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EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS TO NONLINEAR NONLOCAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In the present work we consider a nonlinear nonlocal functional differential equations in a real reflexive Banach space. We apply the method of lines to establish the existence and uniqueness of a strong solution. We consider also some applications of the abstract results.

1. Introduction

Consider the following nonlocal nonlinear functional differential equation in a real reflexive Banach space X,

$$\begin{gathered} u'(t) + Au(t) = f(t, u(t), u(b_1(t)), u(b_2(t)), \dots, u(b_m(t))), \quad t \in (0, T], \\ h(u) = \phi_0, \quad \text{on } [-\tau, 0], \end{gathered} \tag{1.1}$$

where $0 < T < \infty$, $\phi_0 \in \mathcal{C}_0 := C([-\tau, 0]; X)$, the nonlinear operator A is single-valued and m-accretive defined from the domain $D(A) \subset X$ into X, the nonlinear map f is defined from $[0, T] \times X^{m+1}$ into X and the map h is defined from $\mathcal{C}_T := C([-\tau, T]; X)$ into $\mathcal{C}_T$. Here $\mathcal{C}_t := C([-\tau, t]; X)$ for $t \in [0, T]$ is the Banach space of all continuous functions from $[-\tau, t]$ into X endowed with the supremum norm

$$\|\phi\|_t := \sup_{-\tau \le \eta \le t} \|\phi(\eta)\|, \quad \phi \in \mathcal{C}_t,$$

where $\|.\|$ is the norm in X. The existence and uniqueness results for (1.1) may also be applied to the particular case, namely, the retarded functional differential equation,

$$\begin{gathered} u'(t) + Au(t) = f(t, u(t), u(t - \tau_1), u(t - \tau_2), \dots, u(t - \tau_m)), \quad t \in (0, T], \\ u = \phi_0, \quad \text{on } [-\tau, 0], \end{gathered} \tag{1.2}$$

where $\tau_i \ge 0$, and $\tau = \max\{\tau_1, \tau_2, \dots, \tau_m\}$.

The study of the nonlocal functional differential equation of the type (1.1) is motivated by the paper of Byszewski and Akca [6]. In [6] the authors have considered

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the nonlocal Cauchy problem,

$$\begin{gathered} u'(t)+Au(t)=f(t,u(t),u(a_1(t)),u(a_2(t)),\dots,u(a_m(t))),\quad t\in(0,T], \\ u(0)+g(u)=u_0, \end{gathered} \tag{1.3}$$

where $-A$ is the generator of a compact semigroup in X, $g : C([0,T];X)$ into X, $u_0 \in X$ and $a_i : [0,T] \to [0,T]$. Although, in this case we may take $h(u)(t) \equiv u(0)+g(u)$ on $[-\tau,T]$, $\phi_0(t) \equiv u_0$ on $[-\tau,0]$ and $b_i(t)=a_i(t)$, for $t\in[0,T]$ to write it as (1.1), but the analysis presented here will not be applicable to (1.3). We consider here a Volterra type operator h which is assumed to satisfy $h(\phi_1)=h(\phi_2)$ on $[-\tau,0]$ for any ϕ_1 and ϕ_2 in $\mathcal{C}_T$ with $\phi_1=\phi_2$ on $[-\tau,0]$ (cf. (A3) stated below). This condition will not hold in general for the operator $h(u)(t)\equiv u(0)+g(u)$. We shall treat this case differently in our subsequent work.

For the earlier works on existence, uniqueness and stability of various types of solutions of differential and functional differential equations with nonlocal conditions, we refer to Byszewski and Lakshmikantham [7], Byszewski [5], Balachandran and Chandrasekaran [3], Lin and Liu [11] and references cited in these papers.

Our aim is to extend the application of the method of lines to (1.1). For the applications of the method of lines to nonlinear evolution and nonlinear functional evolution equations, we refer to Kartsatos and Parrott [9], Kartsatos [8] Bahuguna and Raghavendra [1] and references cited in these papers.

Let $\tilde{T}$ be any number such that $0<\tilde{T}\le T$. Any function in $\mathcal{C}_T$ is also considered belonging to the space $\mathcal{C}_{\tilde{T}}$ as its restriction on the subinterval $[-\tau,\tilde{T}]$, $0<\tilde{T}\le T$. For any $\phi\in\mathcal{C}_{\tilde{T}}$, we consider the problem,

$$\begin{gathered} u'(t)+Au(t)=f(t,u(t),u(b_1(t)),u(b_2(t)),\dots,u(b_m(t))),\quad t\in(0,\tilde{T}], \\ u=\phi,\quad \text{on } [-\tau,0]. \end{gathered} \tag{1.4}$$

Suppose that there is $\psi_0\in\mathcal{C}_T$ such that $h(\psi_0)=\phi_0$ on $[-\tau,0]$ and $\psi_0(0)\in D(A)$. Let $\mathcal{W}(\psi_0,\tilde{T}) := \{\psi\in\mathcal{C}_{\tilde{T}} : \ \psi=\psi_0, \text{ on } [-\tau,0]\}$. For any $\phi\in\mathcal{W}(\psi_0,\tilde{T})$ we prove the existence and uniqueness of a *strong solution* u of (1.4) under the same assumptions of Theorem 2.1, stated in the next section, in the sense that there exists a unique function $u\in\mathcal{C}_{\tilde{T}}$ such that $u(t)\in D(A)$ for a.e. $t\in[0,\tilde{T}]$, u is differentiable a.e. on $[0,\tilde{T}]$ and

$$\begin{gathered} u'(t)+Au(t)=f(t,u(t),u(b_1(t)),\dots,u(b_m(t))),\quad \text{a.e. } t\in[0,\tilde{T}], \\ u=\phi,\quad \text{on } [-\tau,0]. \end{gathered} \tag{1.5}$$

Let $u_\phi\in\mathcal{C}_{\tilde{T}}$ be the strong solution of (1.4) corresponding to $\phi\in\mathcal{W}(\psi_0,\tilde{T})$. It can be shown that $u_\phi\in\mathcal{W}(\psi_0,\tilde{T})$. We define a map S from $\mathcal{W}(\psi_0,\tilde{T})$ into $\mathcal{W}(\psi_0,\tilde{T})$ given by

$$S\phi=u_\phi,\quad \phi\in\mathcal{W}(\psi_0,\tilde{T}).$$

We then prove that S is constant on $\mathcal{W}(\psi_0,\tilde{T})$ and hence there exists a unique $\chi_0\in\mathcal{W}(\psi_0,\tilde{T})$ such that $\chi_0=S\chi_0=u_{\chi_0}$. We then show that u_{χ_0} is a strong solution of (1.1). Also, we establish that a strong solution $u\in\mathcal{W}(\psi_0,\tilde{T})$ of (1.1) can be continued uniquely to either the whole interval $[-\tau,T]$ or there is the maximal interval $[-\tau,t_{\max})$, $0<t_{\max}\le T$, such that for every $0<\tilde{T}<t_{\max}$, $u\in\mathcal{W}(\psi_0,\tilde{T})$ is a strong solution of (1.1) on $[-\tau,\tilde{T}]$ and in the later case either

$$\lim_{t\to t_{\max}-}\|u(t)\|=\infty,$$

or $u(t)$ goes to the boundary of $D(A)$ as $t \to t_{\max}-$. Finally, we show that u is unique if and only if $\psi_0 \in \mathcal{C}_T$ satisfying $h(\psi_0) = \phi_0$ is unique up to $[-\tau, 0]$. We also consider some applications of the abstract results.

2. Preliminaries and Main Result

Let X be a real Banach space such that its dual X^* is uniformly convex. One of the consequences of the fact that X^* is uniformly convex is that the duality map $F : X \to 2^{X^*}$, given by

$$F(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|_*^2\},$$

is single-valued and is continuous on bounded subsets of X. Here 2^{X^*} denotes the power set of X^*, $\|.\|$ and $\|.\|_*$ are the norms of X and X^*, respectively, $\langle x, x^* \rangle$ is the value of $x^* \in X^*$ at $x \in X$. Further, we assume the following conditions:

(A1) The operator $A : D(A) \subset X \to X$ is m-accretive, i.e., $\langle Ax - Ay, F(x-y) \rangle \geq 0$, for all $x, y \in D(A)$ and $R(I + A) = X$, where $R(.)$ is the range of an operator.

(A2) The nonlinear map $f : [0, T] \times X^{m+1} \to X$ satisfies a local Lipschitz-like condition

$$\begin{aligned} &\|f(t, u_1, u_2, \dots u_{m+1}) - f(s, v_1, v_2, \dots, v_{m+1})\| \\ &\leq L_f(r)[|t - s| + \sum_{i=1}^{m+1} \|u_i - v_i\|], \end{aligned}$$

for all $(u_1, u_2, \dots, u_{m+1})$, $(v_1, v_2, \dots, v_{m+1})$ in $B_r(X^{m+1}, (x_0, x_0, \dots, x_0))$ and $t, s \in [0, T]$ where $L_f : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function and for $x_0 \in X$ and $r > 0$

$$B_r(X^{m+1}, (x_0, x_0, \dots, x_0)) = \{(u_1, \dots, u_{m+1}) \in X^{m+1} : \sum_{i=1}^{m+1} \|u_i - x_0\| \leq r\}.$$

(A3) The nonlinear map $h : \mathcal{C}_T \to \mathcal{C}_T$ is continuous and for any ϕ_1 and ϕ_2 in $\mathcal{C}_T$ with $\phi_1 = \phi_2$ on $[-\tau, 0]$, $h(\phi_1) = h(\phi_2)$ on $[-\tau, 0]$.

(A4) For $i = 1, 2, \dots, m$, the maps $b_i : [0, T] \to [-\tau, T]$ are continuous and $b_i(t) \leq t$ for $t \in [0, T]$.

Theorem 2.1. *Suppose that the conditions (A1)-(A4) are satisfied and there exists $\psi_0 \in \mathcal{C}_T$ such that $h(\psi_0) = \phi_0$ on $[-\tau, 0]$ and $\psi_0(0) \in D(A)$. Then (1.1) has a strong solution u on $[-\tau, \tilde{T}]$, for some $0 < \tilde{T} \leq T$, in the sense that there exists a function $u \in \mathcal{C}_{\tilde{T}}$ such that $u(t) \in D(A)$ for a.e. $t \in [0, \tilde{T}]$, u is differentiable a.e. on $[0, \tilde{T}]$ and*

$$\begin{gathered} u'(t) + Au(t) = f(t, u(t), u(b_1(t)), \dots, u(b_m(t))), \quad \text{a.e. } t \in [0, \tilde{T}], \\ h(u) = \phi_0, \quad \text{on } [-\tau, 0]. \end{gathered} \tag{2.1}$$

Also, u is unique in $\mathcal{W}(\psi_0, \tilde{T})$ and u is Lipschitz continuous on $[0, \tilde{T}]$. Furthermore, u can be continued uniquely either on the whole interval $[-\tau, T]$ or there exists a maximal interval $[0, t_{\max})$, $0 < t_{\max} \leq T$, such that u is a strong solution of (1.1) on every subinterval $[-\tau, \tilde{T}]$, $0 < \tilde{T} < t_{\max}$. A strong solution u of (1.1) is unique on the interval of existence if and only if $\psi_0 \in \mathcal{C}_T$ satisfying $h(\psi_0) = \phi_0$ on $[-\tau, 0]$ is unique up to $[-\tau, 0]$.

3. Discretization Scheme and A Priori Estimates

In this section we establish the existence and uniqueness of a strong solution to (1.4) for a given $\phi \in \mathcal{W}(\psi_0, T)$. Let $\phi \in \mathcal{W}(\psi_0, T)$. Then $x_0 := \phi(0) = \psi_0(0) \in D(A)$. For the application of the method of lines to (1.4), we proceed as follows. We fix $R > 0$ and let $R_0 := R + \sup_{t\in[-\tau,T]} \|\phi(t) - x_0\|$. We choose t_0 such that

$$0 < t_0 \leq T,$$
$$t_0[\|Ax_0\| + 3L_f(R_0)(T + (m+1)R_0) + \|f(0, x_0, x_0, \dots, x_0)\|] \leq R.$$

For $n \in \mathbb{N}$, let $h_n = t_0/n$. We set $u_0^n = x_0$ for all $n \in \mathbb{N}$ and define each of $\{u_j^n\}_{j=1}^n$ as the unique solution of the equation

$$\frac{u - u_{j-1}^n}{h_n} + Au = f(t_j^n, u_{j-1}^n, \tilde{u}_{j-1}^n(b_1(t_j^n)), \dots, \tilde{u}_{j-1}^n(b_m(t_j^n))), \tag{3.1}$$

where $\tilde{u}_0^n(t) = \phi(t)$ for $t \in [-\tau, 0]$, $\tilde{u}_0^n(t) = x_0$ for $t \in [0, t_0]$ and for $2 \leq j \leq n$,

$$\tilde{u}_{j-1}^n(t) = \begin{cases} \phi(t), & t \in [-\tau, 0], \\ u_{i-1}^n + \frac{1}{h_n}(t - t_{i-1}^n)(u_i^n - u_{i-1}^n), & t \in [t_{i-1}^n, t_i^n], \\ & i = 1, 2, \dots, j-1, \\ u_{j-1}^n, & t \in [t_{j-1}^n, t_0]. \end{cases} \tag{3.2}$$

The existence of a unique $u_j^n \in D(A)$ satisfying (3.1) is a consequence of the m-accretivity of A. Using (A2) we first prove that the points $\{u_j^n\}_{j=0}^n$ lie in a ball with its radius independent of the discretization parameters j, h_n and n. We then prove *a priori* estimates on the difference quotients $\{(u_j^n - u_{j-1}^n)/h_n\}$ using (A2). We define the sequence $\{U^n\} \subset \mathcal{C}_{t_0}$ of polygonal functions

$$U^n(t) = \begin{cases} \phi(t), & t \in [-\tau, 0], \\ u_{j-1}^n + \frac{1}{h_n}(t - t_{j-1}^n)(u_j^n - u_{j-1}^n), & t \in (t_{j-1}^n, t_j^n], \end{cases} \tag{3.3}$$

and prove the convergence of $\{U^n\}$ to a unique strong solution u of (1.4) in $\mathcal{C}_{t_0}$ as $n \to \infty$.

Now, we show that $\{u_j^n\}_{j=0}^n$ lie in a ball in X of radius independent of j, h_n and n.

Lemma 3.1. *For $n \in \mathbb{N}$, $j = 1, 2, \dots, n$,*

$$\|u_j^n - x_0\| \leq R.$$

Proof. From (3.1) for $j = 1$ and the accretivity of A, we have

$$\|u_1^n - x_0\| \leq h_n[\|Ax_0\| + 3L_f(R_0)(T + (m+1)R_0) + \|f(0, x_0, x_0, \dots, x_0)\|] \leq R.$$

Assume that $\|u_i^n - x_0\| \leq R$ for $i = 1, 2, \dots, j-1$. Now, for $2 \leq j \leq n$,

$$\begin{aligned} \|u_j^n - x_0\| &\leq \|u_{j-1}^n - x_0\| + h_n[\|Ax_0\| + 3L_f(R_0)(T + (m+1)R_0) \\ &\quad + \|f(0, x_0, x_0, \dots, x_0)\|]. \end{aligned}$$

Repeating the above inequality, we obtain

$$\begin{aligned} \|u_j^n - x_0\| &\leq jh_n[\|Ax_0\| + 3L_f(R_0)(T + (m+1)R_0) \\ &\quad + \|f(0, x_0, x_0, \dots, x_0)\|] \leq R, \end{aligned}$$

as $jh_n \leq t_0$ for $0 \leq j \leq n$. This completes the proof of the lemma. □

Now, we establish *a priori* estimates for the difference quotients $\{\frac{u_j^n - u_{j-1}^n}{h_n}\}$.

Lemma 3.2. *There exists a positive constant K independent of the discretization parameters n, j and h_n such that*
$$\Big\|\frac{u_j^n-u_{j-1}^n}{h_n}\Big\| \le K, \quad j=1,2,\dots,n,\ n=1,2,\dots.$$

Proof. In this proof and subsequently, K will represent a generic constant independent of j, h_n and n. Subtracting $Au_0^n = Ax_0$ from both the sides in (3.1) and applying $F(u_1^n-u_0^n)$, using accretivity of A, we get
$$\Big\|\frac{u_1^n-u_0^n}{h_n}\Big\| \le \|Ax_0\| + \|f(0,x_0,x_0,\dots,x_0)\| + 3L_f(R_0)(T+(m+1)R_0) \le K.$$
Now, for $2\le j\le n$ applying $F(u_j^n-u_{j-1}^n)$ to (3.1) and using accretivity of A, we get
$$\begin{aligned}\Big\|\frac{u_j^n-u_{j-1}^n}{h_n}\Big\| \le \Big\|\frac{u_{j-1}^n-u_{j-2}^n}{h_n}\Big\| &+ \|f(t_j^n,u_{j-1}^n,\tilde u_{j-1}^n(b_1(t_j^n)),\dots,\tilde u_{j-1}^n(b_m(t_j^n)))\\ &- f(t_{j-1}^n,u_{j-2}^n,\tilde u_{j-2}^n(b_1(t_{j-1}^n)),\dots,\tilde u_{j-2}^n(b_m(t_{j-1}^n)))\|.\end{aligned}$$
From the above inequality we get
$$\Big\|\frac{u_j^n-u_{j-1}^n}{h_n}\Big\| \le (1+Ch_n)\Big\|\frac{u_{j-1}^n-u_{j-2}^n}{h_n}\Big\| + Ch_n,$$
where C is a positive constant independent of j, h_n and n. Repeating the above inequality, we get
$$\Big\|\frac{u_j^n-u_{j-1}^n}{h_n}\Big\| \le (1+Ch_n)^j.C_1 \le C_1e^{TC} \le K.$$
This completes the proof of the lemma. □

We introduce another sequence $\{X^n\}$ of step functions from $[-h_n,t_0]$ into X by
$$X^n(t)=\begin{cases}x_0, & t\in[-h_n,0],\\ u_j^n, & t\in(t_{j-1}^n,t_j^n].\end{cases}$$

Remark 3.3. From Lemma 3.2 it follows that the functions U^n and $\tilde u_r^n$, $0\le r\le n-1$, are Lipschitz continuous on $[0,t_0]$ with a uniform Lipschitz constant K. The sequence $U^n(t)-X^n(t)\to 0$ in X as $n\to\infty$ uniformly on $[0,t_0]$. Furthermore, $X^n(t)\in D(A)$ for $t\in[0,t_0]$ and the sequences $\{U^n(t)\}$ and $\{X^n(t)\}$ are bounded in X, uniformly in $n\in\mathbb{N}$ and $t\in[0,t_0]$. The sequence $\{AX^n(t)\}$ is bounded uniformly in $n\in\mathbb{N}$ and $t\in[0,t_0]$.

For notational convenience, let
$$f^n(t)=f(t_j^n,u_{j-1}^n,\tilde u_{j-1}^n(b_1(t_j^n)),\dots,\tilde u_{j-1}^n(b_m(t_j^n))),$$
$t\in(t_{j-1}^n,t_j^n]$, $1\le j\le n$. Then (3.1) may be rewritten as
$$\frac{d^-}{dt}U^n(t)+AX^n(t)=f^n(t), \quad t\in(0,t_0], \tag{3.4}$$
where $\frac{d^-}{dt}$ denotes the left derivative in $(0,t_0]$. Also, for $t\in(0,t_0]$, we have
$$\int_0^t AX^n(s)\,ds = x_0-U^n(t)+\int_0^t f^n(s)\,ds. \tag{3.5}$$

Lemma 3.4. *There exists $u\in\mathcal{C}_{t_0}$ such that $U^n\to u$ in $\mathcal{C}_{t_0}$ as $n\to\infty$. Moreover, u is Lipschitz continuous on $[0,t_0]$.*

Proof. From (3.4) for $t \in (0, t_0]$, we have

$$\Big\langle \frac{d^-}{dt}(U^n(t) - U^k(t)), F(X^n(t) - X^k(t))\Big\rangle \le \big\langle f^n(t) - f^k(t), F(X^n(t) - X^k(t)\big\rangle.$$

From the above inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d^-}{dt}\|U^n(t) - U^k(t)\|^2 \\
&\le \Big\langle \frac{d^-}{dt}(U^n(t) - U^k(t)) - f^n(t) + f^k(t), F(U^n(t) - U^k(t)) - F(X^n(t) - X^k(t))\Big\rangle \\
&\quad + \big\langle f^n(t) - f^k(t), F(U^n(t) - U^k(t))\big\rangle.
\end{aligned}$$

Now,

$$\|f^n(t) - f^k(t)\| \le \epsilon_{nk}(t) + K\|U^n - U^k\|_t,$$

where

$$\begin{aligned}
\epsilon_{nk}(t) = K[&|t_j^n - t_l^k| + (h_n + h_k) + \|X^n(t - h_n) - U^n(t)\| + \|X^k(t - h_k) - U^n(t)\| \\
&+ \sum_{i=1}^m (|b_i(t_j^n) - b_i(t)| + |b_i(t_l^k) - b_i(t)|),
\end{aligned}$$

for $t \in (t_{j-1}^n, t_j^n]$ and $t \in (t_{l-1}^k, t_l^k]$, $1 \le j \le n$, $1 \le l \le k$. Therefore, $\epsilon_{nk}(t) \to 0$ as $n, k \to \infty$ uniformly on $[0, t_0]$. This implies that for a.e. $t \in [0, t_0]$,

$$\frac{d^-}{dt}\|U^n(t) - U^k(t)\|^2 \le K[\epsilon_{nk}^1 + \|U^n - U^k\|_t^2],$$

where ϵ_{nk}^1 is a sequence of numbers such that $\epsilon_{nk}^1 \to 0$ as $n, k \to \infty$. Integrating the above inequality over $(0, s)$, $0 < s \le t \le t_0$, taking the supremum over $(0, t)$ and using the fact that $U^n = \phi$ on $[-\tau, 0]$ for all n, we get

$$\|U^n - U^k\|_t^2 \le K[T\epsilon_{nk}^1 + \int_0^t \|U^n - U^k\|_s^2\, ds].$$

Applying Gronwall's inequality we conclude that there exists $u \in \mathcal{C}_{t_0}$ such that $U^n \to u$ in $\mathcal{C}_{t_0}$. Clearly, $u = \phi$ on $[-\tau, 0]$ and from Remark 3.3 it follows that u is Lipschitz continuous on $[0, t_0]$. This completes the proof of the lemma. □

Proof of Theorem 2.1. First, we prove the existence on $[-\tau, t_0]$ and then prove the unique continuation of the solution on $[-\tau, T]$. Proceeding similarly as in [2], we may show that $u(t) \in D(A)$ for $t \in [0, t_0]$, $AX^n(t) \rightharpoonup Au(t)$ on $[0, t_0]$ and $Au(t)$ is weakly continuous on $[0, t_0]$. Here $\rightharpoonup$ denotes the weak convergence in X. For every $x^* \in X^*$ and $t \in (0, t_0]$, we have

$$\int_0^t \langle AX^n(s), x^*\rangle\, ds = \langle x_0, x^*\rangle - \langle U^n(t), x^*\rangle + \int_0^t < f^n(s), x^*\rangle\, ds.$$

Using Lemma 3.4 and the bounded convergence theorem, we obtain as $n \to \infty$,

$$\begin{aligned}
\int_0^t \langle Au(s), x^*\rangle\, ds &= \langle x_0, x^*\rangle - \langle u(t), x^*\rangle \\
&\quad + \int_0^t \langle f(s, u(s), u(b_1(s)), \dots, u(b_m(s))), x^*\rangle\, ds.
\end{aligned} \tag{3.6}$$

Since $Au(t)$ is Bochner integrable (cf. [2]) on $[0, t_0]$, from (3.6) we get

$$\frac{d}{dt}u(t) + Au(t) = f(t, u(t), u(b_1(t)), \dots, u(b_m(t))), \quad \text{a.e. } t \in [0, t_0]. \tag{3.7}$$

Clearly, u is Lipschitz continuous on $[0, t_0]$ and $u(t) \in D(A)$ for $t \in [0, t_0]$. Now we prove the uniqueness of a function $u \in \mathcal{C}_{t_0}$ which is differentiable a.e. on $[0, t_0]$ with $u(t) \in D(A)$ a.e. on $[0, t_0]$ and $u = \phi$ on $[-\tau, 0]$ satisfying (3.7). Let $u_1, u_2 \in \mathcal{C}_{t_0}$ be two such functions. Let $R = \max\{\|u_1\|_{t_0}, \|u_2\|_{t_0}\}$. Then for $u = u_1 - u_2$, we have

$$\frac{d}{dt}\|u(t)\|^2 \leq C_1(R)\|u\|_t^2, \quad \text{a.e. } t \in [0, t_0],$$

where $C_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function. Integrating over $(0, s)$ for $0 < s \leq t \leq t_0$, taking supremum over $(0, t)$ and using the fact that $u \equiv 0$ on $[-\tau, 0]$, we get

$$\|u\|_t^2 \leq C_1(R) \int_0^t \|u\|_s^2 \, ds.$$

Application of Gronwall's inequality implies that $u \equiv 0$ on $[-\tau, t_0]$.

Now, we prove the unique continuation of the solution u on $[-\tau, T]$. Suppose $t_0 < T$ and consider the problem

$$\begin{gathered} w'(t) + Aw(t) = \tilde{f}(t, w(t), w(\tilde{b}_1(t)), w(\tilde{b}_2(t)), \dots, w(\tilde{b}_m(t))), \quad 0 < t \leq T - t_0, \\ w = \tilde{\phi}_0, \quad \text{on } [-\tau - t_0, 0], \end{gathered} \tag{3.8}$$

where $\tilde{f}(t, u_1, u_2, \dots, u_{m+1}) = f(t + t_0, u_1, u_2, \dots, u_{m+1})$, $0 \leq t \leq T - t_0$,

$$\tilde{\phi}_0(t) = \begin{cases} \phi(t + t_0), & t \in [-\tau - t_0, -t_0], \\ u(t + t_0), & t \in [-t_0, 0], \end{cases}$$

$\tilde{b}_i(t) = b_i(t + t_0) - t_0$, $t \in [0, T - t_0]$ $i = 1, 2, \dots, m$.

Since $\tilde{\phi}_0(0) = u(t_0) \in D(A)$ and $\tilde{f}$ satisfies (A2) and $\tilde{b}_i$, $i = 1, 2, \dots, m$ satisfy (A4) on $[0, T - t_0]$, we may proceed as before and prove the existence of a unique $w \in C([-\tau - t_0, t_1]; X)$, $0 < t_1 \leq T - t_0$, such that w is Lipschitz continuous on $[0, t_1]$, $w(t) \in D(A)$ for $t \in [0, t_1]$ and w satisfies

$$\begin{gathered} w'(t) + Aw(t) = \tilde{f}(t, w(t), w(\tilde{b}_1(t)), w(\tilde{b}_2(t)), \dots, w(\tilde{b}_m(t))), \quad \text{a.e. } t \in [0, t_1], \\ w = \tilde{\phi}_0, \quad \text{on } [-\tau - t_0, 0]. \end{gathered} \tag{3.9}$$

Then the function

$$\bar{u}(t) = \begin{cases} u(t), & t \in [-\tau, t_0], \\ w(t - t_0), & t \in [t_0, t_0 + t_1], \end{cases}$$

is Lipschitz continuous on $[0, t_0 + t_1]$, $\bar{u}(t) \in D(A)$ for $t \in [0, t_0 + t_1]$ and satisfies (1.5) a.e. on $[0, t_0 + t_1]$. Continuing this way we may prove the existence on the whole interval $[-\tau, T]$ or there is the maximal interval $[-\tau, t_{\max})$, $0 < t_{\max} \leq T$, such that u is a strong solution of (1.1) on every subinterval $[-\tau, \tilde{T}]$, $0 < \tilde{T} < t_{\max}$. In the later case, if $\lim_{t \to t_{max^-}} \|u(t)\| < \infty$ and $\lim_{t \to t_{\max}-} u(t) \in D(A)$, then we may continue the solution beyond $t_{\max}$ but this will contradict the definition of maximal interval of existence. Therefore, either $\lim_{t \to t_{max^-}} \|u(t)\| = \infty$ or $u(t)$ goes to the boundary of $D(A)$ as $t \to t_{max^-}$.

Thus, for each $\phi \in \mathcal{W}(\psi_0, \tilde{T})$, we have proved the existence and uniqueness of a strong solution of (1.4).

Now, let u_ϕ be the strong solution of (1.4) corresponding to $\phi \in \mathcal{W}(\psi_0, \tilde{T})$. Since $u_\phi = \phi$ on $[-\tau, 0]$, it follows that $u_\phi \in \mathcal{W}(\psi_0, \tilde{T})$. We define a map $S : \mathcal{W}(\psi_0, \tilde{T}) \to \mathcal{W}(\psi_0, \tilde{T})$ given by $S\phi = u_\phi$ for $\phi \in \mathcal{W}(\psi_0, \tilde{T})$. Using similar arguments as used above in the proof of uniqueness and the fact that $u_\phi = u_\psi = \psi_0$ on $[-\tau, 0]$, we obtain

$$\|S\phi - S\psi\|_t^2 = \|u_\phi - u_\psi\|_t^2 \le C_2(R^{\phi\psi}) \int_0^t \|u_\phi - u_\psi\|_s^2 \, ds,$$

where $R^{\phi\psi} = \max\{\|u_\phi\|_{\tilde{T}}, \|u_\psi\|_{\tilde{T}}\}$ and $C_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function. Applying Gronwall's inequality we obtain that S is constant on $\mathcal{W}(\psi_0, \tilde{T})$ and therefore there exists a unique $\chi_0 \in \mathcal{W}(\psi_0, \tilde{T})$ such that $S\chi_0 = \chi_0 = u_{\chi_0}$. It is easy to verify that u_{χ_0} $(= \chi_0)$ is a strong solution to (1.1). Clearly, if $\psi_0 \in \mathcal{C}_T$ satisfying $h(\psi_0) = \phi_0$ on $[-\tau, 0]$ is unique up to $[-\tau, 0]$ then u is unique. If there are two ψ_0 and $\tilde{\psi}_0$ in $\mathcal{C}_T$ satisfying $h(\psi_0) = h(\tilde{\psi}_0) = \phi_0$ on $[-\tau, 0]$, with $\psi_0 \neq \tilde{\psi}_0$ on $[-\tau, 0]$, then $\mathcal{W}(\psi_0, \tilde{T}) \cap \mathcal{W}(\tilde{\psi}_0, \tilde{T}) = \emptyset$ and hence the solutions u and $\tilde{u}$ of (1.1) belonging to $\mathcal{W}(\psi_0, \tilde{T})$ and $\mathcal{W}(\tilde{\psi}_0, \tilde{T})$, respectively, are different. This completes the proof of Theorem 2.1. □

4. Applications

Theorem 2.1 may be applied to get the existence and uniqueness results for (1.1) in the case when the operator A, with the domain $D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega)$ into $X := L^2(\Omega)$, is associated with the nonlinear partial differential operator

$$Au = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), Du, \dots, D^\alpha u),$$

in a bounded domain Ω in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, where $A_\alpha(x, \xi)$ are real functions defined on $\Omega \times \mathbb{R}^N$ for some $N \in \mathbb{N}$ and satisfying Caratheodory condition of measurability and certain growth conditions (cf. Barbu [4] page 48).

In (1.1), we may take f as the function $f : [0, T] \times (L^2(\Omega))^{m+1} \to L^2(\Omega)$, given by

$$f(t, u_1, u_2, \dots, u_{m+1}) = f_0(t) + a(t) \sum_{i=1}^{m+1} \|u_i\|_{L^2(\Omega)} u_i,$$

where $f_0 : [0, T] \to L^2(\Omega)$, and $a : [0, T] \to \mathbb{R}$ are Lipschitz continuous functions on $[0, T]$ and $\|.\|_{L^2(\Omega)}$ denotes the norm in $L^2(\Omega)$. For the functions b_i, $i = 1, 2, \dots, n$ and h we may have any of the following.

(b1) Let $\tau_i \ge 0$. For $i = 1, 2, \dots, m$, let $b_i(t) = t - \tau_i$, $t \in [0, T]$.

(b2) Let τ_i, $i = 1, 2, \dots, m$ be such that $0 < \tau_i < T$. For $t \in [0, T]$, let

$$b_i(t) = \begin{cases} 0, & t \le \tau_i, \\ t - \tau_i, & t > \tau_i. \end{cases}$$

(b3) For $i = 1, 2, \dots, m$, let $b_i(t) = k_i t$, $t \in [0, T]$, $0 < k_i \le 1$.

(b4) Let $N \in \mathbb{N}$. Let $0 < k_i \le 1/(NT^N)$, $i = 1, 2, \dots, m$. For $i = 1, 2, \dots, m$, let

$$b_i(t) = k_i t^N, \quad t \in [0, T].$$

Let $-\tau \le a_1 < a_2 < \dots < a_r \le 0$, c_i with $C := \sum_{i=1}^r c_i \neq 0$ and $\epsilon_i > 0$, for $i = 1, \dots, r$. Let $x \in D(A)$. Consider the conditions:

(h1) $g_1(\chi) := \int_{-\tau}^0 k(\theta)\chi(\theta) d\theta = x$ for $\chi \in C([-\tau, 0]; X)$, where k is in $L^1(-\tau, 0)$ with $\kappa := \int_{-\tau}^0 k(s) ds \neq 0$

(h2) $g_2(\chi) := \sum_{i=1}^r c_i \chi(a_i) = x$ for $\chi \in C([-\tau, 0]; X)$;

(h3) $g_3(\chi) := \sum_{i=1}^r \frac{c_i}{\epsilon_i} \int_{a_i - \epsilon_i}^{a_i} \chi(s) ds = x$ for $\chi \in C([-\tau, 0]; X)$.

Clearly, $g_i : C([-\tau, 0]; X) \to X$, $i = 1, 2, 3$. For $i = 1, 2, 3$, define $h_i(\psi)(t) \equiv g_i(\psi|_{[-\tau,0]})$ on $[-\tau, T]$ for $\psi \in C([-\tau, T]; X)$ where $\psi|_{[-\tau,0]}$ is the restriction of ψ on $[-\tau, 0]$. Let $\phi_0(t) \equiv x$ on $[-\tau, 0]$. Then conditions (h1), (h2) and (h3) are equivalent to $h_i(\psi) = \phi_0$ on $[-\tau, 0]$, $i = 1, 2, 3$, respectively. For (h1), we may take $\psi_0(t) \equiv x/\kappa$ and for (h2) as well as for (h3), we may take $\psi_0(t) \equiv x/C$ on $[-\tau, T]$.

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