CHAIN SEQUENCES

THESIS

Presented to the Graduate Council of

Southwest Texas State University

in Partial Fulfillment of

the Requirements

For the Degree of

MASTER OF ARTS

By

Joyce Schweinle McCain, B. S.
(San Marcos, Texas)

San Marcos, Texas
August, 1969

ACKNOWLEDGMENTS

The writer wishes to express her appreciation to Dr. Burrell W. Helton, Professor of Mathematics, for his patience, supervision and encouragement during the preparation of this paper. The assistance rendered by Dr. Henry N. McEwen, Associate Professor of Mathematics and Dr. William C. Newberry, Associate Professor of Education, is also gratefully acknowledged.

Acknowledgments are also extended to my husband, Kyle, for his encouragement and understanding.

Joyce S. McCain

San Marcos, Texas
August, 1969

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The purpose of this paper is to examine some of the properties of chain sequences. According to Dr. H. S. Wall [2, p. 79], chain sequences play a fundamental role in the study of continued fractions.

First some general properties of chain sequences will be stated and proved. Properties of constant chain sequences will also be examined. The existence of maximal and minimal parameter sequences for a chain sequence will be established and these parameter sequences will be used to determine the existence of other parameter sequences.

Although the theorems in this paper have been proven in other papers, the proofs given here are original with the author.

CHAPTER II

DEFINITIONS, AXIOMS AND PRELIMINARY THEOREMS

In this paper the following grouping symbols [ ], ( ), ( ], and [ ) will be used to indicate closed, open, open on the left, and open on the right intervals, respectively. In proving theorems, it will be assumed that functions are from real numbers to real numbers. Symbols such as A, B, x, y, etc., will represent numbers unless indicated otherwise. Subscripts will denote nonnegative integers.

Definition 2.1: The number set S has a least upper bound means there is a number M such that

(1) if $x \in S$, then $x \leq M$, and

(2) if $p < M$, then there exists $x \in S$ such that $x > p$.

Notation 2.1: The symbol "l.u.b." means "least upper bound."

Definition 2.2: The number set S has a greatest lower bound means there is a number M such that

(1) if $x \in S$ then $x \geq M$, and

(2) if $p > M$, then there exists $x \in S$ such that $x < p$.

Notation 2.2: The symbol "g.l.b." means "greatest lower bound."

Notation 2.3: The symbol "iff" means "if, and only if."

Definition 2.3: $\{c_n\}_{n=1}^{\infty}$ is a chain sequence iff there exists a number sequence $\{g_n\}_{n=0}^{\infty}$ such that

(1) if n is a positive integer, then $c_n = (1 - g_{n-1})g_n$, and

(2) if $n = 0$ or a positive integer, then $0 \leq g_n \leq 1$.

The sequence $\{g_n\}_{n=0}^{\infty}$ is called a parameter sequence for $\{c_n\}_{n=1}^{\infty}$. It follows from the definition that if $\{c_n\}_{n=1}^{\infty}$ is a chain sequence, then for each positive integer n, $0 \leq c_n \leq 1$.

Definition 2.4: Suppose $\{c_n\}_{n=1}^{\infty}$ is a chain sequence: $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ are minimal and maximal parameter sequences for $\{c_n\}_{n=1}^{\infty}$ means

(1) $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ are parameter sequences for $\{c_n\}_{n=1}^{\infty}$ and

(2) if $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$, then

$m_n \le g_n \le M_n$ for $n = 1, 2, 3, \ldots$.

Definition 2.5: The sequence $\{a_n\}_{n=1}^{\infty}$ is a dense set in the interval $[0,1]$ means, if $p \in [0,1]$ and $\delta > 0$, then there exists a number $a_n \in \{a_n\}_{n=1}^{\infty}$ such that $|a_n - p| < \delta$.

Axiom 2.1: Every non-empty set which is bounded above has a least upper bound.

Theorem 2.1: Every non-empty set which is bounded below has a greatest lower bound.

Theorem 2.2: If $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence which is bounded above, then $\lim_{n\to\infty} a_n$ exists and is the least upper bound of $\{a_n\}_{n=1}^{\infty}$.

Theorem 2.3: If $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence which is bounded below, then $\lim_{n\to\infty} a_n$ exists and is the greatest lower bound of $\{a_n\}_{n=1}^{\infty}$.

Theorem 2.4: If A and B are numbers, the following statements are equivalent:

(1) $A = B$, and

(2) if $\epsilon > 0$, then $|A - B| < \epsilon$.

<u>Theorem 2.5</u>: (Cauchy criterion) If $\{x_n\}_{n=1}^{\infty}$ is a sequence, then the following statements are equivalent:

(1) $\lim_{n\to\infty} x_n$ exists, and

(2) if $\epsilon > 0$, there exists a $N > 0$ such that if $n > N$ and $m > N$, then $|x_n - x_m| < \epsilon$.

<u>Theorem 2.6</u>: If S_n is a subset of $[a,b]$, then the least upper bound and greatest lower bound of S_n belong to $[a,b]$.

<u>Theorem 2.7</u>: (Intermediate Value Theorem) Suppose f is continuous on the closed interval $[a,b]$, $f(a) = A$, $f(b) = B$, and $A \neq B$, then if $A < C < B$, there is a point $p \in [a,b]$ such that $f(p) = C$.

<u>Theorem 2.8</u>: If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences such that $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} y_n$ exists and $x_n \leq y_n$ for each n, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

The preceding theorems will be used without proof in this paper. Proof of these theorems can be found in elementary or advanced calculus books.

CHAPTER III

BASIC PROPERTIES OF CHAIN SEQUENCES

Properties of general chain sequences, constant chain sequences, and special chain sequences of the form $c_1, c_2, c_1, c_2, c_1, c_2, \ldots$ are examined in this chapter.

Theorem 3.1:

Given: $\{c_n\}_{n=1}^{\infty}$ is a chain sequence and for each n, $0 \le b_n \le c_n$.

Conclusion: $\{b_n\}_{n=1}^{\infty}$ is a chain sequence.

Proof:

Suppose that $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$ and define $\{q_n\}_{n=0}^{\infty}$ as the sequence of numbers such that $q_0 = 0$ and if $n \neq 0$ then,

$$q_n = \begin{cases} 0, \text{ if } b_n = 0 \\ \\ \dfrac{b_n}{1 - q_{n-1}}, \text{ if } b_n \neq 0. \end{cases}$$

A proof by induction will be used to show that $q_n \leq g_n$ for each n. Since $q_0 = 0 \leq g_0$, then $q_0 \leq g_0$. If $m \geq 0$, and $q_m \leq g_m$, it will be shown that $q_{m+1} \leq g_{m+1}$. Suppose $b_{m+1} = 0$, then $q_{m+1} = 0 \leq g_{m+1}$. Suppose $b_{m+1} \neq 0$ and assume that $q_{m+1} \leq g_{m+1}$ is false; therefore $q_{m+1} > g_{m+1}$ and since we will assume that $q_m \leq g_m$, then it follows that

$$\begin{aligned} b_{m+1} &= (1 - q_m)q_{m+1} \\ &\geq (1 - g_m)q_{m+1} \\ &> (1 - g_m)g_{m+1} \\ &= c_{m+1}. \end{aligned}$$

Hence $b_{m+1} > c_{m+1}$ which contradicts the hypothesis; therefore, $q_{m+1} \leq g_{m+1}$ and by induction $q_n \leq g_n$ for $n = 0, 1, 2, 3, \ldots$.

Since $q_n \leq g_n \leq 1$, then $q_n \leq 1$ for each n. It will be shown by induction that $0 \leq q_n$ for $n = 0, 1, 2, 3, \ldots$. By definition $0 = q_0$. Assume that $0 \leq q_k$ for $k \geq 1$. Now, if $b_{k+1} = 0$, then $q_{k+1} = 0$; and if $b_{k+1} \neq 0$, then since $q_k \geq 0$, it follows that

$$0 < b_{k+1} = (1 - q_k)\, q_{k+1}$$
$$\leq (1 - 0)\, q_{k+1}$$
$$= q_{k+1}.$$

Therefore $0 < q_{k+1}$ and by induction $0 \leq q_n$ for $n = 0, 1, 2, 3, \ldots$.

Since $0 \leq q_{n-1} \leq 1$ and $b_n = (1 - q_{n-1})q_n$ for $n = 1, 2, 3, \ldots$, then $\{b_n\}_{n=1}^{\infty}$ is a chain sequence.

Corollary 3.1:

Given: $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are chain sequences.

Conclusion: $\{c_n d_n\}_{n=1}^{\infty}$ is a chain sequence.

Proof:

It follows from Definition 2.3, that $0 \leq d_n \leq 1$; therefore, $0 \leq c_n d_n \leq c_n$. Since $\{c_n\}_{n=1}^{\infty}$ is a chain sequence and since for each n, $0 \leq c_n d_n \leq c_n$, then from Theorem 3.1, $\{c_n d_n\}_{n=1}^{\infty}$ is a chain sequence.

Theorem 3.2: Suppose c is a number and $\{c_n\}_{n=1}^{\infty}$ is a sequence such that $c_n = c$ for $n = 1, 2, 3, \ldots$. The following two statements are equivalent:

(1) $\{c_n\}_{n=1}^{\infty}$ is a chain sequence, and

(2) $0 \leq c$ and $c \leq \frac{1}{4}$.

Proof: $1 \rightarrow 2$

Assume the conclusion is false; then either $c < 0$ or $c > \frac{1}{4}$. Since $\{c_n\}_{n=1}^{\infty}$ is the constant chain sequence $c, c, c, \ldots$, then it follows from Definition 2.3, that $c \geq 0$. Therefore, $c \not< 0$.

Assume $c > \frac{1}{4}$, and let $\{g_n\}_{n=0}^{\infty}$ be a parameter sequence for $\{c_n\}_{n=1}^{\infty}$. An indirect proof will be used to show that $g_n > g_{n-1}$ for each n. Suppose there exists a positive integer n such that $g_n \leq g_{n-1}$. Since $0 < \frac{1}{4} < c = (1 - g_{n-1})g_n$, then $(1 - g_{n-1}) \neq 0$ and $\frac{c}{1-g_{n-1}} = g_n \leq g_{n-1}$; therefore $c \leq g_{n-1} - g_{n-1}^2$ and $g_{n-1}^2 - g_{n-1} + c \leq 0$. Since $c > \frac{1}{4}$, it follows that $0 > g_{n-1}^2 - g_{n-1} + \frac{1}{4} = (g_{n-1} - \frac{1}{2})^2 \geq 0$, which is a contradiction. Therefore the assumption that $g_n \leq g_{n-1}$ is false, and it follows that $g_n > g_{n-1}$ for each n.

An indirect proof will be used to show that $g_n - g_{n-1} \geq 2\sqrt{c} - 1$ for $n = 1, 2, 3, \ldots$. Suppose there exists a positive integer n such that

(1) $g_n - g_{n-1} < 2\sqrt{c} - 1$

(2) $g_n - g_{n-1} + 1 < 2\sqrt{c} = 2\sqrt{(1 - g_{n-1})g_n}$.

Since $g_n > g_{n-1}$ for each n, and since $c > \frac{1}{4}$, both sides of inequality 2 are nonnegative. Thus,

(3) $g_n^2 + g_{n-1}^2 + 1 - 2g_n + 2g_n g_{n-1} - 2g_{n-1} < 0$.

However, from inequality 3,

(4) $0 \leq (g_n + g_{n-1} - 1)^2 = g_n^2 + g_{n-1}^2 + 1 - 2g_n + 2g_n g_{n-1} - 2g_{n-1} < 0$.

This gives the contradiction $0 < 0$; hence the assumption in inequality 1 is false and therefore for each n,

(5) $g_n - g_{n-1} \geq 2\sqrt{c} - 1$.

Since $0 \leq g_{n-1} \leq 1$ and $g_n > g_{n-1}$ for $n = 1, 2, 3, \ldots$, then $\{g_n\}_{n=0}^{\infty}$ is a non-decreasing sequence which is bounded above; therefore, by Theorem 2.2, $\lim_{n\to\infty} g_n$ exists and is the least upper bound of $\{g_n\}_{n=0}^{\infty}$.

Since $\lim_{n\to\infty} g_n$ exists and since $2\sqrt{c} - 1 > 0$, then there exists a number $N > 0$ such that if $(n-1) > N$, then $|g_n - g_{n-1}| < 2\sqrt{c} - 1$, (Theorem 2.5). Let $(n-1) > N$. Since $g_n > g_{n-1}$, then

(5) $g_n - g_{n-1} = |g_n - g_{n-1}| < 2\sqrt{c} - 1$.

Inequalities 4 and 5 give the contradiction $2\sqrt{c} - 1 < 2\sqrt{c} - 1$. Therefore, the original assumption is false, and $c \leq \frac{1}{4}$. Hence, $0 \leq c$ and $c \leq \frac{1}{4}$, and Statement 1 implies Statement 2.

Proof: $2 \rightarrow 1$

First we will show that $\{c_n\}_{n=1}^{\infty}$ is a chain sequence if $c_n = \frac{1}{4}$ for each n. Suppose that $c_n = \frac{1}{4}$ for each n and define $\{g_n\}_{n=0}^{\infty}$ as the sequence of numbers such that for each n, $g_{n-1} = \frac{1}{2}$. Since

$$c_n = \frac{1}{4} = \left(1 - \frac{1}{2}\right)\frac{1}{2} = (1 - g_{n-1})g_n,$$

then $c_n = (1 - g_{n-1})g_n$ for each n. Also $0 < \frac{1}{2} = g_{n-1} = \frac{1}{2} < 1$ for each n, hence $0 \leq g_{n-1} \leq 1$. Therefore the constant sequence $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$ is a chain sequence. Since $\{c_n\}_{n=1}^{\infty}$ is a chain sequence when $c_n = \frac{1}{4}$ for each n and since $0 \leq c \leq \frac{1}{4}$, then, from Theorem 3.1, c, c, c, . . . is a chain sequence.

<u>Theorem 3.3</u>:

<u>Given</u>: $\{x_n\}_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} x_n$ exists.

<u>Conclusion</u>: $\lim_{n\to\infty} (x_{n+1} - x_n) = 0$

Proof:

Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n$ exists, then there exists a $N > 0$ such that if $n > N$ and $m > N$ then $|x_n - x_m| < \epsilon$, (Theorem 2.4). Let $n > N$, then

$$|(x_{n+1} - x_n) - 0| = |x_{n+1} - x_n| < \epsilon;$$

therefore $\lim_{n\to\infty} (x_{n+1} - x_n) = 0$.

<u>Theorem 3.4</u>:

<u>Given</u>: Suppose c is a positive number and f is a function such that $f(x) = x - \frac{c}{1-x}$.

<u>Conclusion A</u>: The following statements are equivalent:

(1) there exists a real number α such that $f(\alpha) = 0$, and

(2) $1 - 4c \geq 0$ and $\alpha = \frac{1 \pm \sqrt{1-4c}}{2}$.

<u>Conclusion B</u>: If $1 - 4c > 0$ and $\frac{1-\sqrt{1-4c}}{2} < x < \frac{1+\sqrt{1-4c}}{2}$ then $f(x) > 0$. If $\frac{1-\sqrt{1-4c}}{2} \leq x \leq \frac{1+\sqrt{1-4c}}{2}$, then $f(x) \geq 0$.

<u>Conclusion C</u>: If $1 - 4c \geq 0$ and $x < \frac{1-\sqrt{1-4c}}{2}$ or $x > \frac{1+\sqrt{1-4c}}{2}$ then $f(x) < 0$. If $x \leq \frac{1-\sqrt{1-4c}}{2}$ or $x \geq \frac{1+\sqrt{1-4c}}{2}$, then $f(x) \leq 0$.

Proof of Conclusion A, $1 \rightarrow 2$:

Since $f(\alpha) = 0$, then

$0 = f(\alpha) = \alpha - \frac{c}{1-\alpha}$ and

$0 = \alpha^2 - \alpha + c$. Therefore,

$\alpha = \frac{1 \pm \sqrt{1-4c}}{2}$. Since α is a real number, then $1 - 4c \geq 0$.

$2 \rightarrow 1$

$$f(\alpha) = \alpha - \frac{c}{1-\alpha} = \frac{1 \pm \sqrt{1-4c}}{2} - \frac{c}{1 - \frac{1 \pm \sqrt{1-4c}}{2}}$$

$$= \frac{1 \pm \sqrt{1-4c}}{2} - \frac{2c}{1 \mp \sqrt{1-4c}}$$

$$= \frac{1 \pm \sqrt{1-4c}}{2} - \frac{1 \pm \sqrt{1-4c}}{2}$$

$$= 0.$$

Proof of Conclusion B:

Suppose (1) $1 - 4c > 0$, and

(2) $\frac{1-\sqrt{1-4c}}{2} < x < \frac{1+\sqrt{1-4c}}{2}$.

An indirect proof will be used. Suppose that $f(x) \leq 0$. It follows

that

$x - \frac{c}{1-x} = f(x) \leq 0$; therefore,

$$(3)\ 0 \leq x^2 - x + c = \left(x - \frac{1+\sqrt{1-4c}}{2}\right)\left(x - \frac{1-\sqrt{1-4c}}{2}\right).$$

From inequality 2, $\left(x - \frac{1+\sqrt{1-4c}}{2}\right) < 0$ and $\left(x - \frac{1-\sqrt{1-4c}}{2}\right) > 0$. Thus,

$$(4)\ \left(x - \frac{1+\sqrt{1-4c}}{2}\right)\left(x - \frac{1-\sqrt{1-4c}}{2}\right) < 0.$$

Inequality 3 contradicts inequality 4; therefore the original assumption is false and $f(x) > 0$. Similarly, if $\frac{1-\sqrt{1-4c}}{2} \leq x \leq \frac{1+\sqrt{1-4c}}{2}$, then $f(x) \geq 0$.

Proof of Conclusion C:

Using an indirect proof, we will assume that $1 - 4c \geq 0$ and $x < \frac{1-\sqrt{1-4c}}{2}$ and that $f(x) \geq 0$. Since $c > 0$ and $1 - 4c \geq 0$ then $c \leq \frac{1}{4}$; therefore,

$$(5)\ \frac{1+\sqrt{1-4c}}{2} \geq \frac{1-\sqrt{1-4c}}{2} > x.$$

Since $0 \leq f(x) = x - \frac{c}{1-x}$, then

$$(6)\ 0 \geq x^2 - x + c = \left(x - \frac{1+\sqrt{1-4c}}{2}\right)\left(x - \frac{1-\sqrt{1-4c}}{2}\right).$$

However, from inequality 5, $\left(x - \frac{1+\sqrt{1-4c}}{2}\right) < 0$ and $\left(x - \frac{1-\sqrt{1-4c}}{2}\right) < 0$; therefore the product

(7) $\left(x - \frac{1+\sqrt{1-4c}}{2}\right)\left(x - \frac{1-\sqrt{1-4c}}{2}\right) > 0.$

Inequality 6 contradicts inequality 7; hence the original assumption is false and $f(x) < 0$. Similarly, if $x \leq \frac{1-\sqrt{1-4c}}{2}$, then $f(x) \leq 0$.

If $1 - 4c \geq 0$ and $x > \frac{1+\sqrt{1-4c}}{2}$, a similar indirect argument can be used to show that $f(x) < 0$. Suppose $f(x) \geq 0$, and inequality 6 can be obtained. Since $0 < c \leq \frac{1}{4}$, then, $\frac{1-\sqrt{1-4c}}{2} \leq \frac{1+\sqrt{1-4c}}{2} < x$, and therefore $\left(x - \frac{1+\sqrt{1-4c}}{2}\right) > 0$ and $\left(x - \frac{1-\sqrt{1-4c}}{2}\right) > 0$. Since both factors are positive, the product $\left(x - \frac{1+\sqrt{1-4c}}{2}\right)\left(x - \frac{1-\sqrt{1-4c}}{2}\right) > 0$ which contradicts inequality 6. Therefore, the assumption that $f(x) \geq 0$ is false and $f(x) < 0$. Similarly, if $x \geq \frac{1+\sqrt{1-4c}}{2}$, then $f(x) \leq 0$.

Theorem 3.5:

Given: The number sequence $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for the chain sequence c, c, c, ...

Conclusion A: $g_0 \leq \frac{1+\sqrt{1-4c}}{2}$

Conclusion B: if $c > 0$ and $g_0 = \frac{1\pm\sqrt{1-4c}}{2}$, then $g_n = g_0$ for $n = 1, 2, 3, \ldots,$ and

Conclusion C: if $g_0 < \frac{1+\sqrt{1-4c}}{2}$, then $\lim_{n\to\infty} g_n = \frac{1-\sqrt{1-4c}}{2}$.

Proof:

Since c, c, c, . . . is a chain sequence, it follows from Theorem 3.2 that $0 \leq c \leq \frac{1}{4}$.

Proof for Conclusion A:

An indirect argument will be used to prove Conclusion A, Assume

(1) $g_0 > \frac{1+\sqrt{1-4c}}{2}$.

Since $0 \leq c \leq \frac{1}{4}$, then from inequality 1,

(2) $g_0 > \frac{1+\sqrt{1-4c}}{2} > \frac{1-\sqrt{1-4c}}{2}$.

Induction will be used to show that $g_n > g_{n-1}$ for each n. Define f to be the function such that for each $x \in (0,1)$, $f(x) = x - \frac{c}{1-x}$. From inequality 1 and the fact that $1 \geq g_0$, we can show that $c > 0$. Therefore, from Theorem 3.4, since $g_0 > \frac{1+\sqrt{1-4c}}{2}$, then $0 > f(g_0) = g_0 - \frac{c}{1-g_0} = g_0 - g_1$; hence $g_1 > g_0$. Assume $g_k > g_{k-1}$ for k an integer greater than one and show $g_{k+1} > g_k$. Since $1 \geq g_k > g_{k-1} \geq 0$, then $(1-g_{k-1}) > 0$, and since $g_k > g_{k-1}$, then

(3) $c = (1-g_k)\, g_{k+1} < (1 - g_{k-1})\, g_{k+1}$.

It follows from inequality 3 that

(4) $(1 - g_{k-1})g_k = c < (1 - g_{k-1})g_{k+1}$.

Since $(1 - g_{k-1}) > 0$, then from 4, $g_k < g_{k+1}$. Hence, for each n, $g_n > g_{n-1}$.

Since $\{g_n\}_{n=0}^{\infty}$ is an increasing sequence which is bounded above by 1, then by Theorem 2.2, $\lim_{n\to\infty} g_n$ exists. Since $g_0 > \frac{1+\sqrt{1-4c}}{2}$, there exists a number $\alpha > 0$ such that

(5) $g_0 = \frac{1+\sqrt{1-4c}}{2} + \alpha$.

Since $\lim_{n\to\infty} g_n$ exists and $\alpha^3 > 0$, then there exists a number $N > 0$ such that if $n > N$, then $|g_{n+1} - g_n| < \alpha^3$, (Theorem 2.5). Let $n > N$ and let

(6) $\epsilon = g_{n+1} - g_n$.

From equation 6

$$c = (1 - g_n)\, g_{n+1} = (1 - g_n)\,(\epsilon + g_n)$$

$$= \epsilon\,(1 - g_n) + g_n - g_n^{\,2}.$$

Therefore

$g_n^{\,2} - g_n = \epsilon\,(1 - g_n) - c$ and

$g_n^2 - g_n + \frac{1}{4} = \epsilon\ (1 - g_n) - c + \frac{1}{4}$. Hence,

$g_n - \frac{1}{2} = \pm \frac{\sqrt{1-4c + 4\epsilon(1-g_n)}}{2}$. However, $g_0 > \frac{1+\sqrt{1-4c}}{2} \geq \frac{1}{2}$ and

$\{g_n\}_{n=0}^{\infty}$ is increasing; hence $g_n > g_0 \geq \frac{1}{2}$ and it follows that $\left(g_n - \frac{1}{2}\right) > 0$.

Therefore, since $(1 - g_n) \leq 1$, then

$$(7)\quad \left(g_n - \frac{1}{2}\right) = + \frac{\sqrt{1-4c + 4\epsilon(1-g_n)}}{2} \leq \frac{\sqrt{1-4c + 4\epsilon}}{2}.$$

It follows from equation 4 and inequality 7 that

$$\frac{1+\sqrt{1-4c}}{2} + \alpha = g_0 < g_n \leq \frac{1+\sqrt{1-4c + 4\epsilon}}{2}.$$

Since $\epsilon = g_{n+1} - g_n = |g_{n+1} - g_n| < \alpha^3$, then from inequality 7, we obtain

$$(8)\quad \frac{1+\sqrt{1-4c}}{2} + \alpha < \frac{1+\sqrt{1-4c + 4\epsilon}}{2} < \frac{1+\sqrt{1-4c + 4\alpha^3}}{2}.$$ Therefore,

$\frac{\sqrt{1-4c}}{2} + \alpha < \frac{\sqrt{1-4c + 4\alpha^3}}{2}$ and

$\frac{1-4c}{4} + \alpha\sqrt{1-4c} + \alpha^2 < \frac{1-4c}{4} + \alpha^3$; and since $c \le \frac{1}{4}$,

$$0 \le \alpha\sqrt{1-4c} < \alpha^3 - \alpha^2$$

$$0 < \alpha - 1$$

$$\alpha > 1$$

Now, $1 \ge g_0 = \frac{1+\sqrt{1-4c}}{2} + \alpha > 1$. This is a contradiction; thus the assumption in inequality 1 is false and $g_0 \le \frac{1+\sqrt{1-4c}}{2}$.

Proof of Conclusion B by induction:

Let $c > 0$ and let $g_0 = \frac{1\pm\sqrt{1-4c}}{2}$. Since c, c, c, . . . is a chain sequence and $c > 0$, then $(1 - g_0) \ne 0$; therefore,

$$(9) \quad g_1 = \frac{c}{1-g_0}$$

$$= \frac{c}{1-\frac{1\pm\sqrt{1-4c}}{2}}$$

$$= \frac{2c}{1 \mp \sqrt{1-4c}}$$

$$= \frac{2c \pm 2c\sqrt{1-4c}}{4c}$$

$$= \frac{1 \pm \sqrt{1-4c}}{2}$$

$$= g_0.$$

Assume $g_k = g_0$ for $k > 1$. Therefore,

$$g_{k+1} = \frac{c}{1 - g_k}$$

$$= \frac{c}{1 - g_0}$$

$= g_0$, from equation 9. Hence, for $n = 1, 2, 3, \ldots,$

$g_n = g_0 = \frac{1 \pm \sqrt{1-4c}}{2}$.

Proof of Conclusion C:

Suppose that $c = 0$ and $g_0 < \frac{1 + \sqrt{1-4c}}{2}$; then, $g_0 < 1$. Therefore, for $n = 1, 2, 3, \ldots, g_n = 0$. Hence, $\lim_{n\to\infty} g_n = 0 = \frac{1 - \sqrt{1-4c}}{2}$.

Three cases will be used in order to prove Conclusion C for $c > 0$.

(1) $\frac{1 - \sqrt{1-4c}}{2} < g_0 < \frac{1 + \sqrt{1-4c}}{2}$, and $c < \frac{1}{4}$,

(2) $g_0 < \frac{1 - \sqrt{1-4c}}{2}$ and $c \leq \frac{1}{4}$, and

(3) $g_0 = \frac{1 - \sqrt{1-4c}}{2}$ and $c \leq \frac{1}{4}$.

Case 1:

Let $\frac{1 - \sqrt{1-4c}}{2} < g_0 < \frac{1 + \sqrt{1-4c}}{2}$ and let $0 < c < \frac{1}{4}$. A proof by

induction will be used to show that $g_n < g_{n-1}$ for $n = 1, 2, 3, \ldots$.
Define f to be the function such that for each $x \epsilon (0,1)$, $f(x) = x - \frac{c}{1-x}$.
Since $\frac{1-\sqrt{1-4c}}{2} < g_0 < \frac{1+\sqrt{1-4c}}{2}$, then by Theorem 3.4,
$0 < f(g_0) = g_0 - \frac{c}{1-g_0} = g_0 - g_1$, and therefore $g_1 < g_0$. Assume
$g_k < g_{k-1}$ for $k > 1$. Now,

$$(10) \quad (1 - g_{k-1})\, g_k = c = (1 - g_k)\, g_{k+1}$$

$$> (1 - g_{k-1})\, g_{k+1}.$$

Since $c > 0$, then $(1 - g_{k-1}) \neq 0$; therefore it follows from inequality 10 that $g_k > g_{k+1}$. Hence by induction, $g_n < g_{n-1}$ for each n.

Since $0 \leq g_n$ and $g_n < g_{n-1}$ for each n, then $\{g_n\}_{n=0}^{\infty}$ is a decreasing sequence which is bounded below; therefore, by Theorem 2.2, $\lim_{n\to\infty} g_n$ exists and is the greatest lower bound of $\{g_n\}_{n=0}^{\infty}$.

In order to show that the g.l.b. of $\{g_n\}_{n=0}^{\infty}$ is $\frac{1-\sqrt{1-4c}}{2}$, let p = the g.l.b. of $\{g_n\}_{n=0}^{\infty}$. Since $\lim_{n\to\infty} g_n = p$, it follows from Theorem 3.3, that

$$0 = \lim_{n\to\infty} (g_n - g_{n+1})$$

$$= \lim_{n\to\infty}\left(g_n - \frac{c}{1-g_n}\right)$$

$$= p - \frac{c}{1-p}.$$

Therefore $p = \frac{1 \pm \sqrt{1-4c}}{2}$. Since $g_0 < \frac{1 + \sqrt{1-4c}}{2}$ and $g_n < g_{n-1}$ and g.l.b. of $\{g_n\}_{n=0}^{\infty}$ is p, then $p \leq g_n < g_0 < \frac{1 + \sqrt{1-4c}}{2}$. Hence $p \neq \frac{1 + \sqrt{1-4c}}{2}$ and thus $p = \frac{1 - \sqrt{1-4c}}{2}$.

Case 2:

Let $g_0 < \frac{1 - \sqrt{1-4c}}{2}$ and let $0 < c \leq \frac{1}{4}$. An induction proof will be used to show that $g_n > g_{n-1}$ for each n. Define f to be the function such that for each $x \in (0,1)$, $f(x) = x - \frac{c}{1-x}$. Since $g_0 < \frac{1 - \sqrt{1-4c}}{2}$, and since $c > 0$, then $(1 - g_0) \neq 0$, therefore, by Theorem 3.4,

$$0 > f(g_0) = g_0 - \frac{c}{1-g_0}$$

$$= g_0 - g_1.$$

Hence $g_1 > g_0$. Assume that $g_k > g_{k-1}$ for $k > 1$. Now,

(11) $(1 - g_{k-1})\, g_k = c = (1 - g_k)\, g_{k+1}$

$$< (1 - g_{k-1})\, g_{k+1}.$$

Since $c > 0$, then $(1 - g_{k-1}) \neq 0$ and therefore, from inequality 11, $g_k < g_{k+1}$. Hence, by induction, $g_n < g_{n+1}$ for each n.

Since $g_n \leq 1$ and $g_n < g_{n+1}$ for each n, then $\{g_n\}_{n=0}^{\infty}$ is an increasing sequence which is bounded above; therefore by Theorem 2.1, $\lim_{n\to\infty} g_n$ exists and is the least upper bound of $\{g_n\}_{n=0}^{\infty}$.

We will show that $\frac{1 - \sqrt{1-4c}}{2}$ is the least upper bound of $\{g_n\}_{n=0}^{\infty}$ and then use this fact to show $\lim_{n\to\infty} g_n = \frac{1 - \sqrt{1-4c}}{2}$. An indirect proof will be used to show that if $x \in \{g_n\}_{n=0}^{\infty}$, then $x \leq \frac{1 - \sqrt{1-4c}}{2}$. Suppose there exists a number $x \in \{g_n\}_{n=0}^{\infty}$ such that $x > \frac{1 - \sqrt{1-4c}}{2}$. Since $\{g_n\}_{n=0}^{\infty}$ is an increasing sequence, and since there exists a number $x \in \{g_n\}_{n=0}^{\infty}$ such that $x > \frac{1 - \sqrt{1-4c}}{2}$, then there exists a first number g_n, (where $n \geq 1$) such that $g_n > \frac{1 - \sqrt{1-4c}}{2}$ and $g_{n-1} \leq \frac{1 - \sqrt{1-4c}}{2}$. Since $c > 0$, then $(1 - g_{n-1}) \neq 0$ and therefore,

$$g_n = \frac{c}{1-g_{n-1}}$$

$$\leq \frac{c}{1 - \frac{1-\sqrt{1-4c}}{2}}$$

$$= \frac{1 - \sqrt{1-4c}}{2}$$

$$< g_n.$$

Hence the contradiction $g_n < g_n$ is obtained. Thus, the original assumption must be false and it follows that if $x \in \{g_n\}_{n=0}^{\infty}$, then $x \leq \frac{1 - \sqrt{1-4c}}{2}$.

Let p = l.u.b. of $\{g_n\}_{n=0}^{\infty}$. Suppose $p > \frac{1 - \sqrt{1-4c}}{2}$; from Definition 2.1, there exists a number $g_a \in \{g_n\}_{n=0}^{\infty}$ such that $g_a > \frac{1 - \sqrt{1-4c}}{2}$. However, this contradicts the statement in the preceding paragraph that for each $x \in \{g_n\}_{n=0}^{\infty}$, $x \leq \frac{1 - \sqrt{1-4c}}{2}$. Therefore $p \not> \frac{1 - \sqrt{1-4c}}{2}$.

Suppose $p < \frac{1 - \sqrt{1-4c}}{2}$. Since $\lim_{n\to\infty} g_n$ exists, from Theorem 3.3,

$$0 = \lim_{n\to\infty} (g_{n+1} - g_n)$$

$$= \lim_{n\to\infty}\left(\frac{c}{1-g_n} - g_n\right)$$

$$=\left(\frac{c}{1-p} - p\right)$$

$$= -\left(p - \frac{c}{1-p}\right)$$

$\neq 0$ by Theorem 3.4, since $p < \frac{1 - \sqrt{1-4c}}{2} \leq \frac{1 + \sqrt{1-4c}}{2}$,

then $p \neq \frac{1 \pm \sqrt{1-4c}}{2}$. This gives a contradiction and it follows that

$p = \frac{1 - \sqrt{1-4c}}{2}$; hence $\lim_{n\to\infty} g_n = p = \frac{1 - \sqrt{1-4c}}{2}$.

Case 3:

Let $g_0 = \frac{1 - \sqrt{1-4c}}{2}$. It follows, from Conclusion B of this theorem, that $g_n = g_0 = \frac{1 - \sqrt{1-4c}}{2}$ for each n. Therefore $\lim_{n\to\infty} g_n = \frac{1- \sqrt{1-4c}}{2}$.

<u>Theorem 3.6</u>: If $\lim_{n\to\infty} a_n = A > B$, then there exists a number $N > 0$ such that if $n > N$ then $a_n > \frac{A+B}{2}$.

Proof:

Since $\lim_{n\to\infty} a_n = A$ and since $\frac{A-B}{2} > 0$, then there exists a number $N > 0$ such that if $n > N$, then $|a_n - A| < \frac{A-B}{2}$. Let $n > N$, then

$$\begin{aligned} A = A - a_n + a_n &= (A - a_n) + (a_n) \\ &\leq |A - a_n| + a_n \\ &< \frac{A-B}{2} + a_n. \end{aligned}$$

Therefore, $A < \frac{A-B}{2} + a_n$ and

$A - \frac{A-B}{2} < a_n$; hence

$\frac{A+B}{2} < a_n$.

<u>Lemma 3.1</u>:

<u>Given</u>: (1) $\{c_n\}_{n=1}^{\infty}$ is a chain sequence, and

(2) $\{h_n\}_{n=1}^{\infty}$ is a sequence such that for each n, $h_n = c_{k+n}$ where k is a positive integer.

<u>Conclusion</u>: $\{h_n\}_{n=1}^{\infty}$ is a chain sequence.

Proof:

Since $\{c_n\}_{n=1}^{\infty}$ is a chain sequence, there exists a sequence $\{g_n\}_{n=0}^{\infty}$ such that if n is a positive integer then $c_n = (1 - g_{n-1})g_n$ and $0 \leq g_{n-1} \leq 1$. For each n, $h_n = c_{k+n} = (1 - g_{(k+n)-1})g_{k+n}$, and $0 \leq g_{(k+n)-1} \leq 1$; therefore, there exists a sequence $\{q_n\}_{n=0}^{\infty}$ such that for each n, $q_n = g_{k+n}$. Since $0 \leq q_{n-1} \leq 1$ and $h_n = (1 - q_{n-1})q_n$, then $\{h_n\}_{n=1}^{\infty}$ is a chain sequence.

<u>Theorem 3.7</u>:

<u>Given</u>: $\{c_n\}_{n=1}^{\infty}$ is a chain sequence and $\lim_{n\to\infty} c_n = c$.

Conclusion: $c \leq \frac{1}{4}$.

Assume the conclusion is false; then $c > \frac{1}{4}$. Since $\lim_{n \to \infty} c_n = c > \frac{1}{4}$, then by Theorem 3.6 there exists a number b and $N > 0$, such that if $p > N$ then

$$(1) \quad c_p > \frac{c + \frac{1}{4}}{2} = b > \frac{1}{4}.$$

Let $p > N$. From Lemma 3.1, $\{c_n\}_{n=p}^{\infty}$ is a chain sequence.

Define $\{b_n\}_{n=1}^{\infty}$ as a sequence of numbers such that for $n = 1, 2, 3, \ldots, b_n = b$. Since $0 < \frac{1}{4} < b < c_p$ then for $n = p, p+1, p+2, \ldots, 0 \leq b_n \leq c_n$. It follows from Theorem 3.1 that $\{b_n\}_{n=1}^{\infty}$ is a chain sequence; therefore from Theorem 3.2, $b \leq \frac{1}{4}$ which contradicts inequality 1, $\left(b > \frac{1}{4}\right)$. Hence, the original assumption is false and $c \leq \frac{1}{4}$.

Lemma 3.2: If $0 < a \leq 1$, then $a \leq \sqrt{a}$.

Proof:

An indirect proof will be used. Assume $a > \sqrt{a}$ where $0 < a \leq 1$. Therefore,

$a^2 > a$

$a^2 - a = a(a-1) > 0.$

Since $a > 0$, then $(a-1) > 0$ and $a > 1$ which contradicts the hypothesis $a \leq 1$. Therefore the assumption that $a > \sqrt{a}$ is false and $a \leq \sqrt{a}$.

<u>Theorem 3.8</u>: If c_1 and c_2 are numbers such that $0 < c_1 < c_2 \leq 1$, then the following two statements are equivalent:

(1) $c_1, c_2, c_1, c_2, c_1, c_2, \ldots,$ is a chain sequence, and

(2) $c_1 < \frac{1}{4}$ and $(1 + c_1 - c_2)^2 - 4c_1 \geq 0$.

Proof: $1 \rightarrow 2$

Since $c_1, c_2, c_1, c_2, \ldots$ is a chain sequence and $c_1 \leq c_n$ for each n, then from Theorem 3.1, $c_1, c_1, c_1, c_1, \ldots$ is a chain sequence. From Theorem 3.2, since the constant sequence $c_1, c_1, c_1, c_1, \ldots$ is a chain sequence, then $c_1 \leq \frac{1}{4}$.

An indirect proof will be used to show that $c_1 \neq \frac{1}{4}$. Suppose $c_1 = \frac{1}{4}$ and let $\{g_n\}_{n=0}^{\infty}$ be a parameter sequence for the chain sequence

$c_1, c_2, c_1, c_2, \ldots$. Induction will be used to show that $g_n \geq g_{n-1}$ for each n. Suppose that $g_1 < g_0$. Then

$$\frac{1}{4} = c_1 = (1 - g_0)g_1$$

$$< (1 - g_0)g_0$$

$$= g_0 - g_0^2 .$$

It follows that $0 > g_0^2 - g_0 + \frac{1}{4} = (g_0 - \frac{1}{2})^2 \geq 0$. This is a contradiction; therefore, $g_1 \geq g_0$. Now assume that $g_k \geq g_{k-1}$ for $k \geq 1$. If k is an odd integer, then

$$(1 - g_{k-1})g_k = c_1 < c_2 = (1 - g_k)g_{k+1}$$

$$\leq (1 - g_{k-1})g_{k+1},$$

and since $c_1 > 0$, then $(1 - g_{k-1}) \neq 0$ and $g_k < g_{k+1}$. In order to show that $g_{k+1} \geq g_k$ for each even integer k, we assume an even integer k exists such that $g_{k+1} < g_k$. It follows that

$$\frac{1}{4} = c_1 = (1 - g_k)g_{k+1}$$

$< (1 - g_k)g_k;$

therefore $0 > g_k^2 - g_k + \frac{1}{4} = (g_k - \frac{1}{2})^2 \geq 0$. Since this contradiction is obtained, then $g_{k+1} \geq g_k$, and it follows by induction that $g_n \geq g_{n-1}$ for each n.

For each positive integer n, $\frac{1}{4} \leq c_n = (1 - g_{n-1})g_n$; therefore $g_{n-1} \neq 1$ and $g_n \neq 0$ for $n = 1, 2, 3, \ldots$ and $0 < g_{n-1} < 1$ for each n.

Since $\{g_n\}_{n=0}^{\infty}$ is a non-decreasing sequence which is bounded above, then $\lim_{n\to\infty} g_n$ exists, (Theorem 2.2). Since $\lim_{n\to\infty} g_n$ exists and since $(c_2 - c_1) > 0$, then there exists a number $N > 0$ such that if $n > N$ and $m > N$, then $|g_n - g_m| < (c_2 - c_1)$, (Theorem 2.5). Let r be an even integer such that $(r - 1) > N$; it follows that

$$(1) \quad g_r - g_{r-1} = |g_r - g_{r-1}| < c_2 - c_1.$$

However, since $g_{n-1} < 1$ for each n, then $(1 - g_{n-1}) \neq 0$ and since $g_{r-2} \leq g_{r-1}$, and $(1 - g_{r-1}) < 1$, then

$$g_r - g_{r-1} = \frac{c_2}{1-g_{r-1}} - \frac{c_1}{1-g_{r-2}}$$

$$\geq \frac{c_2}{1-g_{r-1}} - \frac{c_1}{1-g_{r-1}}$$

$$= \frac{c_2 - c_1}{1-g_{r-1}}$$

$$> c_2 - c_1.$$

Therefore, $g_r - g_{r-1} > c_2 - c_1$ which contradicts inequality 1. Hence the assumption that $c_1 = \frac{1}{4}$ is false, and since $c_1 \leq \frac{1}{4}$, then $c_1 < \frac{1}{4}$.

An indirect proof will be used to show that $(1+c_1-c_2)^2 - 4c_1 \geq 0$. Suppose $(1+c_1-c_2)^2 - 4c_1 < 0$, and define h to be the function such that

$$h(x) = x - \frac{c_2}{1-\frac{c_1}{1-x}} \text{ ; then } h(x) = 0 \text{ iff}$$

$$x = \frac{1-c_1 + c_2 \pm \sqrt{(1+c_1-c_2)^2-4c_1}}{2} .$$

Since $(1+c_1-c_2)^2 - 4c_1 < 0$, then if x is a real number, $h(x) \neq 0$. From the Intermediate Value Theorem, Theorem 2.7, since h is continuous on $[0,1-c_1)$ and $h(x) \neq 0$, then for all $x \in [0,1-c_1)$, either $h(x) < 0$ or $h(x) > 0$. Therefore, since $h(0) = -\frac{c_2}{1-c_1} < 0$, then for each $x \in [0,1-c_1)$, $h(x) < 0$.

Since $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$ and since $0 < g_n < 1$, for each n, then,

$$g_{2n+2} = \frac{c_2}{1 - g_{2n+1}}$$

$$= \frac{c_2}{1 - \frac{c_1}{1-g_{2n}}},$$

and for each n such that $g_{2n} \in [0,1-c_1)$, then

$$0 > h(g_{2n}) = g_{2n} - \frac{c_2}{1 - \frac{c_1}{1-g_{2n}}} = g_{2n} - g_{2n+2}.$$

Therefore $g_{2n+2} > g_{2n}$ making $\{g_{2n}\}_{n=0}^{\infty}$ an increasing sequence which is bounded above; hence $\lim_{n\to\infty} g_{2n}$ exists and is the least upper bound p of $\{g_{2n}\}_{n=0}^{\infty}$.

In order to show that the least upper bound p of $\{g_{2n}\}_{n=0}^{\infty}$ belongs to $[0,1-c_1)$, an indirect proof will be used. Suppose $p \notin [0,1-c_1)$, then $p \geq 1-c_1$. Assume $p > 1-c_1$, then from the definition of l.u.b., Definition 2.1, there exists a number $g_{2n} \in \{g_{2n}\}_{n=0}^{\infty}$ such that $g_{2n} > 1-c_1$. Therefore,

$$c_1 > 1 - g_{2n}$$

$$> (1 - g_{2n})g_{2n+1}$$

$$= c_1.$$

Therefore the contradiction $c_1 > c_1$ is obtained. Hence $p \not> 1-c_1$.

Suppose $p = (1-c_1)$. Since $1 - \frac{c_1}{1-c_2} < 1-c_1 = p = $ l.u.b. of $\{g_{2n}\}_{n=0}^{\infty}$, then from Definition 2.1, there exists a number $g_{2a} \in \{g_{2n}\}_{n=0}^{\infty}$ such that $g_{2a} > 1 - \frac{c_1}{1-c_2}$. It follows that

$$g_{2a+2} = \frac{c_2}{1 - \frac{c_1}{1-g_{2a}}}$$

$$> \frac{c_2}{1 - \frac{c_1}{1-\left(1-\frac{c_1}{1-c_2}\right)}}$$

$$= \frac{c_2}{c_2}$$

$$= 1.$$

Therefore, $g_{2a+2} > 1$ which contradicts the fact that $g_{2a+2} \leq 1$ since $g_{2a+2} \in \{g_{2n}\}_{n=0}^{\infty}$. Hence, $p \neq 1-c_1$. Since $p \not\geq 1-c_1$, then $p \in [0,1-c_1)$ and therefore $h(p) < 0$.

From Theorem 3.3, since $\lim_{n\to\infty} g_{2n}$ exists, then $\lim_{n\to\infty} (g_{2n+2} - g_{2n}) = 0$.

However,

$$0 = \lim_{n\to\infty} (g_{2n+2} - g_{2n})$$

$$= \lim_{n\to\infty} \left(\frac{c_2}{1 - \frac{c_1}{1-g_{2n}}} - g_{2n} \right)$$

$$= \frac{c_2}{1 - \frac{c_1}{1-p}} - p$$

$$= - [h(p)]$$

$$\neq 0 \ ,$$

since $h(p) < 0$. Therefore, since the original assumption that $(1 + c_1 - c_2)^2 - 4c_1 < 0$ leads to the contradiction $0 \neq 0$, then $(1 + c_1 - c_2)^2 - 4c_1 \geq 0$.

$2 \to 1$

Define $g_{2n} = \dfrac{1-c_1 + c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}$, and

$$g_{2n+1} = \frac{1+c_1 - c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}, \text{ for } n = 0, 1, 2, \ldots .$$

Since

$$g_{2n} = \frac{1-c_1 + c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}$$

$$> \frac{1-\frac{1}{4} + 0 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}, \text{ since } c_1 < \tfrac{1}{4} \text{ and } c_2 > 0,$$

$$\geq \frac{\frac{3}{4} + 0}{2}, \text{ since } (1 + c_1 - c_2)^2 - 4c_1 \geq 0,$$

$$= \frac{3}{8}$$

> 0, then $g_{2n} > 0$.

An indirect proof will be used to show that $g_{2n} \leq 1$. Suppose

$$g_{2n} = \frac{1-c_1 + c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2} > 1; \text{ then}$$

$$1-c_1 + c_2 + \sqrt{(1+c_1-c_2)^2-4c_1} > 2 \text{ and therefore}$$

$$(1) \quad \sqrt{(1+c_1-c_2)^2-4c_1} > 1 + c_1 - c_2.$$

However, since $c_1 > 0$

$$1+c_1 - c_2 = \sqrt{(1+c_1-c_2)^2} > \sqrt{(1+c_1-c_2)^2-4c_1}$$

$$> 1+c_1 - c_2 \text{ from inequality 1.}$$

This is a contradiction and therefore the assumption that $g_{2n} > 1$ is false and $g_{2n} \leq 1$.

Since $g_{2n+1} = \dfrac{1+c_1 - c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}$

$$\geq \frac{1+c_1 - 1 + 0}{2} \quad \text{, since } c_2 \leq 1 \text{ and } (1+c_1-c_2)^2-4c_1 \geq 0,$$

$$= \frac{c_1}{2}$$

> 0, then $g_{2n+1} > 0$.

An indirect proof will be used to show that $g_{2n+1} \leq 1$. Suppose

$$g_{2n+1} = \frac{1+c_1 - c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2} > 1, \text{ then}$$

(2) $\sqrt{(1+c_1-c_2)^2-4c_1} > 1-c_1 + c_2$.

However, since $(1+c_1 - c_2) > 0$ and $4c_1 > 0$, then

$$1+c_1 - c_2 = \sqrt{(1+c_1-c_2)^2} > \sqrt{(1+c_1-c_2)^2-4c_1}$$

$> 1-c_1 + c_2$, from inequality 2.

It follows that $1+c_1 - c_2 > 1-c_1 + c_2$ and $0 > c_1 - c_2 > - c_1 + c_2 > 0$.

This is a contradiction and therefore $g_{2n+1} > 1$ is false and $g_{2n+1} \leq 1$.

The following will show that $c_n = (1 - g_{n-1})g_n$ for each n:

$$(1 - g_{2n})g_{2n+1}$$

$$= \left(1 - \frac{1-c_1 + c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}\right)\left(\frac{1+c_1 - c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}\right)$$

$$= c_1;$$

$$(1 - g_{2n+1})g_{2n+2}$$

$$= (1 - g_{2n+1})g_{2(n+1)}$$

$$= \left(1 - \frac{1+c_1 - c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}\right)\left(\frac{1-c_1 + c_2 + \sqrt{(1+c_1-c_2)^2-4c_1}}{2}\right)$$

$$= c_2.$$

Since $0 \leq g_{n-1} \leq 1$ for $n = 1, 2, 3, \ldots$, and since $c_n = (1 - g_{n-1})g_n$ for each n, then $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for $c_1, c_2, c_1, c_2, \ldots$ and therefore, $c_1, c_2, c_1, c_2, \ldots$ is a chain sequence.

<u>Theorem 3.9</u>:

<u>Given</u>: $\{c_n\}_{n=1}^{\infty}$ is an increasing chain sequence and $\{g_n\}_{n=0}^{\infty}$ is a

parameter sequence for $\{c_n\}_{n=1}^{\infty}$.

Conclusion: (A) $c_n \leq \frac{1}{4}$ for $n = 1, 2, 3, \ldots,$

(B) $g_{n-1} < \frac{1 + \sqrt{1-4c_n}}{2}$ for $n = 1, 2, 3, \ldots,$

(C) If $g_{n-1} < \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n < \frac{1 + \sqrt{1-4c_n}}{2}$ for

$n = 1, 2, 3, \ldots,$

(D) If $g_{n-1} \leq \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n < \frac{1 - \sqrt{1-4c_{n+1}}}{2}$ for

$n = 1, 2, 3, \ldots,$

(E) If $g_0 \leq \frac{1 - \sqrt{1-4c_1}}{2}$, then $\{g_n\}_{n=0}^{\infty}$ is a non-decreasing

sequence and $\lim_{n\to\infty} g_n$ exists and $\lim_{n\to\infty} g_n \leq \lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}$,

(F) If $\lim_{n\to\infty} c_n = c$ and if $\frac{1 - \sqrt{1-4c}}{2} < g_0 < \frac{1 + \sqrt{1-4c}}{2}$, then

$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}$, and

(G) If $\lim_{n\to\infty} c_n = c$ and if for some n, $0 < g_n < \frac{1 + \sqrt{1-4c}}{2}$,

then $\lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}$.

Proof:

Since $\{c_n\}_{n=1}^{\infty}$ is an increasing chain sequence and since $c_n \leq 1$ for each n, then by Theorem 2.2, $\lim_{n\to\infty} c_n$ exists and is the least upper bound c of $\{c_n\}_{n=1}^{\infty}$.

Proof of Conclusion A:

Since $\lim_{n\to\infty} c_n = c$, as shown above, then by Theorem 3.7, $c \leq \frac{1}{4}$. Also, c is the l.u.b. of $\{c_n\}_{n=1}^{\infty}$ and therefore for each $c_n \in \{c_n\}_{n=1}^{\infty}$, $c_n \leq c \leq \frac{1}{4}$. Hence $c_n \leq \frac{1}{4}$ for n = 1, 2, 3, ...

Proof of Conclusion B:

An indirect proof will be used to show that $g_{n-1} < \dfrac{1 + \sqrt{1-4c_n}}{2}$ for n = 1, 2, 3, ... Assume there exists an integer $k > 1$ such that $g_{k-1} \geq \dfrac{1 + \sqrt{1-4c_k}}{2}$. It will be shown by induction that $g_n \leq g_{n+1}$ if $n \geq k - 1$. Since $g_{k-1} \geq \dfrac{1 + \sqrt{1-4c_k}}{2}$, then by Theorem 3.4C, $0 \geq g_{k-1} - \dfrac{c_k}{1-g_{k-1}} = g_{k-1} - g_k$. Hence, $g_{k-1} \leq g_k$. Assume that $g_{m-1} \leq g_m$ for $m > k$. In order to show that $g_m \leq g_{m+1}$, suppose it is false; then $g_m > g_{m+1}$. It follows that

$$(1 - g_{m-1})g_{m+1} \geq (1 - g_m)g_{m+1}$$

$$= c_{m+1}$$

$$> c_m$$

$$= (1 - g_{m-1})g_m$$

$$> (1 - g_{m-1})g_{m+1}.$$

Therefore $(1 - g_{m-1})g_{m+1} > (1 - g_{m-1})g_{m+1}$ which is a contradiction; hence $g_m \leq g_{m+1}$ and by induction $g_n \leq g_{n+1}$ for $n \geq k - 1$.

Since $g_n \leq 1$ and $g_n \leq g_{n+1}$ for each $n \geq k - 1$, then $\{g_n\}_{n=k-1}^{\infty}$ is a non-decreasing sequence which is bounded above; therefore by Theorem 2.2, $\lim_{n\to\infty} g_n$ exists and is the least upper bound p of $\{g_n\}_{n=k-1}^{\infty}$.

Since $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} g_n = p$ and from Theorem 3.3, it follows that

$$0 = \lim_{n\to\infty} (g_{n+1} - g_n)$$

$$= \lim_{n\to\infty} \left(\frac{c_{n+1}}{1-g_n} - g_n\right)$$

$$= \frac{c}{1-p} - p,$$

and $p = \dfrac{1 \pm \sqrt{1-4c}}{2}$.

Now we will show that $p > \frac{1 \pm \sqrt{1-4c}}{2}$. Since p = l.u.b. of $\{g_n\}_{n=k-1}^{\infty}$,

then

$$p \geq g_{k-1}$$

$$\geq \frac{1 + \sqrt{1-4c_k}}{2}$$

$$> \frac{1 + \sqrt{1-4c}}{2}, \text{ since } c = \text{l.u.b. of } \{c_n\}_{n=1}^{\infty}.$$

Therefore $p > \frac{1 + \sqrt{1-4c}}{2} \geq \frac{1 - \sqrt{1-4c}}{2}$ which contradicts the statement that $p = \frac{1 \pm \sqrt{1-4c}}{2}$. Therefore the original assumption that there exists a $g_{k-1} \geq \frac{1 + \sqrt{1-4c_k}}{2}$ is false and $g_{n-1} < \frac{1 + \sqrt{1-4c_n}}{2}$ for each n.

Proof of Conclusion C:

An indirect proof will be used to show that if $g_{n-1} < \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n < \frac{1 + \sqrt{1-4c_n}}{2}$. Suppose the statement is false, then there exists a number m such that $g_{m-1} < \frac{1 - \sqrt{1-4c_m}}{2}$ and $g_m \geq \frac{1 + \sqrt{1-4c_m}}{2}$; then

$$(1) \quad c_m = (1 - g_{m-1})g_m$$

$$\geq (1 - g_{m-1})\left(\frac{1 + \sqrt{1-4c_m}}{2}\right)$$

Solving inequality 1 for g_{m-1}, we obtain $g_{m-1} \geq \frac{1 - \sqrt{1-4c_m}}{2}$. This is a contradiction of Conclusion B of this theorem; hence, if $g_{n-1} < \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n < \frac{1 + \sqrt{1-4c_n}}{2}$ for $n = 1, 2, 3, \ldots$.

Proof of Conclusion D:

An indirect proof will be used to show that if $g_{n-1} < \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n < \frac{1 - \sqrt{1-4c_{n+1}}}{2}$. Assume Conclusion D is false, then there exists an integer n such that

$$(2) \quad g_{n-1} < \frac{1 - \sqrt{1-4c_n}}{2} \text{ and}$$

$$(3) \quad g_n \geq \frac{1 - \sqrt{1-4c_{n+1}}}{2}.$$

It follows that

$$\begin{aligned}(4) \quad c_n &= (1 - g_{n-1})\, g_n \\ &\geq (1 - g_{n-1})\left(\frac{1 - \sqrt{1-4c_{n+1}}}{2}\right) \\ &= \frac{1 - \sqrt{1-4c_{n+1}}}{2} - g_{n-1}\left(\frac{1 - \sqrt{1-4c_{n+1}}}{2}\right) .\end{aligned}$$

Solving inequality 4 for g_{n-1}, we obtain

$$g_{n-1} \geq 1 - \frac{2c_n}{1 - \sqrt{1-4c_{n+1}}}$$

$$= 1 - \frac{c_n + c_n\sqrt{1-4c_{n+1}}}{2c_{n+1}}$$

$$> 1 - \frac{c_n + c_n\sqrt{1-4c_n}}{2c_n}, \text{ since } c_n < c_{n+1},$$

$$= \frac{1 - \sqrt{1-4c_n}}{2}$$

$$> g_{n-1} \text{ from inequality 2.}$$

This is a contradiction and therefore the assumption in inequality 3 is false and $g_n < \frac{1 - \sqrt{1-4c_{n+1}}}{2}$ for each n.

Proof of Conclusion E:

Let $g_0 \leq \frac{1- \sqrt{1-4c_1}}{2}$ and by induction show that $g_n \leq g_{n+1}$, for $n = 0, 1, 2, 3, \ldots$. Since $g_0 \leq \frac{1 - \sqrt{1-4c_1}}{2}$, then by Theorem 3.4C, $0 \geq g_0 - \frac{c_1}{1-g_0} = g_0 - g_1$. Therefore $g_1 \geq g_0$. Assume that $g_k \geq g_{k-1}$ where the integer $k > 1$, and suppose that $g_{k+1} < g_k$. It follows that

$$(1 - g_{k-1})g_{k+1} \geq (1 - g_k)g_{k+1}$$

$$= c_{k+1}$$

$$> c_k$$

$$= (1 - g_{k-1})g_k$$

$$> (1 - g_{k-1})g_{k+1}.$$

Therefore, $(1 - g_{k-1})g_{k+1} > (1 - g_{k-1})g_{k+1}$, a contradiction; hence $g_{k+1} \geq g_k$ and by induction, $g_{n+1} \geq g_n$ for each n. Thus $\{g_n\}_{n=0}^{\infty}$ is a non-decreasing sequence which is bounded above and by Theorem 2.2, $\lim_{n\to\infty} g_n$ exists and is the least upper bound, p, of $\{g_n\}_{n=0}^{\infty}$.

The following will show that $g_n \leq \frac{1 - \sqrt{1-4c_n}}{2}$ for each n. First an indirect proof will be used to show that $g_{n-1} \leq \frac{1 - \sqrt{1-4c_n}}{2}$ for each n. Suppose the preceding statement is false, then there exists an integer n such that

(5) $g_{n-1} > \frac{1 - \sqrt{1-4c_n}}{2}$. Since $g_n \geq g_{n-1}$, then

$$c_n = (1 - g_{n-1})g_n$$

$$\geq (1 - g_{n-1})g_{n-1} \text{ and therefore,}$$

$$(6) \quad 0 \le g_{n-1}^{\;2} - g_{n-1} + c_n$$

$$= \left(g_{n-1} - \frac{1 + \sqrt{1-4c_n}}{2}\right)\left(g_{n-1} - \frac{1 - \sqrt{1-4c_n}}{2}\right).$$

From Conclusion B of this theorem $g_{n-1} < \frac{1 + \sqrt{1-4c_n}}{2}$; therefore $\left(g_{n-1} - \frac{1 + \sqrt{1-4c_n}}{2}\right) < 0$ and from inequality 5, $\left(g_{n-1} - \frac{1 - \sqrt{1-4c_n}}{2}\right) > 0$. Hence, the product $\left(g_{n-1} - \frac{1 + \sqrt{1-4c_n}}{2}\right)\left(g_{n-1} - \frac{1 - \sqrt{1-4c_n}}{2}\right) < 0$ which contradicts inequality 6. Therefore the assumption in inequality 5 is false and

$$(7) \quad g_{n-1} \le \frac{1 - \sqrt{1-4c_n}}{2} \text{ for each } n.$$

Suppose there exists an integer n such that $g_n > \frac{1 - \sqrt{1-4c_n}}{2}$; then using this and inequality 7,

$$c_n = (1 - g_{n-1})g_n > (1 - g_{n-1})\left(\frac{1 - \sqrt{1-4c_n}}{2}\right)$$

$$\ge \left(1 - \frac{1 - \sqrt{1-4c_n}}{2}\right)\left(\frac{1 - \sqrt{1-4c_n}}{2}\right)$$

$$= c_n.$$

Therefore $c_n > c_n$ which is a contradiction and hence, for each n,

$$g_n \leq \frac{1 - \sqrt{1-4c_n}}{2}.$$

Since $\lim_{n\to\infty} g_n$ exists and $\lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}$ exists, and since $g_n \leq \frac{1 - \sqrt{1-4c_n}}{2}$ for each n, then $\lim_{n\to\infty} g_n \leq \lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}$, (Theorem 2.8).

Proof of Conclusion F:

It has been shown that $\lim_{n\to\infty} c_n = c \leq \frac{1}{4}$.

Let $\frac{1 - \sqrt{1-4c}}{2} < g_0 < \frac{1 + \sqrt{1-4c}}{2}$. From Conclusion B of this theorem, $g_{n-1} < \frac{1 + \sqrt{1-4c_n}}{2}$ for each n. Therefore, either

(8) $\frac{1 - \sqrt{1-4c_n}}{2} < g_{n-1} < \frac{1 + \sqrt{1-4c_n}}{2}$ for each n, or

(9) there exists an integer k such that

$$g_{k-1} \leq \frac{1 - \sqrt{1-4c_k}}{2}.$$

Suppose that inequality 8 is true. From Theorem 3.4B, $0 < g_{n-1} - \frac{c_n}{1-g_n} = g_{n-1} - g_n$. Therefore $g_{n-1} > g_n$ for $n = 1, 2, 3, \ldots$, and $\{g_n\}_{n=0}^{\infty}$ is a decreasing sequence which is bounded below and by Theorem 2.3, $\lim_{n\to\infty} g_n$ exists and is the greatest lower bound, p, of $\{g_n\}_{n=0}^{\infty}$.

Since $\lim_{n\to\infty} g_n$ exists, then from Theorem 3.3,

$$0 = \lim_{n\to\infty} (g_{n-1} - g_n)$$

$$= \lim_{n\to\infty} \left(g_{n-1} - \frac{c_n}{1-g_{n-1}}\right)$$

$= p - \frac{c}{1-p}$. Therefore $p = \frac{1 \pm \sqrt{1-4c}}{2}$. Since p = g.l.b. of $\{g_n\}_{n=0}$, then $p \le g_{n-1} < \frac{1 + \sqrt{1-4c}}{2}$ for each n; hence $p \ne \frac{1 + \sqrt{1-4c}}{2}$ and therefore $p = \frac{1 - \sqrt{1-4c}}{2}$, and $\lim_{n\to\infty} g_n = \frac{1 - \sqrt{1-4c}}{2} = \lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}$.

Suppose the statement made in 9 is true. Since $g_{k-1} < \frac{1 - \sqrt{1-4c_k}}{2}$, then from Conclusion E of this theorem $\lim_{n\to\infty} g_n$ exists. Therefore, let $\alpha = \lim_{n\to\infty} g_n$ and from Theorem 3.3,

$$0 = \lim_{n\to\infty} (g_{n-1} - g_n)$$

$$= \lim_{n\to\infty} (g_{n-1} - \frac{c_n}{1-g_{n-1}}$$

$= \alpha - \frac{c}{1-\alpha}$. It follows that $\alpha = \frac{1 \pm \sqrt{1-4c}}{2}$. Since $g_{k-1} < \frac{1 - \sqrt{1-4c_k}}{2}$, then from Conclusion D of this theorem $g_n < \frac{1 - \sqrt{1-4c_{n+1}}}{2} < \frac{1 - \sqrt{1-4c}}{2}$ for $n \ge k - 1$. Therefore $\frac{1 - \sqrt{1-4c}}{2}$ is an upper bound of $\{g_n\}_{n=0}^{\infty}$, and since $\frac{1 - \sqrt{1-4c}}{2} \le \frac{1 + \sqrt{1-4c}}{2}$, then the least upper bound $\alpha = \frac{1 - \sqrt{1-4c}}{2}$.

Therefore,

$$\lim_{n\to\infty} g_n = \frac{1 - \sqrt{1-4c}}{2} = \lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}.$$

Proof of Conclusion G:

Let m be an integer such that $0 < g_{m-1} < \frac{1 + \sqrt{1-4c}}{2}$. If for $n \geq m$, $\frac{1 - \sqrt{1-4c_n}}{2} < g_{n-1} < \frac{1 + \sqrt{1-4c_n}}{2}$, then by a proof similar to that following from inequality 8 in Conclusion F, $\lim_{n\to\infty} g_n = \frac{1 - \sqrt{1-4c}}{2}$. But if $0 < g_{m-1} < \frac{1 - \sqrt{1-4c}}{2}$, then it follows from a proof similar to that following from statement 9 of Conclusion F that $\lim_{n\to\infty} g_n = \frac{1 - \sqrt{1-4c}}{2}$.

<u>Theorem 3.10</u>:

<u>Given</u>: $\{c_n\}_{n=1}^{\infty}$ is a decreasing sequence and $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$.

<u>Conclusion</u>: (A) If $g_{n-1} < \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n < \frac{1 + \sqrt{1-4c_n}}{2}$ for n = 1, 2, 3, . . . ,

(B) If $\frac{1 - \sqrt{1-4c_1}}{2} \leq g_0 \leq \frac{1 + \sqrt{1-4c_1}}{2}$, then $\{g_n\}_{n=0}^{\infty}$ is a non-increasing sequence and $\lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}$.

(C) If $g_{n-1} > \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n > \frac{1 - \sqrt{1-4c_{n+1}}}{2}$, for $n = 0, 1, 2, 3, \ldots$.

Proof:

Since $\{c_n\}_{n=1}^{\infty}$ is a decreasing sequence which is bounded below by zero, then $\lim_{n\to\infty} c_n$ exists (Theorem 2.3). Let $c = \lim_{n\to\infty} c_n$.

Proof of Conclusion A:

Using an indirect proof, we will assume there exists an integer m such that $g_{m-1} < \frac{1 - \sqrt{1-4c_m}}{2}$ and $g_m \geq \frac{1 + \sqrt{1-4c_m}}{2}$; then

$$\text{(1)} \quad c_m = (1 - g_{m-1})g_m$$

$$\geq (1 - g_{m-1}) \left(\frac{1 + \sqrt{1-4c_m}}{2}\right).$$

Solving inequality 1 for g_{m-1}, we obtain

$$g_{m-1} \geq 1 - \frac{1 - \sqrt{1-4c_m}}{2}$$

$$= \frac{1 + \sqrt{1-4c_m}}{2}$$

$$\geq \frac{1 - \sqrt{1-4c_m}}{2}$$

$> g_{m-1}$. This is a contradiction; hence the assumption that

$g_{m-1} \geq \frac{1 + \sqrt{1-4c_m}}{2}$ is false and if $g_{n-1} < \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n < \frac{1+ \sqrt{1-4c_n}}{2}$,

for each n.

Proof of Conclusion B:

Let (2) $\frac{1 - \sqrt{1-4c_1}}{2} \leq g_0 \leq \frac{1 + \sqrt{1-4c_1}}{2}$. It will be shown by induction that $g_n \leq g_{n-1}$ for each n. From Theorem 3.4,

$0 \leq g_0 - \frac{c_1}{1-g_0} = g_0 - g_1$; therefore $g_1 \leq g_0$. Assume that $g_k \leq g_{k-1}$ for $k \geq 1$. Since $\{c_n\}_{n=1}^{\infty}$ is a decreasing sequence, then for each n, $c_n \neq 0$ for $0 = c_n > c_{n+1} \geq 0$. Since $g_k \leq g_{k-1}$, it follows that

$$(3) \quad (1 - g_k)g_k \geq (1 - g_{k-1})g_k$$

$$= c_k$$

$$> c_{k+1}$$

$$= (1 - g_k)g_{k+1}.$$

Since $c_n \neq 0$ for each n, then $(1 - g_k) \neq 0$ and from inequality 3, $g_k > g_{k+1}$. Therefore, by induction, for each n, $g_n \leq g_{n-1}$.

From the preceding paragraph, $\{g_n\}_{n=0}^{\infty}$ is a non-increasing sequence which is bounded below and therefore $\lim_{n\to\infty} g_n$ exists and is the greatest lower bound, p, of $\{g_n\}_{n=0}^{\infty}$, (Theorem 2.3). From Theorem 3.3, since $\lim_{n\to\infty} g_n$ exists then

$$0 = \lim_{n\to\infty} (g_n - g_{n+1})$$

$$= \lim_{n\to\infty} \left(g_n - \frac{c_{n+1}}{1-g_{n+1}}\right)$$

$= p - \frac{c}{1-p}$. Therefore, $p = \frac{1 \pm \sqrt{1-4c}}{2}$. However, since p = g.l.b. of $\{g_n\}_{n=0}^{\infty}$, by Definition 2.2,

$$p \le g_n$$

$$\le g_0$$

$$\le \frac{1 + \sqrt{1-4c_1}}{2}, \text{ from inequality 1,}$$

$$< \frac{1 + \sqrt{1-4c}}{2},$$

for $c \le c_1$ since $\lim_{n\to\infty} c_n = c =$ g.l.b. of $\{c_n\}_{n=1}^{\infty}$. Therefore $p = \frac{1 - \sqrt{1-4c}}{2}$ and it follows that $\lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{1 - \sqrt{1-4c_n}}{2}$.

Proof of Conclusion C, indirectly:

Assume that Conclusion C is false; then there exists an integer t such that

$$(4)\quad g_{t-1} > \frac{1 - \sqrt{1-4c_t}}{2} \text{ and}$$

$$g_t \leq \frac{1 - \sqrt{1-4c_{t+1}}}{2}.$$

It follows that

$$(5)\quad c_t = (1 - g_{t-1})g_t$$

$$\leq (1 - g_{t-1}) \frac{1 - \sqrt{1-4c_{t+1}}}{2}.$$

Solving inequality 5 for g_{t-1}, we obtain

$$g_{t-1} \leq 1 - \frac{2c}{1 - \sqrt{1-4c_{t+1}}}$$

$$= 1 - \frac{c_t + c_t\sqrt{1-4c_{t+1}}}{2c_{t+1}}$$

$$< 1 - \frac{c_t + c_t\sqrt{1-4c_t}}{2c_t}$$

$$= \frac{1 - \sqrt{1-4c_t}}{2}$$

$< g_{t-1}$, from inequality 4.

Therefore $g_{t-1} < g_{t-1}$, a contradiction. Hence, for each n, if

$g_{n-1} > \frac{1 - \sqrt{1-4c_n}}{2}$, then $g_n > \frac{1 - \sqrt{1-4c_{n+1}}}{2}$.

Theorem 3.11: There exists a chain sequence $\{c_n\}_{n=1}^{\infty}$ and a parameter sequence $\{g_n\}_{n=0}^{\infty}$ such that $\{g_n\}_{n=0}^{\infty}$ has uncountably many cluster points.

Proof:

Define the sequence $\{g_n\}_{n=0}^{\infty}$ as follows: $g_0 = 0$, $g_1 = 1$,

$g_2 = \frac{1}{2}$, $g_3 = \frac{1}{3}$, $g_4 = \frac{2}{3}$, $g_5 = \frac{1}{4}$, $g_6 = \frac{3}{4}$, $g_7 = \frac{1}{5}$, $g_8 = \frac{2}{5}$, $g_9 = \frac{3}{5}$,

$g_{10} = \frac{4}{5}$, $g_{11} = \frac{1}{6}$, $g_{12} = \frac{5}{6}$, $g_{13} = \frac{1}{7}$, $g_{14} = \frac{2}{7}$, $g_{15} = \frac{3}{7}$, $g_{16} = \frac{4}{7}$, ...

Continuing this process yields a sequence such that

(1) $\{g_n\}_{n=0}^{\infty}$ contains all the rational numbers between 0 and 1, and

(2) $0 \leq g_{n-1} \leq$ for each n.

Define $\{c_n\}_{n=1}^{\infty}$ as the sequence obtained by using $\{g_n\}_{n=0}^{\infty}$ as follows:

$c_1 = (1 - g_0)g_1$, $c_2 = (1 - g_1)g_2$, . . . , $c_n = (1 - g_{n-1})g_n$. Therefore $\{c_n\}_{n=1}^{\infty}$ is a chain sequence.

Since the set $\{g_n\}_{n=0}^{\infty}$ is dense in the interval [0,1], then each number of [0,1] is a cluster point of $\{g_n\}_{n=0}^{\infty}$ and therefore the set of cluster points is uncountable.

C H A P T E R   I V

MINIMAL AND MAXIMAL PARAMETER SEQUENCES

The existence of minimal and maximal parameter sequences will be established. Then these sequences will be used to determine other properties of chain sequences.

Lemma 4.1:

Given: a, b, c, and d are numbers such that $0 \leq a < 1$, $0 < b \leq 1$, $0 \leq c < 1$, $0 < d \leq 1$ and $(1-a)b = (1-c)d$.

Conclusion: If $b \geq d$, then $a \geq c$.

Proof:

An indirect proof will be used. Suppose that $b \geq d$ and that $a < c$ Since $a < c$, then

$$\begin{aligned}(1-a)b &> (1-c)b \\ &\geq (1-c)d.\end{aligned}$$

Therefore, $(1-a)b > (1-c)d$, a contradiction of the hypothesis which states that $(1-a)b = (1-c)d$. Hence, if $b \geq d$, then $a \geq c$.

Theorem 4.1:

Given: $\{c_n\}_{n=0}^{\infty}$ is a chain sequence.

Conclusion: There exist minimal and maximal parameter sequences $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ respectively for $\{c_n\}_{n=1}^{\infty}$.

Proof:

Define S_n to be the set of numbers such that $x \in S_n$ iff x is the n^{th} element of some parameter sequence for $\{c_n\}_{n=1}^{\infty}$. Since $\{c_n\}_{n=1}^{\infty}$ is a chain sequence, there exists a parameter sequence $\{g_n\}_{n=0}^{\infty}$ of $\{c_n\}_{n=1}^{\infty}$ and for $n = 1, 2, 3, \ldots, g_{n-1} \in S_n$; therefore S_n is non-empty for $n = 1, 2, 3, \ldots$. Furthermore, S_n is bounded above by 1 and below by 0. Therefore, by Axiom 2.1, S_n has a l.u.b. (t_{n-1}) and by Theorem 2.1, a g.l.b. (s_{n-1}). Since S_n is a subset of $[0,1]$, then $0 \leq s_{n-1} \leq t_{n-1} \leq 1$, (Theorem 2.5).

The following will show that $c_n = (1 - t_{n-1})t_n$ for each n. Let $0 < \epsilon < 1$. Since t_n = l.u.b. of S_{n-1} for each n, there is an element $g_a \epsilon S_{a-1}$ such that $g_a > t_a - \frac{\epsilon}{3}$. Likewise, there is an element $h_{a-1} \epsilon S_{a-2}$ such that $h_{a-1} > t_{a-1} - \frac{\epsilon}{3}$. There exist numbers g_{a-1} and h_a such that $(1 - g_{a-1})g_a = c_a$ and $(1 - h_{a-1})h_a = c_a$. Either $c_a > 0$ or $c_a = 0$. Suppose $c_a > 0$. Then $0 \leq g_{a-1} < 1$, $0 < g_a \leq 1$, $0 \leq h_{a-1} < 1$ and $0 < h_a \leq 1$. Either $g_a > h_a$, $g_a < h_a$, or $g_a = h_a$. For convenience, we will arbitrarily assume that $g_a \geq h_a$. Since $(1 - g_{a-1})g_a = c_a = (1 - h_{a-1})h_a$ then by Lemma 4.1, $g_{a-1} \geq h_{a-1}$. Therefore, $g_{a-1} \geq h_{a-1} > t_{a-1} - \frac{\epsilon}{3}$.

Let $B = t_{a-1} - g_{a-1}$, and since $g_{a-1} > t_{a-1} - \frac{\epsilon}{3}$, then $B = t_{a-1} - g_{a-1} < t_{a-1} - t_{a-1} + \frac{\epsilon}{3}$; hence

(1) $B < \frac{\epsilon}{3}$.

Let $\alpha = t_a - g_a$ and since $g_a > t_a - \frac{\epsilon}{3}$, then $\alpha = t_a - g_a < t_a - t_a + \frac{\epsilon}{3}$; hence

(2) $\alpha < \frac{\epsilon}{3}$. Since $t_{a-1} = B + g_{a-1}$ and $t_a = \alpha + g_a$, then

$$|(1 - t_{a-1})t_a - c_a| = |[1 - (g_{a-1} + B)](g_a + \alpha) - c_a|$$

$$= |(1 - g_{a-1})g_a - Bg_a + (1 - g_{a-1})\alpha - \alpha B - c_a|$$

$$\leq |(1 - g_{a-1})g_a - c_a| + |Bg_a| + |(1 - g_{a-1})\alpha| + |\alpha B|$$

$$= 0 + |Bg_a| + |(1 - g_{a-1})\alpha| + |\alpha B|$$

$$\leq B + \alpha + \alpha B, \text{ since } g_a \leq 1 \text{ and } (1 - g_{a-1}) \leq 1$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon^2}{9}, \text{ from inequalities 1 and 2,}$$

$$< \frac{2\epsilon}{3} + \frac{\epsilon}{3}, \text{ since } 0 < \epsilon < 1, \text{ then } \epsilon^2 < \epsilon,$$

$$= \epsilon.$$

Since $(1 - t_{a-1})t_a$ and c_a are numbers, and since $|(1 - t_{a-1})t_a - c_a| < \epsilon$, then by Theorem 2.4, $(1 - t_{a-1})t_a = c_a$.

Suppose that $c_a = 0$; then since $c_a = (1 - g_{a-1})g_a$, one of the following statements is true:

(A) $g_a = 0$ and $g_{a-1} = 1$ or

(B) $g_{a-1} = 1$ and $g_a \neq 0$, or

(C) $g_{a-1} \neq 1$ and $g_a = 0$.

Suppose A is true. Since $t_{a-1} = $ l.u.b. of S_a, then $1 \geq t_{a-1} \geq g_{a-1} = 1$. Therefore, $t_{a-1} = 1$ and $(1-t_{a-1})t_a = 0 = (1 - g_{a-1})g_a = c_a$.

This same argument holds when $g_a \neq 0$ and $g_{a-1} = 1$, (B).

Suppose C is true. Either all elements belonging to S_{a-1} are zero or at least one element belonging to $S_{a-1} \neq 0$. If all elements in S_{a-1} are zero, then the l.u.b. of $S_{a-1} = t_a = 0$ and $(1 - t_{a-1})t_a = 0 = c_a$. Suppose there exists one element $x_a \in S_{a-1}$ such that $x_a \neq 0$. Since $(1 - x_{a-1})x_a = c_a = 0$, then $x_{a-1} = 1$ and therefore $t_{a-1} = 1$. Hence $c_a = (1 - t_{a-1})t_a = 0$.

Since $c_a = (1 - t_{a-1})t_a$ for $c_a > 0$ or $c_a = 0$, then for each n, $c_n = (1 - t_{n-1})t_n$, and since $0 \leq t_{n-1} \leq 1$ for each n, then $\{t_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$. Also, for each n, t_n = l.u.b. of S_{n-1}; therefore, $t_n \geq b_n$, where $b_n \in \{b_n\}_{n=0}^{\infty}$, (any parameter sequence for $\{c_n\}_{n=1}^{\infty}$). Therefore $\{t_n\}_{n=0}^{\infty}$ is the maximum parameter sequence for $\{c_n\}_{n=1}^{\infty}$.

Using the g.l.b. Theorem 2.1, and similar steps, we can show that $c_n = (1 - s_{n-1})s_n$ and $0 \leq s_{n-1} \leq 1$ for each n. Therefore $\{s_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$. Also, since s_n = g.l.b of S_n

for each n, then $s_n \le g_n$ where $g_n \in \{g_n\}_{n=0}^{\infty}$, (any parameter sequence for $\{c_n\}_{n=1}^{\infty}$).

Theorem 4.2:

Given: $\{c_n\}_{n=0}^{\infty}$ is a chain sequence with minimal and maximal parameter sequences $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$.

Conclusion: If $m_0 \le b \le M_0$, then $\{c_n\}_{n=1}^{\infty}$ has a parameter sequence such that $g_0 = b$.

Proof:

Let $m_0 \le b \le M_0$ and let $\{g_n\}_{n=0}^{\infty}$ be the sequence of numbers such that $g_0 = b$ and if $n \ne 0$ then

$$g_n = \begin{cases} 0, \text{ if } c_n = 0, \\ \dfrac{c_n}{1-g_{n-1}} \text{ if } c_n \ne 0. \end{cases}$$

A proof by induction will be used to show that $0 \le g_n \le M_n$ for each n. Since $0 \le m_0 \le g_0 = b < M_0$, then $0 \le g_0 \le M_0$. Suppose $0 \le g_k \le M_k$ for $k \ge 1$. If $c_{k+1} = 0$, then $g_{k+1} = 0$ and since

$M_{k+1} \geq 0 = g_{k+1}$ then $0 = g_{k+1} \leq M_{k+1}$. If $c_{k+1} \neq 0$, then

$$(1)\quad (1 - g_k)g_{k+1} = c_{k+1} = (1 - M_k)M_{k+1}$$

$$\leq (1 - g_k)M_{k+1}.$$

Therefore, since $c_{k+1} \neq 0$, then $(1 - g_k) \neq 0$ and it follows from inequality 1 that $g_{k+1} \leq M_{k+1}$. Also, since $c_{k+1} \neq 0$, then $g_{k+1} > 0$; therefore $0 < g_{k+1} \leq M_{k+1}$ and by induction $0 \leq g_n \leq M_n$ for each n.

Since $\{M_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$, then for each n, $M_n \leq 1$ and therefore $0 \leq g_n \leq M_n \leq 1$. Hence the sequence $\{g_n\}_{n=0}^{\infty}$ satisfies the conditions that for each n, $c_n = (1 - g_{n-1})g_n$ and $0 \leq g_{n-1} \leq 1$; therefore $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$.

Lemma 4.2:

Given: $\{c_n\}_{n=1}^{\infty}$ is a positive term chain sequence; both $\{g_n\}_{n=0}^{\infty}$ and $\{h_n\}_{n=0}^{\infty}$ are parameter sequences for $\{c_n\}_{n=1}^{\infty}$ and $h_0 = g_0$.

Conclusion: If n is a positive integer, then $h_n = g_n$.

Proof:

An induction proof will be used. From the hypothesis $h_0 = g_0$. Assume $h_k = g_k$ for $k \geq 1$. It follows that

$$(1) \quad (1 - h_k)h_{k+1} = c_{k+1} = (1 - g_k)g_{k+1}$$

$$= (1 - h_k)g_{k+1}.$$

Since $\{c_n\}_{n=1}^{\infty}$ is a positive term chain sequence, then $(1 - h_k) \neq 0$ and from equation 1, $h_{k+1} = g_{k+1}$. It follows by induction that $h_n = g_n$ for each n.

Theorem 4.3: If $\{c_n\}_{n=1}^{\infty}$ is a positive term chain sequence, the following two statements are equivalent:

(1) the maximal parameter M_0 is zero, and

(2) $\{c_n\}_{n=1}^{\infty}$ has exactly one parameter sequence.

Proof: $1 \rightarrow 2$

Since $\{c_n\}_{n=1}^{\infty}$ is a chain sequence, by Theorem 4.1, there exists a parameter sequence $\{m_n\}_{n=0}^{\infty}$ and a parameter sequence $\{M_n\}_{n=0}^{\infty}$ such that if $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$, then

$m_n \leq g_n \leq M_n$ for each n. Let $\{g_n\}_{n=0}^{\infty}$ be a parameter sequence for $\{c_n\}_{n=1}^{\infty}$; then $0 \leq m_0 \leq g_0 \leq M_0 = 0$, and therefore $g_0 = 0$. It follows that for any parameter sequence $\{h_n\}_{n=0}^{\infty}$ of $\{c_n\}_{n=1}^{\infty}$, $h_0 = 0$; therefore, from Lemma 4.3, if n is a positive integer, then $h_n = g_n$; hence $\{c_n\}_{n=1}^{\infty}$ has exactly one parameter sequence.

$2 \rightarrow 1$

An indirect proof will be used to show that Statement 2 implies 1. Suppose that $M_0 \neq 0$; then $M_0 > 0$.

Define $\{g_n\}_{n=0}^{\infty}$ as the sequence of numbers such that $g_0 = 0$ and if $n \neq 0$, then $g_n = \frac{c_n}{1-g_{n-1}}$. (Since $c_n > 0$, then $(1 - g_{n-1}) \neq 0$).

A proof by induction will be used to show that $g_n < M_n$ for each n. By definition, $g_0 = 0$ and from the denial $0 < M_0$; therefore, $g_0 < M_0$. Assume that $g_k < M_k$ for the integer $k \geq 1$. Then

$$(1 - g_k)g_{k+1} = c_{k+1} = (1 - M_k)M_{k+1}$$

$$< (1 - g_k)M_{k+1};$$

therefore, since $c_{k+1} > 0$, then $(1 - g_k) \neq 0$ and $g_{k+1} < M_{k+1}$. Thus by induction, $g_n < M_n$ for each n.

Since $0 < c_n = (1 - g_{n-1})g_n$, then $g_n > 0$ for each n. Therefore, for each n, $0 < g_n < M_n \leq 1$, and $c_n = (1 - g_{n-1})g_n$; hence $\{g_n\}_{n=0}^{\infty}$ is a parameter sequence for $\{c_n\}_{n=1}^{\infty}$.

Since $\{M_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ are parameter sequences for $\{c_n\}_{n=1}^{\infty}$, then $\{c_n\}_{n=1}^{\infty}$ has at least two parameter sequences, which contradicts the statement in the hypothesis that $\{c_n\}_{n=1}^{\infty}$ has exactly one parameter. Therefore the assumption that $M_0 \neq 0$ is false and $M_0 = 0$.

B I B L I O G R A P H Y

1. Fulks, Watson, Advanced Calculus, John Wiley and Sons, Incorporated, New York, 1961.

2. Wall, D. S., Analytic Theory of Continued Fractions, D. Van Nostrand Company, Incorporated, New York, 1948.