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EXISTENCE OF SOLUTIONS TO INFINITE SYSTEMS OF NONLINEAR INTEGRAL EQUATIONS ON THE REAL HALF-AXIS

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Abstract. In this article we study the solvability of an infinite system of integral equations of Volterra-Hammerstein type in the space of functions defined, continuous and bounded on the real half-axis with values in the sequence space l_1. We extend a known existence result for such an infinite system of integral equations and prove a theorem applicable to a wider class of considered infinite systems. We also give an example to show the usefulness of our result.

1. Introduction

The theory of integral equations is a significant branch of mathematical analysis and is closely related to the theory of ordinary and partial differential equations. Both these theories have numerous applications to physics, astronomy, chemistry, engineering and other branches of exact sciences (cf. [11, 13, 16, 20, 22], for example) because most important phenomena in the real world can be described in terms of differential or integral equations. In the study of differential equations we sometimes consider infinite systems of differential equations linked with applications (cf. [8, 11, 12, 19] and references therein). While the theory of infinite systems of differential equations creates rather developed branch of differential equations, the theory of infinite systems of integral equations is quite young field of study. Only a few papers devoted to infinite systems of integral equations have appeared so far, see [3, 4, 6, 7, 9, 18].

The present paper concerns the theory of infinite systems of nonlinear integral equations of Volterra-Hammerstein type. More precisely, we deal with infinite systems of the so-called quadratic integral equations of Volterra-Hammerstein type. We look for solutions of the mentioned infinite systems in the Banach space consisting of functions defined, continuous and bounded on the real half-axis $\mathbb{R}_+$ with values in the Banach sequence space l_1. The main tool used in our study is the technique of measures of noncompactness [1, 2, 5] and the fixed point theorem of Darbo type related to that technique (cf. [5, 10]).

The investigations in this paper continue those contained in papers [3, 9]. In particular, in this paper we generalize results from paper [9].

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2. Preliminaries

At the beginning we establish the notation and recall auxiliary facts used in the paper. Thus, we denote by $\mathbb{R}$ the set of real numbers. The symbol $\mathbb{N}$ stands for the set of natural numbers (positive integers). Apart from this we put $\mathbb{R}_+ = [0, \infty)$. Next, assume that E is a Banach space with the norm $\| \cdot \|_E$ and with the zero vector θ. Denote by $B(x, r)$ the closed ball in E centered at x and with radius r. We write B_r to denote the ball $B(\theta, r)$. Moreover, if X is a subset of the space E then we denote by $\overline{X}$ the closure of X and by $\operatorname{conv} X$ the closed convex hull of the set X. To denote the algebraic operations on subsets X and Y of the space E we use the symbols $X + Y$ and λX ($\lambda \in \mathbb{R}$). If X is a nonempty and bounded subset of E, then the symbol $\operatorname{diam} X$ stands for the diameter of X. Next, denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. We accept the following definition of the concept of a measure of noncompactness (cf. [5]).

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called a measure of noncompactness in E if it satisfies the following conditions:

(i) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(iii) $\mu(\overline{X}) = \mu(X)$.
(iv) $\mu(\operatorname{conv} X) = \mu(X)$.
(v) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
(vi) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The set $\ker \mu$ from axiom (i) is called the kernel of the measure of noncompactness μ. Let us notice that the intersection set X_∞ in axiom (vi) belongs to the family $\ker \mu$ [5]. This observation is important in our further investigations.

To distinguish some classes of measures of noncompactness we say that the measure μ is sublinear if it satisfies the following extra conditions:

(vii) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
(viii) $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.

We say that a measure of noncompactness μ has maximum property if

(ix) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

We say that a measure of noncompactness μ is full if

(x) $\ker \mu = \mathfrak{N}_E$.

A sublinear measure of noncompactness which is full and has maximum property is called a regular measure of noncompactness [5].

The first measure of noncompactness was defined in 1930 by Kuratowski [17] in the following way

$$\begin{aligned} \alpha(X) = \inf \big\{ & \varepsilon > 0 : X \text{ can be covered by a finite family of sets } X_1, X_2, \dots, X_m \\ & \text{such that } \operatorname{diam} X_i \leq \varepsilon \text{ for } i = 1, 2, \dots, m \big\}. \end{aligned}$$

The measure $\alpha(X)$ is called the Kuratowski measure of noncompactness. It is known (see [5]) that the Kuratowski measure of noncompactness is regular. The most useful measure of noncompactness is the Hausdorff measure of noncompactness which was introduced in [14, 15] by the definition

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon - \text{net in } E\}.$$

It is easily seen that the measure $\chi(X)$ is regular and equivalent to the Kuratowski measure $\alpha(X)$ [5]. Additionally, in some Banach spaces such as $C([a,b])$, l_p and c_0 we can write the exact formulas for the Hausdorff measure of noncompactness. Each such a formula involves the structure of a considered space. For example, in the sequence space l_1 consisting of all sequences (x_n) such that $\sum_{n=1}^{\infty} |x_n| < \infty$ we have the formula

$$\chi(X) = \lim_{n\to\infty} \Big\{ \sup \Big\{ \sum_{i=n}^{\infty} |x_i| : x = (x_k) \in X \Big\} \Big\}. \tag{2.1}$$

It is worth mentioning that up to now we do not know exact formulas expressing the Kuratowski measure of noncompactness. This fact shows the importance and usefulness of the Hausdorff measure of noncompactness.

Later on we will apply the fixed point theorem of Darbo type [5, 10] to prove our main result. Fundamental to this theorem is the concept of measure of noncompactness.

Theorem 2.2. *Let μ be an arbitrary measure of noncompactness in the Banach space E. Assume that Ω is a nonempty, bounded, closed and convex subset of E and $Q : \Omega \to \Omega$ is a continuous operator such that there exists a constant $k \in [0,1)$ for which $\mu(QX) \le k\mu(X)$ for an arbitrary nonempty subset X of Ω. Then the operator Q has at least one fixed point in the set Ω.*

Next, we consider the Banach space $BC(\mathbb{R}_+, E)$. This space consists of all functions $x : \mathbb{R}_+ \to E$ which are continuous and bounded on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+, E)$ is equipped with the classical supremum norm defined in the following way

$$\|x\|_\infty = \sup\{\|x(t)\|_E : t \in \mathbb{R}_+\}$$

for $x \in BC(\mathbb{R}_+, E)$. Additionally, we will assume that χ is the Hausdorff measure of noncompactness in the Banach space E. Now let us recall the definition of a measure of noncompactness in the space $BC(\mathbb{R}_+, E)$ introduced in paper [3]. To this end assume that X is an arbitrary nonempty and bounded subset of the space $BC(\mathbb{R}_+, E)$. Fix numbers $\varepsilon > 0$ and $T > 0$. For $x \in X$ we denote by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function x on the interval $[0, T]$ defined by the formula

$$\omega^T(x, \varepsilon) = \sup\{\|x(t) - x(s)\|_E : t, s \in [0, T], |t - s| \le \varepsilon\}.$$

Next, let us put

$$\begin{gathered} \omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\}, \\ \omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X, \varepsilon), \\ \omega_0(X) = \lim_{T\to\infty} \omega_0^T(X). \end{gathered} \tag{2.2}$$

For a fixed $t \in \mathbb{R}_+$ we denote by $X(t) = \{x(t) : x \in X\}$ the so-called cross-section of the set X at the point t. Obviously we have that $X(t) \in \mathfrak{M}_E$. For $T > 0$ we can define

$$\overline{\chi}_T(X) = \sup\{\chi(X(t)) : t \in [0, T]\}$$

and

$$\overline{\chi}_\infty(X) = \lim_{T\to\infty} \overline{\chi}_T(X). \tag{2.3}$$

For a fixed $T > 0$ we define

$$a_T(X) = \sup_{x\in X} \big\{ \sup\{\|x(t)\|_E : t \geq T\}\big\}$$

and next we put

$$a_\infty(X) = \lim_{T\to\infty} a_T(X). \tag{2.4}$$

Finally, gathering (2.2), (2.3) and (2.4) we can define

$$\chi_a(X) = \omega_0(X) + \overline{\chi}_\infty(X) + a_\infty(X). \tag{2.5}$$

It can be shown [3] that the function χ_a defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+,E)}$ by formula (2.5) is a measure of noncompactness in the space $BC(\mathbb{R}_+, E)$. The kernel $\ker \chi_a$ of this measure consists of nonempty and bounded subsets X of the space $BC(\mathbb{R}_+, E)$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and for any $t \in \mathbb{R}_+$ the cross-section $X(t)$ is relatively compact in the space E. Moreover, all functions belonging to X tend to zero at infinity with the same rate i.e, the following condition

$$\forall_{\varepsilon>0}\, \exists_{T>0} \forall_{x\in X} \forall_{t\geq T}\ \|x(t)\|_E \leq \varepsilon$$

is satisfied.

In what follows we will work in the Banach space $BC(\mathbb{R}_+, E)$ for $E = l_1$, where l_1 is a classical sequence space. Thus, our considerations will be carried out in the Banach space $BC(\mathbb{R}_+, l_1)$. This space is denoted by BC_1. The norm in the space BC_1 is defined in the following way

$$\|x\|_{BC_1} = \sup\{\|x(t)\|_{l_1} : t \in \mathbb{R}_+\} = \sup\Big\{\sum_{n=1}^\infty |x_n(t)| : t \in \mathbb{R}_+\Big\},$$

where $x(t) = \big(x_n(t)\big) \in BC_1$. In this case, based on formula (2.1), we can express the measure of noncompactness χ_a in the space BC_1 by formula (2.5), where the components $\omega_0(X)$, $\overline{\chi}_\infty(X)$ and $a_\infty(X)$ are represented in the following way (cf. [3]):

$$\begin{aligned}
&\omega_0(X)\\
&= \lim_{T\to\infty}\Big\{\lim_{\varepsilon\to 0}\Big\{\sup_{x=(x_n)\in X}\Big\{\sup\Big\{\sum_{n=1}^\infty |x_n(t) - x_n(s)| : t, s \in [0,T], |t-s| \leq \varepsilon\Big\}\Big\}\Big\}\Big\},
\end{aligned}$$

$$\overline{\chi}_\infty(X) = \lim_{T\to\infty}\Big\{\sup_{t\in[0,T]}\Big\{\lim_{n\to\infty}\Big\{\sup_{x=(x_k)\in X}\Big\{\sum_{k=n}^\infty |x_k(t)|\Big\}\Big\}\Big\}\Big\}, \tag{2.6}$$

$$a_\infty(X) = \lim_{T\to\infty}\Big\{\sup_{x=(x_n)\in X}\Big\{\sup_{t\geq T}\Big\{\sum_{n=1}^\infty |x_n(t)|\Big\}\Big\}\Big\}. \tag{2.7}$$

3. Solutions of an infinite system of integral equations on the real half-axis

In this section we study the existence of solutions to the infinite system of the nonlinear quadratic integral equations of Volterra-Hammerstein type of the form

$$x_n(t) = a_n(t) + f_n(t, x_1(t), x_2(t), \dots) \int_0^t k_n(t,s) g_n(s, x_1(s), x_2(s), \dots)\, ds \tag{3.1}$$

for $n = 1, 2, \dots$ and for $t \in \mathbb{R}_+$.
It is worth mentioning that in paper [9] we considered a particular case of equation (3.1) in the space BC_1 and in paper [3] we considered also the same particular case of equation (3.1) but in the space $BC_0 = BC(\mathbb{R}_+, c_0)$. Therefore our further result is a generalization of results obtained in [3, 9].

In this section we work in the Banach space $BC_1 = BC(\mathbb{R}_+, l_1)$ described in Section 2. In what follows we formulate assumptions under which infinite system of integral equations (3.1) will be investigated.

(i) The function sequence $(a_n(t))$ is an element of the space BC_1 satisfying $\lim_{t\to\infty} \sum_{n=1}^{\infty} a_n(t) = 0$.

(ii) The functions $k_n(t,s) = k_n : \mathbb{R}_+^2 \to \mathbb{R}$ are continuous on the set $\mathbb{R}_+^2$ ($n = 1, 2, \dots$). Moreover, the functions $t \to k_n(t,s)$ are locally equicontinuous on the set $\mathbb{R}_+$ uniformly with respect to $s \in \mathbb{R}_+$ i.e., the following condition is satisfied

$$\forall_{T>0}\forall_{\varepsilon>0}\,\exists_{\delta>0}\forall_{n\in\mathbb{N}}\forall_{t_1,t_2\in[0,T]}\forall_{s\in\mathbb{R}_+}\,[|t_2 - t_1| \le \delta \Rightarrow |k_n(t_2,s) - k_n(t_1,s)| \le \varepsilon].$$

(iii) There exists a constant $K_1 > 0$ such that

$$\sum_{n=1}^{\infty} \int_0^t |k_n(t,s)| ds \le K_1$$

for any $t \in \mathbb{R}_+$.

(iv) The sequence $(k_n(t,s))$ is equibounded on $\mathbb{R}_+^2$ i.e., there exists a constant $K_2 > 0$ such that $|k_n(t,s)| \le K_2$ for $t, s \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(v) The function $\sum_{n=1}^{\infty} f_n$ is defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and takes real values. Moreover, the function $t \to \sum_{n=1}^{\infty} f_n(t, x_1, x_2, \dots)$ is locally uniformly continuous on $\mathbb{R}_+$ uniformly with respect to $x = (x_n) \in l_1$ i.e., the following condition is satisfied

$$\begin{aligned} &\forall_{\varepsilon>0}\forall_{T>0}\exists_{\delta>0}\forall_{(x_i)\in l_1}\forall_{t,s\in[0,T]}\,[|t-s| \le \delta \\ &\Rightarrow \sum_{n=1}^{\infty} |f_n(t, x_1, x_2, \dots) - f_n(s, x_1, x_2, \dots)| \le \varepsilon]. \end{aligned}$$

(vi) There exists a function $l : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing on $\mathbb{R}_+$, continuous at 0 and there exist a natural number p and a nonnegative integer q such that for any $r > 0$ and for $x = (x_i)$, $y = (y_i) \in l_1$ with $\|x\|_{l_1} \le r$, $\|y\|_{l_1} \le r$ and for $t \in \mathbb{R}_+$, $n \in \mathbb{N}$, $n \ge p + 1$ the following inequality

$$|f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| \le l(r) \sum_{i=n-p}^{n+q} |x_i - y_i|$$

is satisfied. Moreover, the following inequality

$$|f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| \le l(r) \sum_{i=n}^{n+q} |x_i - y_i|$$

holds for $x = (x_i)$, $y = (y_i) \in l_1$ with $\|x\|_{l_1} \le r$, $\|y\|_{l_1} \le r$ and for $t \in \mathbb{R}_+$, $1 \le n \le p$.

(vii) There exists a function $m : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing on $\mathbb{R}_+$, continuous at 0 and there exists a sequence $(\overline{f}_n)$ of nonnegative functions, belonging to the space

BC_1 with the property $\lim_{t\to\infty}\sum_{n=1}^{\infty}\overline{f}_n(t)=0$ and such that for any $r>0$ and for each $x=(x_i)\in l_1$ with $\|x\|_{l_1}\le r$ the following inequality

$$|f_n(t,x_1,x_2,\dots)|\le \overline{f}_n(t)+m(r)\sum_{i=n}^{n+q}|x_i|$$

holds for $n=1,2,\dots$ and for $t\in\mathbb{R}_+$, where q is the number from assumption (vi).

On the basis of assumption (vii) we infer that there exists the finite constant $\overline{F}=\sup\{\sum_{n=1}^{\infty}\overline{f}_n(t):t\in\mathbb{R}_+\}$. In view of assumption (i) we may define the finite constant $A=\sup\{\sum_{n=1}^{\infty}|a_n(t)|:t\in\mathbb{R}_+\}$. Now we can formulate our other assumptions.

(viii) The function $g_n=g_n(t,x_1,x_1,\dots)$ is defined on the set $\mathbb{R}_+\times\mathbb{R}^{\infty}$ and takes real values for $n=1,2,\dots$. Moreover, the operator g defined on the set $\mathbb{R}_+\times l_1$ by the equality

$$(gx)(t)=(g_n(t,x))=(g_1(t,x),g_2(t,x),\dots)$$

transforms the set $\mathbb{R}_+\times l_1$ into l_1 and is bounded on the set $\mathbb{R}_+\times l_1$ by a positive $\overline{G}$ i.e., for any $x\in l_1$ and for each $t\in\mathbb{R}_+$ we have that $\|(gx)(t)\|_{l_1}\le\overline{G}$. Apart from this the family of functions $\{(gx)(t)\}_{t\in\mathbb{R}_+}$ is equicontinuous at every point of the space l_1. More precisely, for any arbitrarily fixed $x\in l_1$ and for a given $\varepsilon>0$ there exists $\delta>0$ such that

$$\|(gy)(t)-(gx)(t)\|_{l_1}\le\varepsilon$$

for every $t\in\mathbb{R}_+$ and for any $y\in l_1$ such that $\|y-x\|_{l_1}\le\delta$.

(ix) There exists a positive solution r_0 of the inequality

$$A+\overline{FG}K_1+(p+1)\overline{G}K_1rm(r)\le r$$

such that

$$\overline{G}K_1(p+q+1)\max\{l(r_0),m(r_0)\}<1.$$

In the sequel we will use in our considerations the following simple lemma (cf. [9]).

Lemma 3.1. *If the sequence (a_n) belongs to the space l_1 then* $\lim_{n\to\infty}\sum_{i=n}^{\infty}|a_i|=0$.

The proof of the above lemma is an immediate consequence of the Cauchy condition for real sequences. Now we formulate the main result of the paper concerning the solvability of infinite system of integral equations (3.1).

Theorem 3.2. *Under assumptions* (i)–(ix) *the infinite system of integral equations* (3.1) *has at least one solution $x(t)=(x_n(t))$ in the space $BC_1=BC(\mathbb{R}_+,l_1)$.*

Proof. We define three operators F, V, Q on the space BC_1 as follows:

$$(Fx)(t)=((F_nx)(t))=(f_n(t,x(t)))=(f_n(t,x_1(t),x_2(t),\dots)),$$
$$(Vx)(t)=((V_nx)(t))=(\int_0^t k_n(t,s)g_n(s,x_1(s),x_2(s),\dots)\,ds),$$
$$(Qx)(t)=((Q_nx)(t))=(a_n(t)+(F_nx)(t)\,(V_nx)(t))$$

for an arbitrary function $x=(x_n)\in BC_1$ and for $t\in\mathbb{R}_+$.

At the beginning we show that operator Q transforms the space BC_1 into itself. Thus, let us fix a function $x = x(t) = (x_n(t)) \in BC_1$. Then, based on assumption (vii) for an arbitrary $t \in \mathbb{R}_+$ we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} |(F_n x)(t)| &= \sum_{n=1}^{\infty} |f_n(t, x_1(t), x_2(t), \dots)| \\
&\leq \sum_{n=1}^{\infty} \big[\overline{f}_n(t) + m(\|x(t)\|_{l_1}) \sum_{i=n}^{n+p} |x_i(t)|\big] \\
&\leq \sum_{n=1}^{\infty} \overline{f}_n(t) + m(\|x\|_{BC_1}) \sum_{n=1}^{\infty} (\sum_{i=n}^{n+p} |x_i(t)|) \\
&\leq \sum_{n=1}^{\infty} \overline{f}_n(t) + (p+1) m(\|x\|_{BC_1}) \sum_{n=1}^{\infty} |x_n(t)|.
\end{aligned}$$

Hence we derive the estimate

$$\|Fx\|_{BC_1} \leq \overline{F} + (p+1) m(\|x\|_{BC_1}) \|x\|_{BC_1}. \tag{3.2}$$

This shows that the function $(Fx)(t)$ is bounded on the set $\mathbb{R}_+$.

Next we show that the function Fx is continuous on the interval $\mathbb{R}_+$. To this end let us fix T_0 and $\varepsilon > 0$. Next, choose a number $\delta > 0$ according to assumption (v). Since the function $x = x(t)$ belongs to the space BC_1 so we can choose $\delta > 0$ in such a way that the following condition

$$\forall_{t,s \in [0,T]} \, [|t - s| \leq \delta \Rightarrow \|x(t) - x(s)\|_{l_1} \leq \varepsilon] \tag{3.3}$$

is satisfied. Thus, taking $t, s \in [0, T]$ such that $|t - s| \leq \delta$ and using assumptions (v) and (vi) we obtain

$$\begin{aligned}
\|(Fx)(t) - (Fx)(s)\|_{l_1} &= \sum_{n=1}^{\infty} |(F_n x)(t) - (F_n x)(s)| \\
&\leq \sum_{n=1}^{\infty} |f_n(t, x_1(t), x_2(t), \dots) - f_n(s, x_1(t), x_2(t), \dots)| \\
&\quad + \sum_{n=1}^{\infty} |f_n(s, x_1(t), x_2(t), \dots) - f_n(s, x_1(s), x_2(s), \dots)| \\
&\leq \varepsilon + \sum_{n=1}^{\infty} l(\|x\|_{BC_1}) \sum_{i=n-p}^{n+q} |x_i(t) - x_i(s)| \\
&= \varepsilon + l(\|x\|_{BC_1}) \sum_{n=1}^{\infty} \big[|x_{n-p}(t) - x_{n-p}(s)| \\
&\quad + |x_{n-p+1}(t) - x_{n-p+1}(s)| + \dots + |x_{n+q}(t) - x_{n+q}(s)|\big] \\
&\leq \varepsilon + l(\|x\|_{BC_1}) \Big[\sum_{n=1}^{\infty} |x_{n-p}(t) - x_{n-p}(s)| \\
&\quad + \sum_{n=1}^{\infty} |x_{n-p+1}(t) - x_{n-p+1}(s)| + \dots
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}|x_{n+q}(t)-x_{n+q}(s)|\Big]\\
&\le \varepsilon + l(\|x\|_{BC_1})(p+q+1)\sum_{n=1}^{\infty}|x_n(t)-x_n(s)|\\
&\le \varepsilon + l(\|x\|_{BC_1}(p+q+1)\|x(t)-x(s)\|_{l_1}.
\end{aligned}
$$

Observe that in the above calculations we assumed that $n \ge p+1$. Obviously in the case when $1 \le n \le p$ we obtain also the same estimate.

Now, choosing the number $\delta > 0$ in such a way that both assumption (v) and condition (3.3) are satisfied we obtain

$$
\|(Fx)(t)-(Fx)(s)\|_{l_1} \le \varepsilon + (p+q+1)l(\|x\|_{BC_1})\varepsilon.
$$

Hence we infer that the function Fx is continuous on the interval $[0,T]$. Based on the arbitrariness of T we obtain the continuity of the function Fx on the interval $\mathbb{R}_+$. Combining the above established facts we conclude that the operator F transforms the space BC_1 into itself.

Next, we intend to show that the operator V transforms the space BC_1 into itself. Thus, let us take an arbitrary function $x = x(t) = (x_n(t)) \in BC_1$. At first we show the boundedness of the function Vx on the interval $\mathbb{R}_+$. Indeed, for an arbitrary number $t \in \mathbb{R}_+$, in virtue of assumptions (iii) and (viii), we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty}|(V_nx)(t)| &\le \sum_{n=1}^{\infty}\int_0^t |k_n(t,s)|\,|g_n(s,x_1(s),x_2(s),\dots)|ds\\
&\le \sum_{n=1}^{\infty}\int_0^t |k_n(t,s)|\overline{G}\,ds\\
&= \overline{G}\sum_{n=1}^{\infty}\int_0^t |k_n(t,s)|ds\\
&\le \overline{G}K_1.
\end{aligned}
\tag{3.4}
$$

Hence we infer that

$$
\|(Vx)(t)\|_{l_1} \le \overline{G}K_1.
$$

This means that the function Vx is bounded on $\mathbb{R}_+$.

To prove the continuity of the function Vx on the interval $\mathbb{R}_+$ let us fix $\varepsilon > 0$ and $T > 0$. Next, let us choose a number $\delta > 0$ according to (3.3) and $t_1, t_2 \in [0,T)$ such that $|t_2 - t_1| \le \delta$. Without loss of generality we may assume that $t_2 > t_1$. Then, keeping in mind assumptions (iv) and (viii) and applying the Lebesgue monotone convergence theorem [21], we obtain the following estimates:

$$
\begin{aligned}
&\sum_{n=1}^{\infty}|(V_nx)(t_2)-(V_nx)(t_1)|\\
&\le \sum_{n=1}^{\infty}\Big|\int_0^{t_2} k_n(t_2,s)g_n(s,x_1(s),x_2(s),\dots)\,ds\\
&\quad - \int_0^{t_2} k_n(t_1,s)g_n(s,x_1(s),x_2(s),\dots)\,ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{n=1}^{\infty} \Big| \int_0^{t_2} k_n(t_1,s) g_n(s, x_1(s), x_2(s), \dots)\, ds \\
&\quad - \int_0^{t_1} k_n(t_1,s) g_n(s, x_1(s), x_2(s), \dots)\, ds \Big| \\
&\le \sum_{n=1}^{\infty} \int_0^{t_2} |k_n(t_2,s) - k_n(t_1,s)|\, |g_n(s, x_1(s), x_2(s), \dots)| ds \\
&\quad + \sum_{n=1}^{\infty} \int_{t_1}^{t_2} |k_n(t_1,s)|\, |g_n(s, x_1(s), x_2(s), \dots)| ds \\
&\le \sum_{n=1}^{\infty} \int_0^{t_2} \omega_k^T(\delta) |g_n(s, x_1(s), x_2(s), \dots)| ds + \sum_{n=1}^{\infty} \int_{t_1}^{t_2} K_2\, |g_n(s, x_1(s), x_2(s), \dots)| ds \\
&\le \omega_k^T(\delta) \sum_{n=1}^{\infty} \int_0^{t_2} |g_n(s, x_1(s), x_2(s), \dots)| ds + K_2 \sum_{n=1}^{\infty} \int_{t_1}^{t_2} |g_n(s, x_1(s), x_2(s), \dots)| ds \\
&= \omega_k^T(\delta) \int_0^{t_2} \sum_{n=1}^{\infty} |g_n(s, x_1(s), x_2(s), \dots)| ds + K_2 \int_{t_1}^{t_2} \sum_{n=1}^{\infty} |g_n(s, x_1(s), x_2(s), \dots)| ds \\
&\le \omega_k^T(\delta) \int_0^{t_2} \|(gx)(s)\|_{l_1} ds + K_2 \int_{t_1}^{t_2} \|(gx)(s)\|_{l_1} ds \\
&\le \omega_k^T(\delta) \overline{G} T + K_2 \overline{G} \delta, \qquad (3.5)
\end{aligned}
$$

where $\omega_k^T(\delta)$ denotes a common modulus of continuity of the function sequence $t \to k_n(t,s)$ on the interval $[0,T]$ (according to assumption (ii)). Obviously $\omega_k^T(\delta) \to 0$ as $\delta \to 0$. Hence, on the basis of (3.5) we obtain the following estimate

$$
\|(Vx)(t_2) - (Vx)(t_1)\|_{l_1} \le \omega_k^T(\delta) \overline{G} T + K_2 \overline{G} \delta,
$$

which allows us to infer that the function Vx is continuous on $[0,T)$. The number T was arbitrarily chosen so we deduce that the function Vx is continuous on the interval $\mathbb{R}_+$. Finally, combining the established properties of the function Vx we deduce that the operator V maps the space BC_1 into itself.

Next, we are going to show that the operator Q transforms the space BC_1 into itself. To prove this fact let us notice that the space BC_1 can be treated as the Banach algebra with respect to the coordinatewise multiplication of sequences of functions. Thus, keeping in mind the definition of the operator Q and taking into account that the functions Fx and Vx are continuous on $\mathbb{R}_+$ we infer that the function Qx is also continuous on $\mathbb{R}_+$. In a similar way we prove the boundedness of the function Qx on $\mathbb{R}_+$. Finally, we conclude that the operator Q transforms the space BC_1 into BC_1.

Next, in view of estimates (3.2), (3.4) and assumption (i) for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} |(Q_n x)(t)| &\le \sum_{n=1}^{\infty} |a_n(t)| + \sum_{n=1}^{\infty} \big[|(F_n x)(t)| |(V_n x)(t)| \big] \\
&\le A + \Big(\sum_{n=1}^{\infty} |(F_n x)(t)| \Big) \Big(\sum_{n=1}^{\infty} |(V_n x)(t)| \Big) \\
&\le A + \|(Fx)(t)\|_{l_1} \|(Vx)(t)\|_{l_1}
\end{aligned}
$$

$$\begin{aligned}&\le A + \|Fx\|_{BC_1}\|Vx\|_{BC_1}\\&\le A + \big[\overline{F} + (p+1)m(\|x\|_{BC_1})\|x\|_{BC_1}\big]\overline{G}K_1.\end{aligned}$$

Hence, we obtain the inequality

$$\|Qx\|_{BC_1} = A + \overline{FG}K_1 + (p+1)\overline{G}K_1 m(\|x\|_{BC_1})\|x\|_{BC_1}.$$

From the above estimate and assumption (ix) we deduce that there exists a number $r_0 > 0$ such that the operator Q transforms the ball B_{r_0} into itself.

Next we prove that the operator Q is continuous on the ball B_{r_0}. To this end fix a number $\varepsilon > 0$ and an element $x \in B_{r_0}$. Next, take an arbitrary function $y \in B_{r_0}$ such that $\|x-y\|_{BC_1} \le \varepsilon$. Fix arbitrarily number $t \in \mathbb{R}_+$. In view of assumption (vi) we obtain the following estimates:

$$\begin{aligned}
&\sum_{n=1}^{\infty} |(F_n x)(t) - (F_n y)(t)|\\
&= \sum_{n=1}^{\infty} |f_n(t, x_1(t), x_2(t), \dots) - f_n(t, y_1(t), y_2(t), \dots)|\\
&\le \sum_{n=1}^{\infty} l(r_0) \sum_{i=n-p}^{n+q} |x_i(t) - y_i(t)|\\
&= \sum_{n=1}^{\infty} l(r_0)\big[|x_{n-p}(t) - y_{n-p}(t)| + |x_{n-p+1}(t) - y_{n-p+1}(t)| + \dots\\
&\quad + |x_{n+q}(t) - y_{n+q}(t)|\big]\\
&\le l(r_0)\Big[\sum_{n=1}^{\infty} |x_{n-p}(t) - y_{n-p}(t)| + \sum_{n=1}^{\infty} |x_{n-p+1}(t) - y_{n-p+1}(t)| + \dots\\
&\quad + \sum_{n=1}^{\infty} |x_{n+q}(t) - y_{n+q}(t)|\Big]\\
&\le l(r_0)(p+q+1) \sum_{n=1}^{\infty} |x_n(t) - y_n(t)|\\
&\le l(r_0)(p+q+1)\, \|x(t) - y(t)\|_{l_1}.
\end{aligned} \tag{3.6}$$

Consequently

$$\|Fx - Fy\|_{BC_1} \le l(r_0)(p+q+1)\varepsilon.$$

This means that the operator F is continuous on the ball B_{r_0}.

Furthermore, we consider the function $\delta = \delta(\varepsilon)$ defined for $\varepsilon > 0$ in the following way

$$\delta(\varepsilon) = \sup\{|g_n(t,x) - g_n(t,y)| : x, y \in l_1, \|x-y\|_{l_1} \le \varepsilon, t \in \mathbb{R}_+, n \in \mathbb{N}\}.$$

Then, by assumption (viii) we deduce that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Take $\varepsilon > 0$ and choose arbitrary $x, y \in B_{r_0}$ such that $\|x-y\|_{BC_1} \le \varepsilon$. For a fixed $t \in \mathbb{R}_+$ we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} |(V_n x)(t) - (V_n y)(t)|\\
&\le \sum_{n=1}^{\infty} \int_0^t |k_n(t,s)| |g_n\big(s, x_1(s), x_2(s), \dots\big) - g_n\big(s, y_1(s), y_2(s), \dots\big)| ds
\end{aligned}$$

$$\begin{aligned}&\le \sum_{n=1}^{\infty}\int_0^t |k_n(t,s)|\delta(\varepsilon)\,ds\\ &\le K_1\delta(\varepsilon).\end{aligned}$$

This implies the estimate

$$\|Vx - Vy\|_{BC_1} \le K_1\delta(\varepsilon)$$

and proves the continuity of the operator V on the ball B_{r_0}.
Finally, linking the continuity of the operators F and V on the ball B_{r_0} and taking into account the representation of the operator Q given at the beginning of the proof, we infer that Q is continuous on B_{r_0}.

Further, we study the behaviour of the operators F, V and Q with respect to measure of noncompactness χ_a defined by (2.5) (cf. also the components of the measure χ_a defined by formulas (2.2)-(2.4)). To this end take a nonempty subset X of the ball B_{r_0}. Next, fix arbitrary numbers $\varepsilon > 0$ and $T > 0$ and choose a function $x = x(t) = (x_n(t)) \in X$. Then, for $t, s \in [0, T]$ such that $|t - s| \le \varepsilon$, making use of assumptions (v) and (vi), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}|(F_nx)(t)-(F_nx)(s)| &\le \sum_{n=1}^{\infty}|f_n(t,x_1(t),x_2(t),\dots)-f_n(s,x_1(t),x_2(t),\dots)|\\ &\quad+\sum_{n=1}^{\infty}|f_n(s,x_1(t),x_2(t),\dots)-f_n(s,x_1(s),x_2(s),\dots)|\\ &\le \omega_1^T(F,\varepsilon)+l(r_0)(p+q+1)\sum_{n=1}^{\infty}|x_n(t)-x_n(s)|\\ &\le \omega_1^T(F,\varepsilon)+l(r_0)(p+q+1)\omega^T(x,\varepsilon),\end{aligned}$$

where we denoted

$$\begin{aligned}\omega_1^T(F,\varepsilon)=\sup\Big\{&\sum_{n=1}^{\infty}|f_n(t,x_1,x_2,\dots)-f_n(s,x_1,x_2,\dots)| : t,s\in[0,T],\\ &|t-s|\le\varepsilon, x\in B_{r_0}\Big\}.\end{aligned}$$

Obviously, in view of assumption (v) we have that $\omega_1^T(F,\varepsilon) \to 0$ as $\varepsilon \to 0$. From the above estimate we obtain

$$\omega^T(Fx,\varepsilon) \le \omega_1^T(F,\varepsilon) + l(r_0)(p+q+1)\omega^T(x,\varepsilon). \tag{3.7}$$

In a similar way as above (cf. estimate (3.5)) we obtain the inequality

$$\omega^T(Vx,\varepsilon) \le \overline{G}T\omega_k^T(\varepsilon) + \overline{G}K_2\varepsilon. \tag{3.8}$$

Finally, take $\varepsilon > 0$ and $T > 0$. Then, for an arbitrary function $x \in X$ and for $t, s \in [0, T]$ such that $|t - s| \le \varepsilon$ based on (3.7), (3.8), (3.2) and (3.4) we obtain

$$\begin{aligned}\|(Qx)(t)-(Qx)(s)\|_{l_1} &\le \|a(t)-a(s)\|_{l_1} + \|(Fx)(t)(Vx)(t)-(Fx)(s)(Vx)(s)\|_{l_1}\\ &\le \|a(t)-a(s)\|_{l_1} + \|(Fx)(t)(Vx)(t)-(Vx)(t)(Fx)(s)\|_{l_1}\\ &\quad + \|(Vx)(t)(Fx)(s)-(Fx)(s)(Vx)(s)\|_{l_1}\\ &\le \|a(t)-a(s)\|_{l_1} + \|(Vx)(t)\|_{l_1}\|(Fx)(t)-(Fx)(s)\|_{l_1}\\ &\quad + \|(Fx)(s)\|_{l_1}\|(Vx)(t)-(Vx)(s)\|_{l_1}\\ &\le \omega^T(a,\varepsilon)+\overline{G}K_1\omega^T(Fx,\varepsilon)\end{aligned}$$

$$
\begin{aligned}
&+ \big[\overline{F} + (p+1)r_0 m(r_0)\big]\omega^T(Vx,\varepsilon) \\
&\le \omega^T(a,\varepsilon) + \overline{G}K_1\big\{\omega_1^T(F,\varepsilon) + l(r_0)(p+q+1)\omega^T(x,\varepsilon)\big\} \\
&\quad + \big[\overline{F} + (p+1)\, r_0 m(r_0)\big]\,\big\{K_2\overline{G}\varepsilon + \overline{G}T\omega_k^T(\varepsilon)\big\},
\end{aligned}
$$

where $\omega_1^T(F,\varepsilon)$ and $\omega_k^T(\varepsilon)$ were defined previously.

Now, from the above inequality, we derive

$$
\begin{aligned}
\omega^T(QX,\varepsilon) &\le \omega^T(a,\varepsilon) + \overline{G}K_1\big\{\omega_1^T(F,\varepsilon) + l(r_0)(p+q+1)\omega^T(X,\varepsilon)\big\} \\
&\quad + \big[\overline{F} + (p+1)r_0 m(r_0)\big]\big\{K_2\overline{G}\varepsilon + \overline{G}T\omega_k^T(\varepsilon)\big\}.
\end{aligned} \tag{3.9}
$$

From assumption (i) we deduce that $\omega^T(a,\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, in view of assumption (v) we have that $\omega_1^T(F,\varepsilon) \to 0$ as $\varepsilon \to 0$. Similarly, keeping in mind assumption (ii) we conclude that $\omega_k^T(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, in view of the above established facts and estimate (3.9) we obtain

$$
\omega_0^T(QX) \le \overline{G}K_1(p+q+1)l(r_0)\omega_0^T(X).
$$

Consequently, we obtain

$$
\omega_0(QX) \le \overline{G}K_1(p+q+1)l(r_0)\omega_0(X). \tag{3.10}
$$

In what follows we estimate the second component expressed by (2.6) of measure of noncompactness χ_a defined by (2.5). More precisely, we will estimate the expression $\overline{\chi}_\infty(QX)$ for an arbitrarily fixed nonempty subset X of the ball B_{r_0}. To this end, fix a function $x \in X$ and a number $T > 0$. Then, for $t \in [0,T]$ and for an arbitrary natural number n, arguing similarly as in estimates (3.2) and (3.4), we obtain

$$
\begin{aligned}
\sum_{i=n}^{\infty} |(Q_i x)(t)| &\le \sum_{i=n}^{\infty} |a_i(t)| + \sum_{i=n}^{\infty} |(F_i x)(t)(V_i x)(t)| \\
&\le \sum_{i=n}^{\infty} |a_i(t)| + \Big[\sum_{i=n}^{\infty} |(F_i x)(t)|\Big]\Big[\sum_{i=n}^{\infty} |(V_i x)(t)|\Big] \\
&\le \sum_{i=n}^{\infty} |a_i(t)| + \Big[\sum_{i=n}^{\infty} \overline{f_i}(t) + (p+1)m(\|x(t)\|_{l_1})\sum_{i=n}^{\infty} |x_i(t)|\Big]\overline{G}K_1 \\
&\le \sum_{i=n}^{\infty} |a_i(t)| + \Big[\sum_{i=n}^{\infty} \overline{f_i}(t) + (p+1)m(r_0)\sum_{i=n}^{\infty} |x_i(t)|\Big]\overline{G}K_1.
\end{aligned}
$$

Hence, we derive

$$
\begin{aligned}
&\sup_{x=(x_i)\in X} \Big\{\sum_{i=n}^{\infty} |(Q_i x)(t)\Big\} \\
&\le \sum_{i=n}^{\infty} |a_i(t)| + \overline{G}K_1\sum_{i=n}^{\infty} \overline{f}_i(t) + \overline{G}K_1(p+1)m(r_0) \sup_{x=(x_i)\in X}\Big\{\sum_{i=n}^{\infty} |x_i(t)|\Big\}.
\end{aligned}
$$

Passing to the limit as $n \to \infty$ and using assumptions (i), (vii) and Lemma 3.1 we obtain

$$
\overline{\chi}_\infty(QX) \le \overline{G}K_1(p+1)m(r_0)\overline{\chi}_\infty(X), \tag{3.11}
$$

where $\overline{\chi}_\infty(X)$ is defined by formula (2.3).

Next we estimate the last component $a_\infty(X)$ of the measure of noncompactness $\chi_a(X)$. Similarly as above, let us assume that X is a nonempty subset of the ball

B_{r_0}. Fix a function $x \in X$ and a number $T > 0$. Next, take an arbitrary number $t \geq T$. Then, in view of (3.2) and (3.4), we obtain

$$\sum_{n=1}^{\infty} |(Q_n x)(t)| \leq \sum_{n=1}^{\infty} |a_n(t)| + (p+1)l(r_0)\overline{G}K_1 \sum_{n=1}^{\infty} |x_n(t)| + \overline{G}K_1 \sum_{n=1}^{\infty} \overline{f}_n(t).$$

Hence, taking supremum over $t \geq T$ and $x = (x_n) \in X$, we obtain

$$\begin{aligned} &\sup_{x=(x_n)\in X} \Big\{ \sup_{t\geq T} \Big\{ \sum_{n=1}^{\infty} |(Q_n x)(t)| \Big\} \Big\} \\ &\leq \sup_{t\geq T} \Big\{ \sum_{n=1}^{\infty} |a_n(t)| \Big\} + (p+1)l(r_0)\overline{G}K_1 \sup_{x=(x_n)\in X} \Big\{ \sup_{t\geq T} \Big\{ \sum_{n=1}^{\infty} |x_n(t)| \Big\} \Big\} \\ &\quad + \overline{G}K_1 \sup_{t\geq T} \Big\{ \sum_{n=1}^{\infty} |\overline{f_n}(t)| \Big\}. \end{aligned}$$

Further, as $T \to \infty$ and applying assumptions (i) and (vii) we obtain the inequality

$$a_\infty(QX) \leq (p+1)l(r_0)\overline{G}K_1 a_\infty(X). \tag{3.12}$$

Finally, combining (3.10), (3.11), (3.12) and keeping in mind formula (2.5), we obtain the inequality

$$\chi_a(QX) \leq \overline{G}K_1(p+q+1)\max\{l(r_0), m(r_0)\}\chi_a(X). \tag{3.13}$$

Now, in view of Theorem 2.2, estimate (3.13) and the second inequality from assumption (ix) we conclude that there exists at least one solution $x(t) = (x_n(t))$ of infinite system of integral equations (3.1) belonging to the ball B_{r_0} in the space $BC_1 = BC(\mathbb{R}_+, l_1)$. The proof is complete. $\square$

4. An example

In this section we illustrate our main result obtained in Theorem 3.2 by an example. We consider the infinite system of nonlinear quadratic Volterra-Hammerstein integral equations

$$\begin{aligned} x_1(t) &= \alpha e^{-2t} + \Big(\frac{\beta}{1+t^2} + \gamma \frac{x_3(t)}{1+x_1^2(t)} \Big) \\ &\quad \times \int_0^t \frac{1}{(1+s^2)[1+(t+s)^2]} \frac{\arctan s}{1^2} \Big[\frac{x_1(s)}{1+x_1^2(s)} + \frac{x_2(s)}{1+2^2 x_2^2(s)} \Big] ds, \\ x_2(t) &= \alpha e^{-2t} + \Big(\frac{\beta}{2(2^2+t^2)} + \gamma \frac{x_3(t)}{1+x_1^2(t)} + \gamma \frac{x_4(t)}{1+x_2^2(t)} \Big) \\ &\quad \times \int_0^t \frac{1}{(2^2+s^2)[1+(t+s)^2]} \frac{\arctan s}{2^2} \Big[\frac{x_2(s)}{1+2^2 x_2^2(s)} + \frac{x_3(s)}{1+3^2 x_3^2(s)} \Big] ds, \\ &\dots \end{aligned} \tag{4.1}$$

$$\begin{aligned} &x_n(t) \\ &= \alpha e^{-2t} \frac{t^{n-1}}{(n-1)!} + \Big(\frac{\beta}{n(n^2+t^2)} + \gamma \frac{x_n(t)}{1+x_{n-2}^2(t)} + \gamma \frac{x_{n+1}(t)}{1+x_{n-1}^2(t)} + \gamma \frac{x_{n+2}(t)}{1+x_n^2(t)} \Big) \\ &\quad \times \int_0^t \frac{1}{(n^2+s^2)[1+(t+s)^2]} \frac{\arctan s}{n^2} \Big[\frac{x_n(s)}{1+n^2 x_n^2(s)} + \frac{x_{n+1}(s)}{1+(n+1)^2 x_{n+1}^2(s)} \Big] ds, \end{aligned}$$

where $t \in \mathbb{R}_+$, $n \in \mathbb{N}$, $n \geq 3$ and α, β, γ are positive constants. Observe that infinite system of integral equations (4.1) is a special case of system (3.1) if we put

$$a_n(t) = \alpha e^{-2t} \frac{t^{n-1}}{(n-1)!}, \tag{4.2}$$

$$f_n(t, x_1, x_2, \dots) = \frac{\beta}{n(n^2 + t^2)} + \gamma \frac{x_n}{1 + x_{n-2}^2} + \gamma \frac{x_{n+1}}{1 + x_{n-1}^2} + \gamma \frac{x_{n+2}}{1 + x_n^2}, \tag{4.3}$$

$$k_n(t, s) = \frac{1}{(n^2 + s^2)[1 + (t + s)^2]}, \tag{4.4}$$

$$g_n(t, x_1, x_2, \dots) = \frac{\arctan t}{n^2} \Big[\frac{x_n}{1 + n^2 x_n^2} + \frac{x_{n+1}}{1 + (n+1)^2 x_{n+1}^2} \Big], \tag{4.5}$$

where $n \in \mathbb{N}$, and $t, s \in \mathbb{R}_+$ in formulas (4.2), (4.4) and (4.5), while in formula (4.3) $n \geq 3$ and $t, s \in \mathbb{R}_+$. In the case $n = 1$ and $n = 2$ formula (4.3) should be replaced by the following ones:

$$f_1(t, x_1, x_2, \dots) = \frac{\beta}{1 + t^2} + \gamma \frac{x_3}{1 + x_1^2}, \tag{4.6}$$

$$f_2(t, x_1, x_2, \dots) = \frac{\beta}{2(2^2 + t^2)} + \gamma \frac{x_3}{1 + x_1^2} + \gamma \frac{x_4}{1 + x_2^2}. \tag{4.7}$$

Notice that in our further investigations we will always consider formula (4.3) in order to express the function $f_n(t, x_1, x_2, \dots)$.

Now we show that infinite system of integral equations(4.1) has a solution in the Banach space BC_1. Our goal is to show that the components of the infinite system of integral equations (4.1) defined by formulas (4.2)-(4.5) satisfy assumptions of Theorem 3.2.

At the beginning observe that the sequence of functions $(a_n(t))$ belongs the space BC_1. Indeed, for arbitrarily fixed $t \in \mathbb{R}_+$ we have

$$\sum_{n=1}^{\infty} a_n(t) = \alpha e^{-2t} \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} = \alpha e^{-2t} e^t = \alpha e^{-t}.$$

Obviously, we have that

$$\lim_{t \to \infty} \sum_{n=1}^{\infty} a_n(t) = \lim_{t \to \infty} \alpha e^{-t} = 0.$$

This shows that assumption (i) is satisfied. We obtain

$$A = \sup \Big\{ \sum_{n=1}^{\infty} |a_n(t)| : t \in \mathbb{R}_+ \Big\} = \alpha.$$

Further, for arbitrarily fixed $t_1, t_2 \in \mathbb{R}_+$, $s \in \mathbb{R}_+$ and for $n \in \mathbb{N}$, we obtain

$$\begin{aligned} |k_n(t_2, s) - k_n(t_1, s)| &= \frac{1}{n^2 + s^2} \Big| \frac{1}{1 + (t_2 + s)^2} - \frac{1}{1 + (t_1 + s)^2} \Big| \\ &\leq \frac{1}{n^2} \frac{|(t_2 + s)^2 - (t_1 + s)^2|}{[1 + (t_2 + s)^2][1 + (t_1 + s)^2]} \\ &\leq \frac{|t_2 - t_1|(t_2 + t_1 + 2s)}{[1 + (t_2 + s)^2][1 + (t_1 + s)^2]} \\ &\leq \frac{1}{n^2} |t_2 - t_1| \Big[\frac{t_2 + s}{1 + (t_2 + s)^2} + \frac{t_1 + s}{1 + (t_1 + s)^2} \Big] \end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{n^2}\,|t_2-t_1|\Big(\frac{1}{2}+\frac{1}{2}\Big)\\
&\le |t_2-t_1|.
\end{aligned}$$

Hence we see that the functions $k_n(t,s)$ are even equicontinuous on the set $\mathbb{R}_+$ uniformly with respect to $s\in\mathbb{R}_+$. Thus the functions $k_n(t,s)$ satisfy assumption (ii). Also, observe that

$$\begin{aligned}
\sum_{n=1}^{\infty}\int_0^t |k_n(t,s)|ds &= \sum_{n=1}^{\infty}\int_0^t \frac{1}{n^2+s^2}\cdot\frac{1}{1+(t+s)^2}ds\\
&\le \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^t \frac{1}{1+(t+s)^2}ds\\
&= \sum_{n=1}^{\infty}\frac{1}{n^2}\int_t^{2t}\frac{du}{1+u^2}\\
&\le \sum_{n=1}^{\infty}\frac{1}{n^2}\arctan 2t\\
&\le \frac{\pi^2}{6}\cdot\frac{\pi}{2}=\frac{\pi^3}{12}.
\end{aligned}$$

From the above estimate we deduce that the functions $k_n(t,s)$ satisfy assumption (iii) with the constant $K_1=\pi^3/12$. Next, taking arbitrary $t,s\in\mathbb{R}_+$ and $n\in\mathbb{N}$, we have

$$|k_n(t,s)|=\frac{1}{n^2+s^2}\cdot\frac{1}{1+(t+s)^2}\le\frac{1}{n^2}\le 1.$$

This shows that assumption (iv) is satisfied with the constant $K_2=1$.

Now we verify assumptions (v), (vi) and (vii) for the function $f_n(t,x_1,x_2,\dots)$ defined by (4.3). The cases $n=1$ and $n=2$ are easier to verify so we consider the case $n\ge 3$. We have

$$f_n(t,x_1,x_2,\dots)=\frac{\beta}{n(n^2+t^2)}+\gamma\frac{x_n}{1+x_{n-2}^2}+\gamma\frac{x_{n+1}}{1+x_{n-1}^2}+\gamma\frac{x_{n+2}}{1+x_n^2}.$$

Hence for an arbitrary $t\in\mathbb{R}_+$ and an arbitrary $x=(x_n)\in l_1$ the series

$$\sum_{n=1}^{\infty} f_n(t,x_1,x_2,\dots)$$

is uniformly convergent on the set $\mathbb{R}_+\times l_1$. This conclusion follows immediately from the standard Weierstrass test. Next, fix arbitrarily $t,s\in\mathbb{R}_+$ and $x=(x_n)\in l_1$. Then we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}|f_n(t,x_1,x_2,\dots)-f_n(s,x_1,x_2,\dots)|\\
&=\sum_{n=1}^{\infty}\frac{1}{n}\Big|\frac{\beta}{n^2+t^2}-\frac{\beta}{n^2+s^2}\Big|\\
&\le \beta\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{|t^2-s^2|}{(n^2+t^2)(n^2+s^2)}
\end{aligned}$$

$$
\begin{aligned}
&\le \beta \sum_{n=1}^{\infty} \frac{1}{n}|t-s|\Big[\frac{t}{(n^2+t^2)(n^2+s^2)} + \frac{s}{(n^2+t^2)(n^2+s^2)}\Big] \\
&\le \beta \sum_{n=1}^{\infty} |t-s|\frac{1}{2}\Big[\frac{1}{n^2+s^2} + \frac{1}{n^2+t^2}\Big] \\
&\le \beta \sum_{n=1}^{\infty} |t-s|\frac{1}{2}\cdot\frac{2}{n^2} \\
&= \beta\frac{\pi^2}{6}|t-s|.
\end{aligned}
$$

This shows that the second part of assumption (v) is satisfied (even more, the function $t \to \sum_{n=1}^{\infty} f_n(t, x_1, x_2, \dots)$ is uniformly continuous on $\mathbb{R}_+$ uniformly with respect to $x = (x_n) \in l_1$). Further on, fix an arbitrary number $r > 0$ and take $x = (x_n), y = (y_n) \in l_1$ such that $\|x\|_{l_1} \le r$, $\|y\|_{l_1} \le r$. Then, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $n \in \mathbb{N}$ ($n \ge 3$), we obtain

$$
\begin{aligned}
&|f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| \\
&\le \gamma\Big|\frac{x_n}{1+x_{n-2}^2} - \frac{y_n}{1+y_{n-2}^2}\Big| + \gamma\Big|\frac{x_{n+1}}{1+x_{n-1}^2} - \frac{y_{n+1}}{1+y_{n-1}^2}\Big| \\
&\quad + \gamma\Big|\frac{x_{n+2}}{1+x_n^2} - \frac{y_{n+2}}{1+y_n^2}\Big|.
\end{aligned} \tag{4.8}
$$

To estimate the right-hand side of (4.8), we consider numbers a, b, c, d such that $|a| \le r$, $|b| \le r$, $|c| \le r$ and $|d| \le r$. We have

$$
\begin{aligned}
&\Big|\frac{a}{1+b^2} - \frac{c}{1+d^2}\Big| \\
&= \frac{|a(1+d^2) - c(1+b^2)|}{(1+b^2)(1+d^2)} \\
&= \frac{|a - c + ad^2 - cb^2|}{(1+b^2)(1+d^2)} \\
&\le \frac{|a-c| + |ad^2 - cb^2|}{(1+b^2)(1+d^2)} \\
&\le \frac{|a-c| + |ad^2 - cd^2| + |cd^2 - cb^2|}{(1+b^2)(1+d^2)} \\
&\le \frac{|a-c| + |a-c|d^2 + |c||b-d|(|b|+|d|)}{(1+b^2)(1+d^2)} \\
&\le |a-c| + |a-c|\frac{d^2}{(1+b^2)(1+b^2)} \\
&\quad + |c||b-d|\Big[\frac{|b|}{(1+b^2)(1+d^2)} + \frac{|d|}{(1+b^2)(1+d^2)}\Big] \\
&\le |a-c| + |a-c|\frac{1}{1+b^2} + |c||b-d| + \Big(\frac{1}{2}\frac{1}{1+d^2} + \frac{1}{2}\frac{1}{1+b^2}\Big) \\
&\le 2|a-c| + r|b-d| \\
&\le \max\{2, r\}(|a-c| + |b-d|).
\end{aligned}
$$

Using this inequality to estimate (4.8), we obtain

$$\begin{aligned}
&|f_n(t,x_1.x_2,\dots)-f_n(t,y_1,y_2,\dots)| \\
&\le \gamma\ \max\{2,r\}\,[|x_{n-2}-y_{n-2}|+|x_n-y_n|] \\
&\quad+\gamma\ \max\{2,r\}\big[|x_{n-1}-y_{n-1}|+|x_{n+1}-y_{n+1}|\big] \\
&\quad+\gamma\ \max\{2,r\}\,[|x_n-y_n|+|x_{n+2}-y_{n+2}|] \\
&\le 2\gamma\ \max\{2,r\}\sum_{i=n-2}^{n+2}|x_i-y_i|.
\end{aligned}$$

The above estimate shows that assumption (vi) is satisfied with $p=q=2$ and $l(r)=2\gamma\max\{2,r\}$.

Assume that $t\in\mathbb{R}_+$ and n is an arbitrarily fixed natural number n ($n\ge 3$). Then for a fixed $r>0$ and for $x=(x_n)\in l_1$ with $\|x\|_{l_1}\le r$ we obtain

$$\begin{aligned}
|f_n(t,x_1,x_2,\dots)| &\le \frac{\beta}{n(n^2+t^2)}+\gamma\frac{|x_n|}{1+x_{n-2}^2}+\gamma\frac{|x_{n+1}|}{1+x_{n-1}^2}+\gamma\frac{|x_{n+2}|}{1+x_n^2} \\
&\le \frac{\beta}{n(n^2+t^2)}+\gamma(|x_n|+|x_{n+1}|+|x_{n+2}|) \\
&= \frac{\beta}{n(n^2+t^2)}+\gamma\sum_{i=n}^{n+2}|x_i|.
\end{aligned}$$

Thus we see that assumption (vii) is satisfied with $m(r)=\gamma$ and $\overline{f}_n(t)=\frac{\beta}{n(n^2+t^2)}$ for $n=1,2,\dots$. Observe that for $t>0$ we have

$$\sum_{n=1}^{\infty}\overline{f}_n(t)=\beta\sum_{n=1}^{\infty}\frac{1}{n(n^2+t^2)}\le\beta\sum_{n=1}^{\infty}\frac{1}{nt^2}=\frac{\beta}{t}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\beta\pi^2}{6t}.$$

This implies that $\lim_{t\to\infty}\sum_{n=1}^{\infty}\overline{f}_n(t)=0$ and shows that the remaining part of assumption (vii) is also satisfied. Obviously, we have that

$$\sup\Big\{\sum_{n=1}^{\infty}\overline{f}_n(t):t\in\mathbb{R}_+\Big\}\le\beta\sum_{n=1}^{\infty}\frac{1}{n^3}\le\beta\frac{\pi^2}{6}.$$

Thus, we can accept that $F=\beta\frac{\pi^2}{6}$, where F is the constant defined as $F=\sup\big\{\sum_{n=1}^{\infty}\overline{f}_n(t):t\in\mathbb{R}_+\big\}$.

To verify assumption (viii) let us first notice that for a fixed natural number n the function $g_n=g_n(t,x_1,x_2,\dots)$ defined on the set $\mathbb{R}_+\times\mathbb{R}^\infty$ by formula (4.5) takes real values for $n=1,2,\dots$. Next, we fix arbitrarily $x=(x_n)\in l_1$. Then, for

$t \in \mathbb{R}_+$ we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} |g_n(t, x_1, x_2, \dots)| &\le \sum_{n=1}^{\infty} \frac{\arctan t}{n^2} \Big[\frac{|x_n|}{1+n^2x_n^2} + \frac{|x_{n+1}|}{1+(n+1)^2x_{n+1}^2} \Big] \\
&\le \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} (\frac{1}{2} + \frac{1}{4}) \\
&= \frac{3\pi}{8} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{\pi^3}{16}.
\end{aligned} \tag{4.9}
$$

It follows from the above estimate that the operator g defined in assumption (viii) transforms the set $\mathbb{R}_+ \times l_1$ into l_1. Moreover, on the basis of (4.9) we infer that the operator g is bounded on $\mathbb{R}_+ \times l_1$ and $\|(gx)(t)\|_{l_1} \le \pi^3/16$. This means that we can accept in our situation that $\overline{G} = \pi^3/16$, where $\overline{G}$ is the constant from assumption (viii).

Now we take a number $\varepsilon > 0$ and choose arbitrary elements $x = (x_n)$, $y = (y_n) \in l_1$ such that $\|x - y\|_{l_1} \le \varepsilon$. Then, for a fixed $t \in \mathbb{R}_+$ we obtain

$$
\begin{aligned}
&\|(gy)(t) - (gx)(t)\|_{l_1} \\
&= \sum_{n=1}^{\infty} |g_n(t, y_1, y_2, \dots) - g_n(t, x_1, x_2, \dots)| \\
&= \sum_{n=1}^{\infty} \frac{\arctan t}{n^2} \Big| \frac{y_n}{1+n^2y_n^2} + \frac{y_{n+1}}{1+(n+1)^2y_{n+1}^2} - \frac{x_n}{1+n^2x_n^2} - \frac{x_{n+1}}{1+(n+1)^2x_{n+1}^2} \Big| \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[\Big| \frac{y_n}{1+n^2y_n^2} - \frac{x_n}{1+n^2x_n^2} \Big| \\
&\quad + \Big| \frac{y_{n+1}}{1+(n+1)^2y_{n+1}^2} - \frac{x_{n+1}}{1+(n+1)^2x_{n+1}^2} \Big| \Big] \\
&\le \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[\frac{|y_n + n^2y_nx_n^2 - x_n - n^2x_ny_n^2|}{(1+n^2y_n^2)(1+n^2x_n^2)} \\
&\quad + \frac{|y_{n+1} + (n+1)^2y_{n+1}x_{n+1}^2 - x_{n+1} - (n+1)^2x_{n+1}y_{n+1}^2|}{(1+(n+1)^2y_{n+1}^2)(1+(n+1)^2x_{n+1}^2)} \Big] \\
&\le \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[\frac{|y_n - x_n| + n^2|x_n||y_n||y_n - x_n|}{(1+n^2y_n^2)(1+n^2x_n^2)} \\
&\quad + \frac{|y_{n+1} - x_{n+1}| + (n+1)^2|x_{n+1}||y_{n+1}||y_{n+1} - x_{n+1}|}{(1+(n+1)^2y_{n+1}^2)(1+(n+1)^2x_{n+1}^2)} \Big] \\
&\le \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Big\{ |y_n - x_n| \Big[\frac{1}{(1+n^2y_n^2)(1+n^2x_n^2)} + \frac{n|y_n|}{1+n^2y_n^2} \cdot \frac{n|x_n|}{1+n^2x_n^2} \Big] \\
&\quad + |y_{n+1} - x_{n+1}| \Big[\frac{1}{(1+(n+1)^2y_{n+1}^2)(1+(n+1)^2x_{n+1}^2)} \\
&\quad + \frac{(n+1)|y_{n+1}|}{1+(n+1)^2y_{n+1}^2} \cdot \frac{(n+1)|x_{n+1}|}{1+(n+1)^2x_{n+1}^2} \Big] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Big[|y_n-x_n|(1+\frac{1}{2}\cdot\frac{1}{2})+|y_{n+1}-x_{n+1}|(1+\frac{1}{2}\cdot\frac{1}{2})\Big]\\
&\le \frac{5\pi}{8}\sum_{n=1}^{\infty}\frac{1}{n^2}\big[|y_n-x_n|+|y_{n+1}-x_{n+1}\big]\\
&\le \frac{5\pi}{8}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}|y_k-x_k|\\
&= \frac{5\pi^3}{48}\|y-x\|_{l_1}\\
&\le \frac{5\pi^3}{8}\varepsilon.
\end{aligned}
$$

This proves the last part of assumption (viii).

Finally, let us consider the first inequality from assumption (ix). Gather all values of the constants $\overline{A}, \overline{F}, \overline{G}, K_1, p, q$ and take into account the formulas for $l(r)$ and $m(r)$, then we see that the mentioned inequality has the form

$$
\alpha+\beta\frac{\pi^8}{1152}+\gamma\frac{\pi^6}{64}r\le r. \tag{4.10}
$$

Thus, taking $\gamma<64/\pi^6$ we infer that any number r such that

$$
r\ge\frac{1}{18}\cdot\frac{1152\alpha+\pi^8\beta}{64-\pi^6\gamma}
$$

satisfies inequality (4.10). Hence it follows that we can accept the value of r_0 as

$$
r_0=\frac{1}{18}\cdot\frac{1152\alpha+\pi^8\beta}{64-\pi^6\gamma}.
$$

The second inequality from (ix) has the form

$$
\frac{5\pi^6}{192}\max\{2\gamma\max\{2,r_0\},\gamma\}<1
$$

and we can choose γ so that the above inequality is satisfied. This proves that assumption (ix) is satisfied. Now we apply Theorem 3.2 and we obtain that infinite system of nonlinear integral equations 3.1 has at least one solution $x=x(t)=(x_n(t))$ in the space $BC_1=BC(\mathbb{R}_+,l_1)$.

Remark 4.1. Let us notice that the functions $k_n(t,s)$ and $g_n(t,x_1,x_2,\dots)$ in equation (4.1) defined by formulas (4.4), (4.5) coincide with the functions $k_n(t,s)$ and $g_n(t,x_1,x_2,\dots)$ from Example 5.1 in [9] (n=1,2,...). However, we repeated calculations concerning those functions to make our paper complete.

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