

## EXISTENCE OF SOLUTIONS FOR IMPLICIT OBSTACLE PROBLEMS INVOLVING NONHOMOGENEOUS PARTIAL DIFFERENTIAL OPERATORS AND MULTIVALUED TERMS

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ABSTRACT. In this article, we study an implicit obstacle problem with a nonlinear nonhomogeneous partial differential operator and a multivalued operator which is described by a generalized gradient. Under quite general assumptions on the data, and employing Kluge's fixed point principle for multivalued operators, Minty technique and a surjectivity theorem, we prove that the set of weak solutions to the problem is nonempty, bounded and weakly closed.

### 1. INTRODUCTION

Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded domain with a  $C^{1,\alpha}$ -boundary  $\partial\Omega$  for some  $0 < \alpha < 1$ . In this paper, we study the following implicit obstacle problem with a nonlinear nonhomogeneous partial differential operator and a multivalued operator which is described by a generalized gradient, namely

$$\begin{aligned} -\operatorname{div} a(x, \nabla u(x)) + \partial j(x, u(x)) &\ni f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ T(u) &\leq U(u). \end{aligned} \tag{1.1}$$

In the above  $f: \Omega \rightarrow \mathbb{R}$  and  $j: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are given two functions, such that  $f \in L^{p'}(\Omega)$  (where  $1 < p < \infty$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ ) and  $j$  is locally Lipschitz with respect to the second variable. By  $\partial j(x, u(x))$  we denote the Clarke's generalized gradient of  $j$  with respect to the last variable. Finally  $T, U: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  are two given functions, which satisfy appropriate assumptions listed in Section 3.

In this article we prove that the set of weak solutions to the problem is nonempty, bounded and weakly closed. In particular we obtain the existence of at least one weak solution to problem (1.1). The main tools used in the proof are the surjectivity theorem for multivalued mappings due to Le [33], Kluge's fixed point principle as well as some techniques of nonsmooth analysis. Problem (1.1) combines several interesting phenomena like a nonhomogeneous operator of  $p$ -Laplacian type, a multivalued mapping provided by the Clarke generalized subdifferential and an implicit obstacle inequality. The latter means that any solution  $u \in W_0^{1,p}(\Omega)$  of (1.1) has to

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2010 *Mathematics Subject Classification.* 35J20, 35J25, 35J60.

*Key words and phrases.* Implicit obstacle problem; Clarke generalized gradient; nonhomogeneous partial differential operator; fixed point theorem; surjectivity theorem.

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Submitted April 1, 2020. Published May 6, 2021.

belong to  $K(u)$ , which is the image of the multivalued map  $K: W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)}$  defined by

$$K(u) := \{v \in W_0^{1,p}(\Omega) : T(v) - U(u) \leq 0\},$$

for some obstacles given by the functions  $T: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  and  $U: W_0^{1,p}(\Omega) \rightarrow (0, +\infty)$ .

For the problems with a nonhomogeneous operator of  $p$ -Laplacian type we refer to Bai-Gasiński-Papageorgiou [2], Candito-Gasiński-Livrea [6], Gasiński-O'Regan-Papageorgiou [20, 21], Gasiński-Papageorgiou [27, 28], Marino-Winkert [35, 36], Papageorgiou-Winkert [39], Papageorgiou-Rădulescu [40], Papageorgiou-Rădulescu-Repovš [41, 42]. In all the aforementioned papers, we find different types of nonhomogeneous operators and boundary value conditions, but we do not have multivalued terms as well as they do not deal with obstacle problems. For the problems dealing with multivalued terms modeled by Clarke's subdifferential we refer to the papers of Averna-Marano-Motreanu [1], Denkowski-Gasiński-Papageorgiou [10, 11, 12, 13], Filippakis-Gasiński-Papageorgiou [15, 16], Gasiński [17, 18], Gasiński-Motreanu-Papageorgiou [19], Gasiński-Papageorgiou [23, 24], Kalita-Kowalski [30], Papageorgiou-Vetro-Vetro [43, 44], Zeng-Liu-Migórski [45]. None of them deals with nonhomogeneous operators and obstacle problems. Finally, for the problems dealing with obstacle problems we refer to the papers of Caffarelli-Salsa-Silvestre [4], Caffarelli-Ros-Oton-Serra [5], Choe [8], Choe-Lewis [9], Feehan-Pop [14], Oberman [38]. As for the paper combining both nonhomogeneous operator and multivalued term provided by a subdifferential we refer to the paper of Gasiński-Papageorgiou [25], although their approach is different from ours and is based on the nonsmooth critical point theory.

This article is organized as follows. In Section 2 we recall some definitions of function spaces needed in the sequel as well as the formulations of the main tools needed for our proofs, in particular the surjectivity results of Le [33] and Kluge's fixed point theorem. In Section 3 we provide the list of assumptions on the data of problem (1.1) and give the definition of the weak solution. In Section 4 we consider an auxiliary problem (see (4.2)) and indicate some properties of its solution set. Finally, in Section 5, we state and prove the main result of the paper (Theorem 5.1), which says that the solution set of (1.1) is a nonempty, bounded and weakly closed subset of  $W_0^{1,p}(\Omega)$ .

## 2. PRELIMINARIES

For a bounded domain  $\Omega \subseteq \mathbb{R}^N$  and  $1 \leq r \leq \infty$ , in what follows, by  $L^r(\Omega)$  and  $L^r(\Omega; \mathbb{R}^N)$  we denote the usual Lebesgue spaces endowed with the norms denoted by  $\|\cdot\|_r$ . Moreover,  $W_0^{1,r}(\Omega)$  stands for the Sobolev space endowed with the norm

$$\|u\| = \|\nabla u\|_r \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

Let us now consider the eigenvalue problem for the  $r$ -Laplacian with homogeneous Dirichlet boundary condition and  $1 < r < \infty$  which is defined by

$$\begin{aligned} -\Delta_r u &= \lambda |u|^{r-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

A number  $\lambda \in \mathbb{R}$  is an eigenvalue of  $(-\Delta_r, W_0^{1,r}(\Omega))$  if problem (2.1) has a nontrivial solution  $u \in W_0^{1,r}(\Omega)$  which is called an eigenfunction corresponding to the eigenvalue  $\lambda$ . We denote by  $\sigma_r$  the set of eigenvalues of  $(-\Delta_r, W_0^{1,r}(\Omega))$ . From Lê

[34] we know that the set  $\sigma_r$  has a smallest element  $\lambda_{1,r}$  which is positive, isolated, simple and it can be variationally characterized through

$$\lambda_{1,r} = \inf \left\{ \frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0 \right\}. \tag{2.2}$$

For  $s > 1$ , we denote by  $s' = \frac{s}{s-1}$  its conjugate, the inner product in  $\mathbb{R}^N$  is denoted by  $\cdot$  and the norm of  $\mathbb{R}^N$  is given by  $|\cdot|$ . Moreover,  $\mathbb{R}_+ = [0, +\infty)$  and the Lebesgue measure in  $\mathbb{R}^N$  is denoted by  $|\cdot|_N$ .

Let  $E$  be a Banach space with its topological dual  $E^*$ . A function  $J : E \rightarrow \mathbb{R}$  is said to be locally Lipschitz at  $u \in E$  if there exist a neighborhood  $N(u)$  of  $u$  and a constant  $L_u > 0$  such that

$$|J(w) - J(v)| \leq L_u \|w - v\|_E \quad \text{for all } w, v \in N(u).$$

**Definition 2.1.** Let  $J : E \rightarrow \mathbb{R}$  be a locally Lipschitz function and let  $u, v \in E$ . The generalized directional derivative  $J^0(u; v)$  of  $J$  at the point  $u$  in the direction  $v$  is defined by

$$J^0(u; v) := \limsup_{w \rightarrow u, t \downarrow 0} \frac{J(w + tv) - J(w)}{t}.$$

The generalized gradient  $\partial J : E \rightarrow 2^{E^*}$  of  $J : E \rightarrow \mathbb{R}$  is defined by

$$\partial J(u) := \{ \xi \in E^* \mid J^0(u; v) \geq \langle \xi, v \rangle_{E^* \times E} \text{ for all } v \in E \} \quad \text{for all } u \in E.$$

The next proposition collects some basic results (see Migórski-Ochal-Sofonea [37, Proposition 3.23]).

**Proposition 2.2.** Let  $J : E \rightarrow \mathbb{R}$  be locally Lipschitz of rank  $L_u > 0$  at  $u \in E$ . Then we have

- (a) the function  $v \mapsto J^0(u; v)$  is positively homogeneous, subadditive, and satisfies

$$|J^0(u; v)| \leq L_u \|v\|_E \text{ for all } v \in E.$$

- (b)  $(u, v) \mapsto J^0(u; v)$  is upper semicontinuous.
- (c) for each  $u \in E$ ,  $\partial J(u)$  is a nonempty, convex, and weak\* compact subset of  $E^*$  with  $\|\xi\|_{E^*} \leq L_u$  for all  $\xi \in \partial J(u)$ .
- (d)  $J^0(u; v) = \max \{ \langle \xi, v \rangle_{E^* \times E} \mid \xi \in \partial J(u) \}$  for all  $v \in E$ .
- (e) the multivalued function  $E \ni u \mapsto \partial J(u) \subset E^*$  is upper semicontinuous from  $E$  into  $w^*-E^*$ .

Next, let  $\vartheta \in C^1(0, \infty)$  be any function satisfying

$$0 < a_1 \leq \frac{t\vartheta'(t)}{\vartheta(t)} \leq a_2 \quad \text{and} \quad a_3 t^{p-1} \leq \vartheta(t) \leq a_4 (t^{q-1} + t^{p-1}) \tag{2.3}$$

for all  $t > 0$ , with some constants  $a_i > 0$ ,  $i \in \{1, 2, 3, 4\}$  and for  $1 < q < p < \infty$ . The hypotheses on  $a : \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  are listed below.

- (H1)  $a(x, \xi) = a_0(x, |\xi|)\xi$  with  $a_0 \in C(\bar{\Omega} \times \mathbb{R}_+)$  for all  $\xi \in \mathbb{R}^N$  and with  $a_0(x, t) > 0$  for all  $x \in \bar{\Omega}$ , for all  $t > 0$  and
  - (i)  $a_0 \in C^1(\bar{\Omega} \times (0, \infty))$ ,  $t \mapsto ta_0(x, t)$  is strictly increasing in  $(0, \infty)$ ,  $\lim_{t \rightarrow 0^+} ta_0(x, t) = 0$  for all  $x \in \bar{\Omega}$  and

$$\lim_{t \rightarrow 0^+} \frac{ta_0'(x, t)}{a_0(x, t)} = c > -1 \quad \text{for all } x \in \bar{\Omega};$$

- (ii)  $|\nabla_{\xi} a(x, \xi)| \leq a_5 \vartheta (|\xi|) / |\xi|$  for all  $x \in \bar{\Omega}$ , for all  $\xi \in \mathbb{R}^N \setminus \{0\}$  and some  $a_5 > 0$ ;
- (iii)  $\nabla_{\xi} a(x, \xi) y \cdot y \geq \vartheta (|\xi|) |y|^2 / |\xi|$  for all  $x \in \bar{\Omega}$ , for all  $\xi \in \mathbb{R}^N \setminus \{0\}$  and all  $y \in \mathbb{R}^N$ .

The following lemma summarizes some properties of the function  $a: \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ .

**Lemma 2.3.** *If hypotheses (H1) hold, then:*

- (i)  $a \in C(\bar{\Omega} \times \mathbb{R}^N; \mathbb{R}^N) \cap C^1(\bar{\Omega} \times (\mathbb{R}^N \setminus \{0\}); \mathbb{R}^N)$  and for all  $x \in \bar{\Omega}$  the map  $\xi \mapsto a(x, \xi)$  is continuous, strictly monotone and so maximal monotone as well;
- (ii) there exists  $a_6 > 0$ , such that  $|a(x, \xi)| \leq a_6 (1 + |\xi|^{p-1})$  for all  $x \in \bar{\Omega}$  and  $\xi \in \mathbb{R}^N$ ;
- (iii)  $a(x, \xi) \cdot \xi \geq \frac{a_3}{p-1} |\xi|^p$  for all  $x \in \bar{\Omega}$  and for all  $\xi \in \mathbb{R}^N$ .

**Lemma 2.4.** *Let  $p \geq 2$ . If the following condition holds,*

(H2)  $t \mapsto a_0(t)t - c_a t^{p-1}$  is increasing on  $[0, \infty)$  with some  $c_a > 0$ ,

then there exists a constant  $m_a > 0$  such that

$$(a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2)_{\mathbb{R}^N} \geq m_a |\xi_1 - \xi_2|^p$$

for all  $\xi_1, \xi_2 \in \mathbb{R}^N$  and a.e.  $x \in \Omega$ .

*Proof.* Since  $p \geq 2$ , it follows from Glowinski-Marrocco [29, Lemma 5.1], that there exists a constant  $m(p) > 0$ , which depends on  $p$  only, such that

$$(|\xi_1|^{p-2} \xi_1 - |\xi_2|^{p-2} \xi_2) \cdot (\xi_1 - \xi_2) \geq m(p) |\xi_1 - \xi_2|^p$$

for all  $\xi_1, \xi_2 \in \mathbb{R}^N$ .

The monotonicity of  $t \mapsto a_0(t)t - c_a t^{p-1}$  ensures that

$$a_0(t)t - a_0(s)s \geq c_a (t^{p-1} - s^{p-1})$$

for all  $t, s \in [0, +\infty)$  with  $t \geq s$ . This inequality leads to

$$\begin{aligned} & (a(\xi_1) - a(\xi_2), \xi_1 - \xi_2)_{\mathbb{R}^N} \\ &= (a_0(|\xi_1|)\xi_1 - a_0(|\xi_2|)\xi_2, \xi_1 - \xi_2) \\ &= [a_0(|\xi_1|)|\xi_1| - a_0(|\xi_2|)|\xi_2|] [|\xi_1| - |\xi_2|] + [a_0(|\xi_1|) + a_0(|\xi_2|)] [|\xi_1||\xi_2| - \xi_1 \cdot \xi_2] \\ &\geq c_a [|\xi_1|^{p-1} - |\xi_2|^{p-1}] [|\xi_1| - |\xi_2|] + c_a [|\xi_1|^{p-2} + |\xi_2|^{p-2}] [|\xi_1||\xi_2| - \xi_1 \cdot \xi_2] \\ &= c_a (|\xi_1|^{p-2} \xi_1 - |\xi_2|^{p-2} \xi_2) \cdot (\xi_1 - \xi_2) \\ &\geq c_a m(p) |\xi_1 - \xi_2|^p \end{aligned}$$

for all  $\xi_1, \xi_2 \in \mathbb{R}^N$ . This means that the desired inequality is satisfied with  $m_a = c_a m(p)$ .  $\square$

Let us introduce the nonlinear operator  $A: W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)^*$  as follows

$$\langle A(u), \phi \rangle = \int_{\Omega} (a(x, \nabla u(x)), \nabla \phi(x))_{\mathbb{R}^N} dx \quad \text{for all } u, \phi \in W_0^{1,p}(\Omega), \quad (2.4)$$

which possesses the following useful properties (see Gasiński-Papageorgiou [26]).

**Proposition 2.5.** *If (H1) hold and the operator  $A: W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)^*$  is defined by (2.4), then  $A$  is bounded, monotone, continuous, hence maximal monotone and of type  $(S_+)$ . Moreover, if the function  $t \mapsto a_0(t)t - c_a t^{p-1}$  is increasing on  $[0, \infty)$*

with some  $c_a > 0$ , then  $A$  is strongly monotone with constant  $m_a > 0$ , where  $m_a$  is given in Lemma 2.4.

The following examples present some operators fitting in our setting.

**Example 2.6.** In the definitions of the operators  $a$ , we drop the dependence on  $x$  just for simplicity. All the following maps satisfy (H1):

- (i) If  $a(\xi) = |\xi|^{p-2}\xi$  with  $1 < p < \infty$ , then the corresponding operator is the classical  $p$ -Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

- (ii) If  $a(\xi) = |\xi|^{p-2}\xi + \mu|\xi|^{q-2}\xi$  with  $1 < q < p < \infty$  and  $\mu > 0$ , then the corresponding operator is the so called weighted  $(p, q)$ -Laplacian defined by  $\Delta_p u + \mu\Delta_q u$  for all  $u \in W^{1,p}(\Omega)$ .
- (iii) If  $a(\xi) = (1 + |\xi|^2)^{\frac{p-2}{2}}\xi$  with  $1 < p < \infty$ , then this map represents the generalized  $p$ -mean curvature differential operator defined by

$$\operatorname{div} \left[ (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right] \quad \text{for all } u \in W^{1,p}(\Omega).$$

Besides, we recall the notion of pseudomonotonicity for multivalued operators (see e.g., Gasiński-Papageorgiou [22, Definition 1.4.8]).

**Definition 2.7.** Let  $X$  be a real reflexive Banach space. The operator  $A: X \rightarrow 2^{X^*}$  is called pseudomonotone if the following conditions hold:

- (i) the set  $A(u)$  is nonempty, bounded, closed and convex for all  $u \in X$ .
- (ii)  $A$  is upper semicontinuous from each finite-dimensional subspace of  $X$  to the weak topology on  $X^*$ .
- (iii) if  $\{u_n\} \subset X$  with  $u_n \rightharpoonup u$  in  $X$  and if  $u_n^* \in A(u_n)$  is such that

$$\limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle_{X^* \times X} \leq 0,$$

then to each element  $v \in X$ , exists  $u^*(v) \in A(u)$  with

$$\langle u^*(v), u - v \rangle_{X^* \times X} \leq \liminf_{n \rightarrow \infty} \langle u_n^*, u_n - v \rangle_{X^* \times X}.$$

Furthermore, we will state the surjectivity theorem for multivalued mappings which are defined as the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping. This theorem was proved in Le [33, Theorem 2.2]. We use the notation  $B_R(0) = \{u \in X : \|u\|_X < R\}$ .

**Theorem 2.8.** Let  $X$  be a real reflexive Banach space, let  $F: D(F) \subset X \rightarrow 2^{X^*}$  be a maximal monotone operator, let  $G: D(G) = X \rightarrow 2^{X^*}$  be a bounded multivalued pseudomonotone operator and let  $L \in X^*$ . Assume that there exist  $u_0 \in X$  and  $R \geq \|u_0\|_X$  such that  $D(F) \cap B_R(0) \neq \emptyset$  and

$$\langle \xi + \eta - L, u - u_0 \rangle_{X^* \times X} > 0$$

for all  $u \in D(F)$  with  $\|u\|_X = R$ , all  $\xi \in F(u)$  and all  $\eta \in G(u)$ . Then there exists  $u \in D(F) \cap D(G)$  such that

$$F(u) + G(u) \ni L.$$

Finally, we recall the fixed point theorem of Kluge [32] which will be used in the proof of our main existence result.

**Theorem 2.9.** *Let  $Z$  be a reflexive Banach space and let  $C \subset Z$  be a nonempty, closed and convex set. Assume that  $\Psi: C \rightarrow 2^C$  is a multivalued mapping such that for every  $u \in C$ , the set  $\Psi(u)$  is nonempty, closed, and convex, and  $\text{Gr } \Psi$  (the graph of  $\Psi$ ) is sequentially weakly closed. If either  $C$  is bounded or  $\Psi(C)$  is bounded, then the map  $\Psi$  has at least one fixed point in  $C$ .*

### 3. ASSUMPTIONS AND DATA PROPERTIES

To obtain the existence of solutions for problem (1.1), we need the following assumptions for the data of problem (1.1).

(H3)  $f \in L^{p'}(\Omega)$ ,

(H4)  $j: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a function such that

(i)  $x \mapsto j(x, r)$  is measurable on  $\Omega$  for all  $r \in \mathbb{R}$  and there exists a function  $l \in L^{q_1}(\Omega)$  with  $q_1 \in (1, p^*)$  such that the function  $x \mapsto j(x, l(x))$  belongs to  $L^1(\Omega)$ .

(ii)  $r \mapsto j(x, r)$  is locally Lipschitz continuous for a.e.  $x \in \Omega$ .

(iii) there exist  $\theta \geq 1$  with  $\theta \leq \min\{q_1, p\}$ ,  $\alpha_j \geq 0$  with  $\alpha_j \lambda_{1,p} < \frac{\alpha_3}{p-1} \delta_\theta$ , and  $\beta_j \in L^1_+(\Omega)$  such that for all  $r \in \mathbb{R}$  and a.e.  $x \in \Omega$  it holds

$$j^0(x, r; -r) \leq \alpha_j |r|^\theta + \beta_j(x),$$

where

$$\delta_\theta = \begin{cases} 1 & \text{if } \theta = p, \\ +\infty & \text{otherwise.} \end{cases}$$

(iv) there exist  $c_j \geq 0$  and  $\gamma_j \in L^{\frac{q_1}{q_1-1}}_+(\Omega)$  such that

$$|\xi| \leq c_j |r|^{q_1-1} + \gamma_j(x) \quad \text{for all } \xi \in \partial j(x, r), \text{ all } r \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where  $\partial j(x, r)$  stands for the generalized gradient of  $j$  with respect to the variable  $r$ .

(v) there exists a constant  $m_j \geq 0$  such that for all  $r_1, r_2 \in \mathbb{R}$  and a.e.  $x \in \Omega$  the inequality is satisfied

$$(\xi_1 - \xi_2)(r_1 - r_2) \geq -m_j |r_1 - r_2|^p$$

whenever  $\xi_1 \in \partial j(x, r_1)$  and  $\xi_2 \in \partial j(x, r_2)$ .

(H5)  $T: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  is a positively homogeneous (i.e.,  $T(tu) = tT(u)$  for all  $t > 0$  and  $u \in W_0^{1,p}(\Omega)$ ) and subadditive function such that

$$T(u) \leq \limsup_{n \rightarrow \infty} T(u_n) \tag{3.1}$$

whenever  $\{u_n\} \subset W_0^{1,p}(\Omega)$  is a sequence such that  $u_n \rightharpoonup u$  in  $W_0^{1,p}(\Omega)$ , as  $n \rightarrow \infty$ , for some  $u \in W_0^{1,p}(\Omega)$ .

(H6)  $U: W_0^{1,p}(\Omega) \rightarrow (0, +\infty)$  is weakly continuous, i.e., for any sequence  $\{u_n\} \subset W_0^{1,p}(\Omega)$  such that  $u_n \rightharpoonup u$ , as  $n \rightarrow \infty$ , for some  $u \in W_0^{1,p}(\Omega)$ , we have

$$U(u_n) \rightarrow U(u), \quad \text{as } n \rightarrow \infty. \tag{3.2}$$

**Remark 3.1.** Assumption (H4)(v) is usually called relaxed monotonicity condition (see e.g. Migórski-Ochal-Sofonea [37]) for the locally Lipschitz function  $r \mapsto j(x, r)$ . It is equivalent to the inequality

$$j^0(x, s_1; s_2 - s_1) + j^0(x, s_2; s_1 - s_2) \leq m_j |s_1 - s_2|^p$$

for all  $s_1, s_2 \in \mathbb{R}$  and for a.e.  $x \in \Omega$ .

Indeed, positive homogeneity and subadditivity of  $T$  confirm that  $T$  is also a convex function. On the other hand, it is not difficult to see that if  $T: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  is lower semicontinuous, then inequality (3.1) holds automatically.

Let us introduce a multivalued map  $K: W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)}$  defined by

$$K(u) = \{v \in W_0^{1,p}(\Omega) : T(v) - U(u) \leq 0\} \tag{3.3}$$

for all  $u \in W_0^{1,p}(\Omega)$ .

**Lemma 3.2.** *Assume that  $T: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  satisfies  $H(T)$  and let  $U: W_0^{1,p}(\Omega) \rightarrow (0, +\infty)$  be any map. Then the map  $K$  defined by (3.3) has nonempty, closed and convex values.*

*Proof.* Let  $u \in W_0^{1,p}(\Omega)$  be fixed. It follows from the positive homogeneity of  $T$  and  $U(u) > 0$ , that  $T(0) = 0 < U(u)$ , namely,  $0 \in K(u) \neq \emptyset$  for each  $u \in W_0^{1,p}(\Omega)$ .

Let  $\{v_n\} \subset K(u)$  be a sequence such that  $v_n \rightarrow v$  in  $W_0^{1,p}(\Omega)$  as  $n \rightarrow \infty$  for some  $v \in W_0^{1,p}(\Omega)$ . Then, for each  $n \in \mathbb{N}$ , we have

$$T(v_n) \leq U(u).$$

Passing to the upper limit as  $n \rightarrow \infty$  in the above inequality and using (3.1) we deduce that

$$T(v) \leq \limsup_{n \rightarrow \infty} T(v_n) \leq U(u).$$

This means that  $v \in K(u)$ , i.e., the set  $K(u)$  is closed.

For any  $v_1, v_2 \in K(u)$  and  $t \in (0, 1)$  fixed, let us set  $v_t = tv_1 + (1 - t)v_2$ . Therefore,  $T(v_i) \leq U(u)$  for  $i = 1, 2$ . However, the convexity of  $T$  (see Remark 3.1) guarantees

$$T(v_t) \leq tT(v_1) + (1 - t)T(v_2) \leq tU(u) + (1 - t)U(u) = U(u),$$

which gives that  $v_t \in K(u)$ . Therefore, we conclude that the set  $K(u)$  is convex in  $W_0^{1,p}(\Omega)$ . □

The weak solutions for problem (1.1) are understood in the following sense.

**Definition 3.3.** We say that  $u \in W_0^{1,p}(\Omega)$  is a weak solution of problem (1.1) if  $u \in K(u)$  and

$$\begin{aligned} & \int_{\Omega} (a(x, \nabla u(x)), \nabla v(x) - \nabla u(x))_{\mathbb{R}^N} dx + \int_{\Omega} j^0(x, u(x); v(x) - u(x)) dx \\ & \geq \int_{\Omega} f(x)[v(x) - u(x)] dx \end{aligned}$$

for all  $v \in K(u)$ , where the multivalued function  $K$  is given by (3.3).

Consider the function  $J: L^{q_1}(\Omega) \rightarrow \mathbb{R}$  defined by

$$J(u) = \int_{\Omega} j(x, u(x)) dx \quad \text{for all } u \in L^{q_1}(\Omega). \tag{3.4}$$

On account of hypotheses (H4) and the definition of  $J$  (see (3.4)), the next lemma is a direct consequence of Migórski-Ochal-Sofonea [37, Theorem 3.47].

**Lemma 3.4.** *Under assumptions (H4)(i)–(iv), we have*

- (i)  $J: L^{q_1}(\Omega) \rightarrow \mathbb{R}$  is locally Lipschitz continuous;

(ii) we have

$$J^0(u; v) \leq \int_{\Omega} j^0(x, u(x); v(x)) dx,$$

$$J^0(u; -u) \leq \alpha_j \|u\|_{\theta}^{\theta} + \|\beta_j\|_1$$

for all  $u, v \in L^{q_1}(\Omega)$ ;

(iii) for each  $u \in L^{q_1}(\Omega)$ , we have

$$\partial J(u) \subset \int_{\Omega} \partial j(x, u(x)) dx,$$

$$\|\xi\|_{q_1'} \leq c_J(1 + \|u\|_{q_1}^{q_1-1}) \quad \text{for all } \xi \in \partial J(u),$$

with some  $c_J > 0$ .

Moreover, if condition (H4)(v) holds, then

$$J^0(u; v - u) + J^0(v; u - v) \leq m_j \|u - v\|_p^p \quad (3.5)$$

for all  $u, v \in W_0^{1,p}(\Omega)$ .

#### 4. AUXILIARY PROBLEMS

Employing Lemma 3.4(ii) we know that if  $u \in W_0^{1,p}(\Omega)$  solves the following problem: Find  $u \in W_0^{1,p}(\Omega)$  such that  $u \in K(u)$  and

$$\int_{\Omega} (a(x, \nabla u(x)), \nabla v(x) - \nabla u(x))_{\mathbb{R}^N} dx + J^0(u; v - u) \geq \int_{\Omega} f(x)[v(x) - u(x)] dx \quad (4.1)$$

for all  $v \in K(u)$ , then  $u$  is a weak solution to problem (1.1) as well. Using this fact, we will prove that problem (4.1) is solvable. To this end, first we investigate the following inequality problem:

Given  $w \in W_0^{1,p}(\Omega)$ , find  $u \in K(w)$  such that

$$\int_{\Omega} (a(x, \nabla u(x)), \nabla v(x) - \nabla u(x))_{\mathbb{R}^N} dx + J^0(u; v - u) \geq \int_{\Omega} f(x)[v(x) - u(x)] dx \quad (4.2)$$

for all  $v \in K(w)$ . Additionally, consider the multivalued map  $\Gamma: W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)}$  given by

$$\Gamma(w) = \{u \in W_0^{1,p}(\Omega) : u \text{ solves problem (4.2) associated with } w\} \quad (4.3)$$

for all  $w \in W_0^{1,p}(\Omega)$ . Indeed, it is not difficult to verify that  $u \in W_0^{1,p}(\Omega)$  is a fixed point of  $\Gamma$ , if and only if  $u$  solves problem (4.1). Motivated by this fact, we shall employ Kluge's fixed point theorem (see Theorem 2.9), to show that the fixed point set of  $\Gamma$  is nonempty.

**Theorem 4.1.** *Let  $U: W_0^{1,p}(\Omega) \rightarrow (0, +\infty)$ . Under the assumptions (H1), (H3), (H4)(i)–(iv) and (H5), we have*

- (i) *for each  $w \in W_0^{1,p}(\Omega)$ , the set of solutions to problem (4.2) is nonempty, bounded, and closed in  $W_0^{1,p}(\Omega)$ , i.e.,  $\Gamma$  has nonempty, bounded, and closed values.*

(ii) if  $p \geq 2$ , hypotheses (H4)(v), H(0), and the smallness condition

$$m_j \lambda_{1,p}^{-1} \leq m_a, \tag{4.4}$$

are fulfilled, then for each  $w \in W_0^{1,p}(\Omega)$ , the set of solutions to problem (4.2) is convex, namely,  $\Gamma(w)$  is convex.

*Proof.* (i) Let  $w \in W_0^{1,p}(\Omega)$  be fixed and  $I_{K(w)}: W_0^{1,p}(\Omega) \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  be the indicator function of  $K(w)$ , i.e.,

$$I_{K(w)}(u) = \begin{cases} 0 & \text{if } u \in K(w), \\ +\infty & \text{otherwise.} \end{cases}$$

Keeping in mind that  $f \in L^{p'}(\Omega) \subset W_0^{1,p}(\Omega)^*$ , problem (4.2) can be rewritten equivalently to the following variational-hemivariational inequality: Find  $u \in W_0^{1,p}(\Omega)$  such that

$$\langle Au, v - u \rangle + J^0(u; v - u) + I_{K(w)}(v) - I_{K(w)}(u) \geq \langle f, v - u \rangle \tag{4.5}$$

for all  $v \in W_0^{1,p}(\Omega)$ , where  $A: W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)^*$  is given by (2.4). However, by the Hahn-Banach Theorem, see e.g. Brezis [3, Theorem 1.6 (the first geometric form)], it is not difficult to prove that problem (4.5) is equivalent to the following inclusion problem: Find  $u \in W_0^{1,p}(\Omega)$  such that

$$Au + \partial J(u) + \partial_C I_{K(w)}(u) \ni f, \tag{4.6}$$

where the notation  $\partial_C I_{K(w)}$  stands for the subdifferential of  $I_{K(w)}$  in the sense of convex analysis.

We shall use the surjectivity result (see Theorem 2.8), to show that problem (4.6) is solvable in  $W_0^{1,p}(\Omega)$ . For this reason, we start with the following claim.

**Claim 1.**  $A + \partial J: W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)^*}$  is a bounded pseudomonotone multivalued operator such that for each  $u \in W_0^{1,p}(\Omega)$ , the set  $A(u) + \partial J(u)$  is closed and convex in  $W_0^{1,p}(\Omega)^*$ .

Directly from Proposition 2.2 and Lemma 3.4 we see the set  $A(u) + \partial J(u)$  is closed and convex in  $W_0^{1,p}(\Omega)^*$  for each  $u \in W_0^{1,p}(\Omega)$ . Moreover, Proposition 2.5, Lemma 3.4(iii) and the fact  $q_1 < p^*$  indicate that  $W_0^{1,p}(\Omega) \ni u \mapsto A(u) + \partial J(u) \subset W_0^{1,p}(\Omega)^*$  is a bounded map.

Next, we assert that  $W_0^{1,p}(\Omega) \ni u \mapsto A(u) + \partial J(u) \subset W_0^{1,p}(\Omega)^*$  is upper semicontinuous from  $W_0^{1,p}(\Omega)$  to  $W_0^{1,p}(\Omega)^*$  with weak topology. By f Migórski-Ochal-Sofonea [37, Proposition 3.8], it is sufficient to show that for any weakly closed subset  $D$  in  $W_0^{1,p}(\Omega)^*$ , the set  $(A + \partial J)^-(D)$  is closed in  $W_0^{1,p}(\Omega)$ . Let  $\{u_n\} \subset (A + \partial J)^-(D)$  be a sequence such that

$$u_n \rightarrow u \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \rightarrow \infty, \text{ for some } u \in W_0^{1,p}(\Omega). \tag{4.7}$$

So, for each  $n \in \mathbb{N}$ , we are able to find  $\xi_n \in \partial J(u_n)$  such that

$$u_n^* = Au_n + \xi_n \in (A(u_n) + \partial J(u_n)) \cap D.$$

But, the continuity of  $A$  (see Proposition 2.5) ensures that  $A(u_n) \rightarrow A(u)$  in  $W_0^{1,p}(\Omega)^*$ , as  $n \rightarrow \infty$ . Taking into account Lemma 3.4(iii) and convergence (4.7), we conclude that the sequence  $\{\xi_n\}$  is bounded in  $W_0^{1,p}(\Omega)^*$ , so, without any loss of generality, we may assume that  $\xi_n \rightharpoonup \xi$  in  $W_0^{1,p}(\Omega)^*$ , as  $n \rightarrow \infty$ , with

some  $\xi \in W_0^{1,p}(\Omega)^*$ . Notice that  $\partial J$  is upper semicontinuous from  $W_0^{1,p}(\Omega)$  to  $w$ - $W_0^{1,p}(\Omega)^*$  and has bounded, convex, closed values (see Proposition 2.2(d)), so, it has a closed graph in  $W_0^{1,p}(\Omega) \times w$ - $W_0^{1,p}(\Omega)^*$  (see Kamenskii-Obukhovskii-Zecca [31, Theorem 1.1.4]). But, thanks to the weak closedness of  $D$ , we derive that  $A(u) + \xi \in D$  and  $\xi \in \partial J(u)$ , which provides that  $u \in (A + \partial J)^-(D)$ . Consequently,  $A + \partial J$  is upper semicontinuous from  $W_0^{1,p}(\Omega)$  to  $W_0^{1,p}(\Omega)^*$  with weak topology.

Next, we show that  $A + \partial J$  is pseudomonotone. Let  $\{u_n\}$  and  $\{u_n^*\}$  be sequences such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \quad (4.8)$$

$$u_n^* \in A(u_n) + \partial J(u_n) \quad \text{with } \limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle \leq 0. \quad (4.9)$$

Our goal is to show that for each  $v \in W_0^{1,p}(\Omega)$  there exists an element  $u^*(v) \in A(u) + \partial J(u)$  such that

$$\liminf_{n \rightarrow \infty} \langle u_n^*, u_n - v \rangle \geq \langle u^*(v), u - v \rangle. \quad (4.10)$$

From (4.9), we are able to find a sequence  $\{\xi_n\} \subset W_0^{1,p}(\Omega)^*$  such that for each  $n \in \mathbb{N}$ ,  $\xi_n \in \partial J(u_n)$  and

$$u_n^* = A(u_n) + \xi_n.$$

The latter combined with the inequality in (4.9) implies

$$\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle + \liminf_{n \rightarrow \infty} \langle \xi_n, u_n - u \rangle \leq 0. \quad (4.11)$$

Applying (4.8) and the compactness of the embedding of  $W_0^{1,p}(\Omega)$  into  $L^{q_1}(\Omega)$ , gives

$$u_n \rightarrow u \quad \text{in } L^{q_1}(\Omega), \quad \text{as } n \rightarrow \infty.$$

On the other hand, employing Chang [7, Theorem 2.2], we have

$$\partial(J|_{W_0^{1,p}(\Omega)})(u) \subset \partial(J|_{L^{q_1}(\Omega)})(u) \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

which implies that

$$\langle \xi_n, u_n - u \rangle = \langle \xi_n, u_n - u \rangle_{L^{q_1}(\Omega)}. \quad (4.12)$$

Additionally, Lemma 3.4(iii) and the boundedness of the sequence  $\{u_n\}$  in  $W_0^{1,p}(\Omega)$  implies that the sequence  $\{\xi_n\}$  is contained in  $L^{q_1}(\Omega)$ . Then, passing to the limit in (4.12) as  $n \rightarrow \infty$  to obtain

$$\lim_{n \rightarrow \infty} \langle \xi_n, u_n - u \rangle = \lim_{n \rightarrow \infty} \langle \xi_n, u_n - u \rangle_{L^{q_1}(\Omega)} = 0.$$

Inserting the above equality into (4.11) yields

$$\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle = \limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle + \liminf_{n \rightarrow \infty} \langle \xi_n, u_n - u \rangle \leq 0.$$

The latter combined with Proposition 2.5 (i.e., the fact that  $A$  is type of  $(S_+)$ ) and (4.8) finds that  $u_n \rightarrow u$  in  $W_0^{1,p}(\Omega)$ , as  $n \rightarrow \infty$ . Moreover, the reflexivity of  $W_0^{1,p}(\Omega)^*$  and boundedness of  $\{\xi_n\} \subset W_0^{1,p}(\Omega)^*$  permit us to conclude that

$$\xi_n \rightharpoonup \xi \quad \text{in } W_0^{1,p}(\Omega)^* \quad \text{for some } \xi \in W_0^{1,p}(\Omega)^*.$$

Now we can assert that  $\xi \in \partial J(u)$  (see, e.g., Kamenskii-Obukhovskii-Zecca [31, Theorem 1.1.4]). Now, because

$$\liminf_{n \rightarrow \infty} \langle u_n^*, u_n - v \rangle = \liminf_{n \rightarrow \infty} \langle A(u_n) + \xi_n, u_n - v \rangle = \langle A(u) + \xi, u - v \rangle,$$

it is clear that (4.10) holds with  $u^* = A(u) + \xi \in A(u) + \partial J(u)$ . Therefore,  $A + \partial J$  is pseudomonotone. This proves Claim 1.

Next, we prove that there exists  $R > 0$  such that

$$\langle Au + \xi + \eta - f, u \rangle > 0 \tag{4.13}$$

for all  $u \in K(w)$  with  $\|u\| = R$ , all  $\xi \in \partial J(u)$  and all  $\eta \in \partial_C(I_{K(w)})(u)$ .

For this purpose, let  $u \in W_0^{1,p}(\Omega)$  be fixed. For any  $\xi \in \partial J(u)$  and  $\eta \in \partial_C(I_{K(w)})(u)$ , since  $0 \in K(w)$  and  $f \in L^{p'}(\Omega) \subset W_0^{1,p}(\Omega)^*$ , we have

$$\begin{aligned} & \langle Au + \xi + \eta - f, u \rangle \\ & \geq \int_{\Omega} (a(x, \nabla u(x)), \nabla u(x))_{\mathbb{R}^N} dx + \int_{\Omega} \xi(x)u(x) dx + I_{K(w)}(u) - I_{K(w)}(0) \\ & \quad - \|f\|_{W_0^{1,p}(\Omega)^*} \|u\| \\ & \geq \frac{a_3}{p-1} \|\nabla u\|_p^p - \int_{\Omega} \xi(x)[-u(x)] dx + I_{K(w)}(u) - \|f\|_{W_0^{1,p}(\Omega)^*} \|u\| \\ & \geq \frac{a_3}{p-1} \|\nabla u\|_p^p - J^0(u; -u) + I_{K(w)}(u) - \|f\|_{W_0^{1,p}(\Omega)^*} \|u\|, \end{aligned} \tag{4.14}$$

where we have used Lemma 2.3(iii). Notice that  $I_{K(w)}: W_0^{1,p}(\Omega) \rightarrow \overline{\mathbb{R}}$  is a proper, convex and lower semicontinuous function, so we now apply Gasiński-Papageorgiou [22, Proposition 1.3.1], for finding  $a_{K(w)}, b_{K(w)} \geq 0$  such that

$$I_{K(w)}(v) \geq -a_{K(w)}\|v\| - b_{K(w)} \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{4.15}$$

Additionally, Lemma 3.4(ii) implies that

$$J^0(u; -u) \leq \alpha_j \|u\|_{\theta}^{\theta} + \|\beta_j\|_1. \tag{4.16}$$

We now distinguish two cases:  $\theta < p$  and  $\theta = p$ . When  $\theta < p$ , let  $c(\theta) > 0$  be such that

$$\|u\|_{\theta} \leq c(\theta)\|u\| \quad \text{for all } u \in W_0^{1,p}(\Omega) \tag{4.17}$$

(its existence is follows from the continuity of the embedding from  $W_0^{1,p}(\Omega)$  to  $L^r(\Omega)$  for any  $r \in (1, p^*)$ ). Inserting (4.15) and (4.16) into (4.14) and using (4.17), we have

$$\begin{aligned} \langle Au + \xi + \eta - f, u \rangle & \geq \frac{a_3}{p-1} \|\nabla u\|_p^p - \alpha_j \|u\|_{\theta}^{\theta} - \|\beta_j\|_1 - a_{K(w)}\|u\| \\ & \quad - b_{K(w)} - \|f\|_{W_0^{1,p}(\Omega)^*} \|u\| \\ & \geq \frac{a_3}{p-1} \|u\|^p - \alpha_j c(\theta)^{\theta} \|u\|^{\theta} - \|\beta_j\|_1 - a_{K(w)}\|u\| \\ & \quad - b_{K(w)} - \|f\|_{W_0^{1,p}(\Omega)^*} \|u\|. \end{aligned} \tag{4.18}$$

Since  $\theta < p$ , we can find a constant  $R_0 > 0$  large enough such that

$$\frac{a_3}{p-1} R_0^p - \alpha_j c(\theta)^{\theta} R_0^{\theta} - \|\beta_j\|_1 - a_{K(w)} R_0 - b_{K(w)} - \|f\|_{W_0^{1,p}(\Omega)^*} R_0 > 0.$$

Therefore, for each  $R \geq R_0$  fixed, the desired inequality (4.13) holds.

Next, if  $\theta = p$ , using variational characterization of  $\lambda_{1,p}$  (see (2.2)), we deduce that

$$\begin{aligned} & \langle Au + \xi + \eta - f, u \rangle \\ & \geq \frac{a_3}{p-1} \|\nabla u\|_p^p - \alpha_j \|u\|_p^p - \|\beta_j\|_1 - a_{K(w)} \|u\| - b_{K(w)} - \|f\|_{W_0^{1,p}(\Omega)^*} \|u\| \quad (4.19) \\ & \geq \left( \frac{a_3}{p-1} - \alpha_j \lambda_{1,p}^{-1} \right) \|\nabla u\|_p^p - \|\beta_j\|_1 - a_{K(w)} \|u\| - b_{K(w)} - \|f\|_{W_0^{1,p}(\Omega)^*} \|u\|. \end{aligned}$$

As  $1 < p$  and  $\alpha_j \lambda_{1,p}^{-1} < \frac{a_3}{p-1}$ , we can take  $R_0 > 0$  large enough such that for all  $R \geq R_0$  it holds

$$\left( \frac{a_3}{p-1} - \alpha_j \lambda_{1,p}^{-1} \right) R^p - \|\beta_j\|_1 - a_{K(w)} R - b_{K(w)} - \|f\|_{W_0^{1,p}(\Omega)^*} R > 0.$$

Therefore, the inequality (4.13) holds.

Recall that  $I_{K(w)}: W_0^{1,p}(\Omega) \rightarrow \overline{\mathbb{R}}$  is a proper, convex and lower semicontinuous function, so,  $\partial_C I_{K(w)}: W_0^{1,p}(\Omega) \rightarrow 2^{W_0^{1,p}(\Omega)^*}$  is maximal monotone. The latter together with Theorem 2.8 implies that there exists  $u_w \in W_0^{1,p}(\Omega)$  resolving inclusion (4.6). Thus,  $\Gamma(w) \neq \emptyset$  for each  $w \in W_0^{1,p}(\Omega)$ .

Next, we demonstrate that  $\Gamma(w)$  is closed in  $W_0^{1,p}(\Omega)$ . Let  $\{u_n\} \subset \Gamma(w)$  be such that

$$u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega) \text{ as } n \rightarrow \infty$$

for some  $u \in W_0^{1,p}(\Omega)$ . So, for each  $n \in \mathbb{N}$ , we have

$$\langle Au_n, v - u_n \rangle + J^0(u_n; v - u_n) + I_{K(w)}(v) - I_{K(w)}(u_n) \geq \langle f, v - u_n \rangle$$

for all  $v \in W_0^{1,p}(\Omega)$ . Passing to the upper limit as  $n \rightarrow \infty$  in the above inequality, we obtain

$$\begin{aligned} & \langle Au, v - u \rangle + J^0(u; v - u) + I_{K(w)}(v) - I_{K(w)}(u) \\ & \geq \limsup_{n \rightarrow \infty} [\langle Au_n, v - u_n \rangle + J^0(u_n; v - u_n) + I_{K(w)}(v) - I_{K(w)}(u_n)] \\ & \geq \limsup_{n \rightarrow \infty} \langle f, v - u_n \rangle \\ & = \langle f, v - u \rangle \end{aligned}$$

for all  $v \in W_0^{1,p}(\Omega)$ , where we have used the continuity of  $A$  (see Proposition 2.5), upper semicontinuity of  $(u, v) \mapsto J^0(u; v)$  (see Proposition 2.2(d)) and lower semicontinuity of  $I_{K(w)}$ . This indicates that  $u \in \Gamma(w)$ , hence  $\Gamma(w)$  is closed.

Finally, we prove that  $\Gamma(w)$  is bounded. Arguing by contradiction, we suppose that  $\Gamma(w)$  is unbounded. Then there exists a sequence  $\{u_n\}$  in  $\Gamma(w)$  such that

$$\|u_n\| \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \quad (4.20)$$

By a simple computation (see (4.18) and (4.19)), we are able to find  $N_0 \in \mathbb{N}$  such that for all  $n \geq N_0$ , it holds

$$0 \geq \langle Au_n, u_n \rangle - J^0(u_n; -u_n) + I_{K(w)}(u_n) > 0,$$

where we have used the fact  $0 \in K(w)$  and (4.20). This leads to a contradiction. Therefore,  $\Gamma(w)$  is bounded.

(ii) Assume that hypothesis (H3)(v) holds. Let  $u_1, u_2 \in W_0^{1,p}(\Omega)$  be two solutions to problem (4.2). Hence

$$\langle Au_i, v - u_i \rangle + J^0(u_i; v - u_i) + I_{K(w)}(v) - I_{K(w)}(u_i) \geq \langle f, v - u_i \rangle$$

for all  $v \in W_0^{1,p}(\Omega)$  and for  $i = 1, 2$ . But, Proposition 2.5 and Lemma 3.4 give

$$\begin{aligned} 0 &\geq \langle Av_1 - Av_2, v_1 - v_2 \rangle - (J^0(v_1; v_2 - v_1) + J^0(v_2; v_1 - v_2)) \\ &\geq m_a \|\nabla v_1 - \nabla v_2\|_p^p - m_j \|v_1 - v_2\|_p^p \\ &\geq (m_a - m_j \lambda_{1,p}^{-1}) \|v_1 - v_2\|_p^p \geq 0 \end{aligned}$$

for all  $v_1, v_2 \in W_0^{1,p}(\Omega)$ . Hence, for  $i = 1, 2$ , we have

$$\langle Av, v - u_i \rangle + J^0(v; v - u_i) + I_{K(w)}(v) - I_{K(w)}(u_i) \geq \langle f, v - u_i \rangle$$

for all  $v \in W_0^{1,p}(\Omega)$ . Let  $t \in (0, 1)$  be arbitrary and let us put  $u_t = tu_1 + (1 - t)u_2$ . Therefore, we have

$$\begin{aligned} &\langle Av, v - u_t \rangle + J^0(v; v - u_t) + I_{K(w)}(v) - I_{K(w)}(u_t) \\ &\geq t [\langle Av, v - u_1 \rangle + J^0(v; v - u_1) + I_{K(w)}(v) - I_{K(w)}(u_1)] \\ &\quad + (1 - t) [\langle Av, v - u_2 \rangle + J^0(v; v - u_2) + I_{K(w)}(v) - I_{K(w)}(u_2)] \\ &\geq \langle f, v - u_t \rangle \end{aligned}$$

for all  $v \in W_0^{1,p}(\Omega)$ .

Now, employing the Minty approach we obtain that  $u_t \in \Gamma(w)$ . Consequently, the set  $\Gamma(w)$  is convex in  $W_0^{1,p}(\Omega)$ .  $\square$

### 5. MAIN RESULT

Now we can state the main result of the paper. Its proof is based on Theorem 4.1 and Kluge’s fixed point theorem (see Theorem 2.9).

**Theorem 5.1.** *Assume that (H1), (H3)–(H5), (H6) hold and  $p \geq 2$ . If, in addition, (H2)) and the smallness condition (4.4) are satisfied, then the set of solutions of problem (1.1), denoted by  $\mathcal{S}$ , is nonempty, bounded and weakly closed.*

*Proof.* As we have already mentioned, the fixed point set of  $\Gamma$  (see (4.3)) is the corresponding set of solutions to problem (4.1). Besides, Lemma 3.2 points out that the set of solutions for problem (4.1) is a subset of the set of solutions for problem (1.1). Consequently, it suffices to prove that the fixed point set of  $\Gamma$  is nonempty.

First we show that

$$\text{Gr } \Gamma \text{ is sequentially weakly closed.} \tag{5.1}$$

For this purpose, let  $\{w_n\}, \{u_n\} \subset W_0^{1,p}(\Omega)$  be two sequences such that  $w_n \rightharpoonup w$  in  $W_0^{1,p}(\Omega)$  and  $u_n \in \Gamma(w_n)$  with  $u_n \rightharpoonup u$  in  $W_0^{1,p}(\Omega)$ , as  $n \rightarrow \infty$ , for some  $w, u \in W_0^{1,p}(\Omega)$ . Then, for each  $n \in \mathbb{N}$ , we have  $u_n \in K(w_n)$  (namely,  $T(u_n) \leq U(u_n)$ ) and

$$\langle Au_n, v - u_n \rangle + J^0(u_n; v - u_n) \geq \langle f, v - u_n \rangle \tag{5.2}$$

for all  $v \in K(w_n)$ .

However, hypotheses (H5) and (H6) imply that

$$T(u) \leq \limsup_{n \rightarrow \infty} T(u_n) \leq \limsup_{n \rightarrow \infty} U(w_n) \leq U(w).$$

This means  $u \in K(w)$ .

For any  $v \in K(w)$  fixed, owing to  $U(w) > 0$ , we now consider the sequence  $\{v_n\}$  constructed by

$$v_n = \frac{U(w_n)}{U(w)}v \quad \text{for all } n \in \mathbb{N}.$$

The non-negativity of  $U$ , positive homogeneity of  $T$  and the fact that  $v \in K(w)$  (thus is,  $T(v) \leq U(w)$ ) give

$$T(v_n) = T\left(\frac{U(w_n)}{U(w)}v\right) = \frac{U(w_n)}{U(w)}T(v) \leq \frac{U(w_n)U(w)}{U(w)} = U(w_n),$$

hence  $v_n \in K(w_n)$ . Moreover, a simple calculating gives

$$\lim_{n \rightarrow \infty} \|v_n - v\| = \lim_{n \rightarrow \infty} |U(w_n) - U(w)| \frac{\|v\|}{U(w)} = 0.$$

Thus, we obtain that  $v_n \rightarrow v$ , as  $n \rightarrow \infty$ .

Since  $u \in K(w)$ , we can take the sequence  $\{\bar{u}_n\} \subset W_0^{1,p}(\Omega)$  such that  $\bar{u}_n = \frac{U(w_n)}{U(w)}u \in K(w_n)$  for each  $n \in \mathbb{N}$  and

$$\bar{u}_n \rightarrow u \quad \text{as } n \rightarrow \infty.$$

Inserting  $v = \bar{u}_n$  into (5.2) gives

$$\langle Au_n, u_n - \bar{u}_n \rangle \leq J^0(u_n; \bar{u}_n - u_n) - \langle f, \bar{u}_n - u_n \rangle. \quad (5.3)$$

It follows from Lemma 3.4 and the convergence  $u_n \rightarrow u$  in  $L^{q_1}(\Omega)$ , as  $n \rightarrow \infty$  that

$$\limsup_{n \rightarrow \infty} J^0(u_n; \bar{u}_n - u_n) \leq 0.$$

Passing to the upper limit as  $n \rightarrow \infty$  into (5.3) and using the above inequality, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle &\leq \limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle + \liminf_{n \rightarrow \infty} \langle Au_n, u - \bar{u}_n \rangle \\ &\leq \limsup_{n \rightarrow \infty} \langle Au_n, u_n - \bar{u}_n \rangle \\ &\leq \limsup_{n \rightarrow \infty} J^0(u_n; \bar{u}_n - u_n) - \liminf_{n \rightarrow \infty} \langle f, \bar{u}_n - u_n \rangle \leq 0. \end{aligned}$$

The latter combined with Proposition 2.5 ( $A$  is of type  $(S_+)$ ) implies  $u_n \rightarrow u$  in  $W_0^{1,p}(\Omega)$ , as  $n \rightarrow \infty$ .

For any  $v \in K(w)$  fixed, let  $\{v_n\} \subset W_0^{1,p}(\Omega)$  be such that  $v_n \in K(w_n)$  for each  $n \in \mathbb{N}$  and  $v_n \rightarrow v$  in  $W_0^{1,p}(\Omega)$ , as  $n \rightarrow \infty$ . We put  $v = v_n$  in (5.2) and then pass to the upper limit as  $n \rightarrow \infty$ , to obtain

$$\begin{aligned} \langle Au, v - u \rangle + J^0(u; v - u) &\geq \limsup_{n \rightarrow \infty} \langle Au_n, v_n - u_n \rangle + \limsup_{n \rightarrow \infty} J^0(u_n; v_n - u_n) \\ &\geq \limsup_{n \rightarrow \infty} [\langle Au_n, v_n - u_n \rangle + J^0(u_n; v_n - u_n)] \\ &\geq \limsup_{n \rightarrow \infty} \langle f, v_n - u_n \rangle = \langle f, v - u \rangle, \end{aligned}$$

where we have used the upper semicontinuity of  $L^{q_1}(\Omega) \times L^{q_1}(\Omega) \ni (v, u) \rightarrow J^0(u; v) \in \mathbb{R}$  (see Proposition 2.2). Hence,  $u \in \Gamma(w)$ . Therefore, we conclude that  $\text{Gr } \Gamma$  is sequentially weakly closed. This proves (5.1).

Next we show that

$$\text{the set } \Gamma(W_0^{1,p}(\Omega)) \text{ is bounded in } W_0^{1,p}(\Omega). \quad (5.4)$$

If the above were not true, then there would exist a sequence  $\{w_n\}$  such that

$$\|u_n\| \rightarrow \infty \quad \text{as } n \rightarrow \infty, \tag{5.5}$$

where  $u_n = \Gamma(w_n)$ . For every  $n \in \mathbb{N}$ , one has (5.2) for all  $v \in K(w_n)$ . Keeping in mind that  $0 \in K(w)$  for each  $w \in W_0^{1,p}(\Omega)$ , we take  $v = 0$  as test function in (5.2) obtaining

$$\langle Au_n, u_n \rangle - J^0(u_n; -u_n) \leq \|f\|_{W_0^{1,p}(\Omega)^*} \|u_n\|.$$

Using the same argument as in the proof of Theorem 4.1 (see (4.18) or (4.19)), we could find  $N_0 \in \mathbb{N}$  large enough such that

$$0 < \langle Au_n, u_n \rangle - J^0(u_n; -u_n) - \|f\|_{W_0^{1,p}(\Omega)^*} \|u_n\| \leq 0$$

for all  $n \geq N_0$ , this gives a contradiction. Therefore, we conclude that the set  $\Gamma(W_0^{1,p}(\Omega))$  is bounded in  $W_0^{1,p}(\Omega)$ . This proves (5.4).

To conclude the proof, we need to verify the conditions of Theorem 2.9 for the mapping  $\Gamma$ . Then,  $\Gamma$  will admit a fixed point in  $W_0^{1,p}(\Omega)$ , which will imply that problem (1.1) has at least one weak solution in  $W_0^{1,p}(\Omega)$ .

Indeed, the boundedness of  $\mathcal{S}$  can be obtained directly via using the analogous arguments as in the proof of (5.4).

It remains to illustrate the weak closedness of  $\mathcal{S}$ . Let  $\{u_n\} \subset \mathcal{S}$  be a sequence such that  $u_n \rightharpoonup u$  in  $W_0^{1,p}(\Omega)$ , as  $n \rightarrow \infty$ , for some  $u \in W_0^{1,p}(\Omega)$ . Hence, for each  $n \in \mathbb{N}$ , it is easy to see that  $u_n \in K(u_n)$  and

$$\langle Av, v - u_n \rangle + \int_{\Omega} j^0(v(x); v(x) - u_n(x)) \, dx \geq \langle f, v - u_n \rangle \tag{5.6}$$

for all  $v \in K(u_n)$ . Because  $\text{Gr } K$  is sequentially weakly closed (see the proof of (5.1)), this implies  $u \in K(u)$ . For any  $v \in K(u)$ , set  $v_n = \frac{U(u_n)}{U(u)}v$ . We have  $v_n \in K(u_n)$  and  $v_n \rightarrow v$  in  $W_0^{1,p}(\Omega)$ , as  $n \rightarrow \infty$ . Putting  $v = v_n$  into (5.6) and passing to the upper limit as  $n \rightarrow \infty$ , we obtain

$$\langle Av, v - u \rangle + \int_{\Omega} j^0(v(x); v(x) - u(x)) \, dx \geq \langle f, v - u \rangle$$

for all  $v \in K(u)$ , where we have applied Fatou's lemma. Invoking Minty approach, we obtain  $u \in \mathcal{S}$ , therefore,  $\mathcal{S}$  is weakly closed in  $W_0^{1,p}(\Omega)$ .  $\square$

**Acknowledgment.** This work was supported by the NNSF of China Grant Nos. 12001478, 12026255 and 12026256, and by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH, by the National Science Center of Poland under Preludium Project No. 2017/25/N /ST1/00611, and by the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07. It was also supported by the Natural Science Foundation of Guangxi Grant No. 2020GXNSFBA297137, and by the Ministry of Science and Higher Education of Republic of Poland under Grants Nos. 4004/GGPJIIH2020/2018/0, 3792/GGPJ/H2020/2017/0, and 440328/PnH2/2019.

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