\phi_p(u(k))=\lambda f(k,u(k))\quad\text{for all } k\in\mathbb{Z}\\ u(k)\to 0\quad\text{as } |k|\to\infty.\end{gathered}\tag{1.1}$$

Here $p>1$ is a real number, λ is a positive real parameter, $\phi_p(t)=|t|^{p-2}t$ for all $t\in\mathbb{R}$, $a,b:\mathbb{Z}\to(0,+\infty)$, while $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Moreover, the forward difference operator is defined as $\Delta u(k-1)=u(k)-u(k-1)$. We say that a solution $u=\{u(k)\}$ of (1.1) is homoclinic if $\lim_{|k|\to\infty}u(k)=0$.

Difference equations represent the discrete counterpart of ordinary differential equations and are usually studies in connection with numerical analysis. We may regard (1.1) as being a discrete analogue of the following second order differential equation

$$-(a(t)\phi_p(x'(t)))'+b(t)\phi_p(x(t))=f(t,x(t)),\quad t\in\mathbb{R}.$$

The case $p=2$ in (1.1) has been motivated in part by searching standing waves for the nonlinear Schrodinger equation

$$i\dot{\psi}_k+\Delta^2\psi_k-\nu_k\psi_k+f(k,\psi_k)=0,\quad k\in\mathbb{Z}.$$

Boundary value problems for difference equations can be studied in several ways. It is well known that variational method in such problems is a powerful tool. Many authors have applied different results of critical point theory to prove existence and multiplicity results for the solutions of discrete nonlinear problems. Studying

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such problems on bounded discrete intervals allows for the search for solutions in a finite-dimensional Banach space (see [1, 2, 9, 10, 19, 20, 21]). The issue of finding solutions on unbounded intervals is more delicate. To study such problems directly by variational methods, [13] and [18] introduced coercive weight functions which allow for preservation of certain compactness properties on l^p-type spaces. That method was used in the following papers [12, 14, 23, 24, 25].

The goal of the present paper is to establish the existence of a sequence of homoclinic solutions for problem (1.1), which has been studied recently in several papers. Infinitely many solutions were obtained in [25] by employing Nehari manifold methods, in [14] by applying a variant of the fountain theorem, in [23] by use of the Ricceri's theorem (see [4, 22]) and in [24] by applying a direct argumentation. In the two latter papers the nonlinearity f has a suitable oscillatory behavior at infinity. In this article we will prove that results analogous to [23] and [24] can be obtained assuming that the nonlinearity f has a suitable oscillatory behavior at zero.

A special case of our contributions reads as follows. For $b:\mathbb{Z}\to\mathbb{R}$ and the continuous mapping $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ define the following conditions:

- (A1) $b(k)\geq\alpha>0$ for all $k\in\mathbb{Z}$, $b(k)\to+\infty$ as $|k|\to+\infty$;
- (A2) there is $T_0>0$ such that $\sup_{|t|\leq T_0}|f(\cdot.t)|\in l_1$;
- (A3) $f(k,0)=0$ for all $k\in\mathbb{Z}$;
- (A4) there are sequences $\{c_m\},\{d_m\}$ such that $0<d_{m+1}<c_m<d_m$, $\lim_{m\to\infty}d_m=0$ and $f(k,t)\leq 0$ for every $k\in\mathbb{Z}$ and $t\in[c_m,d_m], m\in\mathbb{N}$;
- (A5) $\liminf_{t\to0^+}\frac{\sum_{k\in\mathbb{Z}}\max_{|\xi|\leq t}F(k,\xi)}{t^p}=0$;
- (A6) $\limsup_{(k,t)\to(+\infty,0^+)}\frac{F(k,t)}{[a(k+1)+a(k)+b(k)]t^p}=+\infty$;
- (A7) $\limsup_{(k,t)\to(-\infty,0^+)}\frac{F(k,t)}{[a(k+1)+a(k)+b(k)]t^p}=+\infty$;
- (A8) $\sup_{k\in\mathbb{Z}}\big(\limsup_{t\to0^+}\frac{F(k,t)}{[a(k+1)+a(k)+b(k)]t^p}\big)=+\infty$,
  where $F(k,t)$ is the primitive function of $f(k,t)$, i. e. $F(k,t)=\int_0^t f(k,s)\,ds$ for every $t\in\mathbb{R}$ and $k\in\mathbb{Z}$.

The solutions are found in the normed space $(X,\|\cdot\|)$, where

$$X=\big\{u:\mathbb{Z}\to\mathbb{R}\ :\ \sum_{k\in\mathbb{Z}}[a(k)|\Delta u(k-1)|^p+b(k)|u(k)|^p]<\infty\big\},$$

$$\|u\|=\Big(\sum_{k\in\mathbb{Z}}[a(k)|\Delta u(k-1)|^p+b(k)|u(k)|^p]\Big)^{1/p}.$$

Theorem 1.1. *Assume that* (A1)–(A4) *are satisfied. Moreover, assume that at least one of the conditions* (A6)–(A8)*, is satisfied. Then, for any* $\lambda>0$*, problem* (1.1) *admits a sequence of non-negative solutions in* X *whose norms tend to zero.*

Theorem 1.2. *Assume that* (A1), (A2), (A5) *are satisfied. Moreover, assume that at least one of the conditions* (A6)–(A8) *is satisfied. Then, for any* $\lambda>0$*, problem* (1.1) *admits a sequence of solutions in* X *whose norms tend to zero.*

The issue of multiplicity of solutions can be investigated through variational methods, which consist in seeking solutions of a difference equation as critical points of an energy functional defined on a convenient Banach space. In the proof for the first theorem a direct variational approach is used, introduced in [11] and then used in such papers as [8, 15, 16, 17, 24]. In the proof for the second theorem the general variational principle of Ricceri is used, which was applied in [2, 3, 5, 6, 7, 23]. To obtain the differentiability of the energy functional associated with problem (1.1), so far in the literature the following condition has been used

$$\lim_{t\to 0}\frac{|f(k,t)|}{|t|^{p-1}}=0 \quad \text{uniformly for all } k\in\mathbb{Z},$$

following [13, 18] and then used in [23, 24, 25].

We cannot use the condition, as it contradicts each of the conditions (A6)–(A8).

We obtain our results due to a suitable oscillatory behavior of the nonlinearity f. Let us observe that to satisfy the condition (A8) it suffices that a suitable oscillatory behavior is present for just one $k\in\mathbb{Z}$, while for satisfying conditions (A6) or (A7) a suitable behavior of the nonlinearity f needs to be maintained for an infinite number of $k\in\mathbb{Z}$.

The plan of the paper is as follows: Section 2 is devoted to our abstract framework, in Section 3 and Section 4 we prove more general versions of Theorems 1.1 and 1.2 respectively. In Section 5 we give examples and we show that Theorem 1.1 and Theorem 1.2 are independent.

2. Abstract framework

For all $1\le p<+\infty$, we denote by ℓ^p the set of all functions $u:\mathbb{Z}\to\mathbb{R}$ such that

$$\|u\|_p^p=\sum_{k\in\mathbb{Z}}|u(k)|^p<+\infty.$$

Moreover, we denote by ℓ^∞ the set of all functions $u:\mathbb{Z}\to\mathbb{R}$ such that

$$\|u\|_\infty=\sup_{k\in\mathbb{Z}}|u(k)|<+\infty$$

Lemma 2.1. *Let a continuous function $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ satisfies*

$$\sup_{|t|\le T}|f(\cdot,t)|\in l_1 \text{ for all } T>0. \tag{2.1}$$

Then the functional $\Psi: l^p\to\mathbb{R}$ defined by

$$\Psi(u):=\sum_{k\in\mathbb{Z}}F(k,u(k)) \quad \text{for all } u\in l^p, \tag{2.2}$$

where $F(k,s)=\int_0^s f(k,t)dt$ for $s\in\mathbb{R}$ and $k\in\mathbb{Z}$, is continuously differentiable.

Proof. Let us fix $u,v\in l^p$. We will prove that

$$\lim_{\tau\to 0^+}\frac{\Psi(u+\tau v)-\Psi(u)}{\tau}=\sum_{k\in\mathbb{Z}}f(k,u(k))v(k). \tag{2.3}$$

Put $r=\|u\|_\infty+\|v\|_\infty$ and $q(k)=\sup_{|t|\le r}|f(k,t)|$ for all $k\in\mathbb{Z}$. We have $q\in l^1$, by (2.1).

Let us fix arbitrarily $\epsilon>0$. Then, there exists $h\in\mathbb{N}$ such that

$$\sum_{|k|>h}|q(k)|<\frac{\epsilon}{3\|v\|_\infty}.$$

We can find $0<\tau_0<1$ such that for all $0<\tau\le\tau_0$,

$$\sum_{|k|\le h}\Big|\frac{F(k,u(k)+\tau v(k))-F(k,u(k))}{\tau}-f(k,u(k))v(k)\Big|<\frac{\epsilon}{3}.$$

Now fix $0 < \tau < \tau_0$. For all $|k| > h$ we can find $0 \le \tau_k \le \tau$ such that

$$\frac{F(k,u(k)+\tau v(k)) - F(k,u(k))}{\tau} = f(k,u(k)+\tau_k v(k))v(k).$$

We define $w \in l^p$ by putting $w(k) = 0$ for all $|k| \le h$ and $w(k) = u(k) + \tau_k v(k)$ for all $|k| > h$. So $\|w\|_\infty \le r$ and

$$\begin{aligned}
&\Big|\frac{\Psi(u+\tau v) - \Psi(u)}{\tau} - \sum_{k\in\mathbb{Z}} f(k,u(k))v(k)\Big| \\
&\le \frac{\epsilon}{3} + \sum_{|k|>h} |f(k,w(k))v(k)| + \sum_{|k|>h} |f(k,u(k))v(k)| \\
&\le \frac{\epsilon}{3} + 2\|v\|_\infty \sum_{|k|>h} q(k) < \epsilon,
\end{aligned}$$

which proves (2.3). From (2.1) and the continuity of the embeddings $l^p \hookrightarrow l^\infty$ and $l^1 \hookrightarrow l^{p'}$, the linear operator on the right-hand side of (2.3) lies in $l^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, so Ψ is Gateaux differentiable and

$$\langle \Psi'(u), v\rangle = \sum_{k\in\mathbb{Z}} f(k,u(k))v(k).$$

It remains to prove that $\Psi' : l^p \to l^{p'}$ is continuous. Let (u_n) be a sequence such that $u_n \to u$ in l^p. Put $R = \max\{\|u\|_\infty, \sup_{n\in\mathbb{N}} \|u_n\|_\infty\}$ and $Q(k) = \sup_{|t|\le R} |f(k,t)|$ for all $k \in \mathbb{Z}$. We have $Q \in l^1$, by (2.1). Fix an $\epsilon > 0$ arbitrarily. There exists $h \in \mathbb{N}$ such that

$$\sum_{|k|>h} |Q(k)| < \frac{\epsilon}{3} \tag{2.4}$$

and there exists $N \in \mathbb{N}$ such that for all $n > N$ we have

$$\sum_{|k|\le h} |f(k,u_n(k)) - f(k,u(k))| < \frac{\epsilon}{3}. \tag{2.5}$$

Applying (2.4) and (2.5), for every $n > N$ and $v \in l^p$ one has

$$\begin{aligned}
&|\langle \Psi'(u_n) - \Psi'(u), v\rangle| \\
&\le \|v\|_\infty \sum_{k\in\mathbb{Z}} |f(k,u_n(k)) - f(k,u(k))| \\
&\le \|v\|_p \Big(\sum_{|k|\le h} |f(k,u_n(k)) - f(k,u(k))| + \sum_{|k|>h} |f(k,u_n(k))| + \sum_{|k|>h} |f(k,u(k))| \Big) \\
&\le \|v\|_p \Big(\frac{\epsilon}{3} + 2 \sum_{|k|>h} Q(k) \Big) \\
&< \epsilon \|v\|_p,
\end{aligned}$$

thus, $\|\Psi'(u_n) - \Psi'(u)\| < \epsilon$. So, Ψ' is continuous and $\Psi \in C^1(l^p)$. $\square$

Now, we set

$$X = \big\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] < \infty \big\}$$

and

$$\|u\| = \Big(\sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] \Big) 1/p.$$

Clearly we have

$$\|u\|_\infty \le \|u\|_p \le \alpha^{-1/p}\|u\| \quad \text{for all } u \in X. \tag{2.6}$$

As is shown in [13, Propositions 3], $(X, \|\cdot\|)$ is a reflexive Banach space and the embedding $X \hookrightarrow l^p$ is compact. See also [14, Lemma 2.2].

Let $J_\lambda : X \to \mathbb{R}$ be the functional associated with problem (3.3) defined by

$$J_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) := \frac{1}{p}\sum_{k\in\mathbb{Z}}[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] \quad \text{for all } u \in X$$

and Ψ is given by (2.2).

Proposition 2.2. *Assume that* (A1) *and* (2.1) *are satisfied. Then*

(a) $\Psi \in C^1(l^p)$ *and* $\Psi \in C^1(X)$*;*
(b) $\Phi \in C^1(X)$*;*
(c) $J_\lambda \in C^1(X)$ *and every critical point* $u \in X$ *of* J_λ *is a homoclinic solution of problem* (1.1)*;*
(d) J_λ *is sequentially weakly lower semicontinuous functional on* X*.*

Proof. Part (a) follows from Lemma 2.1. Parts (b) and (c) can be proved essentially by the same way as [13, Propositions 5 and 7], where $a(k) \equiv 1$ on $\mathbb{Z}$ and the norm on X is slightly different. See also [14, Lemmas 2.4 and 2.6]. The proof of part (d) is based on the following facts: $\Phi = \frac{1}{p}\|\cdot\|^p$, $\Psi \in C(l^p)$ and the compactness of $X \hookrightarrow l^p$ and it is standard. □

3. Proof of Theorem 1.1

Now we will formulate and prove a stronger form of Theorem 1.1. Let

$$B_\pm := \limsup_{(k,t)\to(\pm\infty,0^+)} \frac{F(k,t)}{[a(k+1)+a(k)+b(k)]t^p}, \tag{3.1}$$

$$B_0 := \sup_{k\in\mathbb{Z}}\Big(\limsup_{t\to 0^+} \frac{F(k,t)}{[a(k+1)+a(k)+b(k)]t^p}\Big). \tag{3.2}$$

Set $B = \max\{B_\pm, B_0\}$. For convenience we put $1/+\infty = 0$.

Theorem 3.1. *Assume that* (A1)–(A4) *are satisfied and assume that* $B > 0$*. Then, for any* $\lambda > \frac{1}{Bp}$*, problem* (1.1) *admits a nonzero sequence of non-negative solutions in* X *whose norms tend to zero.*

Proof. To apply Proposition 2.2, we need to have a nonlinearity which satisfies condition (2.1). Let $T_0 > 0$ be a number satisfying (A2). Define the truncation function

$$\tilde f(k,s) = \begin{cases} 0, & s \le 0 \text{ and } k \in \mathbb{Z},\\ f(x,s), & 0 \le s \le T_0 \text{ and } k \in \mathbb{Z},\\ f(x,T_0), & s \ge T_0 \text{ and } k \in \mathbb{Z}. \end{cases}$$

and consider the problem

$$\begin{gathered} -\Delta\left(a(k)\phi_p(\Delta u(k-1))\right) + b(k)\phi_p(u(k)) = \lambda \tilde f(k,u(k))\\ u(k) \to 0. \end{gathered} \tag{3.3}$$

Clearly, if u is a non-negative solution of problem (3.3) with $\|u\|_\infty \le T_0$, then it is also a non-negative solution of problem (1.1), so it is enough to show that problem (3.3) admits a nonzero sequence of non-negative solutions in X whose norms tend to zero.

Put $\lambda > \frac{1}{Bp}$ and put Φ, Ψ and J_λ as in the previous section. By Proposition 2.2 we need to find a nontrivial sequence $\{u_n\}$ of critical points of J_λ with non-negative terms whose norms tend to zero.

Let $\{c_n\}, \{d_n\}$ be sequences satisfying conditions (A4). Up to subsequence, we may assume that $d_1 < T_0$. For every $n \in \mathbb{N}$ define the set

$$W_n = \{u \in X : \|u\|_\infty \le d_n \text{ for every } k \in \mathbb{Z}\}.$$

Claim 3.2. *For every $n \in \mathbb{N}$, the functional J_λ is bounded from below on W_n and its infimum on W_n is attained.*

The proof of this Claim is essentially the same as the proof of [24, Claim 3.2].

Claim 3.3. *For every $n \in \mathbb{N}$, let $u_n \in W_n$ be such that $J_\lambda(u_n) = \inf_{W_n} J_\lambda$. Then, u_n is a solution of problem* (3.3) *with $0 \le u_n(k) \le c_n$ for all $k \in \mathbb{Z}$.*

Firstly, arguing as in the proof of [24, Claim 3.3], we obtain that if $u_n \in W_n$ is such that $J_\lambda(u_n) = \inf_{W_n} J_\lambda$, then $0 \le u_n(k) \le c_n$ for all $k \in \mathbb{Z}$. Secondly, arguing as in the proof of [24, Claim 3.4], we obtain that u_n is a critical point of J_λ in X, and so is a solution of problem (3.3). This proves Claim 3.3.

Claim 3.4. *For every $n \in \mathbb{N}$, we have $J_\lambda(u_n) < 0$ and $\lim_{n\to+\infty} J_\lambda(u_n) = 0$.*

Firstly, we assume that $B = B_\pm$. Without loss of generality we can assume that $B = B_+$. We begin with $B = +\infty$. Then there exists a number $\sigma > \frac{1}{\lambda p}$, a sequence of positive integers $\{k_n\}$ and a sequence of positive numbers $\{t_n\}$ which tends to 0, such that

$$F(k_n, t_n) > \sigma(a(k_n + 1) + a(k_n) + b(k_n))t_n^p \tag{3.4}$$

for all $n \in \mathbb{N}$. Up to extracting a subsequence, we may assume that $t_n \le d_n$ for all $n \in \mathbb{N}$. Define in X a sequence $\{w_n\}$ such that, for every $n \in \mathbb{N}$, $w_n(k_n) = t_n$ and $w_n(k) = 0$ for every $k \in \mathbb{Z}\backslash\{k_n\}$. It is clear that $w_n \in W_n$. One then has

$$\begin{aligned}
&J_\lambda(w_n)\\
&= \frac{1}{p}\sum_{k\in\mathbb{Z}} (a(k)|\Delta w_n(k-1)|^p + b(k)|w_n(k)|^p) - \lambda\sum_{k\in\mathbb{Z}} F(k, w_n(k))\\
&< \frac{1}{p}\left(a(k_n + 1) + a(k_n)\right) t_n^p + \frac{1}{p} b(k_n)t_n^p - \lambda\sigma(a(k_n + 1) + a(k_n) + b(k_n))t_n^p\\
&= \big(\frac{1}{p} - \lambda\sigma\big)(a(k_n + 1) + a(k_n) + b(k_n))t_n^p < 0
\end{aligned}$$

which gives $J_\lambda(u_n) \le J_\lambda(w_n) < 0$. Next, assume that $B < +\infty$. Since $\lambda > \frac{1}{Bp}$, we can fix $\varepsilon < B - \frac{1}{\lambda p}$. Therefore, also taking $\{k_n\}$ a sequence of positive integers and $\{t_n\}$ a sequence of positive numbers with $\lim_{n\to+\infty} t_n = 0$ and $t_n \le d_n$ for all $n \in \mathbb{N}$ such that

$$F(k_n, t_n) > (B - \varepsilon)(a(k_n + 1) + a(k_n) + b(k_n))t_n^p \tag{3.5}$$

for all $n \in \mathbb{N}$, choosing $\{w_n\}$ in W_n as above, one has

$$J_\lambda(w_n) < \Big(\frac{1}{p} - \lambda(B - \varepsilon)\Big)(a(k_n + 1) + a(k_n) + b(k_n))t_n^p.$$

So, also in this case, $J_\lambda(u_n) < 0$.

Now, assume that $B = B_0$. We begin with $B = +\infty$. Then there exists a number $\sigma > \frac{1}{\lambda p}$ and an index $k_0 \in \mathbb{Z}$ such that

$$\limsup_{t\to 0^+} \frac{F(k_0,t)}{(a(k_0+1)+a(k_0)+b(k_0))|t|^p} > \sigma.$$

Then, there exists a sequence of positive numbers $\{t_n\}$ such that $\lim_{n\to+\infty} t_n = 0$ and

$$F(k_0,t_n) > \sigma(a(k_0+1)+a(k_0)+b(k_0))t_n^p \tag{3.6}$$

for all $n \in \mathbb{N}$. Up to considering a subsequence, we may assume that $t_n \le d_n$ for all $n \in \mathbb{N}$. Thus, take in X a sequence $\{w_n\}$ such that, for every $n \in \mathbb{N}$, $w_n(k_0) = t_n$ and $w_n(k) = 0$ for every $k \in \mathbb{Z}\backslash\{k_0\}$. Then, one has $w_n \in W_n$ and

$$\begin{aligned}
&J_\lambda(w_n)\\
&= \frac{1}{p}\sum_{k\in\mathbb{Z}} (a(k)|\Delta w_n(k-1)|^p + b(k)|w_n(k)|^p) - \lambda \sum_{k\in\mathbb{Z}} F(k,w_n(k))\\
&< \frac{1}{p}\left(a(k_0+1)+a(k_0)\right)t_n^p + \frac{1}{p}b(k_0)t_n^p - \lambda\sigma(a(k_0+1)+a(k_0)+b(k_0))t_n^p\\
&= \big(\frac{1}{p} - \lambda\sigma\big)(a(k_0+1)+a(k_0)+b(k_0))t_n^p < 0
\end{aligned}$$

which gives $J_\lambda(u_n) < 0$. Next, assume that $B < +\infty$. Since $\lambda > \frac{1}{Bp}$, we can fix $\varepsilon > 0$ such that $\varepsilon < B - \frac{1}{\lambda p}$. Therefore, there exists an index $k_0 \in \mathbb{Z}$ such that

$$\limsup_{t\to 0^+} \frac{F(k_0,t)}{(a(k_0+1)+a(k_0)+b(k_0))t^p} > B - \varepsilon.$$

and taking $\{t_n\}$ a sequence of positive numbers with $\lim_{n\to+\infty} t_n = 0$ and $t_n \le d_n$ for all $n \in \mathbb{N}$ and

$$F(k_0,t_n) > (B-\varepsilon)\left(a(k_0+1)+a(k_0)+b(k_0)\right)t_n^p \tag{3.7}$$

for all $n \in \mathbb{N}$, choosing $\{w_n\}$ in W_n as above, one has

$$J_\lambda(w_n) < \big(\frac{1}{p} - \lambda(B-\varepsilon)\big)(a(k_0+1)+a(k_0)+b(k_0))t_n^p < 0.$$

So, also in this case, $J_\lambda(u_n) < 0$.

Moreover, by Claim 3.3, for every $k \in \mathbb{N}$ one has

$$|F(k,u_n(k))| \le \int_0^{c_n} |\tilde{f}(k,t)|dt \le c_n \max_{t\in[0,c_n]} |\tilde{f}(k,t)| \le c_n \max_{t\in[0,T_0]} |\tilde{f}(k,t)| \tag{3.8}$$

Then

$$0 > J_\lambda(u_n) \ge -\sum_{k\in\mathbb{Z}} F(k,u_n(k)) \ge -c_n\|\max_{t\in[0,T_0]} |\tilde{f}(\cdot,t)|\|_1$$

Since the sequence $\{c_n\}$ tends to zero, then $J_\lambda(u_n) \to 0$ as $n \to +\infty$. This proves Claim 3.4.

Now we are ready to end the proof of Theorem 3.1. With Proposition 2.2, Claims 3.3–3.4, up to a subsequence, we have infinitely many pairwise distinct non-negative homoclinic solutions u_n of (3.3). Moreover, due to (3.8), we have

$$\frac{1}{p}\|u_n\|^p = J_\lambda(u_n) + \sum_{k\in\mathbb{Z}} F(k,u_n(k)) < c_n\|\max_{t\in[0,T_0]} |\tilde{f}(\cdot,t)|\|_1,$$

which proves that $\|u_n\|^p \to 0$ as $n \to +\infty$. This concludes our proof. $\square$

We remark that Theorem 1.1 follows now from Theorem 3.1.

4. Proof of Theorem 1.2

Our main tool is a general critical points theorem due to Bonanno and Molica Bisci (see [4]) that is a generalization of a result of Ricceri [22]. Here we state it in a smooth version for the reader's convenience.

Theorem 4.1. *Let $(E, \|\cdot\|)$ be a reflexive real Banach space, let $\Phi, \Psi : E \to \mathbb{R}$ be two continuously differentiable functionals with Φ coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$, and a sequentially weakly lower semicontinuous functional and Ψ a sequentially weakly upper semicontinuous functional. For every $r > \inf_E \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\big(\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)\big) - \Psi(u)}{r - \Phi(u)},$$
$$\delta := \liminf_{r\to(\inf_E \Phi)^+} \varphi(r).$$

Let $J_\lambda := \Phi(u) - \lambda\Psi(u)$ for all $u \in E$. If $\delta < +\infty$ then, for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either

(a) *there is a global minimum of Φ which is a local minimum of J_λ, or*
(b) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of J_λ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_E \Phi$, which weakly converges to a global minimum of Φ.*

Now we formulate and prove a stronger form of Theorem 1.2. Let

$$A := \liminf_{t\to0^+} \frac{\sum_{k\in\mathbb{Z}} \max_{|\xi|\le t} F(k,\xi)}{t^p}.$$

Set $B := \max\{B_\pm, B_0\}$, where $B_\pm$ and B_0 are given by (3.1) and (3.2), respectively. For convenience we put $\frac{1}{0^+} = +\infty$ and $\frac{1}{+\infty} = 0$.

Theorem 4.2. *Assume that* (A1), (A2), (A5) *are satisfied and assume that the following inequality holds $A < \alpha B$. Then, for each $\lambda \in (\frac{1}{Bp}, \frac{\alpha}{Ap})$, problem* (1.1) *admits a sequence of solutions in X whose norms tend to zero.*

Proof. To apply Proposition 2.2, we need to have a nonlinearity which satisfies condition (2.1). Let $T_0 > 0$ be a number satisfying (A2). Define the truncation function

$$\bar f(k,s) = \begin{cases} f(x,-T_0), & s \le -T_0 \text{ and } k \in \mathbb{Z}, \\ f(x,s), & -T_0 \le s \le T_0 \text{ and } k \in \mathbb{Z}, \\ f(x,T_0), & s \ge T_0 \text{ and } k \in \mathbb{Z}. \end{cases}$$

and consider the problem

$$\begin{gathered} -\Delta\left(a(k)\phi_p(\Delta u(k-1))\right) + b(k)\phi_p(u(k)) = \lambda \bar f(k,u(k)) \\ u(k) \to 0. \end{gathered} \tag{4.1}$$

Clearly, if u is a solution of problem (4.1) with $\|u\|_\infty \le T_0$, then it is also a solution of the problem (1.1), so it is enough to show that problem (4.1) admits a nonzero sequence of solutions in X whose norms tend to zero.

It is clear that $A \geq 0$. Put $\lambda \in \left(\frac{1}{Bp}, \frac{\alpha}{Ap}\right)$ and put Φ, Ψ, J_λ as above. Our aim is to apply Theorem 4.1 to function J_λ. By Lemma 2.2, the functional Φ is the continuously differentiable and sequentially weakly lower semicontinuous functional and Ψ is the continuously differentiable and sequentially weakly upper semicontinuous functional. We will show that $\delta < +\infty$. Let $\{c_m\} \subset (0, T_0)$ be a sequence such that $\lim_{m\to\infty} c_m = 0$ and

$$\lim_{m\to+\infty} \frac{\sum_{k\in\mathbb{Z}} \max_{|\xi|\leq c_m} F(k,\xi)}{c_m^p} = A.$$

Set

$$r_m := \frac{\alpha}{p} c_m^p$$

for every $m \in \mathbb{N}$. Then, if $v \in X$ and $\Phi(v) < r_m$, one has

$$\|v\|_\infty \leq \alpha^{-\frac{1}{p}} \|v\| \leq \alpha^{-\frac{1}{p}} \big(p\Phi(v)\big)1/p < c_m$$

which gives

$$\Phi^{-1}\big((-\infty, r_m)\big) \subset \big\{v \in X : \|v\|_\infty \leq c_m\big\}. \tag{4.2}$$

From this and $\Phi(0) = \Psi(0) = 0$ we have

$$\begin{aligned}
\varphi(r_m) &\leq \frac{\sup_{\Phi(v)<r_m} \sum_{k\in\mathbb{Z}} F(k, v(k))}{r_m} \leq \frac{\sum_{k\in\mathbb{Z}} \max_{|t|\leq c_m} F(k,t)}{r_m} \\
&= \frac{p}{\alpha} \cdot \frac{\sum_{k\in\mathbb{Z}} \max_{|t|\leq c_m} F(k,t)}{c_m^p}
\end{aligned}$$

for every $m \in \mathbb{N}$. This gives

$$\delta \leq \lim_{m\to+\infty} \varphi(r_m) \leq \frac{p}{\alpha} \cdot A < \frac{1}{\lambda} < +\infty.$$

Now, we show that the point (a) in Theorem 4.1 does not hold, i.e. we show that the global minimum θ of Φ is not a local minimum of J_λ. Arguing as in the proof of Claim 3.4, we can find a sequence $\{w_n\}$ in X with $\|w_n\|_\infty \to 0$ as $n \to +\infty$, such that $J_\lambda(w_n) < 0$ for $n \in \mathbb{N}$. We have to show that $\|w_n\| \to 0$. Note that

$$\|w_n\| = \big((a(k_n+1) + a(k_n) + b(k_n))t_n^p\big)1/p,$$

where $\{k_n\}$ is a sequence divergent to $+\infty$ or $-\infty$, as in (3.4) and (3.5) or $\{k_n\}$ is a constant sequence, as in (3.6) and (3.7) and $\{t_n\}$ is a sequence convergent to 0^+ from relevant (3.4), (3.5), (3.6) or (3.7). From this

$$\|w_n\| \leq \gamma F(k_n, t_n)$$

for some positive constant γ and all $n \in \mathbb{N}$. Since

$$\lim_{m\to+\infty} \frac{\sum_{k\in\mathbb{Z}} \max_{|\xi|\leq c_m} F(k,\xi)}{c_m^p} < +\infty$$

and $\lim_{m\to+\infty} c_m = 0$, we have

$$\lim_{m\to+\infty} \sum_{k\in\mathbb{Z}} \max_{|\xi|\leq c_m} F(k,\xi) = 0$$

and, as $\max_{|\xi|\leq c_m} F(k,\xi) \geq 0$, we obtain $\lim_{m\to+\infty} \big(\max_{|\xi|\leq c_m} F(k,\xi)\big) = 0$ uniformly for all $k \in \mathbb{Z}$. This and $F(k_n, t_n) > 0$ easily gives $\lim_{n\to+\infty} F(k_n, t_n) = 0$ and so $\lim_{n\to+\infty} \|w_n\| = 0$.

From the above it follows that θ is not a local minimum of J_λ and, by (b), there is a sequence $\{u_n\}$ of pairwise distinct critical points of J_λ with $\lim_{n\to+\infty} \Phi(u_n) =$

$\inf_E \Phi$. This means that $0 = \inf_E \Phi = \lim_{n\to+\infty} \Phi(u_n) = \frac{1}{p}\|u_n\|^p$, and so $\{u_n\}$ strongly converges to zero. The proof is complete. $\square$

We remark that Theorem 1.2 follows now from Theorem 4.2.

5. Examples

Consider the problem

$$\begin{gathered} -\Delta\left(\phi_p(\Delta u(k-1))\right) + |k|\phi_p(u(k)) = \lambda f(k,u(k)) \quad \text{for all } k\in\mathbb{Z} \\ u(k)\to 0 \quad \text{as } |k|\to\infty, \end{gathered} \tag{5.1}$$

where $p>1$ and $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ is defined by

$$f(k,s) = \sum_{m\in\mathbb{N}} e_m\Big(d_m - c_m - 2|s - \frac{1}{2}\left(c_m+d_m\right)|\Big)\cdot \mathbf{1}_{\{m\}\times[c_m,d_m]}(k,s) \tag{5.2}$$

with sequences $\{c_m\},\{d_m\},\{e_m\},\{h_m\}$ defined by

$$\begin{gathered} c_m = 1/2^{2^{2m}} \quad \text{for } m\in\mathbb{N}; \\ d_m = 1/2^{2^{2m-1}} \quad \text{for } m\in\mathbb{N}; \\ h_m = 1/2^{(p+1)2^{2m-2}} \quad \text{for } m\in\mathbb{N}; \\ e_m = 2h_m/(d_m-c_m)^2 \quad \text{for } m\in\mathbb{N}. \end{gathered} \tag{5.3}$$

Here $\mathbf{1}_{A\times B}$ is the indicator of $A\times B$. It is easily seen that f is continuous and conditions (A2), (A3) are satisfied. Set $F(k,t) := \int_0^t f(k,s)ds$ for every $t\in\mathbb{R}$ and $k\in\mathbb{Z}$. Then $F(k,d_k) = \int_{c_k}^{d_k} f(k,t)dt = h_k$ and

$$\begin{aligned} \liminf_{t\to0^+} \frac{\sum_{k\in\mathbb{Z}}\max_{|\xi|\le t}F(k,\xi)}{t^p} &\le \lim_{m\to+\infty} \frac{\sum_{k\in\mathbb{Z}}\max_{|\xi|\le c_m}F(k,\xi)}{c_m^p} \\ &= \lim_{m\to+\infty}\frac{\sum_{k=m+1}^{\infty}F(k,d_k)}{c_m^p} \\ &= \lim_{m\to+\infty}\frac{\sum_{k=m+1}^{\infty}h_k}{c_m^p} \\ &\le \lim_{m\to+\infty}\frac{2h_{m+1}}{c_m^p} = 0 \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \limsup_{(k,t)\to(+\infty,0^+)} \frac{F(k,t)}{(2+k)t^p} &\ge \lim_{m\to+\infty}\frac{F(m,d_m)}{(2+m)d_m^p} \\ &= \lim_{m\to+\infty}\frac{h_m}{(2+m)d_m^p} = +\infty. \end{aligned} \tag{5.5}$$

So, conditions (A4)–(A6) are satisfied and so for any $\lambda>0$, problem (5.1) admits a sequence of non-negative solutions in X whose norms tend to zero, by Theorem 1.1 or Theorem 1.2. Note also that f does not satisfy (A8).

Remark 5.1. For a fixed $k_0\in\mathbb{Z}$, if we define $\tilde f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ by

$$\tilde f(k,s) = \sum_{m\in\mathbb{N}} e_m\big(d_m - c_m - 2|s-\frac{1}{2}(c_m+d_m)|\big)\cdot\mathbf{1}_{\{k_0\}\times[c_m,d_m]}(k,s)$$

with sequences $\{c_m\},\{d_m\},\{e_m\},\{h_m\}$ defined as above, then $\tilde f$ satisfies conditions (A2)–(A5) and (A8), but does not satisfy conditions (A6) and (A7).

Remark 5.2. Theorems 1.1 and 1.2 are independent of each other. Indeed, let us replace h_m in (5.3) by

$$h_m = 1/2^{p2^{2m-2}} \quad \text{for } m \in \mathbb{N}.$$

Then, the function f given by (5.2) is continuous if $p > 2$. It can be seen that the first inequality in (5.4) is in fact equality. Then, an easy computation shows that

$$\liminf_{t\to 0^+} \frac{\sum_{k\in\mathbb{Z}} \max_{|\xi|\le t} F(k,\xi)}{t^p} \ge 1,$$

$$B_+ = \limsup_{(k,t)\to(+\infty,0^+)} \frac{F(k,t)}{(2+k)t^p} = +\infty.$$

This means that we can not apply Theorem 1.2, but Theorem 1.1 works. On the other hand, it is easy to see that we can modify f in the way, that for some (or even infinitely many) k we have $f(k,t) > 0$ for all $t > 0$ and the limits (5.4), (5.5) do not change. Therefore, such an f does not satisfy (A4) and can not be used in Theorem 1.1.

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