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DIRICHLET PROBLEM FOR DEGENERATE ELLIPTIC COMPLEX MONGE-AMPÈRE EQUATION

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Abstract. We consider the Dirichlet problem

$$\det\big(\frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}\big) = g(z,u) \quad \text{in } \Omega\,, \quad u\big|_{\partial\Omega} = \varphi\,,$$

where Ω is a bounded open set of $\mathbb{C}^n$ with regular boundary, g and φ are sufficiently smooth functions, and g is non-negative. We prove that, under additional hypotheses on g and φ, if $|\det \varphi_{i\bar{j}} - g|_{C^{s_*}}$ is sufficiently small the problem has a plurisubharmonic solution.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^{2n}$ with smooth boundary and let $z_i = x_i + ix_{i+n} (1 \le i \le n)$. We shall also denote by Ω the set of $z = (z_1, z_2, \dots, z_n)$ satisfying $(\operatorname{Re} z, \operatorname{Im} z) \in \Omega$. We study the problem of finding a sufficiently smooth plurisubharmonic solution to the degenerate problem

$$\begin{gathered} \det\big(\frac{\partial^2 \phi}{\partial z_i \partial \overline{z_j}}\big) = g(z,\phi) \quad \text{in } \Omega\,, \\ \phi\big|_{\partial\Omega} = \varphi\,. \end{gathered} \tag{1.1}$$

In [8, 9], the author studies local solutions, while, here we consider global solutions.

This problem has received considerable attention both in the non-degenerate case ($g > 0$) and in the degenerate case ($g \ge 0$). In particular, Caffarelli, Kohn, Nirenberg and Spruck [4] established some existence results in strongly pseudoconvex domains based on the construction of a subsolution. The recent work of Guan [6], extends some of these results to arbitrary smooth bounded domains. Guan proved for the nondegenerate case that a sufficient condition for the classical solvability is the existence of a subsolution. Here we are concerned with degenerate problems in an arbitrary smooth bounded domain, which need not be Pseudoconvex.

Counterexamples due to Bedford and Fornaes [2] show that the Dirichlet problem, in general, does not have a regular solution. This implies that we should place some restrictions on g and φ.

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Let us assume that φ is a real function defined in $\overline{\Omega}$, Σ is a finite set of points in Ω, and $g(z,\phi) = K(z)f(\operatorname{Re} z, \operatorname{Im} z, \phi)$. We further assume the following hypotheses.

- (A1) $K \geq 0$ in $\overline{\Omega}$, and $K^{-1}(0) = \Sigma$
- (A2) $f(x,u) > 0$ in $\overline{\Omega} \times \mathbb{R}$, and $\frac{\partial f}{\partial u} \geq -\varrho$ in $\overline{\Omega} \times \mathbb{R}$, with $0 \leq \varrho << 1$
- (A3) $\varphi\big|_{\overline{\Omega}\setminus\Sigma}$ is strictly plurisubharmonic, $(\varphi_{i\bar{j}})\big|_{\Sigma}$ is of rank $(n-1)$, and the eigenvalues of $(\varphi_{i\bar{j}})$ on Σ are distinct.

Our main results are the following theorems:

Theorem 1.1. *Let $s_* \geq 7+2n$ be an integer, $\alpha \in]0,1[$, and $\Gamma > 1$. If $\varphi \in C^{s_*+2,\alpha}(\overline{\Omega})$ satisfies the condition (A3), then one can find a constant $\varepsilon_0 > 0$ (depending on s_*, α, Γ, Ω and φ) such that for any $g = Kf \in C^{s_*}$ satisfying (A1), (A2),*

$$|\det \varphi_{i\bar{j}} - g(\varphi)|_{C^{s_*}} \leq \varepsilon_0 \tag{1.2}$$

and $|\frac{\partial g}{\partial u}|_{C^{s_*}} \leq \Gamma$, then problem (1.1) has a plurisubharmonic (real valued) solution $\phi \in C^{s_*-3-n}(\overline{\Omega})$, which is unique when $\rho = 0$.

Let $l_\alpha(x)$ denote α-th row the matrix of cofactors of $(\varphi_{i\bar{j}})$, and

$$D^k K(x)(l_\alpha(x), l_\beta(x))^{(k)} = D^k K(x)\big(l_\alpha(x), l_\beta(x); \dots; l_\alpha(x), l_\beta(x)\big).$$

Theorem 1.2. *Under the assumptions in Theorem 1.1, suppose that $\varphi \in C^\infty(\overline{\Omega})$ and for any point $x_0 \in \Sigma$ one can find an integer k such that $D^j K(x_0) = 0$ for all $j \leq k-1$ and there exists $\alpha \neq \beta \in \{1,\dots,n\}$ such that $D^k K(x_0)(l_\alpha(x_0), l_\beta(x_0))^{(k)} \neq 0$. Then there exists an integer $s_* > 0$ and a constant $\varepsilon_0 > 0$ such that for any function $g \in C^\infty$ satisfying (A2), (A3) and (1.2), the plurisubharmonic solution ϕ to the problem (1.1) is in $C^\infty(\overline{\Omega})$.*

In Theorem 1.1, the assumption concerning Σ leads to a-priori estimates and the assumption on g and φ ensures the convergence of an iteration scheme of Nash-Moser type. It is to be noted that we do not require demonstrating that a subsolution exists as in [4] and [6].

Under some additional conditions on g, we can prove the smoothness of the solution, using the works of Xu [12] and Xu and Zuily [13].

This paper is organized as follows. In Section 2 we state some preliminary results. In Section 3, we state fundamental global a-priori estimates for degenerate linearized operators that are crucial to establish an iteration scheme of Nash-Moser type. We then prove Theorem 1.1 in Section 4. We prove Theorem 1.2 in Section 5. Finally, we prove the a-priori estimates stated in Section 3.

2. Preliminary results

We shall use the norms

$$|\cdot|_k = \|\cdot\|_{C^k(\overline{\Omega})}, \quad \|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}, \quad |\cdot|_{k,\tau} = \|\cdot\|_{C^{k,\tau}(\overline{\Omega})}$$

where $k \in \mathbb{N}$ and $\tau \in]0,\alpha[$.

In this work, we need some technical lemmas which play important roles in the proof of convergence of our iteration scheme.

Lemma 2.1. *Let s_* be an integer, $s_* \geq 7+2n$. We can find a constant $\beta \geq 2$ such that for any $0 \leq i,j,k \leq s_*+2$, $n_* = n+\tau$ and $u \in C^{s_*+2,\alpha}(\overline{\Omega})$ we have: The Sobolev inequality*

$$|u|_{i,\tau} \leq \beta \|u\|_{i+n_*} \tag{2.1}$$

The Gagliardo-Nirenberg inequality

$$\|u\|_j \le \beta \|u\|_i^{\frac{k-j}{k-i}} \|u\|_k^{\frac{j-i}{k-i}}, \quad i < j < k \tag{2.2}$$

The inequality

$$\|u\|_{s_*} \le \beta |u|_{s_*} \tag{2.3}$$

For any $\lambda \ge 1$, there exists a family of smoothing linear operators $S_\lambda : \cup_{i \ge 0} H^i(\Omega) \to \cap_{j \ge 0} H^j(\Omega)$, satisfying

$$\|S_\lambda u\|_i \le \beta \|u\|_j, \quad \text{if } i \le j \tag{2.4}$$

$$\|S_\lambda u\|_i \le \beta \lambda^{i-j} \|u\|_j, \quad \text{if } i \ge j \tag{2.5}$$

$$\|S_\lambda u - u\|_i \le \beta \lambda^{i-j} \|u\|_j, \quad \text{if } i \le j \tag{2.6}$$

Lemma 2.2 ([1, 7]). *(1) For $t > 0$; if $u, v \in L^\infty \cap H^t$, then $uv \in L^\infty \cap H^t$ and*

$$\|uv\|_t \le K_1(|u|_0 \|v\|_t + \|u\|_t |v|_0), \tag{2.7}$$

where, K_1 is a constant ≥ 1 independent of u and v.
(2) Let $H : \mathbb{R}^m \to \mathbb{C}$ be a function C^∞ of its arguments.
For $s > 0$, if $\omega \in (L^\infty \cap H^s)^m$ and $|\omega|_0 \le M$, then

$$\|H(\omega)\|_s \le K_2(s, H, M)(\|\omega\|_s + 1), \tag{2.8}$$

where $K_2 \ge 1$ and is a constant independent of ω.
If $\omega \in (C^{i,\mu})^m$, $\mu \in]0, 1[$ and $i \in \mathbb{N}$, then $H(\omega) \in C^{i,\mu}$.

If we suppose that $|\omega|_0 \le M$, then we can find a constant $K_3 = K_3(i, \mu, H, M) \ge 1$ such that

$$|H(\omega)|_{i,\mu} \le K_3(|\omega|_{i,\mu} + 1). \tag{2.9}$$

We shall also need the following technical lemma.

Lemma 2.3 ([8, Lemma]). *Let $F(u_{z_i\overline{z_j}}) = \det(u_{z_i\overline{z_j}})$. For $1 \le i, j, a, b \le n$, we have*

$$F \frac{\partial^2 F}{\partial u_{z_a\overline{z_b}} \partial u_{z_i\overline{z_j}}} = \frac{\partial F}{\partial u_{z_a\overline{z_b}}} \frac{\partial F}{\partial u_{z_i\overline{z_j}}} - \frac{\partial F}{\partial u_{z_i\overline{z_b}}} \frac{\partial F}{\partial u_{z_a\overline{z_j}}}. \tag{2.10}$$

3. A priori estimates for the linearized operator

Defining $\phi = \varphi + \varepsilon w$, (1.1) becomes

$$\det(\phi_{z_i\overline{z_j}}) = \det(\varphi_{z_i\overline{z_j}} + \varepsilon w_{z_i\overline{z_j}}) = g. \tag{3.1}$$

Let

$$G(w) = \frac{1}{\varepsilon}[\det \Phi - g]. \tag{3.2}$$

Then the linearization of G at w is

$$L_G(w) = \sum_{i,j=1}^{n} \phi^{ij} \partial_{z_i} \partial_{\overline{z_j}} + b, \tag{3.3}$$

where $\widetilde{\Phi} = (\phi^{ij})$ is the matrix of cofactors of $\Phi = (\phi_{z_i\overline{z_j}}(z, \varepsilon, w))$ and $b = \frac{\partial g}{\partial u}$.

Now we construct linear elliptic operators, maybe degenerate, related to linearized operators. For any smooth real valued function w, the matrix $(\phi_{i\bar{j}})$ is Hermitian and we can find a unitary matrix $T(z, \varepsilon)$ satisfying

$$T(z, \varepsilon)(\phi_{z_i\overline{z_j}})^t \overline{T}(z, \varepsilon) = \operatorname{diag}(\lambda_1, \dots, \lambda_n). \tag{3.4}$$

Without loss of generality we may assume that Σ is reduced to one point, the origin. By means of change of variables we may assume, using (A3), that

$$\varphi_{z_i\overline{z_j}}(0) = \sigma_i\delta_i^j \quad i,j = 1,\dots,n, \tag{3.5}$$

where $\sigma_i > 0$ for $i = 1,\dots,n-1$, $\sigma_n = 0$ and $\sigma_i \neq \sigma_j$ for $i \neq j$. Let $0 < \tau \leq \frac{\alpha}{4}$.

Lemma 3.1. *There exist constants $\varepsilon_1 > 0$, $\delta_1 > 0$ and $M > 0$ depending only on φ, n, Ω such that when*

$$V_0 = \{(z,\varepsilon,w)/|z| \leq \delta_1,\ 0 \leq \varepsilon \leq \varepsilon_1,\ w \in C^{3,\tau}(\overline{\Omega}),\ |w|_{3,\tau} \leq 1\},$$

we have: (i) The eigenvalues λ_i, $i = 1,\dots,n$ of Φ are distinct on V_0 and of class C^1 in $\mathring{V}_0$. Moreover, $\lambda_i > 0$ in V_0, for $i = 1,\dots,n-1$.
(ii) For $(z,\varepsilon,w) \in V_0$,

$$\sum_{i=1}^{n} |\sigma_i - \lambda_i(z,\varepsilon,w)| + |\Phi^{nn}(z,\varepsilon,w) - \prod_{i=1}^{n-1}\sigma_i| \leq M(\varepsilon + |z|). \tag{3.6}$$

(iii) For $(z,\varepsilon,w) \in V_0$ and $i = 1,\dots,n-1$,

$$\lambda_i \geq \inf_{1\leq i\leq n-1}\sigma_i - M\delta_1 - (M+1)\varepsilon_1 > 0 \text{ and } \Phi^{nn} \geq \prod_{i=1}^{n-1}\sigma_i - M\delta_1 - M\varepsilon_1 > 0. \tag{3.7}$$

Proof. Let us consider the function $H(z,\varepsilon,w,\lambda) = \det(\varphi_{z_i\overline{z_j}} + \varepsilon w_{z_i\overline{z_j}} - \lambda\delta_i^j)$. Then $H \in C^1$ and by (3.5), we have

$$H(0,0,0,\sigma_i) = 0 \text{ and } \frac{\partial H}{\partial\lambda}(0,0,0,\sigma_i) \neq 0, \quad \forall i \in \{1,\dots,n\}.$$

By the implicit function theorem, one can find two constants $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that (i) holds. Moreover by (3.5) we have

$$\frac{\partial F}{\partial u_{n\overline{n}}}(\varphi_{i\overline{j}})(0) = \Phi^{nn}(0,0,w) = \prod_{i=1}^{n-1}\sigma_i > 0,$$

which gives (ii) and (iii). $\square$

Lemma 3.2. *There exists a positive constant ε_2 such that for any $0 < \varepsilon < \varepsilon_2$, any real valued function $w \in C^{3,\tau}(\overline{\Omega})$ satisfying $|w|_{3,\tau} \leq 1$ and $\theta = \max_{z\in\overline{\Omega}}|G(w)|$, the operator*

$$L = -L_G(w) - \theta\triangle \tag{3.8}$$

is elliptic, maybe degenerate. (Here $\triangle = \sum_{i=1}^{n}(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2})$)

Proof. Let

$$A = \theta|\xi|^2 + \sum_{i,j=1}^{n}\phi^{ij}\xi_i\overline{\xi_j} \geq 0, \quad \forall (z,\xi) \in \overline{\Omega}\times\mathbb{C}^n. \tag{3.9}$$

If $z \in \overline{\Omega}\backslash\{0\}$, as φ is strictly plurisubharmonic, then $A > 0$ for all $\xi \in \mathbb{C}^n\backslash\{0\}$.

If $z = 0$, for $\xi \in \mathbb{C}^n$, we let $\xi =^t T(\tau,\varepsilon)\widetilde{\xi}$. Then we have

$$A = \theta|\xi|^2 +^t \xi\widetilde{\Phi}\overline{\xi} = \theta|\xi|^2 +^t \widetilde{\xi}T\widetilde{\Phi}^t\overline{T}\overline{\widetilde{\xi}}.$$

Since $\Phi\widetilde{\Phi} = \det\Phi\,\mathrm{Id}$, by (3.4),

$$\det\Phi\,\mathrm{Id} = T\Phi^t\overline{T}T\widetilde{\Phi}^t\overline{T} = \mathrm{diag}(\lambda_i)T\widetilde{\Phi}^t\overline{T},$$

$$T\widetilde{\Phi}^t\overline{T} = \det\Phi\,\mathrm{diag}(\frac{1}{\lambda_i}) = \prod_{i=1}^n \lambda_i\,\mathrm{diag}(\frac{1}{\lambda_i}) = (\varepsilon G + g)\,\mathrm{diag}(\frac{1}{\lambda_i}).$$

Thus,

$$\begin{aligned}
A &= \theta|\widetilde{\xi}|^2 + \det\Phi\sum_{i=1}^n \frac{|\widetilde{\xi}_i|^2}{\lambda_i}\\
&= \theta|\widetilde{\xi}|^2 + \sum_{i=1}^{n-1}\det\Phi\frac{|\widetilde{\xi}_i|^2}{\lambda_i} + \prod_{i=1}^{n-1}\lambda_i|\widetilde{\xi}_n|^2\\
&= (\theta + \prod_{i=1}^{n-1}\lambda_i)|\widetilde{\xi}_n|^2 + \sum_{i=1}^{n-1}\frac{\varepsilon G + g + \theta\lambda_i}{\lambda_i}|\widetilde{\xi}_i|^2.
\end{aligned}$$

By (3.7), for $i = 1,\dots,n-1$, $\varepsilon \le \varepsilon_1$ and $|w|_{3,\tau} \le 1$, we have

$$\varepsilon G + \theta\lambda_i \ge \theta(\sigma_i - M\delta_1 - (M+1)\varepsilon_1) \ge 0.$$

Therefore, $A \ge 0$, which proves the lemma. □

Now we study a boundary-value problem for the degenrate elliptic operator

$$L = -L_G(w) - \theta\triangle = \sum_{i,j=1}^n b^{ij}\partial_{z_i}\partial_{\overline{z_j}} + b,$$

where

$$b^{ij} = -\frac{\partial F}{\partial u_{i\bar{j}}}(\varphi_{i\bar{j}} + \varepsilon w_{i\bar{j}}) - \theta\delta_i^j = -\Phi^{ij} - \theta\delta_i^j$$

and $b = K\frac{\partial f}{\partial u}$. For $k, s \in \mathbb{N}$ we let

$$\begin{gathered}
A(k) = \max(1, \max_{1\le i,j\le n}|b^{ij}|_k, |b|_k)\\
\Lambda_s = \{(i,j) : 0 \le i,j \le s,\ i+j \le s,\ \text{and } i+2 \le \max(s,2)\}
\end{gathered} \tag{3.10}$$

Now from Lemma 3.2 we have the following statement.

Theorem 3.3. *Suppose that $\theta \le 1$ and $A(2) \le M_0$, for some constant $M_0 > 0$. One can find $\varepsilon_3 > 0$ such that for any $\varepsilon \in]0,\varepsilon_3]$, any real valued function $w \in C^{s_*+2,\tau}(\overline{\Omega})$ satisfying the inequality $|w|_{3,\tau} \le 1$ and any real valued function $h \in H^{s_*}$, the problem*

$$\begin{gathered}
Lu = h \quad \text{in } \Omega\\
u\big|_{\partial\Omega} = 0
\end{gathered} \tag{3.11}$$

has a unique solution $u \in H^{s_*}$. Moreover for $0 \le s \le s_*$,

$$\|u\|_0 \le C_0\|h\|_0 \tag{3.12}$$

$$\|u\|_1 \le C_1(\|h\|_1 + \|u\|_0) \tag{3.13}$$

$$\|u\|_s \le C_s\{\|h\|_s + \sum_{j\le s-1,\,(i,j)\in\Lambda_s}(1 + |\varphi + \varepsilon w|_{i+4,\tau})\|u\|_j\}, \quad s \ge 2 \tag{3.14}$$

for some constant $C_s = C_s(\varphi, s, \Omega, M_0, \varepsilon_3)$ independent of w and ε.

For $\nu \in]0,1[$, we denote $L_\nu = L - \nu\triangle$. To solve the Dirichlet problem (3.11), we first establish the following proposition.

Propositon 3.4. *Let $\theta \le 1$ and, for some constant $M_0 > 0$, $A(2) \le M_0$. Then there exists $\varepsilon_3 > 0$ such that for any $\varepsilon \in]0, \varepsilon_3]$, any real valued function $w \in C^{s_*+2,\tau}(\overline{\Omega})$ satisfying the inequality $|w|_{3,\tau} \le 1$ and any real valued function $h \in H^{s_*}(\Omega)$, the regularized problem*

$$\begin{gathered} L_\nu u = h \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \end{gathered} \tag{3.15}$$

has a unique (real valued) solution $u \in H^{s_*+1}(\Omega)$.

Proof. Since $L_G(w)$ is a second order operator with real coefficients, from Lemma 3.2, L_ν is uniformly elliptic with coefficients in $C^{s_*,\tau}(\overline{\Omega})$. Thus by [3, Theorems 6.14 and 8.13] we see that (3.15) has a real valued solution.

If (3.12)–(3.14) hold for the regularized problem (3.15) with an uniform constant C_s independent of $\nu \in]0, 1]$, then by letting ν tend to zero we get a solution $u \in H^{s_*}(\Omega)$ to the original problem which of course satisfies (3.12)–(3.14). □

Using Theorem 3.3, we prove Theorem 1.1 by constructing a sequence of approximating solutions and a priori estimates for linearized operators. The hypothesis (1.2) will play an important role in the proof of the convergence of our iteration scheme of Nash-Moser type.

4. Proof of Theorem 1.1

Part 1: An iteration scheme of Nash-Moser type. In this section, we use the Nash-Moser procedure [7, 10] and the results of Section 3 to prove Theorem 1.1. We construct a sequence which converges to a solution to our problem. We define

$$M_0 = 1 + \max_{H\in\mathcal{F}} K_3(2, \tau, H, (1 + |\varphi|_2))(1 + |\varphi|_{4,\tau}), \tag{4.1}$$

where $\mathcal{F} = \{\frac{\partial F}{\partial u_{i\bar{j}}}, \frac{\partial g}{\partial u}/1 \le i, j \le n\}$ and K_3 is the constant introduced in (2.9). (i.e: $|H(u)|_{j,\mu} \le K_3(j, \mu, H, M)|u|_{j,\mu}$). We also define

$$D = \max\big(\max_{0\le s\le s_*} C_s,\, 1\big). \tag{4.2}$$

Here C_s is the constant (depending only on s, φ, Ω, M_0) given by Theorem 3.3. We let

$$\mu = \max(\beta, 3Ds_*^2(1 + |\varphi|_{s_*+2,\tau}), n, 2^{\frac{1}{\tau}}) \quad \text{and} \quad \widetilde{\mu} = \beta^2\mu^{s_*}, \tag{4.3}$$

$$a_1 = 9K_0\mu^5, \quad a_2 = 5a_1\mu^{s_*+1}, \quad a_3 = 7K_0\mu^5, \tag{4.4}$$

were K_0 is the constant given by Proposition 6.1. Also, we fix $\widetilde{\varepsilon}$ satisfying

$$\widetilde{\varepsilon} \le \min[1, (\varepsilon_i)_{1\le i\le 4}, (3D^2a_2 + 6\widetilde{\mu}D^2)^{-2}], \tag{4.5}$$

were ε_i are given in Lemma 3.2, Theorem 3.3, the proof of Theorem 3.3 and the proof of (3.13).

As a consequence of these inequalities, we have $6\widetilde{\varepsilon}\mu^{s_*} \le 1/4$. Let $g \in C^{s_*}$ satisfy

$$|\det\varphi_{i\bar{j}} - g(\varphi)|_{s_*} \le \widetilde{\varepsilon}^2,$$

with ε_0 in Theorem 1.1 equal to $\widetilde{\varepsilon}^2$. Let $S_n = S_{\mu_n}$ the family of operators given by Lemma 2.1, with $\mu_n = \mu^n$ (μ is given by (4.3)).

Using Theorem 3.3, we construct w_n, $n = 0, 1, \dots$, by induction on n as follows. We let u_0, $w_0 = 0$, and assume $w_0, w_1, \dots, w_n$ have been chosen and define w_{n+1} by

$$w_{n+1} = w_n + u_{n+1}, \tag{4.6}$$

where u_{n+1} is the solution to the Dirichlet problem

$$\begin{gathered} L_G(\widetilde{w}_n)u_{n+1} + \theta_n \triangle u_{n+1} = g_n, \quad \text{in } \Omega \\ u_{n+1}\big|_{\partial\Omega} = 0, \end{gathered} \tag{4.7}$$

given by Theorem 3.3. Here

$$\widetilde{w}_n = S_n w_n, \tag{4.8}$$

$$\theta_n = |G(\widetilde{w}_n)|_0, \tag{4.9}$$

$$g_0 = -S_0 G(0), g_n = S_{n-1}R_{n-1} - S_n R_n + S_{n-1}G(0) - S_n G(0), \tag{4.10}$$

$$R_0 = 0, \quad R_n = \sum_{j=1}^{n} r_j, \tag{4.11}$$

$$r_0 = 0, \quad r_j = [L_G(w_{j-1}) - L_G(\widetilde{w}_{j-1})]u_j + Q_j - \theta_{j-1}\triangle u_j, \quad 1 \le j \le n, \tag{4.12}$$

$$Q_j = G(w_j) - G(w_{j-1}) - L_G(w_{j-1})u_j, \quad 1 \le j \le n. \tag{4.13}$$

To ensure that the w_n's are well defined, we prove the following proposition.

Propositon 4.1. *Let $s \in \mathbb{N}$. If $s_* \ge 7 + 2n$ and $4 + 2n + 2\tau \le \sigma < s_* - 2$, we have*

$$\|u_j\|_s \le \sqrt{\varepsilon}[\max(\mu, \mu_{j-1})]^{s-\sigma}, \quad j \in \mathbb{N}^*,\ 0 \le s \le s_*, \tag{4.14}$$

$$\|w_j\|_s \le \begin{cases} 2\sqrt{\varepsilon}, & \text{for } s \le \sigma - \tau \\ \sqrt{\varepsilon}\mu_j^{s-\sigma}, & \text{for } \sigma - \tau \le s \le s_* \end{cases} \quad j \in \mathbb{N}^*, \tag{4.15}$$

$$|\widetilde{w}_j|_{4,\tau} \le 1, \quad j \in \mathbb{N}^*, \tag{4.16}$$

$$\|w_j - \widetilde{w}_j\|_s \le 2\beta\sqrt{\varepsilon}\mu_j^{s-\sigma}, \quad 0 \le s \le s_*,\ j \in \mathbb{N}^*, \tag{4.17}$$

$$\|r_j\|_s \le \widetilde{\varepsilon} a_1[\max(\mu, \mu_{j-1})]^{s-\sigma}, \quad 0 \le s \le s_* - 2,\ j \in \mathbb{N}^*, \tag{4.18}$$

$$\|g_j\|_s \le \widetilde{\varepsilon} a_2 \mu_j^{s-\sigma}, \quad 0 \le s \le s_*,\ j \in \mathbb{N}, \tag{4.19}$$

$$\theta_j \le a_3\sqrt{\varepsilon}\mu_j^{-2} \le 1, \quad j \in \mathbb{N}, \tag{4.20}$$

$$A_j(2) \le M_0, \quad j \in \mathbb{N}. \tag{4.21}$$

Here, $A_j(k)$ is defined by using the definition of $A(k)$ in (3.10)*, where the coefficients correspond to $\widetilde{w}_j$.*

Let us first show how that Proposition 4.1 implies Theorem 1.1. The proof of this proposition will be given later in Appendix 1.

Part 2: Proof of Theorem 1.1. We prove the convergence of the sequence (w_n) using Proposition 4.1. Set $\sigma = s_* - 2 - \tau$ and $s = \sigma - \tau$. By (4.6) and (4.14), for any $i, k \in \mathbb{N}^*$, $i > k$,

$$\|w_i - w_k\|_s \le \sum_{j=k+1}^{i} \|u_j\|_s \le \beta\sqrt{\widetilde{\varepsilon}} \sum_{j=k+1}^{i} \mu_{j-1}^{-\tau} = \beta\sqrt{\widetilde{\varepsilon}} \sum_{j=k+1}^{i} (\mu^{-\tau})^{j-1}.$$

Since $\mu \ge 2$ and $\tau > 0$, then $\|w_i - w_k\|_s \to 0$ as $i, k \to \infty$. Hence, there is a function $w \in H^{s_*-2-2\tau}(\Omega)$ satisfying $w_n \to w$ in $H^{s_*-2-2\tau}(\Omega)$.

Since $H^{s_*-2-2\tau}(\Omega) \subset C^{s_*-2-n-3\tau}(\overline{\Omega})$, it follows that $w \in C^{s_*-3-n}(\overline{\Omega})$. On the other hand, combining (4.7), (4.12) and (4.13), we obtain

$$r_j = G(w_j) - G(w_{j-1}) - g_{j-1}$$

Taking the sum between $j = 1$ and $j = n$, using (4.10) and (4.11), we get

$$G(w_n) = (I - S_{n-1})R_{n-1} + (I - S_{n-1})G(0) + r_n. \tag{4.22}$$

For $n \geq 2$, using (2.2) and (4.18), we have

$$\|r_n\|_{s_*-2-2\tau} \leq a_1\beta\widetilde{\varepsilon}\mu_{n-1}^{s_*-2-2\tau-\sigma} = a_1\beta\widetilde{\varepsilon}\mu_{n-1}^{-\tau}.$$

Combining (2.3) with (2.6) and (1.2), we get

$$\|(I - S_{n-1})G(0)\|_{s_*-2-2\tau} \leq \beta\mu_{n-1}^{-2-2\tau}\|G(0)\|_{s_*} \leq \beta^2\mu_{n-1}^{-2-2\tau}\widetilde{\varepsilon}.$$

Combining (2.6), (4.11) and (4.18), we can write

$$\begin{aligned}
\|(I - S_{n-1})R_{n-1}\|_{s_*-2-2\tau} &\leq \beta\mu_{n-1}^{-2\tau}\|R_{n-1}\|_{s_*-2} \leq \beta\mu_{n-1}^{-2\tau}\sum_{j=1}^{n-1}\|r_j\|_{s_*-2} \\
&\leq \beta\mu_{n-1}^{-2\tau}\widetilde{\varepsilon}a_1 Big(\mu^{s_*-2-\sigma} + \sum_{j=2}^{n-1}\mu_{j-1}^{s_*-2-\sigma}) \\
&\leq \widetilde{\varepsilon}\beta a_1\mu_{n-1}^{-2\tau}\mu_{n-1}^{s_*-2-\sigma} \leq \beta a_1\widetilde{\varepsilon}\mu_{n-1}^{-\tau}.
\end{aligned}$$

These inequalities imply $G(w_n) \to 0$ in $H^{s_*-2-2\tau}(\Omega)$ as $n \to \infty$.

Since $H^{s_*-2-2\tau}(\Omega) \subset C^2(\overline{\Omega})$ and $w_{n|\partial\Omega} = 0$, we conclude that $G(w) = 0$ and $w\big|_{\partial\Omega} = 0$. That is $u = \varphi + \varepsilon w$ is a solution to the original Monge-Ampère equation which is by Lemma 3.1 plurisubharmonic since g is nonnegative. If we suppose that $\rho = 0$, in (A2), then the uniqueness of the solution follows immediately from [4].

5. Proof of Theorem 1.2

We shall use the result of Xu and Zuily [12, 13] that we recall briefly. Let us consider a non linear partial differential equation

$$F(x, y, u, \nabla u, D^2u) = 0\,,$$

where F is C^∞. To any solution u we can associate the vector fields $X_j = \sum_k \frac{\partial F}{\partial u_{jk}}\partial_k$. Then

Theorem 5.1 ([12]). *Suppose $u \in C^\rho_{\rm loc}(\Omega)$ with $\rho > Max(4, r{+}2)$ for some constant $r \geq 0$ and that the brackets of the X_j, up to the order r, span the tangent space at each point of Ω, then u belongs to $C^\infty(\Omega)$.*

To prove this theorem, it is sufficient to prove that the solution of Theorem 1.1 satisfies Theorem 5.1 at any point in Σ. Suppose $\Sigma = \{0\}$. For $i = 1 \dots n$;

$$X_i = \phi^{ii}\frac{\partial}{\partial x_i} + \sum_{j\neq i,\, j=1}^{n}\frac{\phi^{ij} + \overline{\phi^{ij}}}{2}\frac{\partial}{\partial x_j} + \sum_{j\neq i,\, j=1}^{n}\frac{i\phi^{ij} - i\overline{\phi^{ij}}}{2}\frac{\partial}{\partial x_{j+n}}, \tag{5.1}$$

$$X_{i+n} = \phi^{ii}\frac{\partial}{\partial x_{i+n}} + \sum_{j\neq i,\, j=1}^{n}\frac{\phi^{ij} + \overline{\phi^{ij}}}{2}\frac{\partial}{\partial x_{j+n}} - \sum_{j\neq i,\, j=1}^{n}\frac{i\phi^{ij} - i\overline{\phi^{ij}}}{2}\frac{\partial}{\partial x_j}. \tag{5.2}$$

For computing the Lie algebra generated by the X_i, we need the following result.

Lemma 5.2. *For any integer $1 \le m \le k$,*

$$\begin{aligned}
&(adX_n)^{m-1}[X_n - iX_{2n}, X_i - iX_{i+n}] \\
&= \sum_{l=1}^{2n} \sum_{|\beta| \le m,\, i \ne j} \big[(C_{i\beta p})\partial_x^\beta g + \varepsilon d_{pij}\big]\partial_{x_l} \qquad (5.3)\\
&\quad + [A_n(\varphi_{i\bar{j}})]^{m-1} A_i(\varphi_{i\bar{j}})\big[(\partial_{x_n}^m g + i\partial_{x_n}^{m-1}\partial_{x_{2n}} g)(\partial_{x_i} + i\partial_{x_i+n})\big],
\end{aligned}$$

where $C_{i\beta p}$ and d_{pij} are $C^{s_*-m,\tau}(\Omega)$ (depending on w and φ bounded for ε small enough) satisfying for $|\beta| = m$, $C_{i\beta p}(0) = 0, p = 1, \dots, n$ if $n \ge 3$ and $C_{i\beta 1}(0) = 0$ if $n = 2$. $A_n = \frac{\partial F}{\partial u_{n\bar{n}}}$ and $A_i = \frac{\partial^2 F}{\partial u_{n\bar{n}}\partial u_{i\bar{i}}}$.

Proof. We use induction on the size of the brackets. First we calculate $D_{in} = [X_n + iX_{2n}, X_i + iX_{i+n}]$, for $i \le n-1$.

$$\begin{aligned}
D_{in} &= \Big[\sum_{j=1}^n \Phi^{nj}\partial_{x_j} + i\sum_{j=1}^n \Phi^{nj}\partial_{x_{j+n}}, \sum_{l=1}^n \Phi^{il}\partial_{x_l} + i\sum_{l=1}^n \Phi^{il}\partial_{x_{l+n}}\Big] \\
&= \sum_{l=1}^n \underbrace{\sum_{j=1}^n \{\Phi^{nj}\partial_{x_j}(\Phi^{il}) - \Phi^{ij}\partial_{x_j}(\Phi^{nl})\}\partial_{x_l}}_{(1)} \\
&\quad + i\sum_{l=1}^n \underbrace{\sum_{j=1}^n \{\Phi^{nj}\partial_{x_{j+n}}(\Phi^{il}) - \Phi^{ij}\partial_{x_{j+n}}(\Phi^{nl})\}\partial_{x_l}}_{(2)} \\
&\quad - \sum_{l=1}^n \underbrace{\sum_{j=1}^n \{\Phi^{nj}\partial_{x_{j+n}}(\Phi^{il}) - \Phi^{ij}\partial_{x_{j+n}}(\Phi^{nl})\}\partial_{x_{l+n}}}_{(2)} \\
&\quad + i\sum_{l=1}^n \underbrace{\sum_{j=1}^n \{\Phi^{nj}\partial_{x_j}(\Phi^{il}) - \Phi^{ij}\partial_{x_j}(\Phi^{nl})\}\partial_{x_l+n}}_{(1)},
\end{aligned}$$

where

$$(1) = \sum_{j=1}^n \sum_{p,q=1}^n \{\frac{\partial F}{\partial u_{n\bar{j}}}\frac{\partial^2 F}{\partial u_{i\bar{l}}\partial u_{p\bar{q}}} - \frac{\partial F}{\partial u_{i\bar{j}}}\frac{\partial^2 F}{\partial u_{n\bar{l}}\partial u_{p\bar{q}}}\}\partial_{x_j} u_{p\bar{q}}.$$

Using (2.10), we get

$$\begin{aligned}
F.(1) &= \sum_{j=1}^n \sum_{p,q=1}^n \frac{\partial F}{\partial u_{n\bar{j}}}(\frac{\partial F}{\partial u_{i\bar{l}}}\frac{\partial F}{\partial u_{p\bar{q}}} - \frac{\partial F}{\partial u_{i\bar{q}}}\frac{\partial F}{\partial u_{p\bar{l}}})\partial_{x_j} u_{p\bar{q}} \\
&\quad - \sum_{j=1}^n \sum_{p,q=1}^n \frac{\partial F}{\partial u_{i\bar{j}}}(\frac{\partial F}{\partial u_{n\bar{l}}}\frac{\partial F}{\partial u_{p\bar{q}}} - \frac{\partial F}{\partial u_{n\bar{q}}}\frac{\partial F}{\partial u_{p\bar{l}}})\partial_{x_j} u_{p\bar{q}} \\
&= \sum_{j=1}^n \underbrace{\sum_{p,q=1}^n \frac{\partial F}{\partial u_{p\bar{q}}}\partial_{x_j} u_{p\bar{q}}(\frac{\partial F}{\partial u_{n\bar{j}}}\frac{\partial F}{\partial u_{i\bar{l}}} - \frac{\partial F}{\partial u_{i\bar{j}}}\frac{\partial F}{\partial u_{n\bar{l}}})}_{(5)}
\end{aligned}$$

$$+\underbrace{\sum_{j,p,q=1}^{n}\frac{\partial F}{\partial u_{p\bar{l}}}(\frac{\partial F}{\partial u_{i\bar{j}}}\frac{\partial F}{\partial u_{n\bar{q}}}-\frac{\partial F}{\partial u_{n\bar{j}}}\frac{\partial F}{\partial u_{i\bar{q}}})\partial_{x_j}u_{p\bar{q}}}_{(6)}.$$

Using (2.10), we have

$$(5)=\partial_{x_j}(F)F\frac{\partial^2 F}{\partial u_{n\bar{j}}\partial u_{i\bar{l}}}.$$

Similarly, we prove that

$$\begin{aligned}F.(2)=&\sum_{j=1}^{n}\partial_{x_{j+n}}(F)F\frac{\partial^2 F}{\partial u_{n\bar{j}}\partial u_{i\bar{l}}}\\&+\underbrace{\sum_{j,p,q=1}^{n}\frac{\partial F}{\partial u_{p\bar{l}}}(\frac{\partial F}{\partial u_{i\bar{j}}}\frac{\partial F}{\partial u_{n\bar{q}}}-\frac{\partial F}{\partial u_{n\bar{j}}}\frac{\partial F}{\partial u_{i\bar{q}}})\partial_{x_{j+n}}u_{p\bar{q}}}_{(7)}.\end{aligned}$$

We can easily see that $(6)+i(7)=0$, so,

$$(1)+i(2)=\sum_{j=1}^{n}(\partial_{x_j}(F)+i\partial_{x_{j+n}}(F))\frac{\partial^2 F}{\partial u_{n\bar{j}}\partial u_{i\bar{l}}}$$

and

$$D_{in}=\sum_{l=1}^{n}\sum_{j=1}^{n}(\partial_{x_j}(f)+i\partial_{x_{j+n}}(f))\frac{\partial^2 F}{\partial u_{n\bar{j}}\partial u_{i\bar{l}}}[\partial_{x_l}+i\partial_{x_l+n}].$$

Since F is the determinant function, then, $\frac{\partial F}{\partial u_{i\bar{j}}}$ is independent of $u_{i\bar{l}}$ and $u_{l\bar{j}}$ for $l=1,\dots,n$. Therefore $\frac{\partial^2 F}{\partial u_{i\bar{j}}\partial u_{p\bar{q}}}$ vanishes unless $i\neq p$, $j\neq q$. So,

$$D_{in}=\sum_{(l,j)\neq(i,n),\,l,j\leq n}(\partial_{x_j}(f)+i\partial_{x_{j+n}}(f))\frac{\partial^2 F}{\partial u_{n\bar{j}}\partial u_{i\bar{l}}}[\partial_{x_l}+i\partial_{x_l+n}].$$

We have $\varphi_{i\bar{j}}(0)=(1-\delta_i^n)\sigma_i\delta_i^j$; Therefore, if $n\geq 3$ and $(l,s)\neq(i,n)$,

$$\frac{\partial^2 F}{\partial u_{n\bar{s}}\partial u_{i\bar{l}}}(\varphi_{i\bar{j}})(0)=0.$$

If $n=2$ and $l=1$, then $s=1$ and we also have

$$\frac{\partial^2 F}{\partial u_{2\bar{1}}\partial u_{1\bar{1}}}(\varphi_{i\bar{j}})(0)=0.$$

So, (5.3) is proved for $m=1$. By a recursion on m, we deduce this lemma. □

On the other hand, we have by (3.5)

$$\begin{gathered}\Phi^{ij}(\varphi_{i\bar{j}})(0)=0,\quad\text{for }(i,j)\neq(n,n),\\A_n(\varphi_{i\bar{j}})(0)=\prod_{i=1}^{n-1}\sigma_i>0,\\A_i(\varphi_{i\bar{j}})(0)=\prod_{j\neq i,\,i=1}^{n-1}\sigma_i>0.\end{gathered}\tag{5.4}$$

Or by the hypothesis, $\partial_x^\beta g(0) = 0$ for all $|\beta| < k$, and by (5.4), we can suppose that $\partial_{x_n}^k g(0) \neq 0$ ($\partial_{x_{2n}}^k g(0) \neq 0$ leads to the same result, just consider $(adX_{2n})^{m-1}$ instead of $(adX_n)^{m-1}$).

So, by taking the real and the imaginary parts of (5.3) at the origin, we obtain

$$\begin{aligned}&(adX_n)^{k-1}([X_n, X_i] - [X_{2n}, X_{i+n}])\\&= \sum_{l=1}^{2n}\sum_{j\neq i} \varepsilon d'_{pij}(0)\partial_{x_l} + [A_n(\varphi_{i\bar{j}})(0)]^{k-1}A_i(\varphi_{i\bar{j}})(0)[\partial_{x_n}^k g\partial_{x_i} - \partial_{x_n}^{k-1}\partial_{x_{2n}} g\partial_{x_{i+n}}]\end{aligned}$$

and

$$\begin{aligned}&(adX_n)^{k-1}([X_{2n}, X_i] + [X_n, X_{i+n}])\\&= \sum_{l=1}^{2n}\sum_{j\neq i} \varepsilon d''_{pij}(0)\partial_{x_l} - [A_n(\varphi_{i\bar{j}})(0)]^{k-1}A_i(\varphi_{i\bar{j}})(0)[\partial_{x_n}^{k-1}\partial_{x_{2n}} g\partial_{x_i} + \partial_{x_n}^k g\partial_{x_{i+n}}].\end{aligned}$$

Suppose now that $|w|_{k+2} \leq 1$. We will get at the origin for $\varepsilon \leq \widetilde{\varepsilon}$ small enough the determinant of the vectors

$$\begin{gathered}(adX_n)^{k-1}([X_n, X_i] - [X_{2n}, X_{i+n}]),\\(adX_n)^{k-1}([X_{2n}, X_i] + [X_n, X_{i+n}])_{i=1,\dots,n-1},\\X_n, X_{2n} \text{ is different from zero.}\end{gathered} \tag{5.5}$$

Now, choose s_* so big that $s_* \geq \max(7 + 2n, 6 + k + n)$ by means of Theorem 1.1 there exists $\varepsilon_0 < \widetilde{\varepsilon}^2$ such that for any g satisfying (1.2) there exists a unique solution $u = \varphi + \varepsilon_0^{\frac{1}{2}} w \in C^{k+3}(\Omega)$ to the problem (1.1). Moreover; by (2.1), $|w|_{k+2} \leq \beta\|w\|_{k+2+n+\tau}$. Since $\sigma = s_* - 2 - \tau$, $s_* \geq 6 + k + n$ and $\tau \leq \frac{\alpha}{4} < \frac{1}{4}$, then

$$k + 2 + n + \tau \leq s_* - 4 + \tau = \sigma - 2 + 2\tau \leq \sigma - \tau.$$

We have then, using (4.3), (4.5) and (4.15),

$$|w|_{k+2} \leq 2\beta\sqrt{\widetilde{\varepsilon}} \leq 1.$$

So, by (5.5), we can conclude that for $\widetilde{\varepsilon}$ sufficiently small, the vector fields at the origin; $[(adX_n)^{k-1}([X_{\delta n}, X_i])]_{\delta=1,2;i=1,\dots,2n-1}$, X_n and X_{2n} span all the tangent space. Theorem 1.2 follows then from Theorem 5.1.

6. Appendix 1

To prove proposition 4.1, we need the following result.

Propositon 6.1. *There exists a constant $K_0 \geq 1$ such that for any function $w^i \in C^{s_*+2,\tau}(\overline{\Omega})$, $|w^i|_2 \leq 1$, $i = 1, 2, 3$ and for any $\varepsilon \leq 1$ we have*

$$|G(w^1) - G(w^2)|_0 \leq K_0|w^1 - w^2|_2\big(\|\varphi\|_{2+n_*} + \|w^1\|_{2+n_*} + \|w^2\|_{2+n_*} + 1\big). \tag{6.1}$$

Also for $t \in [0, 1]$, $s \in [0, s_*]$,

$$\begin{aligned}&\|\frac{d}{dt}[L_G(w^1 + tw^2)w^3]\|_s\\&\leq \varepsilon K_0[(\|\varphi\|_{2+s} + \varepsilon\|w^1\|_{2+s} + \varepsilon\|w^2\|_{2+s} + 1)|w^2|_2|w^3|_2\\&\quad + (\|\varphi\|_{2+n_*} + \varepsilon\|w^1\|_{2+n_*} + \varepsilon\|w^2\|_{2+n_*} + 1)(|w^2|_2\|w^3\|_{2+s} + |w^3|_2\|w^2\|_{2+s})].\end{aligned} \tag{6.2}$$

Proof. Just write

$$\begin{aligned}
&G(w^1) - G(w^2)\\
&= \frac{1}{\varepsilon}[\det(\varphi_{i\bar{j}} + \varepsilon w^1_{i\bar{j}}) - \det(\varphi_{i\bar{j}} + \varepsilon w^2_{i\bar{j}}) + g(w^1) - g(w^2)]\\
&= \int_0^1 \sum_{i,j=1}^n \frac{\partial F}{\partial u_{i\bar{j}}}(\varphi_{i\bar{j}} + \varepsilon w^2_{i\bar{j}} + t\varepsilon(w^1_{i\bar{j}} - w^2_{i\bar{j}}))(w^1_{i\bar{j}} - w^2_{i\bar{j}})dt\\
&\quad + \int_0^1 \frac{\partial g}{\partial u}(\varphi + \varepsilon w^2 + t\varepsilon(w^1 - w^2))(w^1 - w^2)\\
&\quad + \int_0^1 \frac{\partial g}{\partial p_i}(\varphi + \varepsilon w^2 + t\varepsilon(w^1 - w^2))(w^1_i - w^2_i),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}[L_G(w^1 + tw^2)w^3]\\
&= \frac{d}{dt}[\sum_{i,j=1}^n \frac{\partial F}{\partial u_{i\bar{j}}}(\varphi_{i\bar{j}} + \varepsilon w^1_{i\bar{j}} + t\varepsilon w^2_{i\bar{j}})w^3_{i\bar{j}} + \frac{\partial g}{\partial u}(\varphi + \varepsilon w^1 + t\varepsilon w^2)w^3 + \dots]\\
&= \varepsilon \sum_{i,j,p,q=1}^n \frac{\partial^2 F}{\partial u_{i\bar{j}} \partial u_{p\bar{q}}}(\varphi_{i\bar{j}} + \varepsilon w^1_{i\bar{j}} + t\varepsilon w^2_{i\bar{j}})w^2_{p\bar{q}} w^3_{i\bar{j}} + \dots.
\end{aligned}$$

Combining (2.1), (2.7), (2.8) and (2.9) with the inequalities

$$|\varphi_{i\bar{j}} + \varepsilon w^2_{i\bar{j}} + t\varepsilon(w^1_{i\bar{j}} - w^2_{i\bar{j}})|_0 \le |\varphi|_2 + 2|w^2|_2 + |w^1|_2 \le 3 + |\varphi|_2$$

and

$$|\varphi_{i\bar{j}} + \varepsilon w^1_{i\bar{j}} + t\varepsilon w^2_{i\bar{j}}|_0 \le |\varphi|_2 + \varepsilon|w^1|_2 + t\varepsilon|w^2|_2 \le 2 + |\varphi|_2,$$

we deduce (6.1) and (6.2). □

Proof of the proposition 4.1. The proposition is proved by induction. We have $u_0 = 0$. Let begin by proving $(4.19)_0$ to $(4.21)_0$. (i.e. (4.19) to (4.21) corresponding to $j = 0$).

(a) $(4.19)_0$: Using (3.2) and (4.10), we have

$$g_0 = -S_0 G(0) \quad \text{and} \quad G(0) = \frac{1}{\widetilde{\varepsilon}}(\det(\varphi_{ij}) - g(\varphi)).$$

But $\varphi \in C^{s_*+2,\alpha}(\overline{\Omega})$, $g \in C^{s_*}$ and S_n are smoothing operators, so $g_0 \in H^{s_*}(\Omega)$. (2.3), (2.4), (3.2) and (1.2) show that

$$\|g_0\|_s \le \beta\|G(0)\|_s \le \frac{\beta}{\widetilde{\varepsilon}}\|\det(\varphi_{ij}) - g(\varphi)\|_{s_*} \le \frac{\beta^2}{\widetilde{\varepsilon}}|\det(\varphi_{ij}) - g(\varphi)|_{s_*} \le \beta^2\widetilde{\varepsilon}.$$

Using (4.4) and $\beta \le \mu$, we get $\|g_0\|_s \le \mu^2\widetilde{\varepsilon} \le a_2\widetilde{\varepsilon}$

(b) $(4.20)_0$: (3.2), (4.4), (4.5) and (1.2) give

$$\theta_0 = |G(0)|_0 \le \frac{1}{\widetilde{\varepsilon}}|\det(\varphi_{i\bar{j}}) - g(\varphi)|_{s_*} \le \widetilde{\varepsilon} \le \sqrt{\widetilde{\varepsilon}}a_3 \le 1.$$

(c) $(4.21)_0$: We have

$$A_0(2) = \max(1, |\frac{\partial g}{\partial u}(\varphi)|_2, \max_{i,j}|\frac{\partial F}{\partial \varphi_{i\bar{j}}}(\varphi_{l\bar{q}})|_2 + \theta_0).$$

Then, by (2.9), (4.1) and $(4.20)_0$, $A_0(2) \le M_0$.

Assume that $u_0, u_1, \dots, u_{n-1} \in H^{s_*}(\Omega)$ satisfy (3.12)–(3.14) and (4.14)–(4.21) for $j \le n-1$. We shall construct $u_n \in H^{s_*}(\Omega)$ satisfying (3.12)–(3.14) and prove that (4.14)–(4.21) are satisfied for $j = n$.

Combining $(4.16)_{n-1}$–$(4.21)_{n-1}$, we have $|\widetilde{w}_{n-1}|_{4,\kappa} \le 1$, $\theta_{n-1} \le 1$, $A_{n-1}(2) \le M_0$ and $g_{n-1} \in H^{s_*}(\Omega)$. We can then apply Theorem 3.3 to get a solution $u_n \in H^{s_*}(\Omega)$ to the problem $(4.7)_n$ satisfying (3.12)–(3.14). Then:
(a) $(4.14)_n$: For $n = 1$, using (1.2), (2.3), (3.2), (3.12), and (4.2), we have

$$\|u_1\|_0 \le D\|g_0\|_0 \le D\beta\|G(0)\|_0 \le D\frac{\beta^2}{\widetilde{\varepsilon}}|\det(\varphi_{ij}) - g(\varphi)|_{s_*} \le D\beta^2\widetilde{\varepsilon}.$$

(4.3), (4.5), and $s_* \ge \sigma$ give

$$\|u_1\|_0 \le \sqrt{\widetilde{\varepsilon}}\mu^{-\sigma}. \tag{6.3}$$

By (3.13), we have $\|u_1\|_1 \le D(\|g_0\|_1 + \|u_1\|_0)$. Therefore, using (1.2), (2.3), (6.3), and $s_* \ge \sigma$, we get

$$\|u_1\|_1 \le D(\beta^2\widetilde{\varepsilon} + \sqrt{\widetilde{\varepsilon}}\mu^{-\sigma}) \le \sqrt{\widetilde{\varepsilon}}\mu^{1-\sigma}.$$

Suppose that for $0 \le l \le s$ and $s \ge 2$ we have

$$\|u_1\|_l \le \sqrt{\widetilde{\varepsilon}}\mu^{l-\sigma}. \tag{6.4}$$

Using (3.14), we have, for $s \ge 2$,

$$\|u_1\|_s \le D\big(\|g_0\|_s + \sum_{l \le s-1,\, (i,l)\in\Lambda_s} (1 + |\varphi|_{i+4,\tau})\|u_1\|_l\big).$$

(1.2), (2.3), (2.4), (4.3), (4.10), and $s_* \ge \sigma$ imply

$$\|g_0\|_s \le \beta\|G(0)\|_s \le \beta^2|G(0)|_s \le \beta^2\widetilde{\varepsilon} \le \widetilde{\mu}\widetilde{\varepsilon}\mu^{s-\sigma},$$

which by (6.3) and (6.4) gives

$$\begin{aligned}\|u_1\|_s &\le D\big(\widetilde{\mu}\widetilde{\varepsilon}\mu^{s-\sigma} + \sum_{l \le s-1,\, (i,l)\in\Lambda_s} (1 + |\varphi|_{i+4,\tau})\sqrt{\widetilde{\varepsilon}}\mu^{l-\sigma}\big)\\ &\le D\big(\widetilde{\mu}\widetilde{\varepsilon}\mu^{s-\sigma} + s_*^2(1 + |\varphi|_{i+4,\tau})\mu^{-1}\sqrt{\widetilde{\varepsilon}}\mu^{s-\sigma}\big),\end{aligned}$$

which by (4.3) and (4.5) shows that $\|u_1\|_s \le \sqrt{\widetilde{\varepsilon}}\mu^{s-\sigma}$.

For $n \ge 2$, (3.12), (4.2), (4.5), and $(4.19)_{n-1}$ imply

$$\|u_n\|_0 \le D\|g_{n-1}\|_0 \le D\widetilde{\varepsilon}a_2\mu_{n-1}^{-\sigma} \le \sqrt{\widetilde{\varepsilon}}\mu_{n-1}^{-\sigma}. \tag{6.5}$$

In the same way; (3.13), (4.2), (4.5), $(4.19)_{n-1}$ and (6.5) give

$$\|u_n\|_1 \le \sqrt{\widetilde{\varepsilon}}\mu_{n-1}^{1-\sigma}.$$

Suppose that, for $0 \le l < s$ and $s \ge 2$, $\|u_n\|_l \le \sqrt{\widetilde{\varepsilon}}\mu_{n-1}^{l-\sigma}$. By (3.14), we have

$$\|u_n\|_s \le D(\|g_{n-1}\|_s + \sum_{l \le s-1,\, (i,l)\in\Lambda_s} (1 + |\varphi + \widetilde{\varepsilon}\widetilde{w}_{n-1}|_{i+4,\tau})\|u_n\|_l).$$

But, (2.1), (2.5), $(4.15)_{n-1}$, and $4 + n_* \le \sigma - \tau$ imply that, for $0 \le i \le s-2$,

$$|\widetilde{w}_{n-1}|_{i+4,\tau} \le \beta\|\widetilde{w}_{n-1}\|_{4+n_*+i} \le \beta^2\mu_{n-1}^i\|\widetilde{w}_{n-1}\|_{4+n_*} \le 2\beta^2\sqrt{\widetilde{\varepsilon}}\mu_{n-1}^i.$$

Therefore, using $(4.19)_{n-1}$, we get

$$\begin{aligned}\|u_n\|_s &\le D\big(\widetilde{\varepsilon}a_2\mu_{n-1}^{s-\sigma} + \sum(1 + |\varphi|_{s_*+2,\tau} + 2\beta^2\sqrt{\widetilde{\varepsilon}}\mu_{n-1}^i)\sqrt{\widetilde{\varepsilon}}\mu_{n-1}^{l-\sigma}\big)\\ &\le D\big(\widetilde{\varepsilon}a_2\mu_{n-1}^{s-\sigma} + 2\beta^2 s_*^2\widetilde{\varepsilon}\mu_{n-1}^{s-\sigma} + (1 + |\varphi|_{s_*+2,\tau})s_*^2\sqrt{\widetilde{\varepsilon}}\mu_{n-1}^{s-1-\sigma}\big),\end{aligned}$$

which combined with (4.4) and (4.5) gives $\|u_n\|_s \le \sqrt{\widetilde{\varepsilon}}\mu_{n-1}^{s-\sigma}$.
(b) $(4.15)_n$: (4.6) shows that $w_n = \sum_{j=1}^n u_j$. By $(4.14)_j$, $1 \le j \le n$, we have

$$\|w_n\|_s \le \sum_{j=1}^n \|u_j\|_s \le \sqrt{\widetilde{\varepsilon}}\mu^{s-\sigma} + \sum_{j=2}^n \sqrt{\widetilde{\varepsilon}}\mu_{j-1}^{s-\sigma} \le \sqrt{\widetilde{\varepsilon}}\mu^{s-\sigma} + \sum_{j=1}^{n-1}\sqrt{\widetilde{\varepsilon}}\mu_j^{s-\sigma}.$$

For $s \le \sigma - \tau$, since $\mu \ge 2^{1/\tau} \ge 2$, we have $\mu_j^{s-\sigma} \le \mu_j^{-\tau} \le \frac{1}{2^j}$ and

$$\|w_n\|_s \le \sum_{j=0}^{n-1} \sqrt{\widetilde{\varepsilon}}\mu_j^{s-\sigma} \le \sqrt{\widetilde{\varepsilon}}\sum_{j=0}^{n-1}\frac{1}{2^j} \le 2\sqrt{\widetilde{\varepsilon}}.$$

For $s \ge \sigma - \tau$, we have

$$\|w_n\|_s \le \sqrt{\widetilde{\varepsilon}}\mu^{s-\sigma} + \sqrt{\widetilde{\varepsilon}}\frac{\mu^{n(s-\sigma)} - \mu^{s-\sigma}}{\mu^{s-\sigma} - 1}.$$

Since $\mu \ge 2^{1/\tau}$, it follows that $\mu^{s-\sigma} \ge \mu^\tau \ge 2$. Therefore, $\|w_n\|_s \le \sqrt{\widetilde{\varepsilon}}\mu_n^{s-\sigma}$.
(c) $(4.16)_n$: Combining (2.1), (2.4), (4.5), $(4.15)_n$ and $4 + n_* \le \sigma - \tau$, we obtain

$$|\widetilde{w}_n|_{4,\tau} \le \beta\|\widetilde{w}_n\|_{4+n_*} \le \beta^2\|w_n\|_{4+n_*} \le 2\beta^2\sqrt{\widetilde{\varepsilon}} \le 1.$$

(d) $(4.17)_n$): In the case $s \le \sigma - \tau$, using (2.6) and $(4.15)_n$, we obtain

$$\|w_n - \widetilde{w}_n\|_s \le \beta\mu_n^{s-[\sigma+\tau]-1}\|w_n\|_{[\sigma+\tau]+1} \le \beta\mu_n^{s-[\sigma+\tau]-1}\sqrt{\widetilde{\varepsilon}}\mu_n^{[\sigma+\tau]+1-\sigma} \le \beta\sqrt{\widetilde{\varepsilon}}\mu_n^{s-\sigma}.$$

In the case $s > \sigma - \tau$, (2.6) $(4.15)_n$) and $\beta \ge 1$ give

$$\|w_n - \widetilde{w}_n\|_s \le \beta\|w_n\|_s \le \beta\sqrt{\widetilde{\varepsilon}}\mu_n^{s-\sigma}.$$

(e) $(4.18)_n$: By (4.12), we have

$$r_n = \underbrace{[L_G(w_{n-1}) - L_G(\widetilde{w}_{n-1})]u_n}_{(1)} - \underbrace{\theta_{n-1}\triangle u_n}_{(2)} + \underbrace{Q_n}_{(3)}$$

When $n = 1$, $(1) = 0$. In the case $n \ge 2$, since

$$(1) = \int_0^1 \frac{d}{dt}[L_G(\widetilde{w}_{n-1} + t(w_{n-1} - \widetilde{w}_{n-1}))u_n]dt,$$

by (2.1) and $(4.17)_{n-1}$, we get

$$|w_{n-1} - \widetilde{w}_{n-1}|_2 \le \beta\|w_{n-1} - \widetilde{w}_{n-1}\|_{2+n_*} \le 2\beta^2\sqrt{\widetilde{\varepsilon}}\mu_{n-1}^{3+n_*-\sigma}.$$

But $2\beta^2\sqrt{\widetilde{\varepsilon}} \le 1$ and $3 + n_* \le 4 + 2n_* \le \sigma$, so, $|w_{n-1} - \widetilde{w}_{n-1}|_2 \le 1$. In the same way, (2.1), (4.5) and $(4.14)_n$ give

$$|u_n|_2 \le \beta\|u_n\|_{2+n_*} \le \beta\sqrt{\widetilde{\varepsilon}}\mu_{n-1}^{3+n_*-\sigma} \le 1.$$

By $(4.16)_{n-1}$, we also have $|\widetilde{w}_{n-1}|_2 \le 1$. Hence, we can apply Proposition 6.1 to get

$$\begin{aligned}\|(1)\|_s \le& \widetilde{\varepsilon}K_0\{[\|\varphi\|_{s+2} + \|\widetilde{w}_{n-1}\|_{s+2} + \|w_{n-1}\|_{s+2} + 1]|w_{n-1} - \widetilde{w}_{n-1}|_2|u_n|_2 \\ &+ (\|\varphi\|_{2+n_*} + \|\widetilde{w}_{n-1}\|_{2+n_*} + \|w_{n-1}\|_{2+n_*} + 1) \\ &\times (|w_{n-1} - \widetilde{w}_{n-1}|_2\|u_n\|_{s+2} + \|w_{n-1} - \widetilde{w}_{n-1}\|_{s+2}|u_n|_2)\}.\end{aligned}$$

Using (2.3) and (4.3), we get for $0 \le s \le s_*$,

$$\|\varphi\|_{s+2} \le \beta|\varphi|_{s_*+2} \le \beta\mu \le \mu^2.$$

By (2.2), it suffices to prove $(4.18)_n$ for $s = 0$ and $s = s_* - 2$.

Case $s=0$: combining (2.1), $(4.14)_n$, $(4.15)_{n-1}$ and $(4.17)_{n-1}$, we have

$$\begin{aligned}\|(1)\|_0 \le \widetilde{\varepsilon}K_0\{(\mu^2+2\beta\sqrt{\widetilde{\varepsilon}}+2\sqrt{\widetilde{\varepsilon}}+1)2\beta^3\widetilde{\varepsilon}\mu_{n-1}^{4+2n_*-2\sigma}\\ +(\mu^2+2\beta\sqrt{\widetilde{\varepsilon}}+2\sqrt{\widetilde{\varepsilon}}+1)4\beta^3\widetilde{\varepsilon}\mu_{n-1}^{4+n_*-2\sigma}\},\end{aligned}$$

which using (4.5) and $\sigma\ge 4+2n_*\ge 4+n_*$ gives $\|(1)\|_0\le \widetilde{\varepsilon}K_0\mu_{n-1}^{-\sigma}$.
Case $s=s_*-2$: (4.5) and $s_*\ge\sigma+\tau$, as in the previous case, imply

$$\|(1)\|_{s_*-2}\le \widetilde{\varepsilon}K_0\mu_{n-1}^{s_*-2-\sigma}.$$

By (2.2), we obtain for $0\le s\le s_*-2$,

$$\|(1)\|_s\le \beta\widetilde{\varepsilon}K_0\mu_{n-1}^{s-\sigma}.$$

Next,

$$\|(2)\|_s\le\theta_{n-1}\|u_n\|_{s+2}.$$

If $n=1$ combining (4.5), (4.9) and $(4.14)_n$, we obtain

$$\|(2)\|_s\le |G(0)|_0\|u_1\|_{s+2}\le\widetilde{\varepsilon}\sqrt{\widetilde{\varepsilon}}\mu^{s+2-\sigma}\le\widetilde{\varepsilon}\mu^{s-\sigma}.$$

In the case $n\ge 2$: $(4.14)_n$ and $(4.20)_{n-1}$ imply

$$\|(2)\|_s\le a_3\widetilde{\varepsilon}\mu_{n-1}^{-2}\mu_{n-1}^{s+2-\sigma}=a_3\widetilde{\varepsilon}\mu_{n-1}^{s-\sigma}.$$

Finally, since by (4.13),

$$\begin{aligned}(3)=Q_n&=G(w_{n-1}+u_n)-G(w_{n-1})-L_G(w_{n-1})u_n\\ &=\int_0^1\Big(\int_0^t\frac{d}{dh}[L_G(w_{n-1}+hu_n)u_n]dh\Big)dt\,.\end{aligned}$$

Then, using (2.1), (4.5) and $(4.15)_{n-1}$, we obtain

$$|w_{n-1}|_2\le\beta\|w_{n-1}\|_{2+n_*}\le 2\beta\sqrt{\widetilde{\varepsilon}}\le 1.$$

Since we proved that $|u_n|_2\le 1$, we can apply proposition 6.1 to have

$$\begin{aligned}\|(3)\|_s\le\widetilde{\varepsilon}K_0[(\|\varphi\|_{s+2}+\|u_n\|_{s+2}+\|w_{n-1}\|_{s+2}+1)|u_n|_2^2\\ +2|u_n|_2\|u_n\|_{s+2}(\|\varphi\|_{2+n_*}+\|u_n\|_{2+n_*}+\|w_{n-1}\|_{2+n_*}+1)].\end{aligned}$$

Combining (2.1), $(4.14)_n$ and $(4.15)_{n-1}$, we get
For $s=0$:

$$\begin{aligned}&\|(3)\|_0\\ &\le\widetilde{\varepsilon}K_0\{(\mu^2+\sqrt{\widetilde{\varepsilon}}[\max(\mu,\mu_{n-1})]^{2-\sigma}+2\sqrt{\widetilde{\varepsilon}}+1)\beta^2\widetilde{\varepsilon}[\max(\mu,\mu_{n-1})]^{4+2n_*-2\sigma}\\ &\quad+8(\mu^2+\sqrt{\widetilde{\varepsilon}}\beta[\max(\mu,\mu_{n-1})]^{2+n_*-\sigma}+2\sqrt{\widetilde{\varepsilon}}+1)\widetilde{\varepsilon}\beta[\max(\mu,\mu_{n-1})]^{4+n_*-2\sigma}\},\end{aligned}$$

which combined with (4.5) and $\sigma\ge 4+2n_*$ gives

$$\|(3)\|_0\le\widetilde{\varepsilon}K_0[\max(\mu,\mu_{n-1})]^{-\sigma}.$$

For $s=s_*-2$; since $\sigma\ge 4+2n_*$, we also get

$$\|(3)\|_{s_*-2}\le\widetilde{\varepsilon}K_0[\max(\mu,\mu_{n-1})]^{s_*-2-\sigma}.$$

Then (2.2) shows that, for $0\le s\le s_*-2$,

$$\|(3)\|_s\le\beta\widetilde{\varepsilon}K_0[\max(\mu,\mu_{n-1})]^{s-\sigma},$$

and we conclude that

$$\|r_n\|_s\le(2\beta K_0+a_3)\widetilde{\varepsilon}[\max(\mu,\mu_{n-1})]^{s-\sigma}$$

$$\begin{aligned} &\le 9K_0\mu^5\widetilde{\varepsilon}[\max(\mu,\mu_{n-1})]^{s-\sigma} \\ &= a_1\widetilde{\varepsilon}[\max(\mu,\mu_{n-1})]^{s-\sigma}. \end{aligned}$$

(f) $(4.19)_n$): By (4.10) and (4.11),

$$\begin{aligned} g_n &= S_{n-1}R_{n-1} - S_nR_n + (S_{n-1}-S_n)G(0) \\ &= \underbrace{(S_{n-1}R_{n-1} - S_nR_{n-1})}_{(4)} - \underbrace{S_nr_n}_{(5)} + \underbrace{(S_{n-1}-S_n)G(0)}_{(6)}. \end{aligned}$$

Case $s=0$: (2.6), (4.11) and $(4.18)_j$, $j \le n-1$, imply

$$\begin{aligned} \|(4)\|_0 &\le \|(I-S_{n-1})R_{n-1}\|_0 + \|(I-S_n)R_{n-1}\|_0 \\ &\le \beta\|R_{n-1}\|_{s_*-2}\mu_{n-1}^{2-s_*} + \beta\mu_n^{2-s_*}\|R_{n-1}\|_{s_*-2} \\ &\le (\beta a_1\widetilde{\varepsilon}\mu_{n-1}^{2-s_*} + \beta a_1\widetilde{\varepsilon}\mu_n^{2-s_*})(\mu^{s_*-2-\sigma} + \sum_{j=2}^{n-1}\mu_{j-1}^{s_*-2-\sigma}). \end{aligned}$$

Since $s_*-2 > \sigma$ and $\beta \le \mu$, then

$$\|(4)\|_0 \le \beta a_1\widetilde{\varepsilon}(\mu_{n-1}^{2-s_*} + \mu_n^{2-s_*})\mu_{n-1}^{s_*-2-\sigma} \le 2a_1\mu^2\widetilde{\varepsilon}\mu_n^{-\sigma}.$$

On the other hand, combining (2.4), $(4.18)_n$, $\sigma < s_*-2$ and $\beta \le \mu$, we obtain

$$\|(5)\|_0 \le \beta\|r_n\|_0 \le \beta a_1\widetilde{\varepsilon}[\max(\mu,\mu_{n-1})]^{-\sigma} \le a_1\mu^2\widetilde{\varepsilon}\mu_n^{-\sigma}.$$

We also have by (1.2), (2.3), (2.6). and $\sigma < s_*-2$,

$$\begin{aligned} \|(6)\|_0 &\le \|(I-S_{n-1})G(0)\|_0 + \|(I-S_n)G(0)\|_0 \\ &\le \beta\mu_{n-1}^{-\sigma}\|G(0)\|_\sigma + \beta\mu_n^{-\sigma}\|G(0)\|_\sigma \\ &\le \beta^2\mu_{n-1}^{-\sigma}|G(0)|_{s_*} + \beta^2\mu_n^{-\sigma}|G(0)|_{s_*} \\ &\le \beta^2\widetilde{\varepsilon}\mu_n^{-\sigma}(\mu^\sigma+1) \le 2\mu^{s_*}\widetilde{\varepsilon}\mu_n^{-\sigma}. \end{aligned}$$

We finally get

$$\|g_n\|_0 \le (2+3a_1)\mu^{s_*}\widetilde{\varepsilon}\mu_n^{-\sigma}.$$

Case $s=s_*$: (2.5), (4.11), $(4.18)_j$, $1 \le j \le n$, and $\sigma < s_*-2$ show that

$$\begin{aligned} &\|(4)+(5)\|_{s_*} \\ &\le \|S_{n-1}R_{n-1}\|_{s_*} + \|S_nR_n\|_{s_*} \\ &\le \beta\mu_{n-1}^2\|R_{n-1}\|_{s_*-2} + \beta\mu_n^2\|R_n\|_{s_*-2} \\ &\le \beta\mu_{n-1}^2a_1\widetilde{\varepsilon}(\mu^{s_*-2-\sigma} + \sum_{j=2}^{n-1}\mu_{j-1}^{s_*-2-\sigma}) + \beta\mu_n^2a_1\widetilde{\varepsilon}(\mu^{s_*-2-\sigma} + \sum_{j=2}^{n}\mu_{j-1}^{s_*-2-\sigma}) \\ &\le \beta a_1\widetilde{\varepsilon}(\mu_{n-1}^2\mu_{n-1}^{s_*-2-\sigma} + \mu_n^2\mu_n^{s_*-2-\sigma}) \\ &\le 2\beta a_1\widetilde{\varepsilon}\mu_n^{s_*-\sigma} \le 2\mu a_1\widetilde{\varepsilon}\mu_n^{s_*-\sigma}. \end{aligned}$$

Next, by (1.2), (2.5), (2.3), and $\beta \le \mu$, we have

$$\begin{aligned} \|(6)\|_{s_*} &\le \|S_nG(0)\|_{s_*} + \|S_{n-1}G(0)\|_{s_*} \\ &\le \beta\mu_n^{s_*-\sigma}\|G(0)\|_\sigma + \beta\mu_{n-1}^{s_*-\sigma}\|G(0)\|_\sigma \\ &\le 2\beta^2\widetilde{\varepsilon}\mu_n^{s_*-\sigma} \le 2\mu^2\widetilde{\varepsilon}\mu_n^{s_*-\sigma}. \end{aligned}$$

Therefore,

$$\|g_n\|_{s_*} \le 2\mu(a_1+\mu)\widetilde{\varepsilon}\mu_n^{s_*-\sigma}.$$

We can finally conclude using (4.4) and $\mu \le a_1$, that

$$\|g_n\|_{s_*} \le 4a_1\mu^2\widetilde{\varepsilon}\mu_n^{s_*-\sigma} \le a_2\widetilde{\varepsilon}\mu_n^{s_*-\sigma}.$$

(g) $(4.20)_n$: By (4.9), we have

$$\theta_n = |G(\widetilde{w}_n)|_0 \le |G(w_n) - G(\widetilde{w}_n)|_0 + |G(w_n)|_0$$

Using (4.22):

$$G(w_n) = (I - S_{n-1})R_{n-1} + (I - S_{n-1})G(0) + r_n.$$

Then

$$\theta_n \le \underbrace{|G(w_n) - G(\widetilde{w}_n)|_0}_{(7)} + \underbrace{|(I - S_{n-1})R_{n-1}|_0}_{(8)} + \underbrace{|(I - S_{n-1})G(0)|_0}_{(9)} + \underbrace{|r_n|_0}_{(10)}.$$

Since we proved that $|w_n|_2 \le 1$ and $|\widetilde{w}_n|_2 \le 1$, we can apply Proposition 6.1 to get

$$(7) \le \beta K_0\|w_n - \widetilde{w}_n\|_{2+n_*}(\|\varphi\|_{2+n_*} + \|w_n\|_{2+n_*} + \|\widetilde{w}_n\|_{2+n_*} + 1).$$

Equations (2.4), $(4.15)_n$, $(4.17)_n$, and $3 + n_* \le 4 + 2n_* - \tau \le \sigma - \tau$ imply

$$(7) \le 2\beta^2K_0\sqrt{\widetilde{\varepsilon}}\mu_n^{2+n_*-\sigma}(\mu^2 + 2\sqrt{\widetilde{\varepsilon}} + 2\beta\sqrt{\widetilde{\varepsilon}} + 1).$$

Since $\widetilde{\varepsilon} \le \frac{1}{(6\beta^2)^2}$, $\beta \le \mu$ and $4 + n_* - \sigma \le 4 + 2n_* - \sigma \le 0$ then

$$(7) \le 4\mu^5K_0\sqrt{\widetilde{\varepsilon}}\mu_n^{-2}.$$

In the case $n = 1$, $(8) = 0$. For $n \ge 2$, since $\beta \le \mu$, $n_* - \sigma \le -2$ and $\mu^4a_1\sqrt{\widetilde{\varepsilon}} \le a_2\sqrt{\widetilde{\varepsilon}} \le 1$, combining (2.1), (2.6), (4.11), and $(4.18)_j$, $j \le n-1$, we obtain

$$\begin{aligned}
(8) &\le \beta\|(I - S_{n-1})R_{n-1}\|_{n_*}\\
&\le \beta^2\mu_{n-1}^{n_*-s_*+2}a_1\widetilde{\varepsilon}\big(\mu^{s_*-2-\sigma} + \sum_{j=2}^{n-1}\mu_{j-1}^{s_*-2-\sigma}\big)\\
&\le \beta^2a_1\widetilde{\varepsilon}\mu_{n-1}^{s_*-\sigma} \le \sqrt{\widetilde{\varepsilon}}\mu_n^{-2}.
\end{aligned}$$

Equations (1.2), (2.1), (2.3), (2.6), (4.5), and $\beta \le \mu$ imply

$$\begin{aligned}
(9) &\le \beta\|(I - S_{n-1})G(0)\|_{n_*} \le \beta^2\mu_{n-1}^{n_*-s_*}\|G(0)\|_{s_*}\\
&\le \beta^3\mu_{n-1}^{-2}\widetilde{\varepsilon} \le \beta^3\mu^2\widetilde{\varepsilon}\mu_n^{-2} \le \sqrt{\widetilde{\varepsilon}}\mu_n^{-2}.
\end{aligned}$$

Finally, by (2.1) and $(4.18)_n$,

$$\begin{aligned}
(10) &\le \beta\|r_n\|_{n_*} \le \beta a_1\widetilde{\varepsilon}[\max(\mu, \mu_{n-1})]^{n_*-\sigma}\\
&\le \mu a_1\widetilde{\varepsilon}[\max(\mu, \mu_{n-1})]^{-2} \le \sqrt{\widetilde{\varepsilon}}\mu_n^{-2}.
\end{aligned}$$

Thus, we conclude that

$$\theta_n \le 7K_0\mu^5\sqrt{\widetilde{\varepsilon}}\mu_n^{-2} = a_3\sqrt{\widetilde{\varepsilon}}\mu_n^{-2} \le 1.$$

(h) (4.21): We have

$$A_n(2) \le \max\Big(1, |\frac{\partial g}{\partial u}(\varphi + \widetilde{\varepsilon}\widetilde{w}_n)|_2, \max_{1\le i,j\le n}|\frac{\partial F}{\partial u_{i\bar{j}}}(\varphi_{k\bar{l}} + \widetilde{\varepsilon}(\widetilde{w}_n)_{k\bar{l}})|_2 + \theta_n\Big).$$

Using (2.9), (4.1), $(4.16)_n$ and $(4.20)_n$, we get $A_n(2) \le M_0$. $\square$

7. Appendix 2

In the rest of this paper, we prove estimates (3.12)–(3.14) for L_ν. We shall need the following result.

Propositon 7.1. *The operator*

$$P = \sum_{i,j=1}^{n} \frac{\partial F}{\partial u_{z_i\overline{z_j}}}(u_{z_i\overline{z_j}})\partial_{z_i}\partial_{\overline{z_j}},$$

where $u \in C^3(\overline{\Omega})$, is formally self-adjoint.

Proof. Let $\Sigma = \{z \in \Omega / F(u_{z_i\overline{z_j}})(z) = 0\}$. Since

$$P = \sum_{i=1}^{n} \partial_{z_i}\Big(\sum_{j=1}^{n} \frac{\partial F}{\partial u_{z_i\overline{z_j}}}\partial_{\overline{z_j}}\Big) - \sum_{i,j=1}^{n} \partial_{z_i}\big(\frac{\partial F}{\partial u_{z_i\overline{z_j}}}(u_{i\bar{j}})\big)\partial_{\overline{z_j}},$$

it is sufficient to prove that for $j = 1, \dots, n$ and $z \in \overline{\Omega}$,

$$\begin{aligned} A_j(z) &= \sum_{i=1}^{n} \partial_{z_i}\big(\frac{\partial F}{\partial u_{z_i\overline{z_j}}}(u_{z_i\overline{z_j}})(z)\big) \\ &= \sum_{i,p,q=1}^{n} \frac{\partial^2 F}{\partial u_{z_i\overline{z_j}}\partial u_{z_p\overline{z_q}}}(u_{z_i\overline{z_j}})u_{z_i z_p\overline{z_q}}(z) = 0. \end{aligned}$$

Using the relation (2.10), we get $A_j(z) = 0$ for any $z \notin \Sigma$. The continuity of the determinant function allow as to have the conclusion when $z \in \Sigma$. $\square$

7.1. Estimates in the elliptic Zone of L.

Let $Q = \sum_{i,j=1}^{2n} b^{ij} D_{x_i} D_{x_j} + b$ be a degenerate elliptic operator with real coefficients b, $b^{ij} = b^{ji} \in C^{s_*,\tau}(\overline{\Omega})$. Assume that there is a continuous function $\lambda(x) \geq 0$ defined in $\overline{\Omega}$ such that

$$\sum_{i,j=1}^{2n} b^{ij}\xi_i\xi_j \geq \lambda(x)|\xi|^2.$$

Let S be a subset of $\overline{\Omega}$ satisfying $\{x \in \overline{\Omega} : \lambda(x) = 0\} \subset S$.

Lemma 7.2. *Assume that Q is uniformly elliptic in $\overline{\Omega}$; that is $\lambda(x) \geq \lambda_0$, λ_0 is a positive constant Then for any integer $1 \leq s \leq s_*$ there exists a constant C'_s depending only on s, λ_0 and $A(0)$ such that for any real function $u \in C^{s_*,\tau}(\Omega) \cap H^1_0(\Omega)$,*

$$\|u\|_1 \leq C'_1(\|Qu\|_0 + A(2)\|u\|_0), \tag{7.1}$$

$$\|u\|_s \leq C'_s(\|Qu\|_{s-1} + \sum_{i\leq s-2,\, i+j\leq s-1} A(i+2)\|u\|_j),\ s \geq 2. \tag{7.2}$$

It is not difficult to prove (7.1). In fact, we need only to apply well-known standard techniques to the linear elliptic operator Q and to calculate several constants precisely. By induction with respect to s and patient calculation, (7.2) follows from (7.1).

For $\delta > 0$, we define the set S_δ by

$$S_\delta = \{x \in \overline{\Omega}, d(x, S) < \delta\}.$$

Lemma 7.3. *Assume that S is a compact C^∞ submanifold of Ω and $\Omega\backslash S$ is connected. Then there exists a function $\mu \in L^\infty(\Omega)$ and a constant $C > 0$ such that $\mu = 0$ on S, $m_\delta = \inf_{\overline{\Omega\backslash S_\delta}} \mu > 0$ for any sufficiently small δ and*

$$\int_\Omega \mu u^2 dx \le C\big\{\|Qu\|_0\|u\|_0 + \frac{1}{2}\sup[b^{ij}_{ij} - 2b]\|u\|_0^2\big\}, \tag{7.3}$$

for $u \in C^{s_*,\tau}(\Omega) \cap H^1_0(\Omega)$.

Proof. Standard techniques of elliptic operators give

$$\int \lambda|Du|^2 dx \le C\big\{\|Qu\|_0\|u\|_0 + \frac{1}{2}\sup[b^{ij}_{ij} - 2b]\|u\|_0^2\big\}.$$

Hence, it suffices to show that $\int \mu u^2 dx \le \int \lambda|Du|^2 dx$. First, let us fix a point $p \in \overline{\Omega\backslash S}$ arbitrarily.

By virtue of the fundamental theorem of ordinary differential equations, we can construct a family of curves $c(t,x) \in C^\infty([0,T_p]\times U_p)$ such that $c(0,x) = x$, $c(t,x) \notin S$ for $0 < t < T_p$ when $x \in \overline{\Omega\backslash S}$, $c(T_p,x) \notin \overline{\Omega}$, $|\dot{c}(t,x)| \equiv 1$, $\sup_{x\in U_p} \tau_x < \infty$, and $c(t,.)$ is a local C^∞ diffeomorphism defined in U_p for any fixed t.

Here, T_p is a positive constant, U_p is a sufficiently small open neighborhood of p, and, $\tau_x = \inf\{t \ge 0 : c(t,x) \notin \Omega\ \}$ We define a function $\mu_p(x)$ by

$$\mu_p(x) = \inf\{\lambda(c(t,x)) : 0 \le t \le \ \tau_x\}.$$

For $u \in C^1(\overline{\Omega})$ satisfying $u\big|_{\partial\Omega} = 0$, since

$$u(x) = u(c(0,x)) - u(c(\tau_x,x)) = -\int_0^{\tau_x} Du(c(t,x)).\dot{c}(t,x)dt,$$

we have

$$|u(x)|^2 \le C\int_0^{\tau_x} |Du(c(t,x))|^2 dt.$$

Multiplying this inequality by μ_p and using its definition, we obtain

$$\mu_p(x)|u(x)|^2 \le C\int_0^{\tau_x} \lambda(c(t,x))|Du(c(t,x))|^2 dt,$$

which implies

$$\int_{U_p} \mu_p|u|^2 \le C\int_\Omega \lambda|Du|^2 dt.$$

Secondly, we note that the above argument ensures the existence of a finite number of points $p_1,\dots,p_N$ such that $\overline{\Omega\backslash S} \subset \cup_{i=1}^N U_{p_i}$ and

$$\int_{U_{p_i}} \mu_{p_i}|u|^2 \le C\int_\Omega \lambda|Du|^2 dt.$$

Therefore, we have only to define μ by

$$\mu(x) = \begin{cases} \min\{\mu_{p_i}(x) : x \in U_{p_i}, 1 \le i \le n\}, & \text{if } x \in \Omega\backslash S,\\ 0, & \text{if } x \in S.\end{cases}$$

$\square$

Lemma 7.4. *For $u \in C_0^1(\Omega)$,*

$$\sum_k \|[\partial_k, Q]u\|_0^2 \le C(A(2)\|Qu\|_1\|u\|_1 + A(2)^2\|u\|_1^2), \tag{7.4}$$

$$\sum_k \|[\partial_k, Q]u\|_s^2 \le C(A(2)\|Qu\|_{s+1}\|u\|_{s+1} + \sum_{(i,j)\in\Lambda_{s+1}} A(i+2)^2\|u\|_j^2) \ s \ge 1. \tag{7.5}$$

Proof. [11, Lemma 1.7.1] shows that

$$(b_k^{ij}u_{ij})^2 \le CA(2)b^{ij}u_{li}u_{lj},$$

which implies

$$\begin{aligned}\sum_k \|[\partial_k, Q]u\|_0^2 &\le C\sum_k \int \{(b_k^{ij}u_{ij})^2 + (b_k u)^2\} \\ &\le CA(2)\sum_k \int b^{ij}u_{li}u_{lj} + CA(1)^2\|u\|_1^2.\end{aligned}$$

Integrating by parts

$$\int b^{ij}u_{li}u_{lj} = -\langle (Qu)_l, u_l\rangle + \langle [\partial_l, Q]u, u_l\rangle + \frac{1}{2}\langle (b_{ij}^{ij} - 2b)u_l, u_l\rangle,$$

which implies

$$\int b^{ij}u_{li}u_{lj} \le C\big(\|Qu\|_1\|u\|_1 + \sum_k \|[\partial_k, Q]u\|_0\|u\|_1 + A(2)\|u\|_1^2\big).$$

From these inequalities, and using the inequality $\alpha\beta \le \varepsilon\alpha^2 + \frac{1}{\varepsilon}\beta^2$ it follows that

$$\sum_k \|[\partial_k, Q]u\|_0^2 \le C(A(2)\|Qu\|_{s+1}\|u\|_{s+1} + A(2)^2\|u\|_1^2).$$

For $s \ge 1$, (6.5) is proved by recursion on s using (6.4). □

Lemma 7.5. *Let $\chi \in C^\infty$ satisfy* $\operatorname{supp}\nabla\chi \subset \Omega$. *For any integer $0 \le s \le s_*$, there exists a constant $C_s > 0$ such that for all $u \in C^{s_*,\tau}(\Omega)$,*

$$\|[\chi, Q]u\|_s^2 \le C_s\big(A(2)\|Qu\|_s\|u\|_s + \sum_{(i,j)\in\Lambda_s} A(i+2)^2\|u\|_j^2\big). \tag{7.6}$$

Proof. Let us consider a cut-off function $\widetilde{\chi} \in C_0^\infty(\Omega)$ satisfying $0 \le \widetilde{\chi} \le 1$ and $\widetilde{\chi} = 1$ on $\cup_i \operatorname{supp}\partial_i\chi$, and define an operator $\widetilde{Q} = \widetilde{b}^{ij}D_{x_i}D_{x_j} + \widetilde{b}$ by $\widetilde{Q} = \widetilde{\chi}Q$. Since $[\chi, \widetilde{Q}]u = [\chi, Q]u$ and $\|\widetilde{Q}u\|_s \le C\|Qu\|_s$, it will suffice to prove (7.6) for $\widetilde{Q}$.

For $s = 0$: The corollary to Lemma 1.7.1 in [11] shows that

$$\big(\sum_{i,j}\widetilde{b}^{ij}u_j\big)^2 \le 2A(0)\widetilde{b}^{ij}u_iu_j.$$

which gives

$$\|[\chi, \widetilde{Q}]u\|_0^2 \le CA(0)\int \widetilde{b}^{ij}u_iu_j + CA(0)^2\|u\|_0^2,$$

Integrating by parts we have

$$\int \widetilde{b}^{ij}u_iu_j = -\langle \widetilde{Q}u, u\rangle + \frac{1}{2}\langle (\widetilde{b}_{ij}^{ij} - 2\widetilde{b})u, u\rangle \le \|\widetilde{Q}u\|_0\|u\|_0 + CA(2)\|u\|_0^2,$$

which implies $(7.6)_0$.

Note that $(7.6)_{s\ge1}$ follows from $(7.6)_0$ by induction with respect to s □

7.2. **Estimates near the degenerate points of L.** For $t \geq t_0 \geq 1$, we define

$$V_t(0) = \{x \in \Omega, |x_n| < \frac{1}{t}\} \cap B(0, \delta_1).$$

Propositon 7.6. *For any integer $0 \leq s \leq s_*$ and any function $u \in C_0^{s_*,\tau}(V_t(0))$, there exists a constant $C''_s = C''_s(n, \Omega, \varphi, \delta_1) > 0$ such that*

$$\|u\|_0 \leq C_0'' t^{-1} \|L_\nu u\|_0, \tag{7.7}$$

$$\|u\|_s \leq C''_s t^{-1} (\|L_\nu u\|_s + \sum_{(i,j)\in\Lambda_s} A(i+2)\|u\|_j), \quad s \geq 1, \tag{7.8}$$

where δ_1 is as in Lemma 3.1.

Proof. Let $v = (T - e^{tx_n})^{-1}u$, and $T > 5e$ a constant. A direct computation gives

$$Qu = (T - e^{tx_n})Qv - te^{tx_n}\{2b^{nj}v_j + tb^{nn}v\},$$

$$\int (T - e^{tx_n})^{-1} Qu.v = -I + II - III - IV,$$

where

$$I = \int b^{ij} v_i v_j, \quad II = \frac{1}{2}\int \{b^{ij}_{ij} - 2b\} v^2,$$

$$III = t^2 \int e^{tx_n} b^{nn} (T - e^{tx_n})^{-1} v^2, \quad IV = 2t \int e^{tx_n} (T - e^{tx_n})^{-1} v b^{nj} v_j.$$

Using the Cauchy-Schwartz inequality, we get

$$|IV| \leq \int b^{ij} v_i v_j + 4t^2 \int e^{2tx_n} (T - e^{tx_n})^{-2} b^{nn} v^2.$$

Since

$$e^{tx_n}(T - e^{tx_n})^{-1} - 4e^{2tx_n}(T - e^{tx_n})^{-2} = e^{tx_n}(T - e^{tx_n})^{-2}(T - 5e^{tx_n}),$$

it follows that

$$t^2 \int e^{2tx_n}(T - e^{tx_n})^{-4}(T - 5e^{tx_n}) b^{nn} u^2 \leq -\int (T - e^{tx_n})^{-2} Qu.u - II.$$

Also

$$e^{-1} \leq e^{tx_n} \leq e, \quad (T - e^{-1})^{-1} \leq (T - e^{tx_n})^{-1} \leq (T - e)^{-1};$$

therefore,

$$C_0 t^2 \inf_{V_t(0)} (b^{nn}) \|u\|_0^2 \leq C\big\{\|Qu\|_0 \|u\|_0 + \frac{1}{2} \sup_{V_t(0)} [b^{ij}_{ij} - 2b] \|u\|_0^2\big\}. \tag{7.9}$$

To prove (7.7), we apply (7.8). So, for $u \in C_0^{s_*,\tau}(V_t(0))$, we can write

$$t\big\{tC_0 \inf_{V_t(0)} (b^{nn}) - \frac{C}{2} \sup_{V_t(0)} |b^{ij}_{ij} - 2b|\big\} \|u\|_0^2 \leq C\|Qu\|_0 \|u\|_0,$$

with $Q = L_\nu$ and $b^{nn} = (\Phi^{nn} + 4(\theta + \nu))$. If $|w|_{3,\tau} \leq 1$, $|x| \leq \delta_0$ and $\varepsilon \leq \varepsilon_1$, we have

$$\Phi^{nn} \geq \prod_{i=1}^{n-1} \sigma_i - M\delta_1 - M\varepsilon_1 = \alpha > 0.$$

Taking $t \geq t_0 = \max(\frac{4(C+1)A(2)}{\alpha C_0}, 1)$, (7.7) is proved. To prove (7.8), we use (7.7) and recursion on s. We now estimate $\|\chi u\|_s$. □

Propositon 7.7. *For any cut-off function* $\chi \in C_0^\infty(V_t(0))$, $u \in C^{s_*,\tau}(\Omega) \cap H_0^1(\Omega)$ *and* $1 \le s \le s_*$,

$$\|\chi u\|_s \le 2C_s''(\|L_\nu u\|_s + \|[\chi, L_\nu]u\|_s + \sum_{j<s,\,(i,j)\in\Lambda_s} (|\varphi + \varepsilon w|_{i+4,\tau} + 1)\|u\|_j). \quad (7.10)$$

Proof. Let us consider a cut-off function $\chi \in C_0^\infty(V_t(0))$. For $u \in C^{s_*,\tau} \cap H_0^1(\Omega)$, since $\operatorname{supp}\chi \subset V_t(0)$, we have by (7.9) for any $1 \le s \le s_*$,

$$\begin{aligned}\|\chi u\|_s &\le C_s'' t^{-1}\big(\|\chi L_\nu u\|_s + \|[\chi, L_\nu]u\|_s + \sum_{j<s,\,(i,j)\in\Lambda_s} A(i+2)\|u\|_j\big)\\ &\quad + C_s'' t^{-1} A(2)\|\chi u\|_s.\end{aligned}$$

We have $A(2) \le M_0$. We fix $t \ge t_0$ such that for $1 \le s \le s_*$, $C_s'' t^{-1} A(2) \le \frac{1}{2}$. On the other hand,

$$A(i+2) = \max\big(1, |\frac{\partial g}{\partial u}(\varphi + \varepsilon w)|_{i+2}, \max_{1\le p,q\le n} |\frac{\partial F}{\partial u_{p\bar q}}(\varphi_{k\bar l} + \varepsilon w_{k\bar l})|_{i+2} + \theta\big).$$

But, for $k \in \{0,1,2\}$, $|\partial^k\varphi + \varepsilon\partial^k w|_0 \le |\varphi|_2 + 1$, then by (2.9), since $\theta \le 1$, we get, for $0 \le i \le s_* - 2$,

$$A(i+2) \le C(\varphi)\big(|\varphi + \varepsilon w|_{i+4,\tau} + 1\big). \quad (7.11)$$

and we deduce (7.10). $\square$

7.3. **Proof of the estimates** (3.12)–(3.14) **for** L_ν**.** . Since $\|u\|_s \le \|(1-\chi)u\|_s + \|\chi u\|_s$, it will suffice to estimate $\|(1-\chi)u\|_s$ and $\|\chi u\|_s$.

Proof of (3.12). Since $\chi = 1$ in a neighborhood of zero in V, then, there exists $\delta > 0$ such that $\mathrm{Supp}(1-\chi) \subset \overline{\Omega}\backslash B(0,\delta)$.

Let us consider the cut-off functions: $\widetilde{\chi}$, $\widetilde{\widetilde{\chi}} \in C_0^\infty(\overline{\Omega}\backslash S)$, $0 \le \widetilde{\chi}, \widetilde{\widetilde{\chi}} \le 1$ and such that $\widetilde{\chi} = 1$ on $\operatorname{supp}\partial_i\chi$ and $\widetilde{\widetilde{\chi}} = 1$ on $\operatorname{supp}\widetilde{\chi}$. Let μ be the function given by Lemma 7.3 (m_δ depends only on φ, Ω, n).

By (7.3), there exists $C_0 = C_0(\varphi, \Omega, n) > 0$ such that

$$\|(1-\chi)u\|_0^2 = \int_{\overline{\Omega}\backslash B(0,\delta)} u^2 dx \le \frac{1}{m_\delta}\int \mu u^2 dx \le C_0(\|u\|_0\|L_\nu u\|_0 + B\|u\|_0^2),$$

where $B = \frac{1}{2}\sup[b_{ij}^{ij} - 2b]$. By proposition 7.1, $\sum_{ij} b_{ij}^{ij} = 0$, and the hypothesis (A2) imply that $-2b \le \varrho$. So, $B \le \varrho$ and we have

$$\|(1-\chi)u\|_0^2 \le C_1(\|u\|_0\|L_\nu u\|_0 + \varrho\|u\|_0^2).$$

Since $\mathrm{Supp}\,\widetilde{\widetilde{\chi}} \subset \overline{\Omega}\backslash\{0\}$, we also have by the same way,

$$\|\widetilde{\widetilde{\chi}} u\|_0^2 \le C_1(\|u\|_0\|L_\nu u\|_0 + \varrho\|u\|_0^2).$$

On the other hand, by (7.8),

$$\|\chi u\|_0^2 \le C_2\|L_\nu \chi u\|_0^2 \le C_2(\|L_\nu u\|_0^2 + \|[\chi, L_\nu]u\|_0^2),$$

but $\widetilde{\chi} L_\nu \widetilde{\widetilde{\chi}} u = \widetilde{\chi} L_\nu u$ and $[\chi, L_\nu]u = [\chi, \widetilde{\chi} L_\nu]\widetilde{\widetilde{\chi}} u$. Since $A(2) \le M_0$ and $\nu \le 1$, using Lemma 7.5, we get

$$\begin{aligned}\|[\chi, L_\nu]u\|_0^2 = \|[\chi, \widetilde{\chi} L_\nu]\widetilde{\widetilde{\chi}} u\|_0^2 &\le C\big[\|\widetilde{\chi} L_\nu \widetilde{\widetilde{\chi}} u\|_0\|\widetilde{\widetilde{\chi}} u\|_0 + (M_0+1)^2\|\widetilde{\widetilde{\chi}} u\|_0^2\big]\\ &\le C'\big(\|L_\nu u\|_0\|\widetilde{\widetilde{\chi}} u\|_0 + \|\widetilde{\widetilde{\chi}} u\|_0^2\big).\end{aligned}$$

Combining these inequalities with the fact that $\varrho << 1$, and using the inequality $\alpha\beta \leq \varepsilon\alpha^2 + \frac{1}{\varepsilon}\beta^2$, we get (3.12) $\square$

Proof of (3.13). We have $\operatorname{supp}(1-\chi) \subset \overline{\Omega}\backslash B(0,\delta)$. Or φ is strictly plurisubharmonic on $E = \operatorname{supp}(1-\chi)$, then for $\varepsilon \leq \varepsilon_4$ small enough, L is uniformly elliptic on E. Using (7.1) and the estimation $A(2) \leq M_0$, we have

$$\|(1-\chi)u\|_1 \leq C_1'(\|L_\nu u\|_0 + (M_0+1)\|u\|_0 + \|[\chi, L_\nu]u\|_0).$$

Applying Lemma 7.5, we get

$$\|[\chi, L_\nu]u\|_0 \leq C_0(\|L_\nu u\|_0 + (M_0+1)\|u\|_0),$$

therefore,

$$\|(1-\chi)u\|_1 \leq C_1(M_0)(\|L_\nu u\|_0 + \|u\|_0).$$

On the other hand, since $A(2) \leq M_0$, we get using (7.10),

$$\|\chi u\|_1 \leq C_1(M_0)(\|L_\nu u\|_1 + \|[\chi, L_\nu]u\|_1 + \|u\|_0)\,.$$

But $\widetilde{\chi}L_\nu\widetilde{\widetilde{\chi}}u = \widetilde{\chi}L_\nu u$ and $[\chi, L_\nu]u = [\chi, \widetilde{\chi}L_\nu]\widetilde{\widetilde{\chi}}u$, so, since $A(2) \leq M_0$, Lemma 7.5 gives

$$\begin{aligned}\|[\chi, L_\nu]u\|_1 &\leq C_1(\|\widetilde{\chi}L_\nu\widetilde{\widetilde{\chi}}u\|_1 + (1+M_0)\|\widetilde{\widetilde{\chi}}u\|_1)\\ &\leq C_1(\|L_\nu u\|_1 + (1+M_0)\|\widetilde{\widetilde{\chi}}u\|_1).\end{aligned}$$

Since L_ν is uniformly elliptic on $\operatorname{supp}\widetilde{\widetilde{\chi}}$ and $A(2) \leq M_0$, then we have by (7.1),

$$\|\widetilde{\widetilde{\chi}}u\|_1 \leq C_1'(\|L_\nu u\|_0 + (M_0+1)\|u\|_0 + \|[\widetilde{\widetilde{\chi}}, L_\nu]u\|_0),$$

which using (7.6) gives

$$\|\widetilde{\widetilde{\chi}}u\|_1 \leq C_1(M_0)(\|L_\nu u\|_1 + \|u\|_0).$$

Combining these inequalities, we get (3.13). $\square$

The proof of (3.14) is identical to that of (3.13) using the inequalities (7.1), (7.2), (7.6), and (7.10).

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