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Cesaro asymptotic equipartition of energy in the coupled case *

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Abstract

It is well known from earlier results that certain types of selfadjoint operators, e.g. operators allowing a representation as operator matrices, show equipartition of energy. In this paper we examine the question whether there are more selfadjoint operators showing equipartition of energy in the Cesaro mean. For this purpose we proof a necessary and sufficient criterion for equipartition of energy and use this criterion to show equipartition for a system of partial differential equations with a coupled boundary condition.

1 Introduction

In this paper we examine the phenomenon of asymptotic equipartition of energy for abstract evolution equations involving selfadjoint operators. This means that if a Hilbert space $\mathcal{H}$ is the direct sum of n Hilbert spaces $\mathcal{H}_i$ $(i = 1, \dots, n)$, then, roughly speaking, each component contributes equal parts to the conserved total energy. More precisely, let π_i be the (orthogonal) projections on the i-th component of $\mathcal{H}$ with respect to this decomposition and let A be be a selfadjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$. Then we consider the following evolution equation:

$$\begin{aligned} D_0 u(t) &= Au(t) \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where $D_0 = \frac{1}{2\pi i}\partial_0$ is the differentiation with respect to the time t. The uniquely determined solution of (1.1) is the unitary group $U(t) := e^{2\pi i A t}$ generated by $2\pi i A$.

Since $U(\cdot)$ is unitary, the square of the norm $\|u(t)\|^2$ (the energy) of $u(t)$ is conserved, i.e.

$$\sum_{i=1}^{n} \|\pi_i u(t)\|^2 = \|u(t)\|^2 = \|U(t)u_0\|^2 = \|u_0\|^2 = \text{const.}$$

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Now we can give the following

Definition 1.1. 1. The selfadjoint operator A admits **asymptotic equipartition of energy**, if for all $u_0 \in \mathcal{H}$ and the corresponding solution $u(t)$ the following asymptotic condition is true

$$\lim_{t\to\infty} \|\pi_i u(t)\|^2 = \frac{1}{n} \|u_0\|^2 . \tag{1.2}$$

(See [11])

2. The selfadjoint operator A admits **asymptotic equipartition of energy in the Cesaro sense**, if for all $u_0 \in \mathcal{H}$ and the corresponding solution $u(t)$ the following asymptotic condition is true

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|\pi_i u(t)\|^2 \, dt = \frac{1}{n} \|u_0\|^2 . \tag{1.3}$$

The earliest results concerning energy equipartition seem to be presented by Brodsky ([2]) and Lax and Phillips ([14, Cor. 2.3, p. 106]). In the following there was a continuing interest in this question. In particular Goldstein and Sandefur contributed a lot to this area (e.g. [7], [8], [11], [9], [12], they treated also more general situations). Goldstein ([8]) and Duffin ([6]) showed results concerning energy equipartition from a finite time on. Also Picard and Seidler (cf. e.g. [19], [17], [18]) examined equipartition results, where they choose matrices of operators as an ansatz for the operator A. For further contributions consult the references. A recent paper is written by Goldstein, de Laubenfels and Sandefur ([13]).

In the known results on equipartition the operator A is assumed to be decomposable in operators with special properties, e.g. as a matrix of closed operators or generators of (regularized) semigroups. This is typically the case for a partial differential equation with decoupled boundary conditions, i.e. when the boundary condition restricts the components separately.

In this paper we want to examine, if there are more operators not belonging to this class, which nevertheless show equipartition of energy. For simplicity we restrict our considerations to the case of energy equipartition in the Cesaro sense for 2-component systems ($n = 2$ in Definition 1.1).

After giving some prerequisites in section 2 we proof a necessary and sufficient condition for asymptotic energy equipartition in section 3. This condition contains some previous results. Further we apply this condition in section 4 to a system of partial differential equations with a coupled boundary condition.

In the last section we give a short outlook on open questions in this area.

2 Prerequisites

As the starting point for our considerations we use the following ansatz given by Picard and Seidler in [19] essentially equivalent to that examined by Goldstein in [7], [8]. They considered 2×2-operator-matrices in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the form

$$A = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}$$

with $B : \mathcal{D}(B) \subset \mathcal{H}_1 \to \mathcal{H}_2$ a densely defined, closed operator and the corresponding initial value problem

$$\begin{aligned} D_0 u &= Au \\ u(0) &= u_0 \in \mathcal{H}. \end{aligned} \tag{2.1}$$

They proved the following theorem concerning equipartition in the Cesaro sense:

Theorem 2.1. *For every initial value $u_0 \in \mathcal{H}$ the following asymptotic relations hold*

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \int_0^T \|\pi_1 u(t)\|^2 \, dt &= \tfrac{1}{2} \|Qu_0\|^2 + \|\pi_1 P u_0\|^2 \\ \lim_{T\to\infty} \frac{1}{T} \int_0^T \|\pi_2 u(t)\|^2 \, dt &= \tfrac{1}{2} \|Qu_0\|^2 + \|\pi_2 P u_0\|^2 , \end{aligned}$$

where P, $Q = I - P$ are the projections to the kernel and the closure of the range of A, respectively.

For the proof of this theorem the following lemma is essential. We will use this lemma later, so we cite it here.

Lemma 2.2. *Let E_λ be the spectral measure of A and $T := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then*

$$TE_I = E_{-I} T$$

for all intervals $I \subset \mathbb{R}$.

As a corollary of Theorem 2.1 we get the following result originally showed by Goldstein in [8, Theorem 4]:

Corollary 2.3. *We have energy equipartition in the Cesaro sense for every initial value $u_0 \in \mathcal{H}$ of (2.1) if and only if 0 is not an eigenvalue of A.*

If looking for a criterion for equipartition of energy in the Cesaro sense we must examine the asymptotic behaviour of

$$S_{u_0}(t) := \big\langle u(t) | T u(t) \big\rangle = \|\pi_1 u(t)\|^2 - \|\pi_2 u(t)\|^2 , \tag{2.2}$$

where $u(t)$ is the solution of (1.1) for the initial value u_0 and T is the operator defined in Lemma 2.2. We see that we have energy equipartition if and only if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S_{u_0}(t) dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \|\pi_1 u(t)\|^2 \, dt - \lim_{T\to\infty} \frac{1}{T} \int_0^T \|\pi_2 u(t)\|^2 \, dt = 0. \tag{2.3}$$

Inserting the unitary group $e^{2\pi iAt}$ generated by $2\pi iA$ in (2.2) we get:

$$S_{u_0}(t) = \big\langle \exp(2\pi iAt)u_0 | T\exp(2\pi iAt)u_0 \big\rangle.$$

For the examination of this expression the following theorem about a functional calculus for scalar products of non-commuting operator functions is quite useful.

Theorem 2.4. *Let A, B be selfadjoint (unbounded) operators on a Hilbert space $\mathcal{H}$, T a bounded operator on $\mathcal{H}$. Let further $f, g : \mathbb{R} \to \mathbb{C}$ be measurable functions (measurable with respect to the Borel-σ-algebras on $\mathbb{R}$ and $\mathbb{C}$, respective), and $x \in \mathcal{D}(f(A))$, $y \in \mathcal{D}(g(B))$. Further let $\overline{f} \otimes g$ be integrable with respect to $\mu_{x,y}$, where $\mu_{x,y}$ is the uniquely determined (complex) measure on the Borel-σ-algebra Σ^2 in $\mathbb{R}^2$, such that for every measurable rectangle $R = M_1 \times M_2$ ($M_1, M_2 \in \Sigma$, Σ the Borel-σ-algebra on $\mathbb{R}$)*

$$\mu_{x,y}(R) = \big\langle E_{M_1}x | TF_{M_2}y \big\rangle$$

is valid. Here E_{M_1} and F_{M_2} are the spectral measures with respect to the two operators A and B, resp. For this measure we have the following result:

$$\big\langle f(A)x | Tg(B)y \big\rangle = \int_{\mathbb{R}\times\mathbb{R}} \overline{f} \otimes g d^2\mu_{x,y}. \tag{2.4}$$

Proof. The proof is standard measure theory, so we omit the details (cf. [1]). □

Corollary 2.5. *The assertion of Theorem 2.4 applies in particular to bounded, measurable functions f, g.*

3 A Criterion for Energy Equipartition for Selfadjoint Operators

We want to use Corollary 2.5 for the examination of (2.2). Let T be the following bounded operator on $\mathcal{H}$:

$$(u, v) \mapsto T(u, v) = (u, -v).$$

According to Corollary 2.5 we have

$$S_{u_0}(t) = \int_{\mathbb{R}\times\mathbb{R}} \exp(-2\pi i(\lambda - \mu)t) d^2\nu_{u_0}(\lambda, \mu), \tag{3.1}$$

where ν_{u_0} is the measure of Theorem 2.4 for $\big\langle E_{M_1}u_0 | TE_{M_2}u_0 \big\rangle$, with E_λ the spectral measure of A. Now we will prove the following theorem giving a necessary and sufficient criterion for energy equipartition.

Theorem 3.1. *If $u_0 \in \mathcal{H}$, S_{u_0} as in (3.1), then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S_{u_0}(t)dt = \int_{\{\lambda=\mu\}} d^2\nu_{u_0}(\lambda, \mu).$$

Proof. Applying Fubini's theorem and the dominated convergence theorem we get

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\int_0^T S_{u_0}(t)dt &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\int_{\mathbb{R}\times\mathbb{R}} \exp(-2\pi i(\lambda-\mu)t)d^2\nu_{u_0}(\lambda,\mu)\right)dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{\{\lambda\neq\mu\}} \left(\int_0^T \exp(-2\pi i(\lambda-\mu)t)dt\right)d^2\nu_{u_0}(\lambda,\mu) \\
&\quad + \lim_{T\to\infty}\frac{1}{T}\int_{\{\lambda=\mu\}} \left(\int_0^T \exp(-2\pi i(\lambda-\mu)t)dt\right)d^2\nu_{u_0}(\lambda,\mu) \\
&= \int_{\{\lambda\neq\mu\}} \lim_{T\to\infty}\frac{\exp(-2\pi i(\lambda-\mu)T)-1}{-2\pi i(\lambda-\mu)T}d^2\nu_{u_0}(\lambda,\mu) \\
&\quad + \int_{\{\lambda=\mu\}} d^2\nu_{u_0}(\lambda,\mu) \\
&= \int_{\{\lambda=\mu\}} d^2\nu_{u_0}(\lambda,\mu),
\end{aligned}$$

which proves the assertion. □

As a consequence of this theorem we can formulate

Theorem 3.2. *Asymptotic energy equipartition in the Cesaro sense for the operator A holds, if and only if*

$$\int_{\{-K<\lambda=\mu\leq K\}} d^2\nu_{u_0}(\lambda,\mu) = 0 \tag{3.2}$$

for every $u_0\in\mathcal{H}$ and every $K\in\mathbb{R}^+$.

Proof. It follows from (2.3) with the help of Theorem 3.1. □

For calculation purposes the next lemma is useful.

Lemma 3.3. *We have*

$$\int_{\{-K<\lambda=\mu\leq K\}} d^2\nu_{u_0}(\lambda,\mu) = \lim_{n\to\infty}\sum_{k=-n}^{n}\langle E_{I_k^{(n)}}u_0|TE_{I_k^{(n)}}u_0\rangle,$$

where

$$I_k^{(n)} := \left\{\lambda\in\mathbb{R}|(k-\tfrac{1}{2})\epsilon_n < \lambda \leq (k+\tfrac{1}{2})\epsilon_n\right\}\quad (k=-n,\dots,n)$$

for $n\in\mathbb{N}$ and $\epsilon_n = \frac{2K}{2n+1}$.

Proof. By the dominated convergence theorem we get

$$\int \chi_{\{-K<\lambda=\mu\leq K\}} d^2\nu_{u_0}(\lambda,\mu) = \lim_{n\to\infty}\sum_{k=-n}^{n}\int \chi_{(I_k^{(n)}\times I_k^{(n)})}(\lambda,\mu) d^2\nu_{u_0}(\lambda,\mu),$$

because

$$\chi_{\{-K<\lambda=\mu\leq K\}} = \lim_{n\to\infty}\chi_{\bigcup_{k=-n}^{k=n}(I_k^{(n)}\times I_k^{(n)})},$$

where χ_M is the characteristic function of the set M. Furthermore we get

$$\begin{aligned}\int \chi_{(I_k^{(n)}\times I_k^{(n)})}(\lambda,\mu) d^2\nu_{u_0}(\lambda,\mu) &= \int \chi_{I_k^{(n)}}(\lambda)\chi_{I_k^{(n)}}(\mu) d^2\nu_{u_0}(\lambda,\mu)\\ &= \langle E_{I_k^{(n)}}u_0 | TE_{I_k^{(n)}}u_0\rangle.\end{aligned}$$

□

Remark 3.1. From the last result we can extract the condition given by Picard and Seidler in [19] using Lemma 2.2.

In the case of operators with compact resolvent we get the following simple criterion.

Corollary 3.4. *Let A be an operator with compact resolvent. Then we have energy equipartition in the Cesaro sense if and only if*

$$\|\pi_1 P_j u_0\|^2 = \|\pi_2 P_j u_0\|^2 \text{ for all } u_0 \in \mathcal{H}.$$

Proof. If A has a compact resolvent there exists an at most countably infinite set of eigenvalues of A with no (finite) accumulation point. Hence there exists an $N \in \mathbb{N}$, such that in every interval $I_k^{(N)}$ of Lemma 3.3 lies at most one eigenvalue, i.e. $E_{I_k^{(n)}} = 0$ or $E_{I_k^{(n)}} = P_j$, where P_j is the (orthogonal) projection on the eigenspace for the eigenvalue, which lies in the interval $I_k^{(n)}$. Using Lemma 3.3 and Theorem 3.2 we get the condition:

$$0 = \lim_{n\to\infty}\sum_{k=-n}^{n}\langle E_{I_k^{(n)}}u_0 | TE_{I_k^{(n)}}u_0\rangle = \sum_j \langle P_j u_0 | TP_j u_0\rangle \text{ for all } u_0 \in \mathcal{H}.$$

Here the summation extends over all j, that lie in the interval $(-K, K]$. Since the condition is true for every u_0 and every $K \in \mathbb{R}^+$, the assertion follows. □

Remark 3.2. The condition of Corollary 3.4 can be formulated also in terms of the eigenvectors of A. Let $B_k := \left\{x_j^{(k)}\right\}$ be a (finite) orthonormal basis of the eigenspace of the k-th eigenvalue λ_k. Then the condition of Corollary 3.4 becomes

$$\langle \pi_i x_j^{(k)} | \pi_i x_l^{(k)}\rangle = \frac{1}{2}\delta_{jl} \quad i = 1, 2, \forall j, k, l.$$

4 Energy Equipartition for a System with Coupled Boundary Condition

In this section we examine as an example for the theorem just proven an operator which can not be regarded as a matrix of operators like in the articles of Picard and Seidler ([19], [18]).

We consider on the Hilbert space $\mathcal{H} := L_{2,p}(I) \oplus L_{2,q}(I)$, with the scalar product

$$\left\langle (u,v)|(u',v')\right\rangle_{\mathcal{H}} := \left\langle u|u'\right\rangle_p + \left\langle v|v'\right\rangle_q := \left\langle p^{-1}u|u'\right\rangle_0 + \left\langle q^{-1}v|v'\right\rangle_0,$$

where $p, q \in L_\infty(I)$ with $p(x), q(x) \geq c_0 > 0$ for almost every $x \in I$ ($I = (a,b)$ with $a, b \in \mathbb{R}$), the following operator:

$$A = \begin{pmatrix} 0 & pD \\ qD & 0 \end{pmatrix} \tag{4.1a}$$

with domain

$$\mathcal{D}(A) := \{(u,v) \in \mathcal{H}_1(I) \oplus \mathcal{H}_1(I) | u(a) + \alpha v(a) = 0 \wedge u(b) + \beta v(b) = 0\}, \tag{4.1b}$$

where $\mathcal{H}_1(I) = W_2^1(I)$ is the Sobolev space of once differentiable L_2-functions ($\mathcal{H}_1(I) \subset \mathbf{C}(\overline{I})$!). By a simple calculation we see that this operator is selfadjoint for α, β being purely imaginary but can not be regarded as an operator matrix (with respect to the given decomposition of $\mathcal{H}$), if $\alpha \neq 0$ or $\beta \neq 0$. Also one can see that $\lambda = 0$ is an eigenvalue if and only if $\alpha = \beta$.

With standard arguments we can now show that A is an operator with compact resolvent (cf. e.g. [15, ch.7.4, p.142ff] or [1]). We get the following

Theorem 4.1. *1. $\mathcal{W}(A)$ is closed, where $\mathcal{W}(A)$ is the range of A.*

2. $A^{-1} : \mathcal{W}(A) \to \mathcal{D}(A) \cap \mathcal{W}(A)$ exists,

3. A^{-1} is compact.

If we restrict A to the range $\mathcal{W}(A)$ we get a selfadjoint operator with compact resolvent, which will be denoted again with A.

Using Corollary 3.4 we prove now the following

Theorem 4.2. *Energy equipartition in the Cesaro sense is true for the operator A, if and only if the components v_i, w_i of the (orthonormal) eigenvectors u_i corresponding to the eigenvalue λ_i fulfill the following condition*

$$\left[\overline{w_i} v_j\right]_a^b = 0 \quad \text{for every } i, j. \tag{4.2}$$

Proof. With the orthonormality of the eigenvectors we get

$$\begin{aligned}\lambda\delta_{ij} &= \lambda\big\langle u_i|u_j\big\rangle_{\mathcal{H}} = \big\langle \lambda u_i|u_j\big\rangle_{\mathcal{H}} = \big\langle Au_i|u_j\big\rangle_{\mathcal{H}} = \\ &= \big\langle pDw_i|v_j\big\rangle_p + \big\langle qDv_i|w_j\big\rangle_q = \big\langle Dw_i|v_j\big\rangle_0 + \big\langle Dv_i|w_j\big\rangle_0.\end{aligned}$$

Integrating by parts and inserting the differential equation we get furthermore

$$\begin{aligned}\big\langle Dw_i|v_j\big\rangle_0 + \big\langle Dv_i|w_j\big\rangle_0 &= \big\langle w_i|Dv_j\big\rangle_0 + \big\langle Dv_i|w_j\big\rangle_0 - \frac{1}{2\pi i}\left[\overline{w_i}v_j\right]_a^b = \\ &= \big\langle w_i|q^{-1}\lambda w_j\big\rangle_0 + \big\langle q^{-1}\lambda w_i|w_j\big\rangle_0 - \frac{1}{2\pi i}\left[\overline{w_i}v_j\right]_a^b = \\ &= 2\lambda\big\langle w_i|w_j\big\rangle_q - \frac{1}{2\pi i}\left[\overline{w_i}v_j\right]_a^b .\end{aligned}$$

Hence

$$\big\langle w_i|w_j\big\rangle_q = \frac{1}{2}\delta_{ij} + \frac{1}{4\pi i\lambda}\left[\overline{w_i}v_j\right]_a^b .$$

Similarly we get

$$\big\langle v_i|v_j\big\rangle_q = \frac{1}{2}\delta_{ij} - \frac{1}{4\pi i\lambda}\left[\overline{w_i}v_j\right]_a^b ,$$

which shows the assertion with the help of Corollary 3.4. □

As an example for operators satisfying the condition of the last theorem but not belonging to the class considered in [19], i.e. $\alpha \neq 0$ or $\beta \neq 0$, we examine now the special case $p = q$ with $p \in \mathbf{C}_1(\overline{I})$. Then the components v, w of the eigenvectors are continuously differentiable and the eigenvalue equation for the operator A can be formulated classically. So we have simple spectrum by the Picard-Lindelöf theorem.

Further we get the following conservation law:

Lemma 4.3. *Let A be as described above and let u, v be the components of the eigenvectors of A, then*

$$|v|^2 + |w|^2 = const.$$

Proof. By multiplication of the differential equation by $\overline{(pD)v}$ and $(pD)w$, resp., we get

$$\begin{aligned}\overline{(pD)v}(pD)w &= \overline{(pD)v}\lambda v\\ (pD)w\overline{(pD)v} &= (pD)w\overline{\lambda w}.\end{aligned}$$

Hence ($0 \neq \lambda \in \mathbb{R}$):

$$-(D\overline{v})v = (Dw)\overline{w}$$

and

$$D\left|v\right|^2 = (Dv)\overline{v} + (D\overline{v})v = -(D\overline{w})w - (Dw)\overline{w} = -D\left|w\right|^2 .$$

□

Now we can derive the following

Theorem 4.4. *Let A be as in Lemma 4.3. We have energy equipartition in the Cesaro sense for A iff $\alpha = \beta$ or $\alpha = -1/\beta$.*

Proof. Inserting the boundary condition in Lemma 4.3 and Theorem 4.2 we get

$$|w(b)|^2 = \frac{-\alpha^2+1}{-\beta^2+1} |w(a)|^2 .$$

and

$$\begin{aligned}\overline{w(b)}v(b) - \overline{w(a)}v(a) = \overline{w(b)}(-\beta)w(b) - \overline{w(a)}(-\alpha)w(a) = \\ = \left(\beta\frac{\alpha^2-1}{-\beta^2+1} + \alpha\right)|w(a)|^2 .\end{aligned}$$

The last expression is 0 if and only if the following quadratic equation is fulfilled:

$$\beta(\alpha^2-1) = \alpha(\beta^2-1).$$

(if $|v(a)|^2 = 0$ we get, by Lemma 4.3, that $v = 0$ which is not an eigensolution.) $\square$

Remark 4.1. It can also be shown that this result remains true, when the interval I is unbounded, e.g. $I = (0, \infty)$, and $p = q = 1$ (see [1]).

5 Outlook

Beginning with the result presented here there a some further questions to solve. First there is the question, when do we have equipartition in the strong sense, i.e. when do the components converge pointwise to the ratio $\frac{1}{n}$ of the total energy, not only in the Cesaro mean. Here in analogy to Theorem 3.1 the behaviour of $\lim_{t\to\infty} S_{u_0}(t)$ has to be examined. Here we expect, that we must state another condition like the *Riemann-Lebesgue-Condition* or the absolute continuity of the spectrum in the decoupled case. It could be, that the absolute continuity of the spectral measure implies also the absolute continuity of the measure ν_{u_0} of Theorem 2.4 on the subsets $D_1 := \{(\lambda,\mu) \in \mathbb{R}^2 | \lambda < \mu\}$ and $D_2 := \{(\lambda,\mu) \in \mathbb{R}^2 | \lambda > \mu\}$, resp. So for absolute continuous spectrum we would have the same condition as in the Cesaro case.

Further we could extend the considerations to bigger systems than $n = 2$. Now it can easily be seen that you can extend the condition of Theorem 3.2 for $n > 2$ to

$$\int_{\{-K<\lambda=\mu\leq K\}} d^2\nu_{u_0,k}(\lambda,\mu) = 0 \text{ for all } u_0 \in \mathcal{H} \text{ and for every } k = 1,\dots,n., \tag{5.1}$$

where the measures $\nu_{u_0,k}$ are constructed like the measure ν_{u_0} with the operators $T_k = (\delta_{i,j} - n\delta_{i,j}\delta_{k,i})_{i,j}$ $(k = 1,\dots,n)$.

Last you could omit the restriction to selfadjoint operators and examine equations as Goldstein et al. did for instance in [9], [12], or [13] where they considered the ratio of the components of the total energy, which is asymptotically constant in certain cases.

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