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# STRUCTURAL STABILITY FOR BRINKMAN-FORCHHEIMER EQUATIONS

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Abstract. In this paper, we obtain the continuous dependence and convergence results for the Brinkman and Forchheimer coefficients of a differential equation that models the flow of fluid in a saturated porous medium.

## 1. Introduction

The concept of structural stability in which the study of continuous dependence (or stability) is on changes in the model itself rather than the initial data, has been the subject of much recent study. Many references to the work of the nature are given in the monograph of Ames and Straughan [1], which stress that continuous dependence on the model itself, or structural stability, is every bit as important as stability with respect to perturbations of the initial data. In particularly, the stability of flow in porous media has attracted much more attention in the literature; see [3, 4, 5, 6, 7, 8] and their references.

In this paper, we are interested in the Brinkman-Forchheimer equations governing the flow of fluid in a saturated porous medium,

$$\begin{gathered}\frac{\partial u_i}{\partial t} = \lambda\Delta u_i - a u_i - b|u|u_i - p_{,i} \\ \frac{\partial u_i}{\partial x_i} = 0\end{gathered} \tag{1.1}$$

where $u_i$ is the average fluid velocity in the porous medium, a is the Darcy coefficient, $\lambda$ is the Brinkman coefficient, $b$ is the Forchheimer coefficient, and $p$ is the pressure. $\lambda$, $a$ and $b$ are positive constants. Here also $\Delta$ is the laplace operator, and $\|\cdot\|$ denotes the norm of $L^2$.

We assume that $\Omega$ is a bounded, simply connected domain with boundary $\partial\Omega$ in $R^3$. Associated with (1.1), we imposed the boundary condition

$$u_i = 0 \quad \text{on } \partial\Omega \times \{t > 0\} \tag{1.2}$$

and the initial condition

$$u_i(x,0) = f_i(x). \tag{1.3}$$

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In this paper, the usual summation convection is employed with repeated Latin subscripts summed from 1 to 3. The comma is used to indicated partial differentiation and the differentiation with respect to the direction $x_k$ is denoted as "$,k$".

## 2. Continuous dependence for the Brinkman coefficient

To study the continuous dependence on $\lambda$, we let $(u_i, p)$ and $(v_i, q)$ solve the following boundary initial-value problems for different Brinkman coefficients $\lambda_1$ and $\lambda_2$,

$$\begin{gathered}\frac{\partial u_i}{\partial t} = \lambda_1 \Delta u_i - a u_i - b|u|u_i - p_{,i} \quad \text{in } \Omega \times \{t > 0\}\\ \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega \times \{t > 0\}\\ u_i = 0 \quad \text{on } \partial\Omega \times \{t > 0\}\\ u_i(x, 0) = f_i(x), \quad x \in \Omega\end{gathered} \tag{2.1}$$

and

$$\begin{gathered}\frac{\partial v_i}{\partial t} = \lambda_2 \Delta v_i - a v_i - b|v|v_i - q_{,i} \quad \text{in } \Omega \times \{t > 0\}\\ \frac{\partial v_i}{\partial x_i} = 0 \quad \text{in } \Omega \times \{t > 0\}\\ v_i = 0 \quad \text{on } \partial\Omega \times \{t > 0\}\\ v_i(x, 0) = f_i(x), \quad x \in \Omega\end{gathered} \tag{2.2}$$

We define the difference variables $w_i$, $\pi$ and $\lambda$ by

$$w_i = u_i - v_i, \pi = p - q, \lambda = \lambda_1 - \lambda_2 \tag{2.3}$$

and then $(w_i, \pi)$ satisfies the boundary initial-value problem

$$\begin{gathered}\frac{\partial w_i}{\partial t} = \lambda_1 \Delta u_i - \lambda_2 \Delta v_i - a w_i - b(|u|u_i - |v|v_i) - \pi_{,i} \quad \text{in } \Omega \times \{t > 0\}\\ \frac{\partial w_i}{\partial x_i} = 0 \quad \text{in } \Omega \times \{t > 0\}\\ w_i = 0 \quad \text{on } \partial\Omega \times \{t > 0\}\\ w_i(x, 0) = 0, \quad x \in \Omega\end{gathered} \tag{2.4}$$

Multiplying $(2.4)_1$ by $w_i$ and integrating over $\Omega$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2\\ &= -\int_\Omega \lambda_1 \nabla u_i \nabla w_i dx + \int_\Omega \lambda_2 \nabla v_i \nabla w_i dx - a\|w\|^2 - b\int_\Omega (|u|u_i - |v|v_i) w_i dx\\ &= -\int_\Omega \lambda \nabla u_i \nabla w_i dx + \int_\Omega \lambda_2 \nabla w_i \nabla w_i dx - a\|w\|^2 - b\int_\Omega (|u|u_i - |v|v_i) w_i dx\end{aligned} \tag{2.5}$$

Since

$$\begin{aligned}&(|u|u_i - |v|v_i)w_i\\ &= \frac{1}{2}|u|(u_i - v_i + v_i)w_i - \frac{1}{2}|v|v_i w_i + \frac{1}{2}|u|u_i w_i + \frac{1}{2}|v|w_i(u_i - v_i - u_i)\\ &= \frac{1}{2}(|u| + |v|)w_i w_i dx + \frac{1}{2}(|u| - |v|)^2(|u| + |v|)\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), using the Cauchy-Schwarz inequality and dropping some negative items, we obtain

$$\frac{d}{dt}\|w\|^2 \le \frac{\lambda^2}{2\lambda_2}\|\nabla u\|^2$$

Integrating from 0 to t, we obtain

$$\|w\|^2 \le \frac{\lambda^2}{2\lambda_2}\int_0^t \|\nabla u\|^2 d\eta \tag{2.7}$$

Our next step is to bound $\int_0^t \|\nabla u\|^2 d\eta$. Multiplying $(2.1)_1$ by $u_i$ and integrating over $\Omega$, we see that

$$\frac{d}{dt}\|u\|^2 + 2\lambda_1\|\nabla u\|^2 \le 0$$

Integrating from 0 to t, we obtain

$$\|u\|^2 + 2\lambda_1\int_0^t \|\nabla u\|^2 d\eta \le \|f\|^2 \tag{2.8}$$

thus

$$\int_0^t \|\nabla u\|^2 d\eta \le \frac{\|f\|^2}{2\lambda_1} \tag{2.9}$$

Combining (2.7) and (2.9), we obtain

$$\|w\|^2 \le \frac{\lambda^2}{4\lambda_1\lambda_2}\|f\|^2 \tag{2.10}$$

Inequality (2.10) shows the continuous dependence on $\lambda$. However, the convergence result can't follow from (2.10) as $\lambda_1 \to 0, \lambda_2 = 0$.

## 3. Convergence as $\lambda_1 \to 0$, $\lambda_2 = 0$

Let $(u_i, p)$ be a solution of (2.1) with $\lambda_1 \to 0$, $(v_i, p)$ be a solution of (2.1) with $\lambda_1 \to 0$, $w_i, \pi$ are defined the same as in section 2.

$$\begin{gathered}
\frac{\partial w_i}{\partial t} = \lambda_1\Delta u_i - aw_i - b(|u|u_i - |v|v_i) - \pi_{,i} \quad \text{in } \Omega\times\{t>0\} \\
\frac{\partial w_i}{\partial x_i} = 0 \quad \text{in } \Omega\times\{t>0\} \\
w_i = 0 \quad \text{on } \partial\Omega\times\{t>0\} \\
w_i(x,0) = 0, \quad x\in\Omega
\end{gathered} \tag{3.1}$$

Multiplying $(3.1)_1$ by $w_i$ and integrating over $\Omega$, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|^2 \\
&= -\lambda_1\int_\Omega \nabla u_i\nabla w_i dx - a\|w\|^2 - b\int_\Omega (|u|u_i - |v|v_i)w_i dx \\
&= -\lambda_1\int_\Omega \nabla u_i\nabla u_i dx + \lambda_1\int_\Omega \nabla u_i\nabla v_i dx - a\|w\|^2 - b\int_\Omega (|u|u_i - |v|v_i)w_i dx \\
&\le \frac{\lambda_1}{4}\int_\Omega \nabla v_i\nabla v_i dx - a\|w\|^2 .
\end{aligned} \tag{3.2}$$

The next step is to bound $\int_\Omega v_{i,j}v_{i,j}dx$. We know

$$\int_\Omega v_{i,j}v_{i,j}dx = \int_\Omega v_{i,j}(v_{i,j} - v_{j,i})dx . \tag{3.3}$$

Using $(2.2)_1$ with $\lambda_2 = 0$, we get

$$\begin{aligned}
&\int_0^t \int_\Omega v_{i,j}(v_{i,j} - v_{j,i}) ds\, d\eta \\
&= \frac{1}{a}\int_0^t \int_\Omega (v_{i,j} - v_{j,i})(-v_{i,jt} - b(|v|v_i)_{,j} - q_{,ij}) dx\, d\eta \\
&= -\frac{1}{a}\int_0^t \int_\Omega (v_{i,j} - v_{j,i}) v_{i,jt} ds\, d\eta - \frac{b}{a}\int_0^t \int_\Omega (v_{i,j} - v_{j,i}) \\
&\quad \times (\frac{v_k v_{k,j}}{|v|} v_i + |v| v_{i,j}) ds\, d\eta + \frac{1}{a}\int_0^t \int_\Omega v_{i,ij} q_{,j} ds\, d\eta \\
&\quad - \frac{1}{a}\int_0^t \int_\Omega v_{j,ji} q_{,i} ds\, d\eta - \frac{1}{a}\int_0^t \int_{\partial\Omega} v_{i,j} q_{,j} n_i ds d\eta + \frac{1}{a}\int_0^t \int_{\partial\Omega} v_{j,i} q_{,i} n_j ds d\eta \\
&= -\frac{1}{2a}\int_\Omega v_{i,j} v_{i,j} dx|_{\eta=t} + \frac{1}{2a}\int_\Omega f_{i,j} f_{i,j} dx \\
&\quad - \frac{b}{a}\int_0^t \int_\Omega \frac{v_k v_{k,j} v_i v_{i,j}}{|v|} ds\, d\eta - \frac{b}{a}\int_0^t \int_\Omega v_{i,j} |v| v_{i,j} ds\, d\eta
\end{aligned} \tag{3.4}$$

thus

$$\int_0^t \int_\Omega v_{i,j} v_{i,j} ds\, d\eta \le \frac{1}{2a}\int_\Omega f_{i,j} f_{i,j} dx \tag{3.5}$$

Combining (3.2), (3.3) and (3.5), we get

$$\|w\|^2 \le \frac{\lambda_1}{8a}\int_\Omega f_{i,j} f_{i,j} ds\, d\eta$$

This inequality demonstrates the convergence $u \to v$ when $\lambda_1 \to 0$, $\lambda_2 = 0$.

## 4. Continuous dependence for the Forchheimer coefficient b

To study continuous dependence on $b$, we let $(u_i, p)$ and $(v_i, q)$ solve the following boundary initial-value problem for different coefficients $b_1$ and $b_2$.

$$\begin{gathered}
\frac{\partial u_i}{\partial t} = \lambda \Delta u_i - a u_i - b_1 |u| u_i - p_{,i} \quad \text{in } \Omega \times \{t > 0\} \\
\frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega \times \{t > 0\} \\
u_i = 0 \quad \text{on } \partial\Omega \times \{t > 0\} \\
u_i(x, 0) = f_i(x), \quad x \in \Omega
\end{gathered} \tag{4.1}$$

and

$$\begin{gathered}
\frac{\partial v_i}{\partial t} = \lambda \Delta v_i - a v_i - b_2 |v| v_i - q_{,i} \quad \text{in } \Omega \times \{t > 0\} \\
\frac{\partial v_i}{\partial x_i} = 0 \quad \text{in } \Omega \times \{t > 0\} \\
v_i = 0 \quad \text{on } \partial\Omega \times \{t > 0\} \\
v_i(x, 0) = f_i(x), \quad x \in \Omega
\end{gathered} \tag{4.2}$$

We define the difference variables

$$w_i = u_i - v_i, \quad \pi = p - q, \quad b = b_1 - b_2\,. \tag{4.3}$$

Then $(w_i, \pi)$ satisfy the boundary initial-value problem

$$\begin{gathered}
\frac{\partial w_i}{\partial t} = \lambda \Delta w_i - a w_i - (b_1|u|u_i - b_2|v|v_i) - \pi_{,i} \quad \text{in } \Omega \times \{t > 0\} \\
\frac{\partial w_i}{\partial x_i} = 0 \quad \text{in } \Omega \times \{t > 0\} \\
w_i = 0 \quad \text{on } \partial\Omega \times \{t > 0\} \\
w_i(x, 0) = 0, \quad x \in \Omega
\end{gathered} \tag{4.4}$$

Multiplying $(4.4)_1$ by $w_i$ and integrating over $\Omega$, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 = -\lambda \int_\Omega |\nabla w|^2 dx - a\|w\|^2 - \int_\Omega (b_1|u|u_i - b_2|v|v_i) w_i dx \tag{4.5}$$

For we have

$$b_1|u|u_i - b_2|v|v_i = \frac{b}{2}(|u|u_i + |v|v_i) + \tilde{b}(|u|u_i - |v|v_i) \tag{4.6}$$

where $\tilde{b} = \frac{b_1+b_2}{2}$. Combining (4.5), (4.6) and (2.6), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|^2 \\
&= -\lambda \int_\Omega |\nabla w|^2 dx - a\|w\|^2 - \frac{b}{2}\int_\Omega (|u|u_i + |v|v_i) w_i dx - \tilde{b}(|u|u_i - |v|v_i) w_i dx \\
&\le -a\|w\|^2 - \frac{b}{2}\int_\Omega (|u|u_i + |v|v_i) w_i dx - \frac{\tilde{b}}{2}\int_\Omega (|u| + |v|) w_i w_i dx
\end{aligned} \tag{4.7}$$

We then use the Cauchy-Schwarz and arithmetic geometric mean inequalities as follows

$$\frac{b}{2}\Big|\int_\Omega (|u|u_i + |v|v_i) w_i dx\Big| \le \frac{b^2}{8\tilde{b}}\int_\Omega (|u|^3 + |v|^3) dx + \frac{1}{2}\tilde{b}\int_\Omega (|u| + |v|) w_i w_i dx \tag{4.8}$$

We now employ (4.8) in (4.7), after an integration, that

$$\frac{1}{2}\|w\|^2 + a\int_0^t \|w\|^2 d\eta \le \frac{b^2}{8\tilde{b}}\int_0^t \int_\Omega (|u|^3 + |v|^3) ds\, d\eta \tag{4.9}$$

From $(4.1)_1$, one deduce that

$$\frac{1}{2}\|u\|^2 + a\int_0^t \|u\|^2 d\eta + b_1 \int_0^t \int_\Omega |u|^3 ds\, d\eta + \lambda \int_0^t \int_\Omega u_{i,j} u_{i,j} ds\, d\eta = \frac{1}{2}\|f\|^2 \tag{4.10}$$

and so

$$\int_0^t \int_\Omega |u|^3 ds\, d\eta \le \frac{1}{2b_1}\|f\|^2 \tag{4.11}$$

Similarly, from $(4.2)_1$, we can also get

$$\int_0^t \int_\Omega |v|^3 ds\, d\eta \le \frac{1}{2b_1}\|f\|^2 \tag{4.12}$$

Inserting (4.11), (4.12) in (4.10), we find

$$\|w\|^2 + 2a\int_0^t \|w\|^2 d\eta \le \frac{b^2}{4b_1 b_2}\|f\|^2 \tag{4.13}$$

This inequality establish continuous dependence on $b$, we note, however, that convergence as $b_1 \to 0$, $b_2 = 0$ is not established from (4.13).

## 5. Convergence as the Forchheimer coefficient $b_1 \to 0$ and $b_2 = 0$

Now, let $(u_i, p)$ be the solution of (4.1), and $(v_i, q)$ be the solution of (4.2) with $b_2 = 0$. The object of this section is to demonstrate convergence of the solution $u_i$ to the solution $v_i$ as $b_1 \to 0$. We also define the variables $w_i$ and $\pi$ by

$$w_i = u_i - v_i, \quad \pi = p - q \tag{5.1}$$

and then $(w_i, \pi)$ satisfy the boundary initial-value problems

$$\begin{gathered}
\frac{\partial w_i}{\partial t} = \lambda \Delta w_i - a w_i - b_1 |u| u_i - \pi_{,i} \quad \text{in } \Omega \times \{t > 0\} \\
\frac{\partial w_i}{\partial x_i} = 0 \quad \text{in } \Omega \times \{t > 0\} \\
w_i = 0 \quad \text{on } \partial\Omega \times \{t > 0\} \\
w_i(x, 0) = 0, \quad x \in \Omega
\end{gathered} \tag{5.2}$$

Multiplying by $w_i$ and integrating over $\Omega$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 = -\lambda \|\nabla w\|^2 - a \|w\|^2 - b_1 \int_\Omega |u| u_i w_i dx \tag{5.3}$$

Using the Hölder inequality, we get

$$\frac{d}{dt} \|w\|^2 \le 2b_1 \Big( \int_\Omega |u|^3 dx \Big)^{2/3} \Big( \int_\Omega |w|^3 dx \Big)^{1/3} - 2\lambda \|\nabla w\|^2 - 2a \|w\|^2 \tag{5.4}$$

For a function $F$ such that $F = 0$ on $\partial\Omega$ (see for example [2]), we have the Sobolev inequality

$$\int_\Omega |F|^4 dx \le c_1 \Big( \int_\Omega |F|^2 dx \Big)^{1/2} \Big( \int_\Omega F_{i,j} F_{i,j} dx \Big)^{3/2}$$

Then, we use the Cauchy-Schwarz inequality, to get

$$\begin{aligned}
\int_\Omega |w|^3 dx &\le \Big( \int_\Omega |w|^2 dx \Big)^{1/2} \Big( \int_\Omega |w|^4 dx \Big)^{1/2} \\
&\le c_1 \Big( \int_\Omega w_i w_i dx \Big)^{3/4} \Big( \int_\Omega w_{i,j} w_{i,j} dx \Big)^{3/4}
\end{aligned} \tag{5.5}$$

Similarly,

$$\int_\Omega |u|^3 dx \le c_1 \Big( \int_\Omega u_i u_i dx \Big)^{3/4} \Big( \int_\Omega u_{i,j} u_{i,j} dx \Big)^{3/4} \tag{5.6}$$

In view of (5.4) and (5.5), (5.3) can be rewritten as

$$\begin{aligned}
\|w\|^2 &\leq 2b_1c_1 \int_0^t \Big[\Big(\int_\Omega w_iw_idx\Big)^{1/4}\Big(\int_\Omega w_{i,j}w_{i,j}dx\Big)^{1/4}\Big(\int_\Omega u_iu_idx\Big)^{1/2} \\
&\quad\times\Big(\int_\Omega u_{i,j}u_{i,j}dx\Big)^{1/2}\Big]d\eta - 2\lambda\int_0^t \|\nabla w\|^2 d\eta - 2a\int_0^t w^2 ds\,d\eta \\
&\leq 2\lambda\int_0^t \|\nabla w\|^2 d\eta - 2a\int_0^t \|w\|^2 d\eta - 2b_1c_1\int_0^t \Big[\Big(\varepsilon_1\int_\Omega w_iw_idx\Big)^{1/4} \\
&\quad\times\Big(\varepsilon_2\int_\Omega w_{i,j}w_{i,j}dx\Big)^{1/4}\Big(\max\int_\Omega u_iu_idx\cdot\int_\Omega(\varepsilon_1\varepsilon_2)^{-1/2}u_{i,j}u_{i,j}dx\Big)^{1/2}\Big]d\eta \\
&\leq \frac{1}{4}\varepsilon_1\int_0^t\int_\Omega w_iw_ids\,d\eta + \frac{1}{4}\varepsilon_2\int_0^t\int_\Omega w_{i,j}w_{i,j}ds\,d\eta \\
&\quad+\frac{4}{2}(b_1c_1)^2\cdot(\varepsilon_1\varepsilon_2)^{-1/2}\int_0^t\int_\Omega u_{i,j}u_{i,j}ds\,d\eta \\
&\quad-2a\max\int_\Omega u_iu_idx\int_0^t\int_\Omega w^2ds\,d\eta - 2\lambda\int_0^t|\nabla w|^2d\eta - 2a\int_0^t\|w\|^2d\eta
\end{aligned}$$

If we choose $\varepsilon_1 = 8a$, $\varepsilon_2 = 8\lambda$, the above expression can be rewritten as

$$\|w\|^2 \leq \frac{b_1^2c_1^2}{4(a\lambda)^{1/2}}\max\int_\Omega u_iu_idx\cdot\int_0^t\int_\Omega u_{i,j}u_{i,j}ds\,d\eta \tag{5.7}$$

From (4.10), we have

$$\max\int_\Omega u_iu_idx \leq \|f\|^2, \quad \int_0^t\int_\Omega u_{i,j}u_{i,j}ds\,d\eta \leq \frac{1}{2\lambda}\|f\|^2;$$

therefore, from (5.7), we have

$$\|w\|^2 \leq \frac{(b_1c_1)^2}{8a^{1/2}\lambda^{3/2}}\|f\|^4,$$

which shows the desired result.

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