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EXISTENCE OF SOLUTIONS FOR NON-LOCAL ELLIPTIC SYSTEMS WITH HARDY-LITTLEWOOD-SOBOLEV CRITICAL NONLINEARITIES

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Abstract. In this work, we establish the existence of solutions for the nonlinear nonlocal system of equations involving the fractional Laplacian,

$$
\begin{gathered}
(-\Delta)^s u = au + bv + \frac{2p}{p+q}\int_\Omega \frac{|v(y)|^q}{|x-y|^\mu}dy|u|^{p-2}u \\
+ 2\xi_1 \int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy|u|^{2^*_\mu-2}u \quad \text{in } \Omega, \\
(-\Delta)^s v = bu + cv + \frac{2q}{p+q}\int_\Omega \frac{|u(y)|^p}{|x-y|^\mu}dy|v|^{q-2}v \\
+ 2\xi_2 \int_\Omega \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu}dy|v|^{2^*_\mu-2}v \quad \text{in } \Omega, \\
u = v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{gathered}
$$

where $(-\Delta)^s$ is the fractional Laplacian operator, Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 < s < 1$, $N > 2s$, $0 < \mu < N$, $\xi_1, \xi_2 \geq 0$, $1 < p, q \leq 2^*_\mu$ and $2^*_\mu = \frac{2N-\mu}{N-2s}$ is the upper critical exponent in the Hardy-Littlewood-Sobolev inequality. The nonlinearities can interact with the spectrum of the fractional Laplacian. More specifically, the interval defined by the two eigenvalues of the real matrix from the linear part contains an eigenvalue of the spectrum of the fractional Laplacian. In this case, resonance phenomena can occur.

Contents

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1. Introduction and statement of main results

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ (at least C^2), $N>2s$ and $s\in(0,1)$. We consider the following nonlinear doubly nonlocal systems involving the fractional Laplacian,

$$
\begin{gathered}
(-\Delta)^s u = au+bv+\frac{2p}{p+q}\int_\Omega \frac{|v(y)|^q}{|x-y|^\mu}dy|u|^{p-2}u \\
+2\xi_1\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u|^{2_\mu^*-2}u \quad \text{in } \Omega, \\
(-\Delta)^s v = bu+cv+\frac{2q}{p+q}\int_\Omega \frac{|u(y)|^p}{|x-y|^\mu}dy|v|^{q-2}v \\
+2\xi_2\int_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu}dy|v|^{2_\mu^*-2}v \quad \text{in } \Omega, \\
u=v=0, \quad \text{in } \mathbb{R}^N\setminus\Omega,
\end{gathered}
\tag{1.1}
$$

where $\mu\in(0,N)$, $\xi_1,\xi_2\geq 0$, $1<p,q\leq 2_\mu^*$ and $2_\mu^*=\frac{2N-\mu}{N-2s}$ is the upper critical exponent in the Hardy-Littlewood-Sobolev inequality. $(-\Delta)^s$ is the fractional Laplacian operator defined as

$$
(-\Delta)^s u(x) = -\,\mathrm{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy
$$

where P. V. denotes the Cauchy principal value. The fractional Laplacian is the infinitesimal generator of Lévy stable diffusion process and appears in physical phenomena, stochastic processes, fluid dynamics, dynamical systems, elasticity, obstacle problems, chemical reactions in liquids and American options in finance. For more details, we refer to [2, 15].

For a measurable function $u:\mathbb{R}^N\to\mathbb{R}$, we define the Gagliardo seminorm

$$
[u]_s := \Big(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}.
$$

Now, we introduce the fractional Sobolev space (which is a Hilbert space)

$$
H^s(\mathbb{R}^N)=\{u\in L^2(\mathbb{R}^N):[u]_s<\infty\},
$$

with the norm $\|u\|_{H^s}=(\|u\|_{L^2}^2+[u]_s^2)^{1/2}$. Let

$$
X(\Omega):=\{u\in H^s(\mathbb{R}^N):u=0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\}.
$$

It holds that $X(\Omega) \hookrightarrow L^r(\Omega)$ continuously for $r \in [1, 2_s^*]$ and compactly for $r \in [1, 2_s^*)$, where $2_s^* = \frac{2N}{N-2s}$. Due to the fractional Sobolev inequality, $X(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_X := \int_{\mathbb{R}^{2N}} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\, dx\, dy,$$

which induces the norm $\|\cdot\|_X = [\cdot]_s$. We shall denote by μ_1 and μ_2 the real eigenvalues of the matrix

$$A := \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, b, c \in \mathbb{R}.$$

Without loss of generality, we will assume $\mu_1 \le \mu_2$. The spectrum of $(-\Delta)^s$, with boundary condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$, will be denoted by $\sigma((-\Delta)^s)$, which consists of the sequence of the eigenvalues $\{\lambda_{k,s}\}$ satisfying

$$0 < \lambda_{1,s} < \lambda_{2,s} \le \lambda_{3,s} \le \dots \le \lambda_{j,s} \le \lambda_{j+1,s} \le \dots, \lambda_{k,s} \to \infty, \quad \text{as } k \to \infty,$$

and are characterized by

$$\lambda_{1,s} = \inf_{u \in X(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy}{\int_{\mathbb{R}^N} |u(x)|^2 dx},$$

$$\lambda_{k+1,s} = \inf_{u \in \mathbb{P}_{k+1}\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy}{\int_{\mathbb{R}^N} |u(x)|^2 dx},$$

where

$$\mathbb{P}_{k+1} = \{u \in X(\Omega) : \langle u, \varphi_{j,s}\rangle_X = 0,\ j = 1, 2, \dots, k\},$$

and $\varphi_{k,s}$ denotes the eigenfunction associated with the eigenvalue $\lambda_{k,s}$, for each $k \in \mathbb{N}$. The following results are true (see [30, 31, 33]).

(i) If $u \in X(\Omega)$ is a $\lambda_{1,s}$-eigenfunction (u is an eigenfunction corresponding to $\lambda_{1,s}$), then either $u(x) > 0$ a.e. in Ω or $u(x) < 0$ a.e. in Ω;

(ii) If $\lambda \in \sigma((-\Delta)^s) \setminus \{\lambda_{1,s}\}$ and u is a λ-eigenfunction, then u changes sign in Ω, and λ has finite multiplicity.

(iii) $\varphi_{k,s} \in C^{0,\sigma}(\Omega)$ for some $\sigma \in (0, 1)$ and the sequence $\{\varphi_{k,s}\}$ is an orthonormal basis in both $L^2(\Omega)$ and $X(\Omega)$.

Remark 1.1. For fixed $k \in \mathbb{N}$ we can assume $\lambda_{k,s} < \lambda_{k+1,s}$, otherwise we can suppose that $\lambda_{k,s}$ has multiplicity $l \in \mathbb{N}$, that is

$$\lambda_{k-1,s} < \lambda_{k,s} = \lambda_{k+1,s} = \dots = \lambda_{k+l-1,s} < \lambda_{k+l,s}\ ,$$

and we denote $\lambda_{k+l,s} = \lambda_{k+1,s}$.

In a pioneering paper [3], Brézis and Nirenberg studied the problem

$$-\Delta u = |u|^{2^*-2} u + \lambda u \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega,$$

where $2^* = \frac{N+2}{N-2}$. They proved the existence of nontrivial solutions for $\lambda > 0, N > 4$ by developing some skillful techniques in estimating the Minimax level. This kind of Brézis-Nirenberg problems has been extensively studied (see, e.g. [4, 6, 7, 5, 10, 17, 20, 21, 19, 18, 37, 36, 39] and references therein). Recently, many well-known Brézis-Nirenberg results in critical local equations have been extended to semilinear equations with fractional Laplacian. Specially, we refer to [31, 32, 34, 35], where the following critical fractional Laplacian problem

$$(-\Delta)^s u = |u|^{2_s^*-2}u + \lambda u \text{ in } \Omega; \quad u = 0 \text{ in } \mathbb{R}^N\backslash\Omega,$$

was investigated, and a nontrivial weak solution was obtained under the following assumptions:

(i) $2s < N < 4s$ and λ is sufficiently large;
(ii) $N = 4s$ and λ is not an eigenvalue of $(-\Delta)^s$ in Ω;
(iii) $N \geq 4s$.

For the Laplacian with nonlocal Choquard nonlinearity, Gao and Yang [13] studied the Brézis-Nirenberg type problem

$$-\Delta u = \lambda u + \Big(\int_\Omega \frac{|u|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u|^{2_\mu^*-2}u \text{ in } \Omega; \quad u = 0, \text{ in } \mathbb{R}^N\backslash\Omega. \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$. They proved the existence, multiplicity and nonexistence results for a range of λ. Moreover, in [14], they also studied a class of critical Choquard equations

$$-\Delta u = \Big(\int_\Omega \frac{|u|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u|^{2_\mu^*-2}u + \lambda f(u), \quad \text{in } \Omega.$$

Some existence and multiplicity results were obtained under suitable assumptions on different types of nonlinearities $f(u)$. For details and recent works we refer to [1, 25] and the references therein. For fractional Laplacian with nonlocal Choquard nonlinearity, D'Avenia, Siciliano and Squassina in [9] considered the following fractional Choquard equation

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\alpha * |u|^q)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $N \geq 3$, $s \in (0,1)$, $\omega \geq 0$, $\alpha \in (0,N)$ and $q \in (\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2s})$. In particularly, when $\omega = 0, \alpha = 4s$ and $q = 2$, then problem (1.3) becomes a fractional Choquard equation with upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality as follows:

$$(-\Delta)^s u = \Big(\int_\Omega \frac{|u|^2}{|x-y|^{4s}}dy\Big)u, \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

They obtained regularity, existence, nonexistence of nontrivial solutions problems (1.3) and (1.4). Mukherjee and Sreenadh [28] extended the study of (1.2) to fractional Laplacian equation.

Regarding a system of equations, in [12, 11, 22, 26], the authors studied elliptic systems involving fractional Laplacian and critical growth nonlinearities, which extended the Brézis and Nirenberg results for variational systems. Particularly, in [26], Miyagaki and Pereira studied the fractional elliptic system

$$\begin{gathered}
(-\Delta)^s u = au + bv + \frac{2p}{p+q}|u|^{p-2}u|v|^q + 2\xi_1 u|u|^{p+q-2} \quad \text{in } \Omega,\\
(-\Delta)^s v = bu + cv + \frac{2q}{p+q}|u|^p|v|^{q-2}v + 2\xi_2 v|v|^{p+q-2} \quad \text{in } \Omega,\\
u = v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{gathered}$$

extending [11] by means of the Linking theorem when

$$\lambda_{k-1,s} \leq \mu_1 < \lambda_{k,s} \leq \mu_2 < \lambda_{k+1,s}, \quad \text{if } k \geq 1.$$

In this case, resonance and double resonance phenomena can occur. Using Nehari manifold techniques, Giacomoni, Mukherjee and Sreenadh [16] established the existence and multiplicity results of weak solutions for the fractional elliptic systems involving Choquard type nonlinearities,

$$\begin{gathered}(-\Delta)^s u=\lambda|u|^{q-2}u+\Big(\int_\Omega\frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|u|^{2^*_\mu-2}u\quad\text{in }\Omega,\\(-\Delta)^s v=\delta|v|^{q-2}v+\Big(\int_\Omega\frac{|u(x)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|v|^{2^*_\mu-2}v\quad\text{in }\Omega,\\u=v=0\quad\text{in }\mathbb{R}^N\setminus\Omega,\end{gathered}$$

where $\lambda,\delta>0$ are real parameters and $1<q<2$.

Motivated by [26, 12], we continue to study the fractional elliptic systems involving Choquard type nonlinearities and focus our attention on the existence results for problem (1.1) under the conditions that (i) $\xi_1=\xi_2=0$, $1<p,q<2^*_\mu$, (ii) $\xi_1=\xi_2=0$, $p=q=2^*_\mu$, (iii) $\xi_1,\xi_2>0$, $p=q=2^*_\mu$ respectively. Our main results are the following:

Theorem 1.2. *Assume that $\xi_1=\xi_2=0$, $1<p,q<2^*_\mu$, $b\ge 0$ and $\mu_2<\lambda_{1,s}$. Then* (1.1) *admits a positive solution.*

Theorem 1.3. *Assume that $\xi_1=\xi_2=0$, $p=q=2^*_\mu$, $b\ge 0$ and $0<\mu_1\le\mu_2<\lambda_{1,s}$. Then* (1.1) *admits a nonnegative solution, provided that either*
(i) *$N\ge 4s$, or*
(ii) *$2s<N<4s$ and μ_1 is large enough.*

Theorem 1.4. *Assume that $\xi_1,\xi_2>0$, $p=q=2^*_\mu$ and $0<\lambda_{k-1,s}<\mu_1<\lambda_{k,s}\le\mu_2<\lambda_{k+1,s}$, for some $k\in\mathbb{N}$. Then* (1.1) *admits a nontrivial solution, if one of the following conditions holds,*
(i) *$N\ge 4s$,*
(ii) *$2s<N<4s$ and μ_1 is large enough.*

The outline of this paper is as follows: Section 2 contains the functional setting and some abstract critical point theorems. In section 3, we obtain a positive solution for problem (1.1) when the nonlinearity is subcritical. In section 4, when the nonlinearity has the critical growth, we obtain a nonnegative solution by the Mountain Pass theorem. In section 5, when the nonlinearity interacts with the fractional Laplacian spectrum, we show a convergence criterion for the $(PS)_c$ sequence and obtain a nontrivial solution by the Linking theorem. We will consider the following notation for the product space $S\times S:=S^2$ and

$$w^+(x):=\max\{w(x),0\},\quad w^-(x):=\min\{w(x),0\},$$

for positive and negative part of a function w. Consequently we obtain $w=w^++w^-$. During chains of inequalities, universal constants will be denoted by the same letter C even if their numerical value may change from line to line.

2. Preliminaries

2.1. **Functional setting.** The starting point to the variational approach to problem (1.1) is the following well-known Hardy-Littlewood-Sobolev inequality, which leads to a new type of critical problem with nonlocal nonlinearities driven by the Riesz potential.

Proposition 2.1 ([24, Theorem 4.3]). *Let $t, r > 1$ and $0 < \mu < N$ with $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f, h such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}\,dx\,dy \le C(t, N, \mu, r)\|f\|_{L^t(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}. \tag{2.1}$$

If $t = r = \frac{2N}{2N-\mu}$ then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \Big\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big\}^{-1+\frac{\mu}{N}}.$$

In this case, there is equality in (2.1) if and only if $f \equiv (constant)h$ and

$$h(x) = A(\gamma^2 + |x-a|^2)^{\frac{-(2N-\mu)}{2}}$$

for some $A \in \mathcal{C}, 0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Remark 2.2. For $u \in H^s(\mathbb{R}^N)$, let $f = h = |u|^p$, by Hardy-Littlewood-Sobolev inequality,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\mu}\,dx\,dy$$

is well defined for all p satisfying

$$2_\mu := \Big(\frac{2N-\mu}{N}\Big) \le p \le \Big(\frac{2N-\mu}{N-2s}\Big) := 2^*_\mu.$$

Now, with Proposition 2.1, we can consider the Hilbert space given by the product space

$$Y(\Omega) := X(\Omega) \times X(\Omega),$$

which is equipped with the inner product

$$\langle (u, v), (\varphi, \psi)\rangle_Y := \langle u, \varphi\rangle_X + \langle v, \psi\rangle_X$$

and the norm

$$\|(u, v)\|_Y := (\|u\|_X^2 + \|v\|_X^2)^{1/2}.$$

$L^m(\Omega) \times L^m(\Omega) (m > 1)$ is a Banach space equipped with the standard product norm

$$\|(u, v)\|_{L^m \times L^m} := (\|u\|_{L^m}^2 + \|v\|_{L^m}^2)^{1/2}.$$

Recall that

$$\mu_1 |U|^2 \le (AU, U)_{R^2} \le \mu_2 |U|^2, \quad \text{for all } U := (u, v) \in \mathbb{R}^2. \tag{2.2}$$

By a solution of (1.1), we mean a weak solution, that is, a pair of functions $(u, v) \in Y(\Omega)$ such that

$$\langle (u, v), (\varphi, \psi)\rangle_Y - \int_\Omega (A(u, v), (\varphi, \psi))_{\mathbb{R}^2} dx - \int_\Omega \frac{\partial F}{\partial u}\varphi dx - \int_\Omega \frac{\partial F}{\partial v}\psi dx = 0,$$

for all $(\varphi, \psi) \in Y(\Omega)$, where

$$\begin{aligned} F(u, v) = \frac{2}{p+q} \int_\Omega \frac{|v(y)|^q}{|x-y|^\mu} dy |u|^p + \frac{1}{2^*_\mu}\Big[\xi_1 \int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy |u|^{2^*_\mu} \\ + \xi_2 \int_\Omega \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu} dy |v|^{2^*_\mu}\Big]. \end{aligned} \tag{2.3}$$

Define the functional $J_s : Y(\Omega) \to \mathbb{R}$ by setting

$$\begin{aligned} J_s(U) \equiv J_s(u,v) =& \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2 + |v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ & - \frac{1}{2}\int_{\mathbb{R}^N} (A(u,v),(u,v))_{\mathbb{R}^2} dx - \int_\Omega F(U)dx, \end{aligned}$$

whose Fréchet derivative is

$$\begin{aligned} & J_s'(u,v)(\varphi,\psi) \\ & = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y)) + (v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy \\ & \quad - \int_\Omega (A(u,v),(\varphi,\psi))_{\mathbb{R}^2} dx - \frac{2p}{p+q}\int_\Omega \frac{|u(x)|^{p-2}u(x)|v(y)|^q}{|x-y|^\mu}\varphi\,dx\,dy \\ & \quad - \frac{2q}{p+q}\int_\Omega \frac{|u(x)|^p|v(y)|^{q-2}v(y)}{|x-y|^\mu}\psi\,dx\,dy \\ & \quad - 2\xi_1\int_\Omega \frac{|u(x)|^{2^*_\mu-2}u(x)|u(y)|^{2^*_\mu}}{|x-y|^\mu}\varphi\,dx\,dy \\ & \quad - 2\xi_2\int_\Omega \frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu-2}v(y)}{|x-y|^\mu}\psi\,dx\,dy, \end{aligned}$$

for every $(\varphi,\psi) \in Y(\Omega)$.

2.2. **Abstract critical point theorems.** We will prove Theorems 1.3 and 1.4 using the following abstract critical point theorems, respectively.

Theorem 2.3 (Mountain Pass theorem [40, Theorem 2.10])**.** *Let X be a Banach space, $J \in C^1(X,\mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} J(u) > J(0) \geq J(e).$$

If J satisfies the $(PS)_c$ condition with

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)), \quad \Gamma := \{\gamma \in C([0,1],X) : \gamma(0)=0, \gamma(1)=e\}.$$

Then c is a critical value of J.

Theorem 2.4 (Linking theorem [40, Theorem 2.12])**.** *Let X be a real Banach space with $X = V \oplus W$, where V is finite dimensional. Suppose $J \in C^1(X,\mathbb{R})$ and*

(i) *There are constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho \cap W} \geq \alpha$, and*
(ii) *There is an $e \in \partial B_\rho \cap W$ and constants $R_1, R_2 > \rho$ such that $J|_{\partial Q} \leq 0$, where*

$$Q = (\overline{B_{R_1}} \cap V) \oplus \{re, 0 < r < R_2\}.$$

Then J possesses a $(PS)_c$ sequence where $c \geq \alpha$ can be characterized as

$$c = \inf_{h\in\Gamma}\max_{u\in Q} J(h(u)),$$

where $\Gamma = \{h \in C(\overline{Q},X) : h = id \text{ on } \partial Q\}$.

Remark 2.5. Here ∂Q is the boundary of Q relative to the space $V \oplus \operatorname{span}\{e\}$, and when $V = \{0\}$, this theorem refers to the usual Mountain Pass theorem. We recall that if $J|_V \le 0$ and $J(u) \le 0$ for all $u \in V \oplus \mathrm{span}\{e\}$ with $\|u\| \ge R$, then J satisfies (ii) for R large enough. Fixed $k \in \mathbb{N}$, define the following subspaces

$$V = \mathrm{span}\{(0, \varphi_{1,s}), (\varphi_{1,s}, 0), (0, \varphi_{2,s}), (\varphi_{2,s}, 0), \dots, (0, \varphi_{k-1,s}), (\varphi_{k-1,s}, 0)\},$$
$$W = V^{\perp} = (\mathbb{P}_k)^2.$$

3. *Case* $1 : \xi_1 = \xi_2 = 0,\ 1 < p, q < 2^*_\mu$

3.1. **Proof of Theorem 1.2.** Let Ω be a bounded domain and suppose that $b \ge 0$ and

$$\mu_2 < \lambda_{1,s}. \tag{3.1}$$

Consider the function $I : Y(\Omega) \to \mathbb{R}$ defined by

$$I(U) := \frac{1}{2}\|U\|_Y^2 - \frac{1}{2}\int_\Omega (AU, U)_{\mathbb{R}^2} dx.$$

We shall minimize the functional I restricted to the set

$$\mathcal{M} := \{U = (u, v) \in Y(\Omega) : \int_\Omega \int_\Omega \frac{|u^+(x)|^p |v^+(y)|^q}{|x-y|^\mu}\, dx\, dy = 1\}.$$

By (3.1) the embedding $X(\Omega) \hookrightarrow L^2(\Omega)$ (with the sharp constant $\lambda_{1,s}$), we have

$$I(U) \ge \frac{1}{2}\min\{1, (1 - \frac{\mu_2}{\lambda_{1,s}})\}\|U\|_Y^2 \ge 0. \tag{3.2}$$

Define

$$I_0 := \inf_{\mathcal{M}} I,$$

and let $(U_n) = (u_n, v_n) \subset \mathcal{M}$ be a minimizing sequence for I_0. Then $I(U_n) = I_0 + o_n(1) \le C$, for some $C > 0$ (where $o_n(1) \to 0$, as $n \to \infty$) and consequently by (3.2), we obtain

$$[u_n]_s^2 + [v_n]_s^2 = \|u_n\|_X^2 + \|v_n\|_X^2 = \|U_n\|_Y^2 \le C'.$$

Hence, there are two subsequences of $\{u_n\} \subset X(\Omega)$ and $\{v_n\} \subset X(\Omega)$ (that we will still label as u_n and v_n) such that $U_n = (u_n, v_n)$ converges to some $U = (u, v)$ in $Y(\Omega)$ weakly and

$$[u]_s^2 \le \liminf_n \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy, \tag{3.3}$$

$$[v]_s^2 \le \liminf_n \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy\,. \tag{3.4}$$

Now we will show that $U := (u, v) \in \mathcal{M}$. Indeed, since $(U_n) \subset \mathcal{M}$, we have

$$\int_\Omega \int_\Omega \frac{|u_n^+(x)|^p |v_n^+(y)|^q}{|x-y|^\mu}\, dx\, dy = 1.$$

In view of the compact embedding $X(\Omega) \hookrightarrow L^r(\Omega)$ for all $r < 2^*_s = \frac{2N}{N-2s}$, as $1 < p, q < 2^*_\mu$, we obtain

$$\int_\Omega \int_\Omega \frac{|u_n^+(x)|^p |v_n^+(y)|^q}{|x-y|^\mu}\, dx\, dy \to \int_\Omega \int_\Omega \frac{|u^+(x)|^p |v^+(y)|^q}{|x-y|^\mu}\, dx\, dy, \quad \text{as } n \to \infty,$$

thus $\int_\Omega \int_\Omega \frac{|u^+(x)|^p |v^+(y)|^q}{|x-y|^\mu}\, dx\, dy = 1$ and consequently $U := (u, v) \in \mathcal{M}$ with $u, v \ne 0$. We now show that $U = (u, v)$ is a minimizer for I on $\mathcal{M}$ and both components u, v are nonnegative. By passing to the limit in $I(U_n) = I_0 + o_n(1)$, where $o_n(1) \to 0$

as $n \to \infty$, using (3.3), (3.4) and the strong convergence of (u_n, v_n) to (u, v) in $(L^2(\Omega))^2$, as $n \to \infty$, we conclude that $I(U) \le I_0$. Moreover, since $U \in \mathcal{M}$ and $I_0 = \inf_{\mathcal{M}} I \le I(U)$, we achieve that $I(U) = I_0$. This proves the minimality of $U \in \mathcal{M}$. On the other hand, we let

$$G(U) = \int_\Omega \int_\Omega \frac{|u^+(x)|^p |v^+(y)|^q}{|x-y|^\mu} \, dx \, dy - 1,$$

where $U = (u, v) \in Y(\Omega)$. Note that $G \in C^1$ and since $U \in \mathcal{M}$,

$$G'(U)U = (p+q) \int_\Omega \int_\Omega \frac{|u^+(x)|^p |v^+(y)|^q}{|x-y|^\mu} \, dx \, dy = p + q \neq 0,$$

hence, by Lagrange Multiplier theorem, there exists a multiplier $\zeta \in \mathbb{R}$ such that

$$I'(U)(\varphi, \psi) = \zeta G'(U)(\varphi, \psi), \quad \forall (\varphi, \psi) \in Y(\Omega). \tag{3.5}$$

Taking $(\varphi, \psi) = (u^-, v^-) := U^-$in (3.5), we obtain

$$\begin{aligned} \|U^-\|_Y^2 = &\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} \, dx \, dy \\ &+ \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2s}} \, dx \, dy + \int_\Omega (AU, U^-)_{\mathbb{R}^2} dx. \end{aligned}$$

Using this formula in the expression of $I(U^-)$, we have

$$\begin{aligned} I(U^-) = &\frac{b}{2} \int_\Omega (v^+u^- + u^+v^-) dx + \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} \, dx \, dy \\ &+ \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2s}} \, dx \, dy \le 0, \end{aligned}$$

since $b \ge 0$, $u^- \le 0$ and $u^+ \ge 0$. Furthermore,

$$I(U^-) \ge \frac{1}{2} \min \big\{1, (1 - \frac{\mu_2}{\lambda_{1,s}})\big\} \|U^-\|_Y^2 \ge 0,$$

we obtain $U^- = (u^-, v^-) = (0, 0)$ and therefore $u, v \ge 0$. We now prove the existence of a positive solution to (1.1). Using again (3.5), we see that

$$\|U\|_Y^2 - \int_\Omega (AU, U)_{\mathbb{R}^2} dx - \zeta(p+q) = 0$$

and since $U \in \mathcal{M}$, we conclude that

$$I_0 = I(U) = \frac{\zeta(p+q)}{2} > 0,$$

Then by (3.5), U satisfies the following system, weakly,

$$\begin{gathered} (-\Delta)^s u = au + bv + \frac{2pI_0}{p+q} \int_\Omega \int_\Omega \frac{|u|^{p-1}|v|^q}{|x-y|^\mu} \, dx \, dy \quad \text{in } \Omega, \\ (-\Delta)^s v = bu + cv + \frac{2qI_0}{p+q} \int_\Omega \int_\Omega \frac{|u|^p |v|^{q-1}}{|x-y|^\mu} \, dx \, dy \quad \text{in } \Omega, \\ u = v = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega. \end{gathered}$$

Now using the homogeneity of the system, we obtain $\tau > 0$ such that $W = (I_0)^\tau U$ is a solution of (1.1). Since $b \ge 0$ and $u, v \ge 0$, we obtain, in the weak sense,

$$(-\Delta)^s u \ge au, \quad \text{in } \Omega;$$

$$
\begin{gathered}
(-\Delta)^s v \geq cv, \quad \text{in } \Omega; \\
u \geq 0, v \geq 0, \quad \text{in } \Omega; \\
u = v = 0, \quad \text{in } \mathbb{R}^N \backslash \Omega.
\end{gathered}
$$

By the strong maximum principle [23, Theorem 2.5] we conclude that $u, v > 0$ in Ω.

4. Case 2: $\xi_1 = \xi_2 = 0$, $p = q = 2^*_\mu$

To obtain a nonnegative solution to the system (1.1), we recall the functional

$$
\begin{aligned}
&J_s(U) \equiv J_s(u, v) \\
&= \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} (A(u, v), (u, v))_{\mathbb{R}^2} dx - \frac{1}{2^*_\mu} \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*_\mu} |v^+(y)|^{2^*_\mu}}{|x - y|^\mu} \, dx \, dy,
\end{aligned}
$$

whose Fréchet derivative is

$$
\begin{aligned}
&J_s'(u, v)(\varphi, \psi) \\
&= \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y)) + (v(x) - v(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} \, dx \, dy \\
&\quad - \int_\Omega (A(u, v), (\varphi, \psi))_{\mathbb{R}^2} dx - \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*_\mu - 1} |v^+(y)|^{2^*_\mu}}{|x - y|^\mu} \varphi \, dx \, dy \\
&\quad - \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*_\mu} |v^+(y)|^{2^*_\mu - 1}}{|x - y|^\mu} \psi \, dx \, dy,
\end{aligned}
\tag{4.1}
$$

for $(\varphi, \psi) \in Y(\Omega)$.

4.1. **Minimizers and some estimates.**

We shall use the definition

$$
S_s := \inf_{u \in X(\Omega) \backslash \{0\}} S_s(u),
$$

where

$$
S_s(u) := \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy}{(\int_{\mathbb{R}^N} |u(x)|^{2^*_s} dx)^{2/2^*_s}}
$$

is the associated Rayleigh quotient. We also define the following related minimizing problems:

$$
\begin{aligned}
S_s^H &= \inf_{u \in X(\Omega) \backslash \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy}{(\int_\Omega \int_\Omega \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x - y|^\mu} \, dx \, dy)^{1/2^*_\mu}}, \\
\widetilde{S}_s^H &= \inf_{(u,v) \in Y(\Omega) \backslash \{(0,0)\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{|x - y|^{N+2s}} \, dx \, dy}{(\int_\Omega \int_\Omega \frac{|u(x)|^{2^*_\mu} |v(y)|^{2^*_\mu}}{|x - y|^\mu} \, dx \, dy)^{1/2^*_\mu}}.
\end{aligned}
$$

Proposition 4.1. (i) ([8, Lemma 2.15]) *The constant S_s^H is achieved by u if and only if u is of the form*

$$
C\Big(\frac{t}{t^2 + |x - x_0|^2}\Big)^{\frac{N-2s}{2}}, \quad x \in \mathbb{R}^N,
$$

for some $x_0 \in \mathbb{R}^N, C > 0$ *and* $t > 0$. *Also it satisfies*

$$(-\Delta)^s u = \Big(\int_{\mathbb{R}^N} \frac{|u|^{2^*_\mu}}{|x-y|^\mu} dy \Big) |u|^{2^*_\mu - 2} u, \quad in\ \mathbb{R}^N.$$

and this characterization of u *also provides the minimizers for* S_s.

(ii) *(*[16, Lemma 2.5]*)*

$$S_s^H = \frac{S_s}{C(N,\mu)^{1/2^*_\mu}}.$$

(iii) *(*[16, Lemma 2.6]*)* $\widetilde{S}_s^H = 2 S_s^H$.

Now we construct auxiliary functions and make some estimates with the help of Proposition 4.1. From [34], consider the family of function $\{U_\epsilon\}$ defined as

$$U_\epsilon(x) = \epsilon^{-\frac{(N-2s)}{2}} u^*(\frac{x}{\epsilon}), \quad x \in \mathbb{R}^N,$$

where $u^*(x) = \overline{u}(\frac{x}{S_s^{\frac{1}{2s}}})$, $\overline{u}(x) = \frac{\widetilde{u}(x)}{\|u\|_{L^{2^*_s}}}$ and $\widetilde{u} = \alpha(\beta^2 + |x|^2)^{-\frac{N-2s}{2}}$ with $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta > 0$ are fixed constants. Then for each $\epsilon > 0$, U_ϵ satisfies

$$(-\Delta)^s u = |u|^{2^*_s - 2} u \quad \text{in } \mathbb{R}^N,$$

in addition,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{|x-y|^{N+2s}} \, dx \, dy = \int_{\mathbb{R}^N} |U_\epsilon|^{2^*_s} dx = S_s^{\frac{N}{2s}}.$$

Without loss of generality, we assume $0 \in \Omega$ and fix $\delta > 0$ such that $B_{4\delta} \subset \Omega$. Let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $0 \le \eta \le 1$ in $\mathbb{R}^N$, $\eta \equiv 1$ in B_δ and $\eta \equiv 0$ in $\mathbb{R}^N \backslash B_{2\delta}$. For $\epsilon > 0$, we define the function

$$u_\epsilon(x) = \eta(x) U_\epsilon(x),$$

for $x \in \mathbb{R}^N$. We have the following results for u_ϵ in [34, Propositions 21, 22] and [31, Proposition 7.2].

Proposition 4.2. *Let* $s \in (0,1)$ *and* $N > 2s$. *Then, the following estimates hold as* $\epsilon \to 0$*:*

$$\int_{\mathbb{R}^{2N}} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \le S_s^{\frac{N}{2s}} + O(\epsilon^{N-2s}), \tag{4.2}$$

$$\int_{\mathbb{R}^N} |u_\epsilon|^{2^*_s} dx = S_s^{\frac{N}{2s}} + O(\epsilon^N), \tag{4.3}$$

$$\int_{\mathbb{R}^N} |u_\epsilon|^2 dx \ge \begin{cases} C_s \epsilon^{2s} + O(\epsilon^{N-2s}) & \text{if } N > 4s, \\ C_s \epsilon^{2s} |\log \epsilon| + O(\epsilon^{2s}) & \text{if } N = 4s, \\ C_s \epsilon^{N-2s} + O(\epsilon^{2s}) & \text{if } 2s < N < 4s, \end{cases} \tag{4.4}$$

for some positive constant C_s *depending on* s,

$$\int_{\mathbb{R}^N} |u_\epsilon| dx = O(\epsilon^{\frac{N-2s}{2}}). \tag{4.5}$$

Remark 4.3. Using Proposition 4.1(ii), Inequality (4.2) can be written as

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{|x-y|^{N+2s}} \, dx \, dy &\le S_s^{\frac{N}{2s}} + O(\epsilon^{N-2s}) \\ &= C(N,\mu)^{\frac{N-2s}{2N-\mu} \cdot \frac{N}{2s}} (S_s^H)^{\frac{N}{2s}} + O(\epsilon^{N-2s}). \end{aligned} \tag{4.6}$$

Proposition 4.4 ([16, Proposition 2.8]). *The following estimate holds*

$$\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2^*_\mu}|u_\epsilon(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \geq C(N,\mu)^{\frac{N}{2s}}(S_s^H)^{\frac{2N-\mu}{2s}} - O(\epsilon^N). \tag{4.7}$$

Now consider the minimization problem

$$S_{s,\lambda} = \inf_{v\in X(\Omega)\setminus\{0\}} S_{s,\lambda}(v),$$

where

$$S_{s,\lambda}(v) = \frac{\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_{\mathbb{R}^N}|v(x)|^2dx}{\big(\int_\Omega\int_\Omega \frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy\big)^{1/2^*_\mu}}.$$

Lemma 4.5. *Let $N > 2s$ and $s \in (0,1)$. Then the following facts hold.*

(i) *For $N \geq 4s$, we have $S_{s,\lambda}(u_\epsilon) < S_s^H$ for all $\lambda > 0$, provided $\epsilon > 0$ is sufficiently small.*

(ii) *For $2s < N < 4s$, there exists $\lambda_s > 0$ such that for all $\lambda > \lambda_s$, we have $S_{s,\lambda}(u_\epsilon) < S_s^H$, provided $\epsilon > 0$ is sufficiently small.*

Proof. Case 1: $N > 4s$. By (4.4), (4.6) and (4.7), we infer that

$$\begin{aligned}
S_{s,\lambda}(u_\epsilon) &\leq \frac{C(N,\mu)^{\frac{N-2s}{2N-\mu}\cdot\frac{N}{2s}}(S_s^H)^{\frac{N}{2s}} + O(\epsilon^{N-2s}) - \lambda C_s\epsilon^{2s} + O(\epsilon^{N-2s})}{\big(C(N,\mu)^{\frac{N}{2s}}(S_s^H)^{\frac{2N-\mu}{2s}} - O(\epsilon^N)\big)^{1/2^*_\mu}} \\
&\leq S_s^H - \lambda C_s\epsilon^{2s} + O(\epsilon^{N-2s}) \\
&< S_s^H, \quad \text{if } \lambda > 0, \text{ and } \epsilon > 0 \text{ is sufficiently small.}
\end{aligned}$$

Case2: $N = 4s$.

$$\begin{aligned}
S_{s,\lambda}(u_\epsilon) &\leq \frac{C(N,\mu)^{\frac{N-2s}{2N-\mu}\cdot\frac{N}{2s}}(S_s^H)^{\frac{N}{2s}} + O(\epsilon^{N-2s}) - \lambda C_s\epsilon^{2s}|\log\epsilon| + O(\epsilon^{2s})}{\big(C(N,\mu)^{\frac{N}{2s}}(S_s^H)^{\frac{2N-\mu}{2s}} - O(\epsilon^N)\big)^{1/2^*_\mu}} \\
&\leq S_s^H - \lambda C_s\epsilon^{2s}|\log\epsilon| + O(\epsilon^{2s}) \\
&< S_s^H, \quad \text{if } \lambda > 0, \text{ and } \epsilon > 0 \text{ is sufficiently small.}
\end{aligned}$$

Case 3: $2s < N < 4s$.

$$\begin{aligned}
S_{s,\lambda}(u_\epsilon) &\leq \frac{C(N,\mu)^{\frac{N-2s}{2N-\mu}\cdot\frac{N}{2s}}(S_s^H)^{\frac{N}{2s}} + O(\epsilon^{N-2s}) - \lambda C_s\epsilon^{N-2s} + O(\epsilon^{2s})}{\big(C(N,\mu)^{\frac{N}{2s}}(S_s^H)^{\frac{2N-\mu}{2s}} - O(\epsilon^N)\big)^{1/2^*_\mu}} \\
&\leq S_s^H + \epsilon^{N-2s}(O(1) - \lambda C_s) + O(\epsilon^{2s}), \\
&< S_s^H,
\end{aligned}$$

for all $\lambda > 0$ large enough ($\lambda \geq \lambda_s$), $\epsilon > 0$ sufficiently small. This completes the proof. □

4.2. **Compactness convergence.**

Lemma 4.6 (Boundedness). *The $(PS)_c$ sequence $\{(u_n,v_n)\} \subset Y(\Omega)$ is bounded.*

Proof. From (2.2) and the definition of $\lambda_{1,s}$, we have

$$C + C\|(u_n,v_n)\|_Y \geq J_s(u_n,v_n) - \frac{1}{2\cdot 2^*_\mu}J_s'(u_n,v_n)(u_n,v_n)$$

$$
\begin{aligned}
&= \Big(\frac{1}{2} - \frac{1}{2\cdot 2^*_\mu}\Big)\| (u_n, v_n) \|_Y^2 \\
&\quad - \Big(\frac{1}{2} - \frac{1}{2\cdot 2^*_\mu}\Big) \int_{\mathbb{R}^N} (A(u_n, v_n), (u_n, v_n))_{\mathbb{R}^2}\, dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{2\cdot 2^*_\mu}\Big)(1 - \frac{\mu_2}{\lambda_{1,s}})\|(u_n, v_n)\|_Y^2.
\end{aligned}
$$

Since $\mu_2 < \lambda_{1,s}$, the assertion follows. $\square$

Proposition 4.7. *Let $s \in (0,1), N > 2s$ and $0 < \mu < N$. If $\{u_n\}, \{v_n\}$ are bounded sequences in $L^{\frac{2N}{N-2s}}(\Omega)$ such that $u_n \to u, v_n \to v$ almost everywhere in Ω as $n \to \infty$, we have*

$$
\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\mu}|v_n(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy - \int_\Omega\int_\Omega \frac{|(u_n-u)(x)|^{2^*_\mu}|(v_n-v)(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy \\
&\to \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy,
\end{aligned}
$$

as $n \to \infty$.

Proof. From fractional Sobolev embedding,

$$
|u_n|^{2^*_\mu} - |u_n - u|^{2^*_\mu} \rightharpoonup |u|^{2^*_\mu}, \tag{4.8}
$$

$$
|v_n|^{2^*_\mu} - |v_n - v|^{2^*_\mu} \rightharpoonup |u|^{2^*_\mu}, \tag{4.9}
$$

in $L^{\frac{2N}{2N-\mu}}(\Omega)$ as $n \to \infty$. By Proposition 2.1, we have

$$
\int_\Omega \frac{|u_n(y)|^{2^*_\mu} - |(u_n-u)(y)|^{2^*_\mu}}{|x-y|^\mu} dy \rightharpoonup \int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy, \tag{4.10}
$$

$$
\int_\Omega \frac{|v_n(y)|^{2^*_\mu} - |(v_n-v)(y)|^{2^*_\mu}}{|x-y|^\mu} dy \rightharpoonup \int_\Omega \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu} dy, \tag{4.11}
$$

in $L^{\frac{2N}{\mu}}(\Omega)$ as $n \to \infty$. On the other hand, notice that

$$
\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\mu}|v_n(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy - \int_\Omega\int_\Omega \frac{|(u_n-u)(x)|^{2^*_\mu}|(v_n-v)(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy \\
&= \int_\Omega\int_\Omega \frac{(|u_n(x)|^{2^*_\mu} - |(u_n-u)(x)|^{2^*_\mu})(|v_n(y)|^{2^*_\mu} - |(v_n-v)(y)|^{2^*_\mu})}{|x-y|^\mu}\, dx\, dy \\
&\quad + \int_\Omega\int_\Omega \frac{(|u_n(x)|^{2^*_\mu} - |(u_n-u)(x)|^{2^*_\mu})|(v_n-v)(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy \\
&\quad + \int_\Omega\int_\Omega \frac{(|v_n(x)|^{2^*_\mu} - |(v_n-v)(x)|^{2^*_\mu})|(u_n-u)(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy.
\end{aligned} \tag{4.12}
$$

From boundness of $\{u_n\}$ and $\{v_n\}$ in $L^{\frac{2N}{2N-\mu}}(\Omega)$, we have $|u_n - u|^{2^*_\mu} \rightharpoonup 0, |v_n - v|^{2^*_\mu} \rightharpoonup 0$ in $L^{\frac{2N}{2N-\mu}}(\Omega)$ as $n \to \infty$. From (4.8)–(4.12), the result follows. $\square$

Next we give a compactness result, which is crucial for applying Theorem 2.3 to our functional J_s.

Lemma 4.8. *If $\{(u_n, v_n)\} \subset Y(\Omega)$ is a $(PS)_c$ sequence for the functional J_s with*

$$c < \frac{N+2s-\mu}{2N-\mu}(S_s^H)^{\frac{2N-\mu}{N+2s-\mu}},$$

then $\{(u_n, v_n)\}$ has a convergent subsequence.

Proof. Let (u_0, v_0) be the weak limit of $\{(u_n, v_n)\}$ and define $w_n := u_n - u_0$, $z_n := v_n - v_0$, then we know $w_n \rightharpoonup 0, z_n \rightharpoonup 0$ in $X(\Omega)$ and $w_n \to 0$ a.e. in $\Omega, z_n \to 0$ a.e. in Ω as $n \to \infty$. Moreover, by [29, Lemma 5] and the Brézis-Lieb lemma, we know that

$$\|u_n\|_X^2 = \|w_n\|_X^2 + \|u_0\|_X^2 + o_n(1), \quad \|v_n\|_X^2 = \|z_n\|_X^2 + \|v_0\|_X^2 + o_n(1),$$
$$\|u_n\|_{L^2}^2 = \|w_n\|_{L^2}^2 + \|u_0\|_{L^2}^2 + o_n(1), \quad \|v_n\|_{L^2}^2 = \|z_n\|_{L^2}^2 + \|v_0\|_{L^2}^2 + o_n(1).$$

By Proposition 4.7, we obtain

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|u_n^+(x)|^{2_\mu^*}|v_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy &= \int_\Omega\int_\Omega \frac{|w_n^+(x)|^{2_\mu^*}|z_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \\ &\quad + \int_\Omega\int_\Omega \frac{|u_0^+(x)|^{2_\mu^*}|v_0^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \\ &\quad + o_n(1).\end{aligned}$$

Consequently,

$$\begin{aligned}
&c \leftarrow J_s(u_n, v_n)\\
&= \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2 + |v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} (A(u_n,v_n),(u_n,v_n))_{\mathbb{R}^2}dx - \frac{1}{2_\mu^*}\int_\Omega\int_\Omega \frac{|u_n^+(x)|^{2_\mu^*}|v_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\\
&\ge \frac{1}{2}\Big(\int_{\mathbb{R}^{2N}} \frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad + \int_{\mathbb{R}^{2N}} \frac{|z_n(x)-z_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^{2N}} \frac{|v_0(x)-v_0(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)\\
&\quad - \frac{\mu_2}{2}\Big(\int_{\mathbb{R}^N}|w_n|^2dx + \int_{\mathbb{R}^N}|z_n|^2dx + \int_{\mathbb{R}^N}|u_0|^2dx + \int_{\mathbb{R}^N}|v_0|^2dx\Big)\\
&\quad - \frac{1}{2_\mu^*}\Big(\int_\Omega\int_\Omega \frac{|w_n^+(x)|^{2_\mu^*}|z_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy + \int_\Omega\int_\Omega \frac{|u_0^+(x)|^{2_\mu^*}|v_0^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big) + o_n(1);
\end{aligned}$$

therefore,

$$\begin{aligned}
c &\ge J_s(u_0,v_0) + \frac{1}{2}\Big(\int_{\mathbb{R}^{2N}} \frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad + \int_{\mathbb{R}^{2N}} \frac{|z_n(x)-z_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big) - \frac{\mu_2}{2}\Big(\int_{\mathbb{R}^N}|w_n|^2dx + \int_{\mathbb{R}^N}|z_n|^2dx\Big)\\
&\quad - \frac{1}{2_\mu^*}\int_\Omega\int_\Omega \frac{|w_n^+(x)|^{2_\mu^*}|z_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy + o_n(1).
\end{aligned} \tag{4.13}$$

From the boundedness of Palais-Smale sequences (see Lemma 4.6) and compact embedding theorems, we have (u_0, v_0) weakly in $Y(\Omega)$, $(u_n, v_n) \to (u_0, v_0)$ a.e. in

Ω and strongly in $L^r(\Omega)\times L^r(\Omega)$ for $1\le r<2_s^*$. Then

$$
\begin{gathered}
|u_n^+|^{2_\mu^*} \rightharpoonup |u_0^+|^{2_\mu^*} \quad \text{in } L^{\frac{2N}{2N-\mu}}(\Omega),\\
|v_n^+|^{2_\mu^*} \rightharpoonup |v_0^+|^{2_\mu^*} \quad \text{in } L^{\frac{2N}{2N-\mu}}(\Omega)
\end{gathered}
$$

and

$$
\begin{gathered}
|u_n^+|^{2_\mu^*-1} \rightharpoonup |u_0^+|^{2_\mu^*-1} \quad \text{in } L^{\frac{2N}{N+2s-\mu}}(\Omega),\\
|v_n^+|^{2_\mu^*-1} \rightharpoonup |v_0^+|^{2_\mu^*-1} \quad \text{in } L^{\frac{2N}{N+2s-\mu}}(\Omega),
\end{gathered}
$$

as $n\to\infty$. By Proposition 2.1, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\Omega)$ to $L^{\frac{2N}{\mu}}(\Omega)$ which gives

$$
\begin{gathered}
\int_\Omega \frac{|u_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}dy \rightharpoonup \int_\Omega \frac{|u_0^+(y)|^{2_\mu^*}}{|x-y|^\mu}dy \quad \text{in } L^{\frac{2N}{\mu}}(\Omega),\\
\int_\Omega \frac{|v_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}dy \rightharpoonup \int_\Omega \frac{|v_0^+(y)|^{2_\mu^*}}{|x-y|^\mu}dy \quad \text{in } L^{\frac{2N}{\mu}}(\Omega),
\end{gathered}
$$

as $n\to\infty$. Combining all these, we obtain

$$
\begin{gathered}
\int_\Omega \frac{|u_n^+(y)|^{2_\mu^*}|v_n^+(x)|^{2_\mu^*-1}}{|x-y|^\mu}dy \rightharpoonup \int_\Omega \frac{|u_0^+(y)|^{2_\mu^*}|v_0^+(x)|^{2_\mu^*-1}}{|x-y|^\mu}dy \text{ in } L^{\frac{2N}{N+2s}}(\Omega),\\
\int_\Omega \frac{|v_n^+(y)|^{2_\mu^*}|u_n^+(x)|^{2_\mu^*-1}}{|x-y|^\mu}dy \rightharpoonup \int_\Omega \frac{|v_0^+(y)|^{2_\mu^*}|u_0^+(x)|^{2_\mu^*-1}}{|x-y|^\mu}dy \text{ in } L^{\frac{2N}{N+2s}}(\Omega),
\end{gathered} \tag{4.14}
$$

as $n\to\infty$. Since, for any $\varphi,\psi\subset X(\Omega)$,

$$
\begin{aligned}
&0\leftarrow J_s'(u_n,v_n)(\varphi,\psi)\\
&=\int_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))+(v_n(x)-v_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad-\int_\Omega (A(u_n,v_n),(\varphi,\psi))_{\mathbb{R}^2}dx-\int_\Omega\int_\Omega \frac{|v_n^+(y)|^{2_\mu^*}|u_n^+(x)|^{2_\mu^*-1}\varphi(x)}{|x-y|^\mu}\,dx\,dy\\
&-\int_\Omega\int_\Omega \frac{|u_n^+(x)|^{2_\mu^*}|v_n^+(y)|^{2_\mu^*-1}\psi(y)}{|x-y|^\mu}\,dx\,dy
\end{aligned}
$$

Passing to the limit as $n\to\infty$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(\varphi(x)-\varphi(y))+(v_0(x)-v_0(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&-\int_\Omega (A(u_0,v_0),(\varphi,\psi))_{\mathbb{R}^2}dx-\int_\Omega\int_\Omega \frac{|v_0^+(y)|^{2_\mu^*}|u_0^+(x)|^{2_\mu^*-1}\varphi(x)}{|x-y|^\mu}\,dx\,dy\\
&-\int_\Omega\int_\Omega \frac{|u_0^+(x)|^{2_\mu^*}|v_0^+(y)|^{2_\mu^*-1}\psi(y)}{|x-y|^\mu}\,dx\,dy=0,
\end{aligned} \tag{4.15}
$$

which means that (u_0,v_0) is a weak solution of (1.1).

Taking $\varphi=u_0,\psi=v_0$ as a test function in equation (4.15), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^2+|v_0(x)-v_0(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&=\int_\Omega (A(u_0,v_0),(u_0,v_0))_{\mathbb{R}^2}dx+2\int_\Omega\int_\Omega \frac{|u_0^+(x)|^{2_\mu^*}|v_0^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy,
\end{aligned}
$$

so for $0<\mu<N$,

$$J_s(u_0,v_0)=\frac{N+2s-\mu}{2N-\mu}\int_\Omega\int_\Omega\frac{|u_0^+(x)|^{2_\mu^*}|v_0^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\geq 0. \tag{4.16}$$

Using (4.13), (4.16) and $\int_{\mathbb{R}^N}|w_n|^2dx\to 0, \int_{\mathbb{R}^N}|z_n|^2dx\to 0$, as $n\to\infty$, we obtain

$$\begin{aligned}c\geq{}&\frac12\Big(\int_{\mathbb{R}^{2N}}\frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathbb{R}^{2N}}\frac{|z_n(x)-z_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)\\&-\frac{1}{2_\mu^*}\int_\Omega\int_\Omega\frac{|w_n^+(x)|^{2_\mu^*}|z_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy+o_n(1).\end{aligned} \tag{4.17}$$

Since (u_0,v_0) is a weak solution of (1.1), (u_0,v_0) must be a critical point of J_s which gives $\langle J_s'(u_0,v_0),(u_0,v_0)\rangle=0$, hence

$$\begin{aligned}o_n(1)&=\langle J_s'(u_n,v_n),(u_n,v_n)\rangle\\&=\langle J_s'(u_0,v_0),(u_0,v_0)\rangle+\int_{\mathbb{R}^{2N}}\frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\&\quad+\int_{\mathbb{R}^{2N}}\frac{|z_n(x)-z_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy-2\int_\Omega\int_\Omega\frac{|w_n^+(x)|^{2_\mu^*}|z_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\\&\quad+o_n(1)\\&=\int_{\mathbb{R}^{2N}}\frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathbb{R}^{2N}}\frac{|z_n(x)-z_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\&\quad-2\int_\Omega\int_\Omega\frac{|w_n^+(x)|^{2_\mu^*}|z_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy+o_n(1).\end{aligned} \tag{4.18}$$

From (4.18), there exists a nonnegative constant l such that

$$\int_{\mathbb{R}^{2N}}\frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy+\int_{\mathbb{R}^{2N}}\frac{|z_n(x)-z_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\to l,$$

$$\int_\Omega\int_\Omega\frac{|w_n^+(x)|^{2_\mu^*}|z_n^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\to\frac{l}{2},$$

as $n\to\infty$. Thus from (4.17), we obtain

$$c\geq\frac{N+2s-\mu}{4N-2\mu}l. \tag{4.19}$$

By the definition of the best constant $\widetilde{S}_s^H$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2+|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\&\geq\widetilde{S}_s^H\Big(\int_\Omega\int_\Omega\frac{|u(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big)^{\frac{N-2s}{2N-\mu}},\end{aligned}$$

which yields $l\geq\widetilde{S}_s^H(\frac{l}{2})^{\frac{N-2s}{2N-\mu}}$. Thus we have either $l=0$ or

$$l\geq\frac{1}{2^{\frac{N-2s}{N+2s-\mu}}}(\widetilde{S}_s^H)^{\frac{2N-\mu}{N+2s-\mu}}.$$

In the latter case, by Proposition 4.1(iii), from (4.19) we obtain

$$c\geq\frac{N+2s-\mu}{2N-\mu}(S_s^H)^{\frac{2N-\mu}{N+2s-\mu}}$$

which contradicts with the fact that $c < \frac{N+2s-\mu}{2N-\mu}(S_s^H)^{\frac{2N-\mu}{N+2s-\mu}}$. Thus $l=0$, and

$$\|(u_n-u_0, v_n-v_0)\|_Y \to 0,$$

as $n\to\infty$. This completes the proof. □

4.3. **Mountain pass geometry.**

Lemma 4.9. *Suppose $\mu_2 < \lambda_{1,s}$. The functional J_s satisfies*

- (i) *There exist $\beta, \rho > 0$ such that $J_s(u,v) \geq \beta$, if $\|(u,v)\|_Y = \rho$;*
- (ii) *There exists $(e_1, e_2) \in Y(\Omega)\backslash\{(0,0)\}$ with $\|(e_1,e_2)\|_Y > \rho$ such that $J_s(e_1,e_2) \leq 0$.*

Proof. (i) From the definition of $\widetilde{S}_s^H$, we obtain

$$\int_\Omega\int_\Omega \frac{|u^+(x)|^{2_\mu^*}|v^+(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \leq \frac{1}{(\widetilde{S}_s^H)^{2_\mu^*}}\|(u,v)\|_Y^{2\cdot 2_\mu^*}. \tag{4.20}$$

Combining this with (2.2) and the definition of $\lambda_{1,s}$, we obtain

$$J_s(u,v) \geq \frac{1}{2}(1-\frac{\mu_2}{\lambda_{1,s}})\|(u,v)\|_Y^2 - \frac{1}{2_\mu^*(\widetilde{S}_s^H)^{2_\mu^*}}\|(u,v)\|_Y^{2\cdot 2_\mu^*}.$$

Since $2 < 2\cdot 2_\mu^*$ and thus, some $\beta,\rho>0$ can be chosen such that $J_s(u,v)\geq\beta$ for $\|(u,v)\|_Y=\rho$.

(ii) Choose $(\widetilde{u_0},\widetilde{v_0}) \in Y(\Omega)\backslash\{(0,0)\}$ with $\widetilde{u_0}>0, \widetilde{v_0}>0$ a.e. Then

$$\begin{aligned}
J_s(t\widetilde{u_0}, t\widetilde{v_0}) &= \frac{t^2}{2}\int_{\mathbb{R}^{2N}} \frac{|\widetilde{u_0}(x)-\widetilde{u_0}(y)|^2+|\widetilde{v_0}(x)-\widetilde{v_0}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad - \frac{t^2}{2}\int_{\mathbb{R}^N}(A(\widetilde{u_0},\widetilde{v_0}),(\widetilde{u_0},\widetilde{v_0}))dx \\
&\quad - \frac{t^{2\cdot 2_\mu^*}}{2_\mu^*}\int_\Omega\int_\Omega \frac{|\widetilde{u_0}(x)|^{2_\mu^*}|\widetilde{v_0}(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy.
\end{aligned}$$

Choosinig $t>0$ sufficiently large, the assertion follows. □

4.4. **Proof of Theorem 1.3.**

Lemma 4.10. *If $(u,v)\subset Y(\Omega)$ is a critical point of J_s, then $(u^-,v^-)=(0,0)$.*

Proof. By choosing $\varphi := u^- \in X(\Omega)$ and $\psi := v^- \in X(\Omega)$ as test functions in (4.1) and using the elementary inequality

$$(w_1-w_2)(w_1^- - w_2^-) \geq (w_1^- - w_2^-)^2 \quad \text{for all } w_1, w_2\in\mathbb{R},$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(u^-(x)-u^-(y)) + (v(x)-v(y))(v^-(x)-v^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\geq \int_{\mathbb{R}^{2N}} \frac{(u^-(x)-u^-(y))^2+(v^-(x)-v^-(y))^2}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned}$$

Now, note that, since $b\geq 0$ and $w^-\leq 0$ and $w^+\geq 0$, it holds

$$\int_{\mathbb{R}^N}(A(u,v),(u^-,v^-))_{\mathbb{R}^2}dx \leq \int_{\mathbb{R}^N}(A(u^-,v^-),(u^-,v^-))_{\mathbb{R}^2}dx.$$

In fact, it follows that

$$(A(u,v),(u^-,v^-))_{\mathbb{R}^2} = (A(u^-,v^-),(u^-,v^-))_{\mathbb{R}^2} + b(v^+u^- + u^+v^-),$$

$$\leq (A(u^-, v^-), (u^-, v^-))_{\mathbb{R}^2}.$$

In turn, from the formula for $J_s'(u,v)(u^-,v^-)$, it follows that

$$\begin{aligned} J_s'(u,v)(u^-,v^-) &\geq \int_{\mathbb{R}^{2N}} \frac{(u^-(x)-u^-(y))^2 + (v^-(x)-v^-(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad - \int_\Omega (A(u^-,v^-),(u^-,v^-))_{\mathbb{R}^2}dx \\ &\geq I(u^-) + I(v^-), \end{aligned}$$

where we have set

$$I(w) := \int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \mu_2 \int_\Omega |w|^2 dx = \|w\|_X^2 - \mu_2\|w\|_{L^2(\Omega)}^2.$$

On the other hand, by the definition of $\lambda_{1,s}$, we have

$$I(w) \geq (1 - \frac{\mu_2}{\lambda_{1,s}})\|w\|_X^2,$$

which finally yields the inequality

$$J_s'(u,v)(u^-,v^-) \geq (1 - \frac{\mu_2}{\lambda_{1,s}})(\|u^-\|_X^2 + \|v^-\|_X^2).$$

Since $\{(u,v)\} \subset Y(\Omega)$ is a critical point of J_s, we obtain $J_s'(u,v)(u^-,v^-) = 0$, from which that assertion immediately follows. $\square$

From Lemma 4.9 and the Mountain Pass theorem, there exists a $(PS)_c$ sequence $\{(u_n,v_n)\} \subset Y(\Omega)$ such that $J_s(u_n,v_n) \to c$ and $J_s'(u_n,v_n) \to 0$ in $Y(\Omega)$, at the minimax level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_s(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], Y(\Omega)) : \gamma(0) = (0,0) \text{ and } J_s(\gamma(1)) \leq 0\}.$$

Let $u_\epsilon \geq 0$ as in Proposition 4.2, fix $\epsilon > 0$ sufficiently small such that Lemma 4.5 holds, by (2.2), for every $t \geq 0$, we obtain

$$J_s(tu_\epsilon, tu_\epsilon) \leq t^2\|u_\epsilon\|_X^2 - \mu_1 t^2 \|u_\epsilon\|_{L^2}^2 - \frac{t^{2\cdot 2_\mu^*}}{2_\mu^*} \int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2_\mu^*}|u_\epsilon(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy := f(t).$$

It is easy to verify that $f(t)$ attains its maximum at

$$t_* = \Big[\frac{\|u_\epsilon\|_X^2 - \mu_1\|u_\epsilon\|_{L^2}^2}{\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2_\mu^*}|u_\epsilon(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy}\Big]^{\frac{1}{2\cdot 2_\mu^* - 2}}.$$

By the definition of $S_{s,\lambda}(v)$ and Lemma 4.5, we have

$$\begin{aligned} c &\leq \sup_{t\geq 0} J_s(tu_\epsilon, tu_\epsilon) \\ &\leq f(t_*) \\ &= \frac{N+2s-\mu}{2N-\mu}\Big[\frac{(\|u_\epsilon\|_X^2 - \mu_1\|u_\epsilon\|_{L^2}^2)}{(\int_\Omega\int_\Omega \frac{|u_\epsilon(x)|^{2_\mu^*}|u_\epsilon(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy)^{1/2_\mu^*}}\Big]^{\frac{2N-\mu}{N+2s-\mu}} \\ &= \frac{N+2s-\mu}{2N-\mu}(S_{s,\mu_1}(u_\epsilon))^{\frac{2N-\mu}{N+2s-\mu}} \end{aligned}$$

$$< \frac{N+2s-\mu}{2N-\mu}(S_s^H)^{\frac{2N-\mu}{N+2s-\mu}},$$

if one of the following two conditions holds,

(i) $N \geq 4s$ and $\mu_1 > 0$, or
(ii) $2s < N < 4s$ and μ_1 is large enough.

Therefore, from Lemma 4.8, $\{(u_n, v_n)\}$ has a convergent subsequence, and J_s has a critical value $c \in (0, \frac{N+2s-\mu}{2N-\mu}(S_s^H)^{\frac{2N-\mu}{N+2s-\mu}})$. Moreover, from Lemma 4.10, we conclude that the solution is nonnegative.

5. Case 3: $\xi_1, \xi_2 > 0$, $p = q = 2_\mu^*$

In this case, the function $J_s : Y(\Omega) \to \mathbb{R}$ is

$$\begin{aligned}
&J_s(U) \equiv J_s(u,v)\\
&= \frac{1}{2}\Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)^2 + |v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} (A(u,v),(u,v))_{\mathbb{R}^2}dx - \frac{1}{2_\mu^*}\Big(\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\\
&\quad + \xi_1\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy + \xi_2\int_\Omega\int_\Omega \frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big),
\end{aligned}$$

whose Fréchet derivative is

$$\begin{aligned}
&J_s'(u,v)(\varphi,\psi)\\
&= \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y)) + (v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&- \int_\Omega (A(u,v),(\varphi,\psi))_{\mathbb{R}^2}dx - \int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*-2}u(x)|v(y)|^{2_\mu^*}}{|x-y|^\mu}\varphi\,dx\,dy\\
&- \int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|v(y)|^{2_\mu^*-2}v(y)}{|x-y|^\mu}\psi\,dx\,dy - 2\xi_1\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*-2}u(x)|u(y)|^{2_\mu^*}}{|x-y|^\mu}\varphi\,dx\,dy\\
&- 2\xi_2\int_\Omega\int_\Omega \frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*-2}v(y)}{|x-y|^\mu}\psi\,dx\,dy,
\end{aligned}$$

for $(\varphi,\psi) \in Y(\Omega)$. Meanwhile,

$$F(u,v) = \frac{1}{2_\mu^*}\Big[\int_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u|^{2_\mu^*} + \xi_1\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u|^{2_\mu^*} + \xi_2\int_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu}dy|v|^{2_\mu^*}\Big].$$

5.1. **Minimizers.**

For notational convenience, if $(u,v) \in Y(\Omega)$, we set

$$B(u,v) := \int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy,$$

and let

$$\widetilde{S}_\xi^H = \inf_{(u,v)\in Y(\Omega)\setminus\{(0,0)\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2+|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\big(B(u,v)+\xi_1 B(u,u)+\xi_2 B(v,v)\big)^{1/2_\mu^*}}. \tag{5.1}$$

Remark 5.1. Let $T(u,v) := |u|^{2^*_\mu}|v|^{2^*_\mu} + \xi_1|u|^{2\cdot 2^*_\mu} + \xi_2|v|^{2\cdot 2^*_\mu}$. It is clear that $T(u,v)^{1/2^*_\mu}$ is 2-homogeneous, i.e.

$$T(\varpi U) = \varpi^{2\cdot 2^*_\mu} T(U), \quad \forall U \in \mathbb{R}^2,\ \forall \varpi \geq 0.$$

There exists a constant $M > 0$ satisfying

$$T(u,v)^{1/2^*_\mu} \leq M(|u|^2 + |v|^2), \text{ for all } u, v \in \mathbb{R}, \tag{5.2}$$

where M is the maximum of the function $T(u,v)^{1/2^*_\mu}$ attained in some (s_0,t_0) of the compact set $\{(s,t) : s,t \in \mathbb{R}, |s|^2 + |t|^2 = 2\}$. Let $m = M^{-1}$, we have that

$$T(s_0,t_0)^{1/2^*_\mu} = m^{-1}(s_0^2 + t_0^2). \tag{5.3}$$

The following basic inequality is proved in [16, Lemma 2.3].

Proposition 5.2. *For $u, v \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$, we have*

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|v(y)|^p}{|x-y|^\mu}\,dx\,dy \\ &\leq \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\mu}\,dx\,dy\Big)^{1/2}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^p|v(y)|^p}{|x-y|^\mu}\,dx\,dy\Big)^{1/2}, \end{aligned}$$

where $\mu \in (0,N)$ and $p \in [2_\mu, 2^*_\mu]$.

The following result shows the relation between S^H_s and $\widetilde{S}^H_\xi$. The proof is similar to [26, Lemma 2.3].

Lemma 5.3. *Let Ω be a smooth bounded domain, then $\widetilde{S}^H_\xi = mS^H_s$. Moreover, if g_0 realizes S^H_s, then (s_0g_0, t_0g_0) realizes $\widetilde{S}^H_\xi$, for some $s_0, t_0 > 0$.*

Proof. Let $\{g_n\} \subset X(\Omega)\backslash\{0\}$ be a minimizing sequence for S^H_s and consider the sequence $(\widetilde{u_n}, \widetilde{v_n}) = (s_0g_n, t_0g_n)$. Substituting $(\widetilde{u_n}, \widetilde{v_n})$ in quotient (5.1), we obtain

$$\frac{(s_0^2 + t_0^2)\|g_n\|_X^2}{(s_0^{2^*_\mu}t_0^{2^*_\mu} + \xi_1 s_0^{2\cdot 2^*_\mu} + \xi_2 t_0^{2\cdot 2^*_\mu})^{1/2^*_\mu} B(g_n,g_n)^{1/2^*_\mu}} \geq \widetilde{S}^H_\xi.$$

Consequently by (5.3), it follows that

$$m\frac{\|g_n\|_X^2}{B(g_n,g_n)^{1/2^*_\mu}} \geq \widetilde{S}^H_\xi. \tag{5.4}$$

Taking the limit in (5.4), we obtain

$$mS^H_s \geq \widetilde{S}^H_\xi.$$

To prove the reversed inequality, let $\{(u_n,v_n)\}$ be a minimizing sequence for $\widetilde{S}^H_\xi$. We set $u_n = r_nv_n$ for $r_n > 0$. By Proposition 5.2, we obtain

$$\frac{\|(u_n,v_n)\|_Y^2}{\big(B(u_n,v_n) + \xi_1B(u_n,u_n) + \xi_2B(v_n,v_n)\big)^{1/2^*_\mu}} \geq \frac{(1+\frac{1}{r_n^2})S^H_s}{\big(\frac{1}{r_n^{2^*_\mu}} + \xi_1 + \xi_2\frac{1}{r_n^{2\cdot 2^*_\mu}}\big)^{1/2^*_\mu}}. \tag{5.5}$$

Now, by inequality (5.2), we obtain

$$m\Big(\frac{1}{r_n^{2^*_\mu}} + \xi_1 + \xi_2\frac{1}{r_n^{2\cdot 2^*_\mu}}\Big)^{1/2^*_\mu} \leq 1 + \frac{1}{r_n^2}. \tag{5.6}$$

Hence, using inequalities (5.5) and (5.6), we have

$$\frac{\|(u_n,v_n)\|_Y^2}{\left(B(u_n,v_n)+\xi_1 B(u_n,u_n)+\xi_2 B(v_n,v_n)\right)^{1/2_\mu^*}} \geq m S_s^H.$$

Therefore, passing to the limit in the above inequality, the desired reversed inequality is obtained. □

5.2. **Compactness convergence.**

Lemma 5.4 (Boundedness)**.** *The $(PS)_c$ sequence $\{(u_n,v_n)\} \subset Y(\Omega)$ is bounded.*

Proof. Let $U_n \in Y(\Omega)$ be a $(PS)_c$ sequence, we have

$$J_s(U_n) - \frac{1}{2}\langle J_s'(U_n), U_n\rangle = (2_\mu^* - 1)\int_\Omega F(U_n)dx \leq \widetilde{C_1}(1+\|U_n\|_Y), \tag{5.7}$$

for some positive constant $\widetilde{C_1}$. From (2.2),

$$\begin{aligned} &J_s(U_n) + \frac{1}{2}\langle J_s'(U_n), U_n\rangle \\ &= \|U_n\|_Y^2 - \int_\Omega (A(u,v),(u,v))_{\mathbb{R}^2}dx - (2_\mu^*+1)\int_\Omega F(U_n)dx \\ &\leq \|U_n\|_Y^2 - \mu_1\|U_n\|_{L^2}^2 - (2_\mu^*+1)\int_\Omega F(U_n)dx \\ &\leq \widetilde{C_2}(1+\|U_n\|_Y), \end{aligned} \tag{5.8}$$

for some positive constant $\widetilde{C_2}$. Recalling that $2_s^* > 2$, by Hölder's inequality and [38, Lemma 2.2], we obtain

$$\|u_n\|_{L^2}^2 \leq |\Omega|^{\frac{2s}{N}}\|u_n\|_{L^{2_s^*}}^2 \leq \widetilde{C_3}\Big(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big)^{1/2_\mu^*},$$

$$\|v_n\|_{L^2}^2 \leq |\Omega|^{\frac{2s}{N}}\|v_n\|_{L^{2_s^*}}^2 \leq \widetilde{C_4}\Big(\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big)^{1/2_\mu^*},$$

for some positive constant $\widetilde{C_3}$ and $\widetilde{C_4}$. Combining with (5.7), we can obtain

$$\begin{aligned} \|U_n\|_{L^2}^2 &\leq \widetilde{C_5}\Big(\Big(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big)^{1/2_\mu^*} \\ &\quad + \Big(\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big)^{1/2_\mu^*}\Big) \\ &\leq \widetilde{C_5}\Big(\int_\Omega F(U_n)dx\Big)^{1/2_\mu^*} \\ &\leq \widetilde{C_6}(1+\|U_n\|_Y)^{1/2_\mu^*}, \end{aligned} \tag{5.9}$$

for some positive constants $\widetilde{C_5}$ and $\widetilde{C_6}$. Hence, by (5.7)-(5.9),

$$\|U_n\|_Y^2 \leq \widetilde{C_7}(1+\|U_n\|_Y) + \widetilde{C_8}(1+\|U_n\|_Y)^{1/2_\mu^*},$$

for some positive constant $\widetilde{C_7}$ and $\widetilde{C_8}$. Therefore, the sequence $\{U_n\}$ is bounded. □

Since $\{U_n\}$ is bounded in $Y(\Omega)$, up to a subsequence, still denoted by U_n, there exists $U = (u_0, v_0) \in Y(\Omega)$ such that

$$U_n \rightharpoonup U \quad \text{in } Y(\Omega),$$

$$U_n \rightharpoonup U \quad \text{in } L^{2_s^*}(\Omega) \times L^{2_s^*}(\Omega), \tag{5.10}$$
$$U_n \to U \text{ a.e in } \Omega,$$
$$U_n \to U \quad \text{in } L^r(\Omega) \times L^r(\Omega), \text{ for all } r \in [1, 2_s^*). \tag{5.11}$$

In addition, we have the following relations:

Lemma 5.5. (i) $J_s(U) = (2_\mu^* - 1) \int_\Omega F(U)dx \geq 0$.
(ii) $J_s(U_n) = J_s(U) + \frac{1}{2}\|U_n - U\|_Y^2 - \int_\Omega F(U_n - U)dx + o(1)$.
(iii) $\|U_n - U\|_Y^2 = 2 \cdot 2_\mu^* \int_\Omega F(U_n - U)dx + o(1)$.

Proof. (i) Since $|u_n|^{2_\mu^*} \rightharpoonup |u_0|^{2_\mu^*}, |v_n|^{2_\mu^*} \rightharpoonup |v_0|^{2_\mu^*}$ in $L^{\frac{2N}{2N-\mu}}(\Omega)$ as $n \to \infty$, by (4.14), we obtain

$$\nabla F(U_n) \rightharpoonup \nabla F(U) \quad \text{in } L^{\frac{2N}{N+2s}}(\Omega) \times L^{\frac{2N}{N+2s}}(\Omega). \tag{5.12}$$

So for any $\Theta \in Y(\Omega)$, $\int_\Omega (\nabla F(U_n), \Theta)_{\mathbb{R}^2} dx \to \int_\Omega (\nabla F(U), \Theta)_{\mathbb{R}^2} dx$, we have

$$J_s'(U_n)(\Theta) = o(1). \tag{5.13}$$

Passing to the limit in (5.13) as $n \to \infty$, and combining with the above convergences, we obtain

$$\langle U, \Theta \rangle_Y - \int_\Omega (AU, \Theta)_{\mathbb{R}^2} dx - \int_\Omega (\nabla F(U), \Theta)_{\mathbb{R}^2} dx = 0, \quad \forall \Theta \in Y(\Omega),$$

which means U is a weak solution of problem (1.1).

Notice that the nonlinearity F is $2 \cdot 2_\mu^*$-homogeneous, particularly, we have

$$(\nabla F(U), U)_{\mathbb{R}^2} = uF_u(U) + vF_v(U) = 2 \cdot 2_\mu^* F(U), \quad \forall U = (u, v) \in \mathbb{R}^2. \tag{5.14}$$

Combining this with $J_s'(U)U = 0$, we reach conclusion (i).

(ii) By Lemma 5.4 and the Brézis-Lieb Lemma, we have

$$\|U_n\|_Y^2 = \|U_n - U\|_Y^2 + \|U\|_Y^2 + o(1), \tag{5.15}$$
$$\|U_n\|_{L^{2_s^*}}^2 = \|U_n - U\|_{L^{2_s^*}}^2 + \|U\|_{L^{2_s^*}}^2 + o(1). \tag{5.16}$$

By Proposition 4.7, we obtain

$$\int_\Omega F(U_n)dx = \int_\Omega F(U)dx + \int_\Omega F(U_n - U)dx + o(1), \quad \text{as } n \to \infty. \tag{5.17}$$

Therefore, using that $U_n \to U$ in $L^r(\Omega) \times L^r(\Omega)$, for all $r \in [1, 2_s^*)$, by the definition of J_s, (5.15), (5.16) and (5.17), we deduce (ii).

(iii) By (5.10), (5.12) and (5.14), we obtain

$$\begin{aligned}
&\int_\Omega (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2} dx \\
&= \int_\Omega (\nabla F(U_n), U_n)_{\mathbb{R}^2} dx - \int_\Omega (\nabla F(U), U)_{\mathbb{R}^2} dx + o(1) \\
&= 2 \cdot 2_\mu^* \int_\Omega F(U_n)dx - 2 \cdot 2_\mu^* \int_\Omega F(U)dx + o(1).
\end{aligned}$$

Therefore, using (5.17), we obtain

$$\int_\Omega (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2} dx = 2 \cdot 2_\mu^* \int_\Omega F(U_n - U)dx + o(1). \tag{5.18}$$

On the other hand,

$$o(1) = J_s'(U_n)(U_n - U) = J_s'(U_n)(U_n - U) - J_s'(U)(U_n - U)$$

$$
\begin{aligned}
&= \langle U_n, U_n - U\rangle_Y - \int_\Omega (AU_n, U_n - U)_{\mathbb{R}^2} dx - \int_\Omega (\nabla F(U_n), U_n - U)_{\mathbb{R}^2} dx \\
&\quad - \langle U, U_n - U\rangle_Y + \int_\Omega (AU, U_n - U)_{\mathbb{R}^2} dx + \int_\Omega (\nabla F(U), U_n - U)_{\mathbb{R}^2} dx \\
&= \langle U_n - U, U_n - U\rangle_Y - \int_\Omega (A(U_n - U), U_n - U)_{\mathbb{R}^2} dx \\
&\quad - \int_\Omega (\nabla F(U_n) - \nabla F(U), U_n - U)_{\mathbb{R}^2} dx.
\end{aligned}
$$

Hence, from (5.11) and (5.18), it follows that

$$
\|U_n - U\|_Y^2 = 2 \cdot 2_\mu^* \int_\Omega F(U_n - U) dx + o(1), \quad \text{as } n \to \infty.
$$

This completes the proof. $\square$

In the next lemma, we prove a convergence criterion for the $(PS)_c$ sequences which will play an important role in applying Theorem 2.4.

Lemma 5.6. *Let $N > 2s$, $0 < \mu < N$ and $\{U_n\}$ be a $(PS)_c$ sequence of J_s with*

$$
c < \frac{N + 2s - \mu}{2N - \mu} \Big(\frac{\widetilde{S}_\xi^H}{2} \Big)^{\frac{2N-\mu}{N+2s-\mu}}. \tag{5.19}
$$

Then, $\{U_n\}$ has a convergent subsequence.

Proof. We assume that

$$
\|U_n - U\|_Y^2 \to L, \ \text{as } n \to \infty. \tag{5.20}
$$

From Lemma 5.5(iii)

$$
2 \cdot 2_\mu^* \int_\Omega F(U_n - U) dx \to L, \ \text{as } n \to \infty,
$$

and it is clear that $L \in [0, \infty)$. By the definition of $\widetilde{S}_\xi^H$, we have

$$
L \geq \widetilde{S}_\xi^H \Big(\frac{L}{2} \Big)^{\frac{1}{2_\mu^*}}
$$

and consequently, either $L = 0$ or

$$
L \geq \Big(\frac{1}{2} \Big)^{\frac{N-2s}{N+2s-\mu}} \big(\widetilde{S}_\xi^H \big)^{\frac{2N-\mu}{N+2s-\mu}}.
$$

In the latter case, from Lemma 5.5(iii), it follows that

$$
\frac{1}{2} \|U_n - U\|_Y^2 - \int_\Omega F(U_n - U) dx = \frac{N + 2s - \mu}{2(2N - \mu)} \|U_n - U\|_Y^2 + o(1).
$$

Therefore, using Lemma 5.5(ii) and the above equality, we see that

$$
\begin{aligned}
J_s(U) + \frac{N + 2s - \mu}{2(2N - \mu)} \|U_n - U\|_Y^2 &= J_s(U) + \frac{1}{2} \|U_n - U\|_Y^2 - \int_\Omega F(U_n - U) dx + o(1) \\
&= J_s(U_n) + o(1) = c + o(1), \quad \text{as } n \to \infty.
\end{aligned}
$$

Then

$$
c = J_s(U) + \frac{N + 2s - \mu}{2(2N - \mu)} L \geq \frac{N + 2s - \mu}{2(2N - \mu)} L
$$

$$\geq \frac{N+2s-\mu}{2(2N-\mu)}\Big(\frac{1}{2}\Big)^{\frac{N-2s}{N+2s-\mu}}(\widetilde{S}_\xi^H)^{\frac{2N-\mu}{N+2s-\mu}} = \frac{N+2s-\mu}{2N-\mu}\Big(\frac{\widetilde{S}_\xi^H}{2}\Big)^{\frac{2N-\mu}{N+2s-\mu}},$$

which contradicts (5.19). Thus $L = 0$ and therefore, by (5.20), we have

$$\|U_n - U\|_Y^2 \to 0, \text{ as } n \to \infty$$

and the assertion of Lemma 5.6 follows. □

5.3. **Linking geometry.**

Lemma 5.7. *If $\mathbb{F}$ is a finite dimensional subspace of $Y(\Omega)$, there exists $R > 0$ large enough such that $J_s(u, v) \leq 0$, for all $(u, v) \in \mathbb{F}$ with $\|(u, v)\|_Y \geq R$ and $uv \neq 0$.*

Proof. Choose $(\widetilde{u_0}, \widetilde{v_0}) \in \mathbb{F}$ with $\widetilde{u_0}\widetilde{v_0} \neq 0$, then

$$\begin{aligned}
&J_s(t\widetilde{u_0}, t\widetilde{v_0})\\
&= \frac{t^2}{2}\int_{\mathbb{R}^{2N}} \frac{|\widetilde{u_0}(x) - \widetilde{u_0}(y)|^2 + |\widetilde{v_0}(x) - \widetilde{v_0}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad - \frac{t^2}{2}\int_{\mathbb{R}^N} (A(\widetilde{u_0}, \widetilde{v_0}), (\widetilde{u_0}, \widetilde{v_0}))dx - \frac{t^{2\cdot 2_\mu^*}}{2_\mu^*}\Big(\int_\Omega\int_\Omega \frac{|\widetilde{u_0}(x)|^{2_\mu^*}|\widetilde{v_0}(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\\
&\quad + \xi_1\int_\Omega\int_\Omega \frac{|\widetilde{u_0}(x)|^{2_\mu^*}|\widetilde{u_0}(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy + \xi_2\int_\Omega\int_\Omega \frac{|\widetilde{v_0}(x)|^{2_\mu^*}|\widetilde{v_0}(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big),
\end{aligned}$$

by choosing $t > 0$ large enough, the assertion follows. This concludes the proof. □

Lemma 5.8. *If*

$$\lambda_{k-1,s} < \mu_1 < \lambda_{k,s} \leq \mu_2 < \lambda_{k+1,s}, \quad \text{for some } k \geq 1, \tag{5.21}$$

then the functional J_s satisfies:

(i) *there exist $\alpha, \rho > 0$ such that $J_s(u, v) \geq \alpha$ for all $(u, v) \in W$ with $\|(u, v)\|_Y = \rho$;*
(ii) *if $Q = (V \cap \overline{B}_R(0)) \oplus [0, R]e$, where $e \in W \cap \partial B_1(0)$ is a fixed vector, $J_s(u, v) < 0$ for all $(u, v) \in \partial Q$ and $R > \rho$ large enough.*

Proof. Consider following subspace $W = Z_k \oplus H$, where

$$Z_k = \operatorname{span}\{(\varphi_{k,s}, 0), (0, \varphi_{k,s})\}, \quad H = \operatorname{span}\{\overline{(\varphi_{k+1,s}, 0), (0, \varphi_{k+1,s}), \dots}\}.$$

If $U \in W$, we have that $U = U^k + \overline{U}$ with $U^k \in Z_k$ and $\overline{U} \in H$. Since $\|U\|_Y^2 = \|U^k\|_Y^2 + \|\overline{U}\|_Y^2$, by (2.2) and (4.20), we have

$$J_s(U) \geq \frac{1}{2}(\|U^k\|_Y^2 + \|\overline{U}\|_Y^2) - \frac{\mu_2}{2}(\|U^k\|_{(L^2)^2}^2 + \|\overline{U}\|_{(L^2)^2}^2) - C(\|U^k\|_Y^2 + \|\overline{U}\|_Y^2)^{2_\mu^*},$$

where $U = (u, v)$ and $C := C(\xi_1, \xi_2) > 0$ is a constant. Therefore, using that $U^k \in Z_k \subset W$ and $\overline{U} \in H$, we obtain

$$\|U^k\|_{(L^2)^2}^2 \leq \frac{1}{\lambda_{k,s}}\|U^k\|_Y^2 \text{ and } \|\overline{U}\|_{(L^2)^2}^2 \leq \frac{1}{\lambda_{k+1,s}}\|\overline{U}\|_Y^2.$$

Consequently,

$$\begin{aligned} J_s(U) &\geq \Big(\frac{1}{2}\|\overline{U}\|_Y^2 - \frac{\mu_2}{2}\|\overline{U}\|_{(L^2)^2}^2\Big) + \Big(\frac{1}{2}\|U^k\|_Y^2 - \frac{\mu_2}{2}\|U^k\|_{(L^2)^2}^2\Big) \\ &\quad - C(\|U^k\|_Y^2 + \|\overline{U}\|_Y^2)^{2^*_\mu} \\ &\geq \frac{1}{2}\Big(1 - \frac{\mu_2}{\lambda_{k+1,s}}\Big)\|\overline{U}\|_Y^2 + \frac{1}{2}\Big(1 - \frac{\mu_2}{\lambda_{k,s}}\Big)\|U^k\|_Y^2 - C\|U^k\|_Y^{2\cdot 2^*_\mu} \\ &\quad - C\|\overline{U}\|_Y^{2\cdot 2^*_\mu}. \end{aligned} \tag{5.22}$$

Taking $\|U\|_Y = \rho$ small enough, since $\|U\|_Y^2 = \|U^k\|_Y^2 + \|\overline{U}\|_Y^2$, we obtain that $\|U^k\|_Y := y(\rho)$ and $\|\overline{U}\|_Y := z(\rho) \equiv z$ are small enough. Now consider the function

$$\begin{aligned} \alpha(z) &= \frac{1}{2}\Big(1 - \frac{\mu_2}{\lambda_{k+1,s}}\Big)z^2 + \frac{1}{2}\Big(1 - \frac{\mu_2}{\lambda_{k,s}}\Big)y(\rho)^2 - C(y(\rho)^{2\cdot 2^*_\mu} + z^{2\cdot 2^*_\mu}) \\ &= h(z) + \frac{1}{2}\Big(1 - \frac{\mu_2}{\lambda_{k,s}}\Big)y(\rho)^2 - Cy(\rho)^{2\cdot 2^*_\mu}, \end{aligned}$$

where $h(z) = \frac{1}{2}\big(1 - \frac{\mu_2}{\lambda_{k+1,s}}\big)z^2 - Cz^{2\cdot 2^*_\mu}$. By (5.21), the maximum value of $h(z)$, for ρ sufficiently small, is given by

$$\overline{h} := \frac{N+2s-\mu}{2N-\mu}\Big(\frac{1}{2^*_\mu C}\Big)^{\frac{N-2s}{N+2s-\mu}}\Big(\frac{1}{2}(1 - \frac{\mu_2}{\lambda_{k+1,s}})\Big)^{\frac{2N-\mu}{N+2s-\mu}} > 0,$$

which is independent of ρ and it is assumed at

$$\overline{z} := \Big(\frac{1}{2\cdot 2^*_\mu C}\Big)^{\frac{N-2s}{2(N+2s-\mu)}}\Big(1 - \frac{\mu_2}{\lambda_{k+1,s}}\Big)^{\frac{N-2s}{2(N+2s-\mu)}}.$$

Therefore, it is possible to choose $y(\rho)$ small enough, such that

$$\alpha(\overline{z}) = \overline{h} - cy(\rho)^2 - Cy(\rho)^{2\cdot 2^*_\mu} \geq \overline{h} - (c+C)y(\rho)^2 > 0,$$

where $c = \frac{1}{2}(\frac{\mu_2}{\lambda_k} - 1) \geq 0$. Hence, by the estimate (5.22) and by the above information, for $\|U\|_Y = \rho$ small enough, there exists $\alpha > 0$ such that $J_s(U) \geq \alpha$. This proves item (i).

To prove item (ii), we take $U = (u,v) \in V$, where $u = \Sigma_{i=1}^{k-1} u_i\varphi_{i,s}$, $v = \Sigma_{i=1}^{k-1} v_i\varphi_{i,s}$. Using [30, Proposition 9], we obtain

$$\int_{\mathbb{R}^N} |u|^2 dx = \Sigma_{i=1}^{k-1} u_i^2, \int_{\mathbb{R}^N} |v|^2 dx = \Sigma_{i=1}^{k-1} v_i^2,$$

also

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2 + |v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy &= \Sigma_{i=1}^{k-1}(u_i^2 + v_i^2)\|\varphi_{i,s}\|_X^2 \\ &= \Sigma_{i=1}^{k-1}(u_i^2 + v_i^2)\lambda_{i,s}. \end{aligned}$$

Using (2.2), we prove that $J_s(U) < 0$ on V. Let $U = (u,v) \in V$, since $\lambda_{k-1,s} < \mu_1 < \lambda_{k,s} \leq \mu_2 < \lambda_{k+1,s}$, we have that

$$\begin{aligned} J_s(u,v) &\leq \frac{1}{2}\Sigma_{i=1}^{k-1}(u_i^2 + v_i^2)\lambda_{i,s} - \frac{\mu_1}{2}\Sigma_{i=1}^{k-1}(u_i^2 + v_i^2) \\ &\quad - \frac{1}{2^*_\mu}\Big(\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|v(x)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy + \xi_1\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \end{aligned}$$

$$\begin{aligned}&+\xi_2\int_\Omega\int_\Omega\frac{|v(x)|^{2^*_\mu}|v(x)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy\Big)\\&\le\frac{1}{2}\Sigma_{i=1}^{k-1}(u_i^2+v_i^2)(\lambda_{i,s}-\mu_1)<0.\end{aligned}$$

Now, to complete the proof, it is sufficient to apply Lemma 5.7 to the finite dimensional subspace $V\oplus\operatorname{span}\{e\}$ containing $Q=(V\cap\overline{B}_R(0))\cap[0,R]e$, for some $e\in W\cap\partial B_1(0)$ and $R>\rho$. $\square$

Remark 5.9. Note that in Lemma 5.8 we can choose the finite dimensional subspace $\mathbb{F}$ of $Y(\Omega)$ as

$$\mathbb{F}_\epsilon=V\oplus\operatorname{span}\{e\}=V\oplus\operatorname{span}\{(\widetilde{z_\epsilon},0)\},$$

where $V=\operatorname{span}\{(0,\varphi_{1,s}),(\varphi_{1,s},0),(0,\varphi_{2,s}),(\varphi_{2,s},0),\dots,(0,\varphi_{k-1,s}),(\varphi_{k-1,s},0)\}$, $\widetilde{z_\epsilon}=\frac{z_\epsilon}{\|z_\epsilon\|_X}$, with $z_\epsilon=u_\epsilon-\Sigma_{j=1}^{k-1}(\int_\Omega u_\epsilon\varphi_{j,s}dx)\varphi_{j,s}$.

Lemma 5.10. *Let $s\in(0,1)$, $N>2s$ and $M_\epsilon:=\max_{u\in G}S_{s,\mu_1}$, where $G:=\{u\in\mathbb{F}_\epsilon:\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy=1\}$. Suppose $\lambda_{k-1,s}<\mu_1<\lambda_{k,s}\le\mu_2<\lambda_{k+1,s}$, for some $k\in\mathbb{N}$, we have*

(i) *M_ϵ is achieved by $u_M\in\mathbb{F}_\epsilon$ and u_M can be written as follows*

$$u_M=\nu+tu_\epsilon,\ \text{ with } \nu\in\operatorname{span}\{\varphi_{1,s},\varphi_{2,s},\dots,\varphi_{k-1,s}\}\ \text{ and } t\ge0;$$

(ii) *$M_\epsilon<S_s^H$, provided: (a) $N\ge4s$ and $\mu_1>0$, or (b) $2s<N<4s$ and μ_1 is large enough.*

Proof. (1) Thanks to the Weierstrass theorem, M_ϵ is achieved at u_M. Since $u_M\in\mathbb{F}_\epsilon$ and by the definition of $\mathbb{F}_\epsilon$, we have that $u_M=\widetilde{\nu}+tz_\epsilon$, for some $\widetilde{\nu}\in\operatorname{span}\{\varphi_{1,s},\varphi_{2,s},\dots,\varphi_{k-1,s}\}$ and $t\in\mathbb{R}$. We can suppose that $t\ge0$ (otherwise, if $t\le0$, we can replace u_M with $-u_M$). From the definition of z_ϵ in Remark 5.9, we have

$$u_M=\nu+tu_\epsilon,\tag{5.23}$$

where

$$\nu=\widetilde{\nu}-t\Sigma_{i=1}^{k-1}(\int_\Omega u_\epsilon\varphi_{i,s}dx)\varphi_{i,s}\in\operatorname{span}\{\varphi_{1,s},\varphi_{2,s},\dots,\varphi_{k-1,s}\}.$$

(ii) First let $t=0$, then $u_M=\nu$ and

$$M_\epsilon=\|\nu\|^2-\mu_1\int_{\mathbb{R}^N}|\nu|^2dx\le(\lambda_{k-1,s}-\mu_1)\|\nu\|^2_{L^2(\Omega)}<0<S_s^H.$$

Now, suppose $t>0$, we find that $\widetilde{\nu}$ and z_ϵ are orthogonal in $L^2(\Omega)$, then $\|u_M\|^2_{L^2(\Omega)}=\|\widetilde{\nu}\|^2_{L^2(\Omega)}+t^2\|z_\epsilon\|^2_{L^2(\Omega)}$. Since $\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy=1$, using [27, Lemma 4.7], we obtain a constant $C_0>0$ (independent of ϵ) such that $\|u_M\|_{L^{2^*_\mu}(\Omega)}\le C_0$. Subsequently, using Hölder inequality, we obtain a constant $C_1>0$ (also independent of ϵ) such that $\|u_M\|^2_{L^2(\Omega)}\le C_1$. Therefore, we can find $C_2>0$ such that $\|u_M\|^2_{L^2(\Omega)}$

and $\|\widetilde{\nu}\|_{L^2(\Omega)}^2$ are both uniformly bounded in ϵ. By computations, we obtain

$$
\begin{aligned}
\|u_\epsilon\|_{L^{\frac{N(3N-2\mu+2s)}{(2N-\mu)(N-2s)}}(\Omega)}^{\frac{3N-2\mu+2s}{N-2s}} &= \Big(\int_\Omega |u_\epsilon|^{\frac{N(3N-2\mu+2s)}{(2N-\mu)(N-2s)}}\,dx\Big)^{\frac{2N-\mu}{N}} \\
&\le \Big(\int_{B_{2\delta}} |U_\epsilon|^{\frac{N(3N-2\mu+2s)}{(2N-\mu)(N-2s)}}\,dx\Big)^{\frac{2N-\mu}{N}} \\
&\le C_3\epsilon^{\frac{N-2s}{2}}\Big(\int_0^{\frac{2\delta}{\epsilon}} \frac{r^{N-1}}{(1+r^2)^{\frac{N(3N-2\mu+2s)}{(2N-\mu)(N-2s)}}}\,dr\Big)^{\frac{2N-\mu}{N}} \\
&\le O(\epsilon^{\frac{N-2s}{2}}),
\end{aligned}
\tag{5.24}
$$

where $C_3 > 0$ is a constant. Since $\varphi_{1,s}, \varphi_{2,s}, \dots, \varphi_{k-1,s} \in L^\infty(\Omega)$, we have $\widetilde{\nu} \in L^\infty(\Omega)$. Using that the map $t \mapsto t^{2\cdot 2^*_\mu}$ is convex, for $t > 0$ and $\operatorname{span}\{\varphi_{1,s}, \varphi_{2,s}, \dots, \varphi_{k-1,s}\}$ is a finite dimensional space, all norms are equivalent, we obtain

$$
\begin{aligned}
1 &= \int_\Omega\int_\Omega \frac{|u_M(x)|^{2^*_\mu}|u_M(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\
&= \int_\Omega\int_\Omega \frac{|(\nu+tu_\epsilon)(x)|^{2\cdot 2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\
&\ge \int_\Omega\int_\Omega \frac{|tu_\epsilon(x)|^{2\cdot 2^*_\mu}}{|x-y|^\mu}\,dx\,dy + 2\cdot 2^*_\mu\int_\Omega\int_\Omega \frac{|tu_\epsilon(x)|^{2\cdot 2^*_\mu-1}|\nu(x)|}{|x-y|^\mu}\,dx\,dy \\
&\ge \int_\Omega\int_\Omega \frac{|tu_\epsilon(x)|^{2^*_\mu}|tu_\epsilon(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\
&\quad - 2\cdot 2^*_\mu\|\nu\|_{L^\infty(\Omega)}\int_\Omega\int_\Omega \frac{|tu_\epsilon(x)|^{\frac{2\cdot 2^*_\mu-1}{2}}|tu_\epsilon(y)|^{\frac{2\cdot 2^*_\mu-1}{2}}}{|x-y|^\mu}\,dx\,dy \\
&\ge \int_\Omega\int_\Omega \frac{|tu_\epsilon(x)|^{2^*_\mu}|tu_\epsilon(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy - C_4\|\nu\|_{L^2(\Omega)}\|u_\epsilon\|_{L^{\frac{N(3N-2\mu+2s)}{(2N-\mu)(N-2s)}}(\Omega)}^{\frac{3N-2\mu+2s}{N-2s}}.
\end{aligned}
$$

Combining (5.24) with the above inequality, we obtain

$$
\int_\Omega\int_\Omega \frac{|tu_\epsilon(x)|^{2^*_\mu}|tu_\epsilon(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \le 1 + C_4\|\nu\|_{L^2(\Omega)}O(\epsilon^{\frac{N-2s}{2}}). \tag{5.25}
$$

Hence, using the definition of S_{s,μ_1} and (5.23), we obtain

$$
\begin{aligned}
M_\epsilon &= \int_{\mathbb{R}^{2N}} \frac{|u_M(x)-u_M(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \mu_1\int_{\mathbb{R}^N}|u_M(x)|^2dx\\
&= \int_{\mathbb{R}^{2N}} \frac{|(\nu(x)+tu_\epsilon(x))-(\nu(y)+tu_\epsilon(y))|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad -\mu_1\int_{\mathbb{R}^N}|\nu(x)+tu_\epsilon(x)|^2dx\\
&= \int_{\mathbb{R}^{2N}} \frac{|\nu(x)-\nu(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + t^2\int_{\mathbb{R}^{2N}} \frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad + 2t\int_{\mathbb{R}^{2N}} \frac{|(\nu(x)-\nu(y))(u_\epsilon(x)-u_\epsilon(y))|}{|x-y|^{N+2s}}\,dx\,dy - \mu_1\int_{\mathbb{R}^N}|\nu(x)|^2dx\\
&\quad -\mu_1t^2\int_{\mathbb{R}^N}|u_\epsilon(x)|^2dx - 2\mu_1t\int_{\mathbb{R}^N}|u_\epsilon(x)\nu(x)|dx\\
&\le (\lambda_{k-1,s}-\mu_1)\|\nu\|^2_{L^2(\Omega)}\\
&\quad + S_{s,\mu_1}(u_\epsilon)\Big(\int_\Omega\int_\Omega \frac{|tu_\epsilon(x)|^{2^*_\mu}|tu_\epsilon(y)|^{2^*_\mu}}{|x-u|^\mu}\,dx\,dy\Big)^{\frac{N-2s}{2N-\mu}}\\
&\quad + 2t\int_{\mathbb{R}^{2N}} \frac{|(\nu(x)-\nu(y))(u_\epsilon(x)-u_\epsilon(y))|}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad - 2\mu_1t\int_{\mathbb{R}^N}|u_\epsilon(x)\nu(x)|dx.
\end{aligned}
\tag{5.26}
$$

Now we write $\nu=\Sigma_{i=1}^{k-1}\nu_i\varphi_{i,s}$ for some $\nu_i\in\mathbb{R}$, such that $\|\nu\|^2_{L^2(\Omega)}=\Sigma_{i=1}^{k-1}\nu_i^2$. By the Hölder inequality and the equivalence of the norms in a finite dimensional space,

$$
\begin{aligned}
|\langle u_\epsilon,\nu\rangle_X| &= \Sigma_{i=1}^{k-1}\lambda_{i,s}\nu_i\int_\Omega u_\epsilon(x)\varphi_{i,s}(x)dx\\
&\le \Sigma_{i=1}^{k-1}\lambda_{i,s}|\nu_i|\|u_\epsilon\|_{L^1(\Omega)}\|\varphi_{i,s}\|_{L^\infty(\Omega)}\\
&\le \widetilde{k}\lambda_{k,s}\|u_\epsilon\|_{L^1(\Omega)}\|\nu\|_{L^\infty(\Omega)}\\
&\le \overline{k}\|u_\epsilon\|_{L^1(\Omega)}\|\nu\|_{L^2(\Omega)},
\end{aligned}
$$

for suitable $\widetilde{k}$ and $\overline{k}>0$. More explicitly,

$$
\Big|\int_{\mathbb{R}^{2N}} \frac{(\nu(x)-\nu(y))(u_\epsilon(x)-u_\epsilon(y))}{|x-y|^{N+2s}}\,dx\,dy\Big| \le \overline{k}\|u_\epsilon\|_{L^1(\Omega)}\|\nu\|_{L^2(\Omega)}. \tag{5.27}
$$

Gathering the results in (5.25), (5.26) and (5.27), using again the Hölder inequality and (4.5), we obtain

$$
\begin{aligned}
M_\epsilon &\le (\lambda_{k-1,s}-\mu_1)\|\nu\|^2_{L^2(\Omega)} + S_{s,\mu_1}(u_\epsilon)\Big(1+C_4\|\nu\|_{L^2(\Omega)}O(\epsilon^{\frac{N-2s}{2}})\Big)^{\frac{N-2s}{2N-\mu}}\\
&\quad + 2t\overline{k}\|u_\epsilon\|_{L^1(\Omega)}\|\nu\|_{L^2(\Omega)} - 2\mu_1t\|u_\epsilon\|_{L^1(\Omega)}\|\nu\|_{L^\infty(\Omega)}\\
&\le (\lambda_{k-1,s}-\mu_1)\|\nu\|^2_{L^2(\Omega)} + S_{s,\mu_1}(u_\epsilon)\Big(1+C_4\|\nu\|_{L^2(\Omega)}O(\epsilon^{\frac{N-2s}{2}})\Big)^{\frac{N-2s}{2N-\mu}}\\
&\quad + \kappa\|u_\epsilon\|_{L^1(\Omega)}\|\nu\|_{L^2(\Omega)}\\
&\le (\lambda_{k-1,s}-\mu_1)\|\nu\|^2_{L^2(\Omega)} + S_{s,\mu_1}(u_\epsilon)(1+C_4\|\nu\|_{L^2(\Omega)}O(\epsilon^{\frac{N-2s}{2}}))
\end{aligned}
$$

$$+ O(\epsilon^{\frac{N-2s}{2}})\|\nu\|_{L^2(\Omega)}.$$

Since the parabola $(\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + O(\epsilon^{\frac{N-2s}{2}})\|\nu\|_{L^2(\Omega)}$ stays always below its vertex, that is

$$(\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + O(\epsilon^{\frac{N-2s}{2}})\|\nu\|_{L^2(\Omega)} \le \frac{1}{4(\lambda_{k-1,s} - \mu_1)} O(\epsilon^{N-2s}) = O(\epsilon^{N-2s}).$$

From Lemma 4.5, we obtain three cases:

Case 1: $N > 4s$,

$$\begin{aligned} M_\epsilon &\le \left(S_s^H - \mu_1 C_s \epsilon^{2s} + O(\epsilon^{N-2s})\right)\Big(1 + C_4\|\nu\|_{L^2(\Omega)} O(\epsilon^{\frac{N-2s}{2}})\Big) \\ &\quad + (\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + O(\epsilon^{\frac{N-2s}{2}})\|\nu\|_{L^2(\Omega)} \\ &\le S_s^H - \mu_1 C_s \epsilon^{2s} + O(\epsilon^{N-2s}) < S_s^H, \end{aligned}$$

for sufficiently small $\epsilon > 0$ and $\mu_1 > 0$.

Case 2: $N = 4s$,

$$\begin{aligned} M_\epsilon &\le \left(S_s^H - \mu_1 C_s \epsilon^{2s}|log\epsilon| + O(\epsilon^{2s})\right)\Big(1 + C_4\|\nu\|_{L^2(\Omega)} O(\epsilon^{\frac{N-2s}{2}})\Big) \\ &\quad + (\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + O(\epsilon^{\frac{N-2s}{2}})\|\nu\|_{L^2(\Omega)} \\ &\le S_s^H - \mu_1 C_s \epsilon^{2s}|log\epsilon| + O(\epsilon^{2s}) < S_s^H, \end{aligned}$$

for sufficiently small $\epsilon > 0$ and $\mu_1 > 0$.

Case 3: $2s < N < 4s$,

$$\begin{aligned} M_\epsilon &\le \left(S_s^H + \epsilon^{N-2s}(O(1) - \mu_1 C_s) + O(\epsilon^{2s})\right)\Big(1 + C_4\|\nu\|_{L^2(\Omega)} O(\epsilon^{\frac{N-2s}{2}})\Big) \\ &\quad + (\lambda_{k-1,s} - \mu_1)\|\nu\|^2_{L^2(\Omega)} + O(\epsilon^{\frac{N-2s}{2}})\|\nu\|_{L^2(\Omega)} \\ &\le S_s^H + \epsilon^{N-2s}(O(1) - \mu_1 C_s) + O(\epsilon^{2s}) < S_s^H, \end{aligned}$$

for sufficiently small $\epsilon > 0$ and μ_1 large enough. This completes the proof. □

5.4. **Proof of Theorem 1.4.** By Lemmas 5.7 and 5.8, J_s satisfies the geometric structure of the Linking theorem. Now we apply Theorem 2.4 for the functional J_s with

$$Q = (\overline{B}_R \cap V) \oplus \{r(\widetilde{z}_\epsilon, 0) : 0 < r < R\},$$

and the critical level is characterized as

$$c = \inf_{h \in \Gamma} \max_{(u,v) \in Q} J_s(h(u,v)),$$

where $\Gamma = \{h \in C(\overline{Q}, Y) : h = id \text{ on } \partial Q\}$. Note that, for all $h \in \Gamma$,

$$c = \inf_{h \in \Gamma} \max_{(u,v) \in Q} J_s(h(u,v)) \le \max_{(u,v) \in Q} J_s(h(u,v)).$$

Let $\mathbb{F}_\epsilon$ be as in Remark 5.9 with ϵ sufficiently small. Since $Q \subset (\mathbb{F}_\epsilon)^2$, taking $h = id$ and recalling that $(\mathbb{F}_\epsilon)^2$ is a linear subspace, we obtain

$$\begin{aligned} c &\le \max_{(u,v) \in (\mathbb{F}_\epsilon)^2, (u,v) \ne (0,0)} J_s(h(u,v)) \\ &= \max_{(u,v) \in (\mathbb{F}_\epsilon)^2, \eta \ne 0} J_s\big(|\eta|(\frac{u}{|\eta|}, \frac{v}{|\eta|})\big) \\ &= \max_{(u,v) \in (\mathbb{F}_\epsilon)^2, \eta > 0} J_s(\eta(u,v)) \le \max_{(u,v) \in (\mathbb{F}_\epsilon)^2, \eta \ge 0} J_s(\eta(u,v)). \end{aligned}$$

Now we claim that

$$\max_{(u,v)\in(\mathbb{F}_\epsilon)^2,\eta\geq 0} J_s(\eta(u,v)) < \frac{N+2s-\mu}{2N-\mu}\Big(\frac{\widetilde{S_\xi^H}}{2}\Big)^{\frac{2N-\mu}{N+2s-\mu}}.$$

To verify this claim, fix $U=(u,v)\in(\mathbb{F}_\epsilon)^2$ such that $uv\neq 0$. Then by (2.2), for all $r\geq 0$, we infer that

$$\begin{aligned}
J_s(rU) &\leq \frac{r^2}{2}(\|U\|_Y^2-\mu_1\|U\|^2_{(L^2(\Omega))^2}) - \frac{r^{2\cdot 2_\mu^*}}{2_\mu^*}\Big(\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|v(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy \\
&\quad + \xi_1\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy + \xi_2\int_\Omega\int_\Omega \frac{|v(x)|^{2_\mu^*}|v(x)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\Big) \\
&=: \frac{Ar^2}{2} - \frac{r^{2\cdot 2_\mu^*}B}{2_\mu^*} =: g(r).
\end{aligned}$$

Note that $r_0=\big(\frac{A}{2B}\big)^{\frac{1}{2\cdot 2_\mu^*-2}}$ is the maximum point of $g(r)$, which maximum value is given by

$$\frac{N+2s-\mu}{2N-\mu}\Big(\frac{A}{2B^{1/2_\mu^*}}\Big)^{\frac{2_\mu^*}{2_\mu^*-1}}.$$

Then

$$\max_{r\geq 0} J_s(rU) \leq \frac{N+2s-\mu}{2N-\mu}\Big\{\frac{\|U\|_Y^2-\mu_1\|U\|^2_{(L^2)^2}}{2(B(u,v)+\xi_1B(u,u)+\xi_2B(v,v))^{1/2_\mu^*}}\Big\}^{\frac{2_\mu^*}{2_\mu^*-1}}.$$

Therefore, it is sufficient to show that

$$\widetilde{M_\epsilon} := \max_{(u,v)\in(\mathbb{F}_\epsilon)^2} \frac{\|U\|_Y^2-\mu_1\|U\|^2_{(L^2)^2}}{2(B(u,v)+\xi_1B(u,u)+\xi_2B(v,v))^{1/2_\mu^*}} < \frac{1}{2}\widetilde{S_\xi^H}.$$

Define

$$\begin{aligned}
M_\epsilon &:= \max_{u\in\mathbb{F}_\epsilon\setminus\{0\}} \frac{\|u\|_X^2-\mu_1\|u\|_{L^2}^2}{\big(\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy\big)^{1/2_\mu^*}} \\
&= \max_{u\in\mathbb{F}_\epsilon,\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,dx\,dy=1} (\|u\|_X^2-\mu_1\|u\|_{L^2}^2).
\end{aligned}$$

Taking $s_0,t_0>0$ as in Remark 5.1 and u_M as in Lemma 5.10, $\widetilde{M_\epsilon}$ is achieved by function $U_M=(s_0u_M,t_0u_M)$. Therefore, from Lemmas 5.10 and 5.3, and using (5.3), we conclude that

$$\begin{aligned}
\widetilde{M_\epsilon} &= \frac{1}{2}\frac{(s_0^2+t_0^2)\big(\|u_M\|_X^2-\mu_1\|u_M\|_{L^2}^2\big)}{\big(s_0^{2_\mu^*}t_0^{2_\mu^*}+\xi_1s_0^{2\cdot 2_\mu^*}+\xi_2t_0^{2\cdot 2_\mu^*}\big)^{1/2_\mu^*}B(u_M,u_M)^{1/2_\mu^*}} \\
&= \frac{1}{2}mM_\epsilon < \frac{1}{2}mS_s^H = \frac{1}{2}\widetilde{S_\xi^H},
\end{aligned}$$

if one of the following conditions holds:

(i) $N\geq 4s$ and $\mu_1>0$,
(ii) $2s<N<4s$ and μ_1 is large enough ($\mu_1>\lambda_{k-1,s}>0$).

Now, using the Linking theorem and Lemma 5.6, we conclude that (1.1) has a nontrivial solution with critical value $c\geq\alpha$.

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