Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 07, pp. 1–23.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

EXISTENCE AND CONTROLLABILITY FOR NEUTRAL PARTIAL DIFFERENTIAL INCLUSIONS NONDENSELLY DEFINED ON A HALF-LINE

NGUYEN THI VAN ANH, BUI THI HAI YEN

Abstract. In this article, we study the existence of the integral solution to the neutral functional differential inclusion

$$\frac{d}{dt}\mathcal{D}y_t - A\mathcal{D}y_t - Ly_t \in F(t, y_t), \quad \text{for a.e. } t \in J := [0,\infty),$$
$$y_0 = \phi \in C_E = C([-r, 0]; E), \quad r > 0,$$

and the controllability of the corresponding neutral inclusion

$$\frac{d}{dt}\mathcal{D}y_t - A\mathcal{D}y_t - Ly_t \in F(t, y_t) + Bu(t), \quad \text{for a.e. } t \in J,$$
$$y_0 = \phi \in C_E,$$

on a half-line via the nonlinear alternative of Leray-Schauder type for contractive multivalued mappings given by Frigon. We illustrate our results with applications to a neutral partial differential inclusion with diffusion, and to a neutral functional partial differential equation with obstacle constrains.

1. Introduction

Neutral differential equations arise in many areas of applied mathematics, and their general theory was laid in the nineteen sixties. For extensive studies of neutral differential equations, we refer the reader to the monographs [13, 24]. In the case of partial neutral functional differential equations, the first result was obtained by Datko [18]. Since then, a wide range of neutral problems have been investigated, such as: Bohr–Neugebauer type theorems in [2], spectral decomposition problems in [5], existence of decay integral solutions in [6], Hopf bifurcation and stability/instability of periodic orbits in [22, 23], conditional stability for periodic partial neutral differential equations in [26], partial neutral functional differential-difference equations on the unit circle in [35], and regularity of solutions under nonlocal conditions in [38].

It should be mentioned that, besides the neutral functional differential equations, the inclusion version appears as an essential requirement when we consider neutral differential equations with discontinuous right-hand sides or in control problems. Accordingly, functional-differential inclusions of neutral type have been attracting the attention of many authors. In recent years, the neutral functional differential

2020 *Mathematics Subject Classification.* 34G25, 34K35, 34K40, 93B05.
Key words and phrases. Hille-Yosida operators; neutral differential inclusions; multivalued maps; fixed point arguments; controllability.

Submitted January 17, 2022. Published January 20, 2023.

inclusions were addressed; see. e.g. [15, 20, 39], which included the impulsive factors, nonlocal conditions or stochastic process.

Conforming to the trend, in this article, we are interested in the following parabolic differential inclusions of neutral type

$$\frac{d}{dt}\mathcal{D}y_t - A\mathcal{D}y_t - Ly_t \in F(t, y_t), \quad \text{for a.e. } t \in J := [0, \infty), \tag{1.1}$$

$$y_0 = \phi, \tag{1.2}$$

where A is a Hille-Yosida operator (nondensely defined in general) on a real Banach space $(E, \|\cdot\|)$; $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t, namely $y_t(\theta) = y(t+\theta)$, for all $\theta \in [-r, 0]$; $F : J \times C_E \to \mathcal{P}(E)$ is a multivalued map with compact values (here, $\mathcal{P}(E)$ denotes the family of all nonempty subsets of E); $L, P : C_E \to E$ are bounded linear operators and $\mathcal{D} : C_E \to E$ is also a linear continuous operator, defined by

$$\mathcal{D}\phi = \phi(0) - P\phi, \quad \forall \phi \in C_E,$$

and we study the controllability of the partial neutral functional differential inclusion

$$\frac{d}{dt}\mathcal{D}y_t - A\mathcal{D}y_t - Ly_t \in F(t, y_t) + Bu(t), \quad \text{for a.e. } t \in J, \tag{1.3}$$

$$y_0 = \phi. \tag{1.4}$$

where the given control function $u(\cdot)$ belongs to $L^2_{\rm loc}([0,\infty); \mathcal{U})$, where $\mathcal{U}$ is a Banach space of admissible control functions. B is a bounded linear from $\mathcal{U}$ to E.

Many works have been devoted to neutral functional differential equations both denselly defined and non-denselly defined conditions of the linear part A. More specifically, when A is the infinitesimal generator of a strongly continuous semigroup on E, we refer to [31], while when A is a Hille-Yosida operator, we refer to [1, 3, 27].

For considering problems (1.1)-(1.2) and (1.3)-(1.4), we briefly review some related articles. In [20], the authors investigated the problems (1.1)-(1.2) and (1.3)-(1.4) in the case that $J := [0, a]$, $\mathcal{D}\phi = \phi(0) - f(\cdot, \phi)$ and $L \equiv 0$. The existence and controllability of integral solutions were proved under sufficient conditions by the framework of addmissible multivalued contractions and Leray-Schauder type fixed point theorem given by Frigon. In case of equation situations (that is, F is a single-valued map), the system (1.1)-(1.2) was studied by Henríquez and Cuevas in [25]. To be precise, the following equation was considered on the whole line

$$\frac{d}{dt}D(x_t) = AD(x_t) + L(x_t) + f(t), \quad t \in \mathbb{R}.$$

The authors proved the existence of an almost automorphic solution for above equation and applied abstract results to a neutral wave equation with delay.

It should be mention that, while there are several works considering the neutral functional differential equations/inclusions on finite intervals (see e.g., [20, 29]) as well as the neutral functional differential equations on infinite intervals [1, 2, 3, 6], the existence and controllability for the neutral functional differential inclusions (1.1)-(1.2) and (1.3)-(1.4) on half-lines are, up to our knowledge, completely open. Regarding applications, we investigate the solvability and controlability for the explicit neutral partial differential inclusions. We also apply our abstract results to a new class of neutral functional differential variational inequalities.

The main tools used to investigate the existence results for initial-valued problem (1.1)-(1.2) and the controllability of the control inclusion (1.3)-(1.4) are based on the technique associated with integrated semigroups [10, 19, 28], multivalued analysis and nonlinear alternative arguments [21]. In this spirit, we have to impose that the multivalued part F is of Carathéodory class, the linear operator P has norm less than 1 and some technical conditions are supposed. To assure the controllability of (1.3)-(1.4), we require the Cauchy operator formed by the semigroup generated by the part of A together operator B has an bounded invertible operator.

The remainder of this article is organized as follows. The precise analytic framework is given in Section 2 below. Section 3 is devoted to the existence result of integral solutions on a half-line for (1.1)-(1.2), while Section 4 presents the controllability at infinity for (1.3)-(1.4). Our main results are contained in Theorem 3.3 and Theorem 4.4. Two examples of these abstract results to a class of neutral partial differential inclusions with the Hille-Yosida linear parts and neutral functional partial differential equations with obstacle constrains are given in Section 5.

2. Preliminaries

2.1. **Multivalued analysis.** In this section, we recall some results on multivalued analysis and on the nonlinear alternative for multivalued admissible contractions in Fréchet spaces due to [16] and [21]. Let X be a metric space, $\wedge$ be a directed set and d_α, $\alpha \in \wedge$ be a metric in X; we define

$$\begin{gathered}\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\},\\ \mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\},\\ \mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\},\\ \mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\},\end{gathered}$$

and denote by $D_\alpha, \alpha \in \wedge$ the Hausdorff pseudometric induced by the metric d_α,

$$\begin{aligned}D_\alpha(A,B) = \inf\big\{&\epsilon > 0 : \forall x \in A, y \in B, \exists \bar{x} \in A, \bar{y} \in B\\ &\text{such that } d_\alpha(x,\bar{y}) < \epsilon,\ d_\alpha(\bar{x},y) < \epsilon\},\end{aligned}$$

with $\inf\emptyset = \infty$. In the particular case, that X is a complete locally convex space, we say that a subset $A \subset X$ is bounded if $D_\alpha(\{0\},A) < \infty$ for every $\alpha \in \wedge$.

Definition 2.1. Let X, E be two Fréchet spaces. A multivalued map $F : X \to \mathcal{P}(E)$ is called an admissible contraction with constants $\{k_\alpha\}_{\alpha\in\wedge}$ if for each $\alpha \in \wedge$ there exists $k_\alpha \in (0,1)$ such that the following statements hold

(i) $D_\alpha(F(x),F(y)) \le k_\alpha d_\alpha(x,y)$ for all $x, y \in X$.
(ii) for every $x \in X$ and every $\epsilon \in (0,\infty)^\wedge$, there exists $y \in F(x)$ such that
$$d_\alpha(x,y) \le d_\alpha(x,F(x)) + \epsilon_\alpha \quad \text{for every } \alpha \in \wedge.$$

We recall the following result which gives sufficient conditions ensuring the existence of a fixed point for admissible multivalued contractions, due to Frigon [21, Corollary 3.5]. The following alternative of Leray-Schauder type theorem will play a critical role in exploring the existence of solutions to our problems.

Theorem 2.2 (Nonlinear Alternative)**.** *Let X be a Fréchet space, U be an open neighborhood with its origin in X, and $N : \overline{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements holds:*

(i) *N has a fixed point;*
(ii) *there exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.*

To use nonlinear alternative argument given by Theorem2.2, we would like to apply the Hausdorff distance between two subsets in E defined by $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$,

$$H_d(A,B) = \max\Big(\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\Big),$$

where $d(a,B) = \inf_{b\in B} d(a,b)$, $d(A,b) = \inf_{a\in A} d(a,b)$. We note that the space $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

For compact valued measurable multifunctions, we obtain the following result.

Proposition 2.3. *If Γ_1 and Γ_2 are compact valued measurable multifunctions then the multifunction $t \multimap \Gamma_1(t) \cap \Gamma_2(t)$ is measurable.*

The following theorem gives a criterion for the existence of the measurable selection of the multifunction.

Theorem 2.4. *Let X be a separable metric space, (T,σ) be a measurable space, Γ be a multifunction from T to complete nonempty subsets of X. If for each open set V in X, $\Gamma^-(V) = \{t : \Gamma(t) \cap V \neq \emptyset\}$ belongs to σ, then Γ admits a measurable selection.*

In the last part of this subsection, we recall the definition of a multivalued mapping of the Carathéodory class which acts on $[0,\infty) \times X$ to $\mathcal{P}(Y)$, here Y stands for another Banach space. Throughout the paper, we always assume that the multivalued parts are Carathéodory functions.

Definition 2.5. *The multifunction $F : [0,\infty) \times X \to \mathcal{P}(Y)$ is of L_1-Carathéodory class if it satisfies the following assertions:*
((i) *$t \mapsto F(t,y)$ is measurable for each $y \in X$.*
(ii) *$x \mapsto F(t,x)$ is continuous for almost all $t \in [0,\infty)$.*
(iii) *For each $q > 0$, there exists $h_q \in L^1_{\rm loc}([0,\infty), \mathbb{R}_+)$ such that*

$$\|F(t,y)\| \le h_q(t) \quad \text{for } \|y\| \le q, \text{ and for almost all } t \in [0,\infty).$$

2.2. **Integrated semigroup.** We recall some essential concepts and results related to integrated semigroups that will be used to give the variation of constants formula for integral solutions of (1.1)-(1.2). For further details, we refer to [10, 19, 30, 36].

Definition 2.6. An integrated semigroup is a family $\{S(t)\}_{t\ge 0}$ of bounded linear operators on X with the following properties:
(i) $S(0) = 0$;
(ii) $t \mapsto S(t)v$ is continuous for each $v \in X$;
(iii) $S(s)S(t)v = \int_0^s (S(t+r) - S(r))v dr$, for all $t,s \ge 0$, $v \in X$.

The integrated semigroup $\{S(t)\}_{t\ge0}$ is called nondegenerate if $S(t)v = 0$ for all $t \ge 0$ implies that $v = 0$.

Definition 2.7. *An operator A is said to be a generator of the integrated semigroup $\{S(t)\}_{t\ge0}$ on X if there exists $\omega \in \mathbb{R}$ such that $(\omega; +\infty) \subset \rho(A)$ and*

$$R(\lambda, A)v := (\lambda I - A)^{-1} v = \lambda \int_0^{+\infty} e^{-\lambda t} S(t) v dt$$

for all $\lambda > \omega$ and $v \in X$.

We have the following relations between an integrated semigroup and its generator (see, e.g, [10, 11]).

Proposition 2.8. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geq 0}$. Then*

(1) *for all $x \in X$ and $t \geq 0$, one gets*
$$\int_0^t S(\tau)x d\tau \in D(A) \quad \text{and} \quad S(t)x = A\Big(\int_0^t S(\tau)x\, d\tau\Big) + tx;$$
(2) *for all $x \in D(A)$ and $t \geq 0$, we have*
$$S(t)x \in D(A), \quad AS(t)x = S(t)Ax, \quad S(t)x = \int_0^t S(\tau)Ax\, d\tau + tx;$$
(3) *$R(\lambda, A)S(t) = S(t)R(\lambda, A)$, for all $t \geq 0, \lambda > \omega$.*

We next give the precise concept of a locally Lipschitz continuous integrated semigroup in the following definition (see [28]).

Definition 2.9. An integrated semigroup $\{S(t)\}_{t\geq 0}$ is called locally Lipschitz continuous, if for all $\tau > 0$, there exists a constant $L_\tau > 0$ such that
$$\|S(t) - S(s)\|_{\mathcal{L}(X)} \leq L_\tau |t - s|, \text{ for all } t, s \in [0, \tau].$$

It is known (see [28]) that a Lipschitz continuous integrated semigroup is exponentially bounded. The below lemma addresses equivalent relations between the generator of a nondegenerate, the locally Lipschitz continuous integratedsemigroup and the Hille-Yosida operator.

Lemma 2.10. *The following assertions are equivalent:*

(i) *A is the generator of a nondegenerate, locally Lipschitz continuous integrated semigroup;*
(ii) *A satisfies the Hille-Yosida condition, that is, there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega; +\infty) \subset \rho(A)$ and*
$$\sup\{(\lambda - \omega)^n \|(\lambda I - A)^{-n}\|_{\mathcal{L}(X)} : n \in \mathbb{N}, \lambda > \omega\} \leq M.$$

We mention that if $\{S(t)\}_{t\geq 0}$ is an integrated semigroup generated by a Hille-Yosida operator A, then $t \mapsto S(t)v$ is differentiable for each $v \in \overline{D(A)}$ and $\{S'(t)\}_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by the part A_0 of A, which is defined by
$$\begin{gathered} D(A_0) = \{v \in D(A) : Av \in \overline{D(A)}\}, \\ A_0 v = Av, \quad \text{for } v \in D(A_0). \end{gathered}$$
The following proposition provides the continuous differentiability of the function defined by an integrated semigroup and an estimate of its derivative (see [28]).

Proposition 2.11. *Let $\{S(t)\}_{t\geq 0}$ be a locally Lipschitz continuous integrated semigroup on X and $f : [0, T] \to X$ be a Bochner integrable function. Then the function $V : [0, T] \to X$,*
$$V(t) = \int_0^t S(t - s)f(s)ds$$
is continuously differentiable and, moreover,
$$\|V'(t)\| \leq 2L_T \int_0^t \|f(s)\|ds$$

for all $t \in [0,T]$, where L_T is the Lipschitz constants of S on $[0,T]$ given by Definition 2.9.

In addition, we have the relation between $V'(t)$ and $S'(t)$, that

$$R(\lambda, A)V'(t) = \int_0^t S'(t-s)R(\lambda, A)f(s)ds.$$

This implies that

$$V'(t) = \lim_{\lambda\to\infty}\int_0^t S'(t-s)\lambda R(\lambda, A)f(s)ds,$$

thanks to the facts that $\lim_{\lambda\to\infty}\lambda R(\lambda, A)v = v$ for all $v \in X$.

We consider the linear problem involving Hille-Yosida operator

$$u'(t) = Au(t) + f(t), \quad t > 0, \tag{2.1}$$

$$u(0) = \xi, \quad t \in \mathbb{R}. \tag{2.2}$$

where $f \in L^1_{\rm loc}(\mathbb{R}^+; X)$, here $f \in L^1_{\rm loc}(\mathbb{R}^+; X)$ the space of integrable functions in the sense of Bochner.

By an integral solution to (2.1)-(2.2) we mean a continuous function $u : \mathbb{R}^+ \to \overline{D(A)}$ satisfying the integral equation

$$u(t) = S'(t)\xi + \lim_{\lambda\to+\infty}\int_0^t S'(t-\tau)\lambda(\lambda I - A)^{-1}f(\tau)\,d\tau,$$

for all $t \in \mathbb{R}^+$.

3. Integral solutions

In this section, we establish sufficient conditions for the existence of integral solutions to the problem (1.1)-(1.2). Assume that A, P and F satisfy the following hypotheses:

- (H1) A is a Hille-Yosida operator generating an integrated semigroup $\{S(t)\}_{t\geq 0}$.
- (H2) $\|P\| < 1$.
- (H3) F is of Carathéodory class. Moreover,
  - (1) there exist a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ and a function $p \in L^1_{\rm loc}([0,\infty), \mathbb{R}_+)$ such that
  $$\|F(t,y)\| \leq p(t)\psi(\|y\|) \text{ for a.e. } t \in J, \forall y \in C([-r,0], \overline{D(A)}),$$
  where $\psi : J \to J$ satisfies
  $$\int_1^\infty \frac{ds}{s+\psi(s)} = \infty;$$
  - (2) for all $R > 0$, there exists $l_R \in L^1_{\rm loc}([-r,\infty), \mathbb{R}_+)$ such that
  $$H_d(F(t,y), F(t,\bar{y})) \leq l_R(t)\|y-\bar{y}\|,$$
  for all $y, \bar{y} \in C_E$ with $\|y\|, \|\bar{y}\| \leq R$, and
  $$d(0, F(t,0)) \leq l_R(t) \quad \text{for a.e. } t \in J.$$

For each $v \in C([-r,\infty); X)$ we denote the selection of F by

$$S_{F,v} = \{g \in L^1_{\rm loc}(J,E) : g(t) \in F(t,v_t) \text{ for a.e. } t \in J\}.$$

Definition 3.1. A continuous function $y \in C([-r,\infty),\overline{D(A)})$ is called an integral solution of (1.1) if there exists a function $g \in S_{F,y}$ such that the following assertions hold

(i) $\int_0^t \mathcal{D}y_s ds \in \overline{D(A)}$, for all $t \geq 0$,

(ii)
$$\mathcal{D}y_t = \mathcal{D}\phi + A\int_0^t \mathcal{D}y_s ds + L\int_0^t \mathcal{D}y_s ds + \int_0^t g(s)ds, \quad \text{for all } t \geq 0,$$

(iii) $y_0 = \phi$ on $[-r,0]$.

Remark 3.2. If an integral solution of (1.1) exists, then it is given as in [4] by the variation of constants formula
$$\mathcal{D}y_t = S'(t)\mathcal{D}\phi(0) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L y_s ds + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds.$$
where $R_\lambda = \lambda R(\lambda, A)$.

For each $n \in \mathbb{N}$ we define on $C([-r,\infty),\overline{D(A)})$ a semi-norm
$$\|y\|_n := \sup_{t\leq n} e^{-(wt+\tau L_n(t))}\|y(t)\|,$$
for a suitable choice of $L_n(t)$ in the main theorems. Then, $C([-r,\infty),\overline{D(A)})$ is a Fréchet space with the family of semi-norm $\{\|\cdot\|_n\}_{n\geq 1}$. In order to apply the nonlinear altenative argument given in Theorem 2.2, we shall choose τ sufficiently large. We now provide the solvability result for problem (1.1)-(1.2).

Theorem 3.3. *Suppose that hypotheses* (H1)–(H3) *are satisfied. Then problem* (1.1)-(1.2) *has at least one integral solution on* $[-r,\infty)$.

Proof. Consider the operator $N : C([-r,\infty),\overline{D(A)}) \to \mathcal{P}(C([-r,\infty),\overline{D(A)}))$ defined by,
$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ S'(t)\mathcal{D}\phi(0) + Py_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L y_s ds \\ +\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds, \quad g \in S_{F,y}, & \text{if } t \in J. \end{cases}$$
It is clear that the fixed points of the operator N are integral solutions of the problem (1.1)-(1.2). Thus, we shall show that N has a fixed point. The proof of this is long and technical and is therefore divided into several steps.

Step 1: Estimate the possible solutions to (1.1)**-**(1.2)**.** Let y be a solution of the initial valued problem (1.1)-(1.2). Then one gets $y \in N(y)$ and there exists $g \in S_{F,y}$ such that, for each $t \in [0,\infty)$, we have
$$y(t) = S'(t)\mathcal{D}\phi(0) + Py_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L y_s ds + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds.$$
By the strong continuity of semigroup $\{S'(t)\}_{t\geq 0}$, there are $M \geq 1$ and $\omega \in \mathbb{R}$ so that
$$\|S'(t)\|_{\mathcal{L}(E)} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

Moreover, taking into account that $\lim_{\lambda\to\infty} R_\lambda v = v$ for all $v \in E$, we have $\lim_{\lambda\to\infty} \|R_\lambda\| = 1$. Hence,

$$\begin{aligned} \|y(t)\| &\le Me^{wt}\|\mathcal{D}\phi\| + \|Py_t\| + Me^{wt}\|L\| \int_0^t e^{-ws}\|y_s\|\,ds \\ &\quad + Me^{wt}\int_0^t e^{-ws}p(s)\psi(\|y\|)\,ds. \end{aligned} \tag{3.1}$$

For a given $n \in \mathbb{N}$, we define $\mu : [0, n] \to \mathbb{R}^+$as

$$\mu(t) = \sup\{\|y(s)\| : -r \le s \le t\}.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = \|y(t^*)\|$. If $t^* \in [0, n]$, by estimate (3.1) we have

$$\begin{aligned} e^{-wt}\mu(t) &\le \frac{M(1+\|P\|)\|\phi\|}{1-\|P\|} + \frac{M\|L\|}{1-\|P\|}\int_0^t e^{-ws}\|y_s\|\,ds \\ &\quad + \frac{M}{1-\|P\|}\int_0^t e^{-ws}p(s)\psi(\mu(s))\,ds, \quad \text{for } t \in [0, n]. \end{aligned} \tag{3.2}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$. This implies that the inequality (3.2) holds. Let

$$v(t) = \frac{M(1+\|P\|)\|\phi\|}{1-\|P\|} + \frac{M\|L\|}{1-\|P\|}\int_0^t e^{-ws}\|y_s\|\,ds + \frac{M}{1-\|P\|}\int_0^t e^{-ws}p(s)\psi(\mu(s))\,ds.$$

Thus,

$$\mu(t) \le e^{wt}v(t) \text{ for all } t \in [0, n],$$

Then it is easy to see that

$$\begin{gathered} v(0) = \frac{M(1+\|P\|)\|\phi\|}{1-\|P\|}, \\ v'(t) = \frac{M\|L\|}{1-\|P\|}e^{-wt}\|y_t\| + \frac{M}{1-\|P\|}e^{-wt}p(t)\psi(\mu(t)). \end{gathered}$$

Since ψ is an increasing function, we obtain

$$\begin{aligned} v'(t) &\le \frac{M\|L\|}{1-\|P\|}e^{-wt}\mu(t) + \frac{M}{1-\|P\|}e^{-wt}p(t)\psi(e^{wt}v(t)) \\ &\le \frac{M\|L\|}{1-\|P\|}e^{-wt}e^{wt}v(t) + \frac{M}{1-\|P\|}e^{-wt}p(t)\psi(e^{wt}v(t)) \text{ for a.e. } t \in [0, n]. \end{aligned}$$

Therefore,

$$e^{wt}v'(t) \le \frac{M\|L\|}{1-\|P\|}e^{wt}v(t) + \frac{M}{1-\|P\|}p(t)\psi(e^{wt}v(t)) \quad \text{for a.e. } t \in [0, n].$$

So

$$\begin{aligned} (e^{wt}v(t))' &= we^{wt}v(t) + e^{wt}v'(t) \\ &\le \Big(w + \frac{M\|L\|}{1-\|P\|}\Big)e^{wt}v(t) + \frac{M}{1-\|P\|}p(t)\psi(e^{wt}v(t)) \\ &\le m(t)\left(e^{wt}v(t) + \psi(e^{wt}v(t))\right), \quad \text{for a.e. } t \in [0, n]. \end{aligned}$$

where

$$m(t) = \max\Big\{w + \frac{M\|L\|}{1-\|P\|}; \frac{M}{1-\|P\|}p(t)\Big\}.$$

Thus,
$$\int_{v(0)}^{e^{wt}v(t)} \frac{d\xi}{\xi+\psi(\xi)} \le \int_0^n m(s)\,ds < \infty$$
Consequently, from (H3)(1) there exists a constant d_n such that $e^{wt}v(t) \le d_n, t \in [0,n]$ and hence $\sup_{-r\le s\le n}\|y_s\| \le \max\{\|\phi\|, d_n\} := M_n$.

Step 2: Construct an open neighborhood with its origin in $C([-r,\infty),E)$. Set
$$U = \{y \in C([-r,\infty),E) : \sup\{\|y(t)\| : t \le n\} < M_n + 1, \text{ for all } n \in \mathbb{N}\}.$$
Clearly, U is an open neighbourhood of the origin in $C([-r,\infty),E)$. We consider the multifunction $N : \overline{U} \to \mathcal{P}(C([-r,\infty),\overline{D(A)}))$.

Step 3: N is a contraction mapping. We prove that there exists $0 < \gamma < 1$ ensuring
$$H_d(N(y),N(\overline{y})) \le \gamma\|y-\overline{y}\|_n \text{ for all } y,\overline{y} \in \overline{U}.$$
Let $y,\overline{y} \in \overline{U}$ and $h \in N(y)$. By the definition of N, there exists $g \in S_{F,y}$ such that
$$\begin{aligned} h(t) &= S'(t)\mathcal{D}\phi(0) + Py_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L y_s ds \\ &\quad + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds, \quad t \ge 0. \end{aligned}$$
From (H3)(2) and the definition of U, it follows that
$$H_d(F(t,y_t),F(t,\overline{y}_t)) \le \ell_n(t)\|y_t-\overline{y}_t\|, \quad \forall t \in [0,n].$$
where $\ell_n(t) = l_{M_n+1}(t)$. Hence, there is $w \in F(t,y_t)$ such that
$$\|g(t)-w\| \le \ell_n(t)\|y_t-\overline{y}_t\|, \quad \forall t \in [0,n].$$
We introduce the mapping $U_* : [0,n] \to \mathcal{P}(E)$ given by
$$U_*(t) = \{w \in E : \|g(t)-w\| \le \ell_n(t)\|y_t-\overline{y}_t\|\}.$$
By applying Proposition 2.3, we deduce that the multivalued operator $V_*(t) = U_*(t) \cap F(t,\overline{y}_t)$ is measurable. Thus, by Theorem 2.4, there exists a function $\overline{g}$, which is a measurable selection for V_*. So, $\overline{g} \in F(t,\overline{y}_t)$ and
$$\|g(t)-\overline{g}(t)\| \le l_n(t)\|y_t-\overline{y}_t\|, \quad \forall t \in [0,n].$$
Let us define
$$\begin{aligned} \overline{h}(t) &= S'(t)\mathcal{D}\phi(0) + P\overline{y}_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L\overline{y}_t ds \\ &\quad + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda \overline{g}(s)ds. \end{aligned}$$
Additionally, a simple calculation gives
$$\begin{aligned} \|h(t)-\overline{h}(t)\| &\le \|Py_t - P\overline{y}_t\| + \|\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda(Ly_s - L\overline{y}_t)ds\| \\ &\quad + \|\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda(g(s)-\overline{g}(s))ds\|. \end{aligned} \tag{3.3}$$
Put
$$\overline{L}_n(t) = \int_0^t M\overline{k}_n(s)ds, \quad \overline{k}_n(t) = \max\{\|L\|, \ell_n(t)\}.$$

Estimates for the terms on the right-hand side of (3.3) are shown below. For the first one, we have

$$\begin{aligned}\|Py_t - P\overline{y}_t\| &\le \|P\|\|y_t - \overline{y}_t\| \\ &\le \|P\| e^{wt+\tau\overline{L}_n(t)} e^{-(wt+\tau\overline{L}_n(t))}\|y - \overline{y}\| \\ &\le \|P\| e^{wt+\tau\overline{L}_n(t)}\|y - \overline{y}\|_n,\end{aligned}$$

and for the second term, we have

$$\begin{aligned}&\big\| \lim_{\lambda\to\infty} \int_0^t S'(t-s)R_\lambda(Ly_s - L\overline{y}_t)ds\big\| \\ &\le Me^{wt}\|L\| \int_0^t e^{-ws}\|y_s - \overline{y}_s\|\,ds \\ &\le Me^{wt}\int_0^t k_n(s)e^{\tau\overline{L}_n(s)}e^{-(ws+\tau\overline{L}_n(s))}\|y_s - \overline{y}_s\|\,ds \\ &\le Me^{wt}\int_0^t k_n(s)e^{\tau\overline{L}_n(s)}\|y - \overline{y}\|_n\,ds \\ &\le \frac{1}{\tau}e^{wt+\tau\overline{L}_n(t)}\|y - \overline{y}\|_n.\end{aligned}$$

For the last term of (3.3), one has

$$\begin{aligned}\big\| \lim_{\lambda\to\infty} \int_0^t S'(t-s)R_\lambda(g(s) - \overline{g}(s))ds\big\| &\le Me^{wt}\int_0^t e^{-ws}\|g(s) - \overline{g}(s)\|\,ds \\ &\le Me^{wt}\int_0^t e^{-ws}\ell_n(s)\|y_s - \overline{y}_s\|\,ds \\ &\le Me^{wt}\int_0^t k_n(s)e^{\tau\overline{L}_n(s)}\|y - \overline{y}\|_n\,ds \\ &\le \frac{1}{\tau}e^{wt+\tau\overline{L}_n(t)}\|y - \overline{y}\|_n.\end{aligned}$$

Then, we obtain

$$|h(t) - \overline{h}(t)| \le \big(\|P\| + \frac{2}{\tau}\big)e^{wt+\tau\overline{L}_n(t)}\|y - \overline{y}\|_n, \quad \forall t \in [0,n],$$

which yields

$$\|h - \overline{h}\|_n \le \big(\|P\| + \frac{2}{\tau}\big)\|y - \overline{y}\|_n.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \le \big(\|P\| + \frac{2}{\tau}\big)\|y - \overline{y}\|_n.$$

Since $\|P\| < 1$, we can choose τ sufficiently large such that $\|P\| + \frac{2}{\tau} < 1$. Then, N is a contraction mapping.

Step 4: N is an admissible multivalued map. We consider the map $N : C([-r,n], \overline{D(A)}) \to \mathcal{P}_{cl}(C([-r,n], \overline{D(A)}))$, given by

$$N(y) = \{h \in C([-r,n], \overline{D(A)}) :$$

$$h(t)=\begin{cases}\phi(t), \quad \text{if } t\in[-r,0]\\ S'(t)\mathcal{D}\phi(0)+Py_t+\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda Ly_s ds\\ +\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds, \quad \text{if } t\in[0,n]\end{cases}$$
$$\}.$$

where, $g\in S^n_{F,y}=\{h\in L^1([0,n],\overline{D(A)}): h(t)\in F(t,y_t) \text{ for a.e. } t\in[0,n]\}$.

From Assumptions (H1)–(H3) and that F is a multivalued map with compact values, for every $y\in C([-r,n],\overline{D(A)})$, we have $N(y)\in\mathcal{P}_{cp}(C([-r,n],\overline{D(A)}))$, and that there exists $y_*\in C([-r,n],\overline{D(A)})$ such that $y_*\in N(y_*)$ (for a proof in detail, see Benchohra-Ouahab [14]).

Now let $h\in C([-r,n],\overline{D(A)}),\overline{y}\in\overline{U}$ and $\epsilon>0$ be an arbitrary small. If $y_*\in N(\overline{y})$, we have $\|y^n_*-N_n(\overline{y})\|=0$ and

$$\begin{aligned}|\overline{y}(t)-y_*(t)|&\le|\overline{y}(t)-h(t)|+|y_*(t)-h(t)|\\&\le\|\overline{y}-N(\overline{y})\|_n e^{\omega t+\tau\overline{L}_n(t)}+|y_*(t)-h(t)|,\quad t\in[0,n].\end{aligned}$$

Since h is arbitrary we may suppose that $h\in B(y_*,\epsilon)$. Therefore,

$$\|\overline{y}-y_*\|_n\le\|\overline{y}-N(\overline{y})\|_n+\epsilon.$$

On the other hand, if $y_*\notin N(\overline{y})$, then $\|y_*-N(\overline{y})\|\neq0$. By the compactness of $N(\overline{y})$, there exists $x\in N(\overline{y})$ such that $\|y_*-N(\overline{y})\|=\|y_*-x\|$. Thus,

$$\begin{aligned}|\overline{y}(t)-x(t)|&\le|\overline{y}(t)-h(t)|+|x(t)-h(t)|\\&\le\|\overline{y}-N(\overline{y})\|_n e^{\omega t+\tau\overline{L}_n(t)}+|x(t)-h(t)|,\quad t\in[0,n].\end{aligned}$$

Since h is arbitrary, we may suppose that $h\in B(x,\epsilon)$. Then

$$\|\overline{y}-x\|_n\le\|\overline{y}-N\overline{y}\|_n+\epsilon.$$

Hence, N is an admissible operator contraction. In addition, because of the choice of U, there is no element $y\in\partial U$ such that $y\in\lambda N(y)$ for some $\lambda\in[0,1)$. We deduce from Theorem 2.2 that N has at least one fixed point which is an integral solution of (1.1)-(1.2), which completes the proof of our assertions. □

4. Controllability

This section concerns the controllability of problem (1.3)-(1.4). We start by introducing the following definitions. First, we mention the definition of the integral solution for (1.3)-(1.4).

Definition 4.1. A continuous function $y\in C([-r,\infty),\overline{D(A)})$ is called an integral solution of (1.3)-(1.4) if there exist a selection $g\in S_{F,y}$ and a control $u\in L^2_{\rm loc}(\mathbb{R}^+;\mathcal{U})$ such that

(i) $\int_0^t\mathcal{D}y_s ds\in\overline{D(A)}, t\ge0$,

(ii) $\mathcal{D}y_t=\mathcal{D}\phi+A\int_0^t\mathcal{D}y_s ds+L\int_0^t\mathcal{D}y_s ds+\int_0^t g(s)ds+\int_0^t(Bu)(s)ds, t\ge0$,

(iii) $y_0=\phi,\forall t\in[-r,0]$.

We also have that the integral solution y of (1.3)-(1.4) satisfies the variation of constant formula

$$\begin{aligned}\mathcal{D}y_t &= S'(t)\mathcal{D}\phi(0) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L y_s ds\\ &\quad + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda Bu(s)ds, \quad \text{for a.e. } t\in J.\end{aligned}$$

where $g \in S_{F,y}$ and $u \in L^2_{\rm loc}(\mathbb{R}^+;\mathcal{U})$.

Definition 4.2. The neutral problem (1.3) is said to be controllable on a half-line $\mathbb{R}^+$ (or controllable at infinity) if for any continuous function ϕ on $[-r,0]$, for any $x_1 \in E$ and for each $n \in \mathbb{N}$ there exists a control $u(\cdot) \in L^2([0,n];\mathcal{U})$ such that the integral solution y of (1.3) with initial valued $y_0 = \phi$ satisfies $y(n) = x_1$.

Let us introduce the following assumption imposed on B.

(H4) *For each $n > 0$ the linear operator $W : L^2([0,n],\mathcal{U}) \to E$ defined by*

$$Wu = \int_0^n S'(n-s)Bu(s)ds.$$

has an invertible operator W^{-1} which takes values in $L^2([0,n],\mathcal{U}) \setminus \ker W$ and there exist positive constants M_1, M_2 such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$.

Remark 4.3. *The question of the existence of the operator of W and its inverse was discussed in the paper by N. Carmichael and M.D. Quinn* [32].

We are now in a position to state the main result in this section.

Theorem 4.4. *Assume that hypotheses* (H1)–(H4) *hold. Then the partial neutral differential inclusion* (1.3) *is controllable at infinity.*

Proof. Let $\phi \in C_E$ and $x_1 \in E$. Using hypothesis (H4), for each $y(\cdot)$, $g \in S_{F,y}$ and for each $n \in \mathbb{N}$, we define the control

$$\begin{aligned}u_y^n(t) &= W^{-1}\Big[x_1 - Py_n - S'(n)\mathcal{D}\phi(0) - \lim_{\lambda\to\infty}\int_0^n S'(n-s)R_\lambda L y_s\,ds\\ &\quad - \lim_{\lambda\to\infty}\int_0^n S'(n-s)R_\lambda g(s)ds\Big](t)\\ &= W^{-1}\Big[\mathcal{D}y_n - S'(n)\mathcal{D}\phi(0) - \lim_{\lambda\to\infty}\int_0^n S'(n-s)R_\lambda L y_s ds\\ &\quad - \lim_{\lambda\to\infty}\int_0^n S'(n-s)R_\lambda g(s)ds\Big](t)\end{aligned}$$

Consider the operator $\mathcal{Q} : C([-r,\infty),\overline{D(A)}) \to \mathcal{P}(C([-r,\infty),\overline{D(A)}))$ defined by:

$$\mathcal{Q}(y) = \{h \in C([-r,\infty],\overline{D(A)}) :$$

$$h(t) = \begin{cases} \phi(t), \quad \text{if } t\in[-r,0],\\ S'(t)\mathcal{D}\phi(0) + Py_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L y_s ds\\ +\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds\\ +\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda Bu_y^n(s)ds, g\in S_{F,y}\\ \quad \text{if } t\in[0,\infty)\end{cases}$$

}.

It is clear that the fixed points of $\mathcal{Q}$ are integral solutions to problem (1.3)-(1.4).

Let $y \in C([-r,\infty),\overline{D(A)})$ be a solution of the problem (1.3)-(1.4). Then there exists $g \in S_{F,y}$ such that for each $t \in [0,\infty)$, one has

$$\begin{aligned} y(t) &= S'(t)\mathcal{D}\phi(0) + Py_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L y_s ds \\ &\quad + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda B u_y^n(s)ds. \end{aligned}$$

This implies by (H3)(2) and (H4) that, for each $t \in [0,n]$, we have

$$\begin{aligned} \|y(t)\| &\le Me^{wt}\|\mathcal{D}\phi\| + \|Py_t\| + Me^{\omega t}\int_0^t \|Ly_s\|ds \\ &\quad + Me^{wt}\int_0^t e^{-ws}p(s)\psi(\|y_s\|)\,ds + Me^{wt}\int_0^t e^{-ws}\|Bu_y^n(s)\|\,ds. \end{aligned} \tag{4.1}$$

We evaluate the estimate for the last term of (4.1).

$$\begin{aligned} &Me^{wt}\int_0^t e^{-ws}\|Bu_y^n(s)\|\,ds \\ &\le Me^{wt}M_1M_2n\Big[\|\mathcal{D}y_n\| + Me^{wn}\|\phi\| + Me^{wn}\int_0^n e^{-ws}p(s)\psi(\|y_s\|)\,ds \\ &\quad + Me^{wn}\|L\|\int_0^n e^{-ws}\|y_s\|\,ds\Big] \\ &\le Me^{wt}M_1M_2n\Big[|x_1| + \|P\|\|y_n\| + Me^{wn}\|\phi\| \\ &\quad + Me^{wn}\int_0^n e^{-ws}p(s)\psi(\|y_s\|)\,ds + Me^{wn}\|L\|\int_0^n e^{-ws}\|y_s\|\,ds\Big]. \end{aligned} \tag{4.2}$$

Combining (4.1) with (4.2), one has

$$\begin{aligned} \|y(t)\| &\le \|P\|\|y_t\| + Me^{wt}(1+\|P\|)\|\phi\| + Me^{\omega t}\int_0^t \|Ly_s\|ds \\ &\quad + Me^{wt}\int_0^t e^{-ws}p(s)\psi(\|y_s\|)\,ds \\ &\quad + Me^{wt}M_1M_2nMe^{wn}\|L\|\int_0^n e^{-ws}\|y_s\|\,ds \\ &\quad + Me^{wt}M_1M_2n\Big[|x_1| + \|P\|\|y_n\| + Me^{wn}\|\phi\| \\ &\quad + Me^{wn}\int_0^n e^{-ws}p(s)\psi(\|y_s\|)\,ds\Big] \end{aligned} \tag{4.3}$$

We consider the function

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le n.$$

Let $t^* \in [-r, t]$ such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, n]$, by estimate (4.3), we have for $t \in [0, n]$,

$$\begin{aligned} e^{-wt}\mu(t) \le \frac{M(1+\|P\|)\|\phi\|}{1-\|P\|} + \frac{M\|L\|}{1-\|P\|}\int_0^t e^{-ws}\|y_s\|\,ds \\ + \frac{M}{1-\|P\|}\int_0^t e^{-ws}p(s)\psi(\mu(s))\,ds \\ + \frac{MM_1M_2nMe^{wn}\|L\|}{1-\|P\|}\int_0^n e^{-ws}\|y_s\|\,ds \\ + \frac{MM_1M_2n}{1-\|P\|}\Big[|x_1| + \|P\|\|y_n\| + Me^{wn}\|\phi\| \\ + Me^{wn}\int_0^n e^{-ws}p(s)\psi(\|y_s\|)\,ds\Big]. \end{aligned} \tag{4.4}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$, one also has estimate (4.4).

Let us take the right-hand side of estimate (4.4) as $v(t)$. Then we have

$$\mu(t) \le e^{wt}v(t) \text{ for all } t \in [0, n],$$

$$\begin{aligned} v(0) = \frac{M(1+\|P\|)\|\phi\|}{1-\|P\|} + \frac{MM_1M_2nMe^{wn}\|L\|}{1-\|P\|}\int_0^n e^{-ws}\|y_s\|\,ds \\ + \frac{MM_1M_2n}{1-\|P\|}\Big[|x_1| + \|P\|\|y_n\| + Me^{wn}\|\phi\| \\ + Me^{wn}\int_0^n e^{-ws}p(s)\psi(\|y_s\|)\,ds\Big], \end{aligned}$$

and

$$v'(t) = \frac{M\|L\|}{1-\|P\|}e^{-wt}\|y_t\| + \frac{M}{1-\|P\|}e^{-wt}p(t)\psi(\mu(t)).$$

Using the increasing character of ψ we obtain

$$\begin{aligned} v'(t) &\le \frac{M\|L\|}{1-\|P\|}e^{-wt}\mu(t) + \frac{M}{1-\|P\|}e^{-wt}p(t)\psi(e^{wt}v(t)) \\ &\le \frac{M\|L\|}{1-\|P\|}e^{-wt}e^{wt}v(t) + \frac{M}{1-\|P\|}e^{-wt}p(t)\psi(e^{wt}v(t)). \end{aligned}$$

Then for each $t \in [0, n]$ we have

$$e^{wt}v'(t) \le \frac{M\|L\|}{1-\|P\|}e^{wt}v(t) + \frac{M}{1-\|P\|}p(t)\psi(e^{wt}v(t)).$$

Thus,

$$\begin{aligned} (e^{wt}v(t))' &= we^{wt}v(t) + e^{wt}v'(t) \\ &\le \Big(w + \frac{M\|L\|}{1-\|P\|}\Big)e^{wt}v(t) + \frac{M}{1-\|P\|}p(t)\psi(e^{wt}v(t)) \\ &\le m(t)\left(e^{wt}v(t) + \psi(e^{wt}v(t))\right). \end{aligned}$$

where $m(t) = \max\left\{w + \frac{M\|L\|}{1-\|P\|}; \frac{M}{1-\|P\|}p(t)\right\}$. So,

$$\int_{v(0)}^{e^{wt}v(t)} \frac{du}{u+\psi(u)} \le \int_0^n m(s)\,ds < \infty.$$

Consequently, by (H3), there exists a constant $\overline{D}_n$ such that $e^{wt}v(t) \leq \overline{D}_n, t \in [0,n]$ and hence

$$\sup_{-r\leq s\leq n} \|y_s\| \leq \max\{\|\phi\|, \overline{D_n}\} := K_n.$$

Set $U_1 = \{y \in C([-r,\infty), E) : \sup\{\|y(t)\| : 0 \leq t \leq n\} < K_n + 1, \forall n \in \mathbb{N}\}$. It is clear that U_1 is a open subset of $C([-r,\infty), E)$.

We shall prove that $\mathcal{Q} : \overline{U_1} \to \mathcal{P}(C([-r,\infty), \overline{D(A)}))$ is a contraction and admissible operator. First, we prove that $\mathcal{Q}$ is a contraction, means that there exists $\gamma < 1$ such that

$$H_d(\mathcal{Q}(y), \mathcal{Q}(\overline{y})) \leq \gamma\|y - \overline{y}\|_n, \forall y, \overline{y} \in C([-r,\infty), \overline{D(A)}).$$

Indeed, let $y, \overline{y} \in \overline{U_1}$. For given $n \in \mathbb{N}$ and $h \in \mathcal{Q}(y)$, there exists $g(t) \in F(t, y_t)$ such that

$$\begin{aligned} h(t) &= S'(t)\mathcal{D}\phi(0) + Py_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L y_s ds \\ &\quad + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda Bu_y^n(s)ds. \end{aligned}$$

Thanks to (H3)(1), it follows that

$$H_d(F(t,y_t), F(t,\overline{y}_t)) \leq \kappa_n(t)\|y_t - \overline{y}_t\|, \quad t \in [0,n],$$

where $\kappa_n(t) = l_{K_n+1}(t)$. Hence there is $w \in F(t, y_t)$ such that

$$\|g(t) - w\| \leq \kappa_n(t)\|y_t - \overline{y}_t\|, \quad \text{for } t \in [0,n].$$

Consider $\mathcal{U}_* : [0,n] \to \mathcal{P}(E)$ given by

$$\mathcal{U}_*(t) = \{w \in E : \|g(t) - w\| \leq \kappa_n(t)\|y_t - \overline{y}_t\|, \quad t \in [0,n]\}.$$

Since the multi-valued operator $\mathcal{V}_*(t) = \mathcal{U}_*(t) \cap F(t, \overline{y}_t)$ is measurable (see Proposition 2.3), there exists a function $\overline{g}(t)$, which is a measurable selection of $\mathcal{V}_*$. Then, $\overline{g} \in F(t, \overline{y}_t)$ and one has

$$\|g(t) - \overline{g}(t)\| \leq \kappa_n(t)\|y_t - \overline{y}_t\|, \quad t \in [0,n].$$

Let us define

$$\begin{aligned} \overline{h}(t) &= S'(t)\mathcal{D}\phi(0) + P\overline{y}_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda L\overline{y}_s ds \\ &\quad + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda \overline{g}(s)ds + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda Bu_{\overline{y}}n(s)ds. \end{aligned}$$

Then

$$\begin{aligned} \|h(t) - \overline{h}(t)\| &\leq \|Py_t - P\overline{y}_t\| + \|\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda(Ly_s - L\overline{y}_s)ds\| \\ &\quad + \|\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda(g(s) - \overline{g}(s))ds\| \\ &\quad + \|\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda(Bu_y^n(s) - Bu_{\overline{y}}^n(s))ds\|. \end{aligned} \tag{4.5}$$

By a process similar to that in the proof of Theorem 3.3, in order to construct the family of semi-norms in $C([-r,\infty);\overline{D(A)})$, we put

$$
\begin{aligned}
&\widehat{l}_n(t)\\
&= \max\left\{\kappa_n(t), nMM_1M_2e^{wn}\|L\|, nMM_1M_2e^{wn}\|P\|, MM_1M_2e^{wn}\|\kappa_n\|_{L^1([0,n])}\right\},
\end{aligned}
$$

$$
\widehat{L}_n(t)=\int_0^t M\widehat{l}_n(s)ds\,.
$$

Also we estimate the terms on the right-hand side of (4.5) in the following lines. For the first term, we have

$$
\begin{aligned}
\|Py_t-P\overline{y}_t\| &\le \|P\|\|y_t-\overline{y}_t\|\\
&\le \|P\|e^{wt+\tau\widehat{L}_n(t)}e^{-(wt+\tau\widehat{L}_n(t))}\|y_t-\overline{y}_t\|\\
&\le \|P\|e^{wt+\tau\widehat{L}_n(t)}\|y-\overline{y}\|_n.
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
&\|\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda(Ly_s-L\overline{y}_s)ds\|\\
&\le Me^{wt}\|L\|\int_0^t e^{-ws}\|y_s-\overline{y}_s\|\,ds\\
&\le Me^{wt}\int_0^t \widehat{l}_n(s)e^{\tau\widehat{L}_n(s)}e^{-(ws+\tau\widehat{L}_n(s))}\|y_s-\overline{y}_s\|\,ds\\
&\le Me^{wt}\int_0^t \widehat{l}_n(s)e^{\tau\widehat{L}_n(s)}\|y-\overline{y}\|_n\,ds\\
&\le \frac{1}{\tau}e^{ws+\tau\widehat{L}_n(s)}\|y-\overline{y}\|_n,
\end{aligned}
$$

Using (H3)(1), we evaluate the third term as follows

$$
\begin{aligned}
\|\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda(g(s)-\overline{g}(s))ds\| &\le Me^{wt}\int_0^t e^{-ws}\|g(s)-\overline{g}(s)\|\,ds\\
&\le Me^{wt}\int_0^t e^{-ws}\kappa_n(s)\|y_s-\overline{y}_s\|\,ds\\
&\le Me^{wt}\int_0^t \widehat{l}_n(s)e^{\tau\widehat{L}_n(s)}\|y-\overline{y}\|_n\,ds\\
&\le \frac{1}{\tau}e^{ws+\tau\widehat{L}_n(s)}\|y-\overline{y}\|_n,
\end{aligned}
$$

The last term is estimated due to assumption (H4). Specifically, one has

$$
\begin{aligned}
&I:=\|\lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda(Bu_y^n(s)-Bu_{\overline{y}}^n(s))ds\|\\
&\le Me^{wt}\int_0^t e^{-ws}\|B\|\|u_y^n(s)-u_{\overline{y}}^n(s)\|\,ds\\
&\le MM_1e^{wt}\int_0^t e^{-ws}\|W^{-1}\Big[\mathcal{D}y_n-S'(t)\mathcal{D}\phi(0)-\lim_{\lambda\to\infty}\int_0^n S'(n-r)R_\lambda Ly_rdr\\
&\quad -\lim_{\lambda\to\infty}\int_0^n S'(n-r)R_\lambda g(r)dr\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- W^{-1}\Big[\mathcal{D}\overline{y}_n - S'(t)\mathcal{D}\phi(0) - \lim_{\lambda\to\infty}\int_0^n S'(n-r)R_\lambda L\overline{y}_r dr \\
&- \lim_{\lambda\to\infty}\int_0^n S'(n-r)R_\lambda\overline{g}(r)dr\Big]\| \, ds \\
\le\ & MM_1M_2e^{wt}\int_0^t e^{-ws}\|\mathcal{D}y_n - \mathcal{D}\overline{y}_n\| \, ds \\
&+ M^2M_1M_2e^{wt}e^{wn}\|L\|\int_0^t \Big(e^{-ws}\int_0^n \|y_r - \overline{y}_r\|dr\Big)\, ds \\
&+ M^2M_1M_2e^{wt}e^{wn}\int_0^t \Big(e^{-ws}\int_0^n \kappa_n(r)\|y_r - \overline{y}_r\|dr\Big)\, ds
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I &\le MM_1M_2e^{wt}\|P\|\int_0^t e^{-ws}\|y_s - \overline{y}_s\| \, ds \\
&\quad + M^2M_1M_2e^{wt}e^{wn}n\|L\|\int_0^t e^{-ws}\|y_s - \overline{y}_s\| \, ds \\
&\quad + M^2M_1M_2e^{wt}e^{wn}n\int_0^t e^{-ws}\kappa_n(r)\|y_s - \overline{y}_s\| \, ds \\
&\le \frac{3}{\tau}e^{ws+\tau\widehat{L}_n(s)}\|y - \overline{y}\|_n .
\end{aligned}
$$

Therefore,

$$
\|h - \overline{h}\|_n \le \Big(\|P\| + \frac{5}{\tau}\Big)\|y - \overline{y}\|_n .
$$

We interchange the roles of y and $\overline{y}$ to obtain

$$
H_d(\mathcal{Q}(y), \mathcal{Q}(\overline{y})) \le \Big(\|P\| + \frac{5}{\tau}\Big)\|y - \overline{y}\|_n .
$$

Since $\|P\| < 1$, then we can choose τ sufficiently large such that $\|P\| + \frac{5}{\tau} < 1$. So, $\mathcal{Q}$ is a contraction.

Now, we shall show that $\mathcal{Q}$ is an admissible multivalued map. We consider the multivalued mapping

$$
\mathcal{Q} : C([-r,n], \overline{D(A)}) \to \mathcal{P}_{cl}(C([-r,n], \overline{D(A)})),
$$

$$
\mathcal{Q}(y) = \Big\{h \in C([-r,n], \overline{D(A)}) :
$$

$$
h(t) = \begin{cases} \phi(t), \quad \text{if } t \in [-r,0], \\ S'(t)\mathcal{D}\phi(0) + Py_t + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda Ly_s ds \\ + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda g(s)ds \\ + \lim_{\lambda\to\infty}\int_0^t S'(t-s)R_\lambda Bu_y^n(s)ds, \\ \quad \text{if } t \in [0,n], g \in S^n_{F,y} \end{cases}
$$

$$
\Big\}.
$$

where $S^n_{F,y} = \{h \in L^1([0,n], \overline{D(A)}) : h(t) \in F(t, y_t) \text{ for a.e. } t \in [0,n]\}$.

By (H1)–(H4), and by using the fact that F is a multivalued map with compact values, for every $y \in C([-r,n], \overline{D(A)})$, we can show that $\mathcal{Q}(y) \in \mathcal{P}_{cp}(C([-r,n], \overline{D(A)}))$. Moreover, there exists $z_* \in C([-r,n], E)$ such that $z_* \in \mathcal{Q}(z_*)$. We let $h \in$

$C([-r,n],\overline{D(A)}), \overline{y} \in \overline{U}_1$ and $\epsilon > 0$. If $z_* \in \mathcal{Q}(\overline{y})$ then $\|z_* - \mathcal{Q}(\overline{y})\| = 0$ and we observe that

$$\begin{aligned} |\overline{y}(t) - y_*(t)| &\le |\overline{y}(t) - h(t)| + |y_*(t) - h(t)| \\ &\le \|\overline{y} - \mathcal{Q}(\overline{y})\|_n e^{\omega t + \tau \widehat{L}_n(t)} + |z_*(t) - h(t)|, \quad t \in [0,n]. \end{aligned}$$

Since h is arbitrary, we may suppose that $h \in B(z_*, \epsilon)$. Therefore,

$$\|\overline{y} - z_*\|_n \le \|\overline{y} - \mathcal{Q}(\overline{y})\|_n + \epsilon.$$

If $z_* \notin \mathcal{Q}(\overline{y})$, then $\|z_* - \mathcal{Q}(\overline{y})\| \neq 0$. By the compactness of $\mathcal{Q}(\overline{y})$, there exists $x \in \mathcal{Q}(\overline{y})$ such that $\|z_* - \mathcal{Q}(\overline{y})\| = \|z_* - x\|$. Thus,

$$\begin{aligned} |\overline{y}(t) - x(t)| &\le |\overline{y}(t) - h(t)| + |x(t) - h(t)| \\ &\le \|\overline{y} - \mathcal{Q}(\overline{y})\|_n e^{\omega t + \tau \widehat{L}_n(t)} + |x(t) - h(t)|, \quad t \in [0,n]. \end{aligned}$$

Since h is arbitrary, we may suppose that $h \in B(x, \epsilon)$. Then

$$\|\overline{y} - x\|_n \le \|\overline{y} - \mathcal{Q}\overline{y}\|_n + \epsilon.$$

In both the above cases, we have for every $\overline{y} \in \overline{U}_1$ and for every $n \in \mathbb{N}$, there exists $\overline{x} \in \mathcal{Q}(\overline{y})$ such that

$$\|\overline{y} - \overline{x}\|_n \le \|\overline{y} - \mathcal{Q}\overline{y}\|_n + \epsilon.$$

So, $\mathcal{Q}$ is an admissible operator contraction. Proceeding as in the proof of Theorem 3.3, due to the choice of U_1, there is no $y \in \partial U_1$ such that $y \in \lambda \mathcal{Q}(y)$ for some $\lambda \in [0,1)$. We deduce from Theorem 2.2 that $\mathcal{Q}$ has at least one fixed point which is an integral solution of (1.3) with initial condition $y_0 = \phi$ and $y(n) = x_1$. This completes the proof. $\square$

5. Applications

5.1. **Parabolic differential inclusions of neutral type.** As an application of our results we consider the following partial neutral functional differential inclusion

$$\begin{gathered} \frac{\partial}{\partial t}[y(t,x) - P(y_t(\cdot,x))] - \Delta[y(t,x) - P(y_t(\cdot,x))] \\ - L y_t \in Q(t, y(t-r,x)), \quad t \in [0,\infty), x \in [0,\pi], \\ y(t,0) = y(t,\pi), \quad t \in [0,\infty), \\ y(t,x) = \phi(t,x), \quad t \in [-r,0],\ x \in [0,\pi], \end{gathered} \tag{5.1}$$

where Δ is the Laplacian operator on $[0,\pi]$, $\phi \in C([-r,0]; C([0,\pi];\mathbb{R}))$; $P : C([-r,0]; C([0,\pi];\mathbb{R})) \to C([0,\pi];\mathbb{R})$ is a linear bounded operator, and $\|P\| < 1$, which follows that the condition (H2) holds. The operator $L : C([-r,0]; C([0,\pi];\mathbb{R})) \to C([0,\pi];\mathbb{R})$ is linear bounded. The multivalued map $Q : [0,\infty) \times [0,\pi] \to \mathcal{P}(\mathbb{R})$ has compact values satisfying

$$\exists k > 0 : H_d(Q(t,x_1), Q(t,x_2)) \le k\|x_1 - x_2\|, \quad t \in [0,\infty),\ x_1, x_2 \in [0,\pi]$$

and $d(0, Q(t,0)) \le k, t \in [0,\infty)$.

It is well known from [17] that Δ possesses the following properties:

$$\begin{gathered} \overline{D(A)} = \{u \in C([0,\pi];\mathbb{R}) : u(0) = u(\pi) = 0\}, \\ (0,\infty) \subset \rho(\Delta), \\ \|(\lambda - \Delta)^{-1}\| \le \frac{1}{\lambda}, \quad \lambda > 0. \end{gathered}$$

Then, A generates an integrated semigroup on $C([0,\pi];\mathbb{R})$. Assumption (H1) is verified. Consider $E = C([0,\pi];\mathbb{R})$, operator A from E given by $Az = \Delta z$, where

$$D(A) = \{z \in C([0,\pi];\mathbb{R}) : z(0) = z(\pi) = 0, \Delta z \in C([0,\pi];\mathbb{R})\},$$
$$\overline{D(A)} = C_0([0,\pi];\mathbb{R}) = \{z \in C([0,\pi];\mathbb{R}) : z(0) = z(\pi) = 0\}.$$

In addition, assume that there exists an integrable function $\eta : J \to J$ satisfying

$$|Q(t, w(t-x)) \le \eta(t)\Psi(\|w\|),$$

where $\Psi : J \to J$ is continuous and nondecreasing with

$$\int_1^\infty \frac{ds}{s + \Psi(s)} = \infty.$$

We define on $[0,\infty) \times C([-r,0]; C([0,\pi];\mathbb{R}))$ a multivalued operator F as

$$F(t, w_t)(x) = Q(t, w(t-x)), 0 \le x \le \pi.$$

Therefore, (H3) is satified.

Then, problem (5.1) takes the abstract form (1.1). We can easily check that all the hypotheses of Theorem 3.3 are satisfied. Hence, by employing Theorem 3.3, problem (5.1) has at least one integral solution on $[0,\infty)$.

At the end of this subsection, we consider the control problem associated with (5.1) as follows

$$\begin{gathered}
\frac{\partial}{\partial t}[y(t,x) - P(y_t(\cdot,x))] - \Delta[y(t,x) - P(y_t(\cdot,x))] \\
- Ly_t \in Q(t, y(t-r,x)) + B\boldsymbol{u}(t), \quad t \in [0,\infty), x \in [0,\pi], \\
y(t,0) = y(t,\pi), \quad t \in [0,\infty), \\
y(t,x) = \phi(t,x), \quad t \in [-r,0],\ x \in [0,\pi],
\end{gathered} \tag{5.2}$$

where $B : \mathcal{U} \to C([0,\pi];\mathbb{R})$ is a bounded linear operator defined on a Banach space $\mathcal{U}$ and $u \in L^2_{\rm loc}(J;\mathcal{U})$. Let us assume that $W : L^2_{\rm loc}(J;\mathcal{U}) \to E$ given by

$$W\boldsymbol{u} = \lim_{\lambda\to\infty} \int_0^n T_0(n-s) R_\lambda B\boldsymbol{u}(s) ds$$

has the bounded inverse W^{-1}, where $T_0(t)$ is C_0-semigroup generated by the part of Δ in $\overline{D(A)}$.

By the above description, we can apply Theorem 4.4 to get the controllability for (5.2) given below.

Theorem 5.1. *If $\phi(0) - P(\phi) \in \overline{D(\Delta)}$, then the partial neutral functional differential inclusion* (5.2) *is controllable at infinity.*

5.2. **Neutral functional partial differential equations with obstacle constrains.** In this subsection, we consider the neutral functional partial differential equation mixed an elliptic variational inequality. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. We consider the problem

$$\begin{aligned}
&\frac{\partial}{\partial t}[Y(t,x) - P(Y_t(\cdot,x))] - \Delta_x[Y(t,x) - P(Y_t(\cdot,x))] \\
&= f(t,x,Y(t,x),u(t,x)), \quad \forall t > 0,\ x \in \Omega
\end{aligned} \tag{5.3}$$

$$Y(t,x) = 0, \quad \forall t \ge 0,\ x \in \partial\Omega, \tag{5.4}$$

$$-\Delta_x u(t,x) + \beta(u(t,x) - \psi(x)) \ni h(t,x,Y(t,x),u(t,x)), \quad \forall t > 0,\ x \in \Omega, \tag{5.5}$$

with the initial condition

$$Y(t,x)=\phi(t)(x), \quad \forall t\in[-\tau,0],\ x\in\Omega.$$

where $\phi\in C([-\tau,0];C(\overline{\Omega}))$, Δ_x is the Laplacian with respect to variable x, $f,h:\mathbb{R}\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, ψ is in $H^2(\Omega)$ and $\beta:\mathbb{R}\to 2^{\mathbb{R}}$ is a maximal monotone graph

$$\beta(r)=\begin{cases}0 & \text{if } r>0,\\ \mathbb{R}^- & \text{if } r=0,\\ \emptyset & \text{if } r<0.\end{cases}$$

Note that, the elliptic variational inequality (5.5) reads as follows

$$\begin{gathered}-\Delta_x u(t,x)=h(t,x,Y(t,x),u(t,x))\\ \text{in } \{(t,x)\in Q:=(0,T)\times\Omega: u(t,x)\geq\psi(x)\},\end{gathered} \tag{5.6}$$

$$-\Delta_x u(t,x)\geq h(t,x,Y(t,x),u(t,x)), \quad \text{in } Q, \tag{5.7}$$

which represents a rigorous and efficient way to treat diffusion problems with a free or moving boundary. A simple physical model for the obstacle problem is that of an elastic membrane that occupies a plane domain Ω and is limited from below by a rigid obstacle ψ while it is under the pressure of a vertical force field of density h. This model is called the *obstacle elliptic problem* (see [12, 33]). With the further observation, the obstacle problem has been also discussed in [37] and the references therein.

System (5.3)-(5.5) was investigated in the case without neutral terms $P(Y_t(\cdot,x))$ in [7, 8, 9]. System (5.3)-(5.5) is called a system of neutral differential variational inequalities, which consists of the neutral differential inclusion and the constraint satisfying a variational inequality. In our consideration, we study the solvability of (5.3)-(5.5) as well as the controlability at infinity of associated system

$$\begin{aligned}&\frac{\partial}{\partial t}[Y(t,x)-P(Y_t(\cdot,x))]-\Delta_x[Y(t,x)-P(Y_t(\cdot,x))]\\ &=f(t,x,Y(t,x),u(t,x))+\mathcal{B}\mathbf{u}(t,x), \quad \forall t>0, x\in\Omega\end{aligned} \tag{5.8}$$

$$Y(t,x)=0, \quad \forall t\geq 0,\ x\in\partial\Omega, \tag{5.9}$$

$$-\Delta_x u(t,x)+\beta(u(t,x)-\psi(x))\ni h(t,x,Y(t,x),u(t,x)), \quad \forall t>0,\ x\in\Omega, \tag{5.10}$$

$$Y(t,x)=\phi(t)(x), \quad \forall t\in[-\tau,0],\ x\in\Omega, \tag{5.11}$$

where $\mathcal{B}:\mathcal{U}\to C(\overline{\Omega};\mathbb{R})$ is given as in the Section 5.1.

Let

$$E=C(\overline{\Omega}), \quad E_0=C_0(\overline{\Omega})=\{v\in C(\overline{\Omega}): v=0 \text{ on } \partial\Omega\},$$

here E and E_0 are endowed with the supremum norm $\|v\|=\sup_{x\in\Omega}|v(x)|$. We define

$$\begin{gathered}Av=\Delta_x v,\\ v\in D(A)=\{v\in C_0(\overline{\Omega})\cap H_0^1(\Omega):\Delta v\in C_0(\overline{\Omega})\}.\end{gathered}$$

It is easily seen that

$$\overline{D(A)}=E_0\neq E.$$

Following [34], A satisfies the Hille-Yosida condition on X, then it generates an integrated semigroup $\{S(t)\}_{t\geq 0}$ on E.

By the same argument as in [7, Section 5], one has the equivalent system to (5.3)-(5.5) as follows

$$
\begin{gathered}
\frac{\partial}{\partial t}[Y(t,x) - P(Y_t(\cdot,x))] - \Delta_x[Y(t,x) - P(Y_t(\cdot,x))] \\
= f(t,x,Y(t,x),Q(Y(t,x))), \quad \forall t > 0,\ x \in \Omega \\
Y(t,x) = 0, \quad \forall t \geq 0,\ x \in \partial\Omega, \\
u(t,x) = Q(Y(t,x)), \quad \forall t \geq 0,\ x \in \Omega,
\end{gathered}
$$

where Q is unique solution of elliptic variational inequality (5.6)-(5.7) for each given Y. Thanks to Theorem 3.3 and Theorem 4.4, we obtain a theorem, which uses the following assumptions:

(A1) There are positive functions $a(\cdot), b(\cdot) \in C(\overline{\Omega})$ such that
$$|f(x,p,q) - f(x,p',q')| \leq a(x)|p-p'| + b(x)|q-q'|,$$
for all $x \in \Omega, p, q, p', q' \in \mathbb{R}$.

(A2) There are positive functions $\tilde{a}(\cdot), \tilde{b}(\cdot) \in C(\overline{\Omega})$ such that
$$|f(x,p,q) - f(x,p',q')| \leq \tilde{a}(x)|p-p'| + \tilde{b}(x)|q-q'|,$$
for all $x \in \Omega, p, q, p', q' \in \mathbb{R}$.

(A3) $\|\tilde{b}\|_{L^\infty(\Omega)} < \lambda_1$, where λ_1 is the first eigenvalue of the Laplacian on Ω with homogeneous Dirichlet boundary condition.

Theorem 5.2. *Under assumptions* (A1)–(A3) *we have the following:*

(1) *For each initial function $\phi \in C([-\tau,0];L^2(\Omega))$, system* (1.1)-(1.3) *has an integral solution on $[0,\infty)$.*

(2) *If $\phi(0) - P(\phi) \in \overline{D(A)}$ and*
$$W\boldsymbol{u} = \lim_{\lambda\to\infty} \int_0^n T_0(n-s)R_\lambda \mathcal{B}\boldsymbol{u}(s)ds$$
has the bounded inverse W^{-1}, where $T_0(t)$ is C_0-semigroup generated by the part of Δ in $\overline{D(A)}$, then the associated system (5.8)-(5.11) *is controllable at infinity.*

Acknowledgments. The authors are grateful with referee for the careful reading and constructive comments and suggestions, that help to improve the original manuscript. N. T. V. Anh wants to thank Hanoi National University of Education for providing a fruitful working environment. This work is funded by the Simons Foundation Grant Targeted for Institute of Mathematics (Vietnam Academy of Science and Technology) and by the Vietnam Ministry of Education and Training under grant number B2023–SPH–13.

References

[1] M. Adimy, M. Alia, K. Ezzinbi; Functional differential equations with unbounded delay in extrapolation spaces, *Electronic Journal of Differential Equations*, **2014** (2014) No. 180, 1–16.

[2] M. Adimy, A. Elazzouzi, K. Ezzinbi; Bohr–Neugebauer type theorem for some partial neutral functional differential equations, *Nonlinear Analysis: Theory, Methods & Applications*, **66**(5) (2007), 1145-1160.

[3] M. Adimy, K. Ezzinbi; A class of linear partial neutral functional differential equations with non-dense domain, *J. Differential Equations*, **147** (1998), 285–332.

[4] M. Adimy, K., Ezzinbi; Existence and linearized stability for partial neutral functional differential equations with non-dense domain, *Differ. Equ. Dyn. Syst.*, **7** (1999), 371–417.
[5] M. Adimy, K. Ezzinbi, M. Laklach; Spectral decomposition for partial neutral functional differential equations.*Can. Appl. Math. Q.*, **9**(1) (2001), 1–34.
[6] N.T. Anh, T. D. Ke; Decay integral solutions for neutral fractional differential equations with infinite delays. *Math. Methods Appl. Sci.*, **38**(8) (2015), 1601–1622. ,
[7] N. T. V. Anh, T. D. Ke; On the differential variational inequalities of parabolic-elliptic type, *Math. Meth. Appl. Sci.*, **40** (2017), 4683–4695.
[8] N. T. V. Anh, T. V. Thuy; On the delay differential variational inequalities of parabolic–elliptic type, *Complex Variables and Elliptic Equations*, **67**(12) 2022, 3048–3073.
[9] N. T. V. Anh; Periodic solutions to differential variational inequalities of parabolic-elliptic type. *Taiwanese J. Math.*, **24**(6) (2020), 1497–1527.
[10] W. Arendt; Resolvent positive operators, *Proc. Lond. Math. Soc.*, s3 **54**(2) (1987), 321–349.
[11] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander; Vector-valued Laplace Transforms and Cauchy Problems, *Monographs in Mathematics*, **96** (2001), Birkhauser Verlag, Basel.
[12] V. Barbu; Nonlinear Differential Equations of Monotone Types in Banach Spaces, *Springer Monographs in Mathematics, London,* 2010.
[13] R. Bellman, K. L. Cooke; *Differential-Difference Equations*, Academic Press, New York, 1963.
[14] M. Benchohra, A. Ouahab; Controllability results for functional semilinear differential inclusions in Fréchet spaces. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods* **61** (2005), 405–423.
[15] M. Benchohra, J. Henderson, S. Ntouyas: *Impulsive Differential Equations and Inclusions*, Contemporary Mathematics and Its Applications, Volume 2, 2006.
[16] C. Castaing, M. Valadier; *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, vol. 580, Springer - Verlag, Berlin-Heidelberg-New York, 1977.
[17] D. P. Giuseppe, E. Sinestrari; Differential operators with non-dense domain. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, **14** (1987), 285–344.
[18] R. Datko; Linear autonomous neutral differential equations in a Banach space, *J. Differential Equations 25* (1977), 258–274.
[19] K. L. Engel, R. Nagel; *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.
[20] K. Ezzinbi, S. L. Rhali; Existence and controllability for nondensely defined partial neutral functional differential inclusions, *Appl. Math.*, **60**(3) (2015), 321–340.
[21] M. Frigon; Fixed point resultats for multivalued contractions on gauge spaces, *SIMAA*, **4** (2002), 175–181.
[22] J. K. Hale; Partial neutral functional differential equations, *Rev. Roumaine Math. Pure Appl.*, **39** (1994), 339–344.
[23] J. K. Hale; Coupled oscillators on a circle, *Resenhas IME-USP*, **1**(4) (1994), 441–457.
[24] J. K. Hale, K. R. Meyer; *A class of functional equations of neutral type*, Memoirs of the American Mathematical Society, **76** American Mathematical Society, Providence, RI, 1967 iii+65 pp.
[25] H. R. Henríquez, C. Cuevas; Almost automorphy for abstract neutral differential equations via control theory. *Ann. Mat. Pura Appl.*, **192**(3) (2013), 393–405.
[26] N. T. Huy, N. T. Loan, V. T. N. Ha; Periodic solutions and their conditional stability for partial neutral functional differential equations, *J. Evol. Equ.*, **19**(4) (2019), 1091–1110.
[27] C. Jendoubi; Periodic solutions for partial neutral nondensely differential equations, *Applicable Analysis*, **100**(8) (2021), 1752–1773.
[28] H. Kellerman, M. Hieber; Integrated semigroup, *J. Funct. Anal.*, **84**, 160-180 (1989).
[29] B. Liu; Controllability of neutral functional differential and integrodifferential inclusions with infinite delay. *J. Optim. Theory Appl.*, **123**(3) (2004), 573–593.
[30] A. Pazy; Semigroups of Linear Operators and Applications to Partial Differential Equations, *Springer-Verlag, New York* (1983).
[31] L. Qiang, H. Zhang; Existence and regularity of periodic solutions for neutral evolution equations with delays, *Adv. Difference Equ.*, **330**, (2019).
[32] M. D. Quinn, N. Carmichael; An approach to nonlinear control problem using fixed point methods, degre theory, pseudo-inverse, *Numer. Funct. Anal. Optim.*, **7** (1985), 197–219.

[33] M. Sofonea Y-B. Xiao; Tykhonov well-posedness of elliptic variational-hemivariational inequalities. *Electron. J. Differential Equations*, **2019** (2019) No. 64, 1–19.

[34] I. I. Vrabie; *C_0-semigroups and applications*, North-Holland Mathematics Studies, 191. North-Holland Publishing Co., Amsterdam, 2003.

[35] J. Wu, H. Xia; Self-sustained oscillations in a ring array of coupled lossless transmission lines, *J. Differential Equations*, **124** (1996), 247–278.

[36] K. Yosida; *Functional Analysis*, 6^{th} edn. Springer-Verlag, Berlin, 1980.

[37] S. Zeng, Y. Bai, L. Gasinski, I. Krech; Existence of solutions for implicit obstacle problems involving nonhomogeneous partial differential operators and multivalued terms, *Electronic J. of Differential Equations*, **2021** (2021) no. 37, 1–17.

[38] J. Zhu, X. Fu; Existence and regularity of solutions for neutral partial integro-differential equations with nonlocal conditions. *J. Fixed Point Theory Appl.*, **22** (2020), Article number 34.

[39] Y. Zuomao: Approximate Controllability of Fractional Impulsive Partial Neutral Stochastic Differential Inclusions with State-Dependent Delay and Fractional Sectorial Operators, *Numer. Funct. Anal. Optim.*, **37**(12) (2016), 1590–1639.

Nguyen Thi Van Anh

Department of Mathematics, Hanoi National University of Education, No. 136 Xuan Thuy, Cau Giay, Hanoi, Vietnam.

Institute of Mathematics, Vietnam Academy of Science and Technology, No. 18 Hoang Quoc Viet, Hanoi, Vietnam

Email address: `anhntv.ktt@hnue.edu.vn`

Bui Thi Hai Yen

Department of Mathematics, Hoa Lu University, Ninh Nhat, Ninh Binh, Vietnam

Email address: `bthyen.ktn@hluv.edu.vn`