EXPLORING THE INSTRUCTIONAL COHERENCE OF INTRODUCTION TO FRACTIONS IN CHINESE CLASSROOMS

by

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DEDICATION

To my father

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ABSTRACT

The study investigated the instructional coherence of Chinese teachers' teaching in grade 3 mathematics classroom from Shandong, China. The study illustrates how the three Chinese teachers provide students with opportunity of learning, thinking and understanding with their previous knowledge conceptually and procedurally on the topic of introduction of fractions along with the instructional resources that teachers used to support their instructional coherence. The result of this study provides the teachers and curriculum developers in the U.S. an international perspective in addition to a new perspective on the instruction of the first stage of teaching fractions. The participants of my study were three Chinese teachers from two elementary schools from a city in Shandong province, China. The data collection tools that I used in the study were observations and interviews. The data of the study, including the videotaped lessons, interviews and teaching resources comprised of textbooks, teachers' guiding books, and teachers' interviews, were analyzed using qualitative method.

All three teachers used instructional tools such as concrete (paper), semi-concrete (diagrams) models, and contextual examples to connect students' previous knowledge to the new knowledge. They also frequently referred to the knowledge learned in different lessons when imparting new knowledge. Procedurally, all three teachers were observed to keep a routine of reviewing at the beginning of each lesson and ending with a summary. From the analysis of teachers' interviews and the instructional resources, it appeared that the coherence manifested in the teachers' instruction reflected the textbook and suggestions by the teaching guide.

The evidence in this study provides some revelations on how Chinese teachers' teaching manifest a coherence on the topic of introduction of fractions, although the differences of teachers' teaching experience and teaching environments account for the variations in teachers' enactments of the class activities. There is no significant gap between the three teachers in terms of the instructional coherence, which could be explained by teachers' high fidelity to the textbook with a curriculum that is mandatory and suggests a sequence of activities to follow.

Keywords: instructional coherence, introduction of fractions, teaching resources

1. INTRODUCTION

Among all the topics in the school mathematics curriculum, the topic of fractions has been widely recognized by some scholars as "the most protracted regarding development, the most difficult to teach, the most mathematically complex, and the most cognitively challenging" (Lamon, 2007, p. 629). The National Mathematics Advisory Panel (NMAP, 2008), which consisted of eminent mathematics educators and researchers including Deborah Ball, Liping Ma and Douglas Clements, indicated the fluency with fractions as one of the “critical foundations of Algebra (p.17)” and an essential prerequisite knowledge for students to learn algebra. Therefore, considering the challenge and importance of learning fractions, it is significant to investigate how to support the students’ learning of fractions by examining the teaching of fractions.

Several researchers have widely studied teaching as a central factor that impacts the quality of a mathematics lesson (Hiebert & Grouws, 2007). Many factors influence teachers' instruction, from the teachers' knowledge (Hill, Sleep, Lewis, & Ball, 2007) to the teachers' beliefs (Philipp, 2007). However, teaching is not a single-sided activity. The ultimate goal is to engage students with a productive learning experience. Therefore, the springboard of good teaching should be aiming toward students' cognitive learning. In the same way that engaging stories or movies present a connected and coherent plot, an accessible lesson should also be presented as connected and coherent instruction (Stigler & Hiebert, 1999).

Statement of the Problems

Existing studies have revealed that coherence is an essential characteristic of mathematics classroom instruction in Asian countries like Japan and China (e.g., Hiebert et al. 2003; Shimizu, 2007; Wang & Murphy, 2004), and that coherent mathematics lessons can promote students' connected and coherent conceptual understanding of mathematics (Fernandez, Yoshida & Stigler, 1992). The results from international mathematics tests such as Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA) provide evidence that students participants from Asian countries, including China, score higher than their counterparts in other countries. For example, Shanghai, China came out on top with the highest score in mathematics in both the 2009 and 2012 PISA Test (PISA, 2012). Mainland China and other areas such as Hong Kong and Taipei also ranked within the top 10 on the 2015 PISA mathematics test. Furthermore, Chinese student participant outscored their U.S. peers in the area of fractions in several cross-cultural research studies (Zhou, Peverly, & Lin, 2005; Wang & Lin, 2009). The high performance Chinese students participant displayed in these international assessments and research studies raised a great deal of interest among researchers in how students learn and teachers teach in China. Of particular importance was the idea of instructional coherence, which was a significant characteristic in Chinese classrooms that has been studied in a considerable body of research (e.g., Chen & Li, 2010; Ding & Anthorny, 2011; Mik, 2013; Su, 2013; Wang & Murphy, 2004; Wang, Cai & Hwang, 2015). Nearly all those studies examine the instructional coherence of Chinese teachers using exemplary classes. They also tended to select experienced teachers as the participants in their studies. It appears that the researchers perceived the exemplary classes or an experienced teacher’s classes as the target to examine the instructional coherence with the assumption that the experienced teachers' instruction or the instruction in a model class is assuredly coherent.

However, the attainability of instructional coherence in a typical classroom remains to be explored. Do novice teachers' classes manifest instructional coherence as well as experienced teachers' classes? Moreover, do the daily lessons that students receive at school also show coherence like the well-designed model classes? To answer these questions, I plan to explore and fully describe how actual Chinese daily classes are conducted.

Positionality

As an international student from China, I worked as an instructional assistant in a mathematics department in the United States as I pursued a Ph.D. in mathematics education. As a graduate assistant, I have taught developmental math courses, pre-calculus, and calculus courses. Through my teaching and learning experiences, I gradually noticed some differences in the ways of teaching and learning mathematics in U.S. and Chinese classrooms from the lens of a person who was educated in China for approximately twenty years. For instance, I noticed the challenge faced by students on the topic of fractions as I never did before I came to the United States. And it is the different understanding of fractions between me and my students that intrigued me and left me to ponder what caused the discrepancy on cognition of fractions. My first thought was that the learning experience makes a difference.

Although I no longer have problems with fractions, I remember being bothered by the names of the numerator and denominator when I was first introduced to fractions. In Chinese, the name of the denominator (分母) includes a character that means mother and the name of the numerator (分子) has a character related to the child. It was confusing for me to match their name and their position. Finally, I figured out the notation by imagining a fraction as a child carried by his mother, meaning, the denominator is the one about mother since it is on the bottom, and the numerator is like a child, so it is on the top. While this was a personal story, there was some implication of the confusion that students may have when they tried to interpret fractions. Furthermore, my teacher was unable to perceive my struggles since those struggles are not shown in my homework or test.

From my experience of learning mathematics in China, there were several pivotal principles that I took towards learning and teaching mathematics. I want to present those ideas through a neutral lens. First of all, the education I received emphasized the systematic and consistent nature of the foundational mathematical knowledge and the construction of such knowledge. The consolidation and reinforcement of previous knowledge promoted learning new knowledge, and the teachers always used the metaphor that when building a house, a steady and robust foundation made the construction proceed smoothly.

Another principle that I perceived in China was that learning was a progressive process. Students might not steadily grasp the knowledge they initially received; even students who understood the material right away might need more time to learn how to apply it proficiently. With this in mind, review and practice play an important role in Chinese classes. In order to gain better performance in the college entrance exam, it is not rare to see that some high schools in China finish teaching all the required curricular material within two years and use one more year to review what students learned in the previous two years. The adage "practice makes perfect" is another principle that my educational background instilled in me, especially for learning mathematics.

Moreover, I realized that my learning experience makes me think of mathematics as a whole. Instead of dividing mathematics based on the content area, as is done in the U.S., the curriculum of mathematics in China has only one subject just called mathematics. I perceive subjects such as geometry, algebra, and pre-calculus as a part of one whole, which is mathematics. For example, I am not always able to think about geometry and algebra separately since sometimes I solve geometry problems by applying the ideas and techniques acquired from algebra content.

In summary, my learning experiences in China enabled me to learn mathematics as a coherent and unified subject. In contrast, in the U.S. mathematics is more of a general term that contains several independent courses such as geometry and algebra. It is the distinct interpretation of mathematics that inspired me to reflect on my learning experience, especially on the topic of fractions. Thus, I will conduct this study in China to investigate in depth how a sample of Chinese students and teachers learn and teach fractions.

Purpose of the Study

I used a case study to gain insight into four different Chinese teachers' instruction on the topic of introducing fractions. Specifically, by conducting this study, I examined and contrasted in depth the instructional coherence in different teachers' classrooms in the region of Dongying, China. In particular, I observed and analyzed how teachers implemented a sequence of lessons when they introduced the topic of fractions. Additionally, I explored the instructional resources teachers used and I investigated how these resources that teachers brought to the classroom impacted the instructional coherence in their respective classrooms.

As mentioned above, instructional coherence is a significant feature of Chinese teachers' teaching, and contributes to students' learning by promoting their understanding (Stigler & Hiebert, 1999, Wang & Murphy, 2004). Therefore, it is valuable to disclose how Chinese teachers' teaching manifests the instructional coherence. To better examine the instructional coherence, I selected the topic on the introduction of fractions as the focus of the mathematics content for this study. I provide two main reasons for my choice of this topic. Firstly, the knowledge covered on the topic of introduction of fractions is well structured and it typically takes a series of classes to teach this topic. The instructional coherence is therefore more significant between and throughout each class and could be tracked through the teaching of the entire sequence of lessons. Secondly, the understanding of basic fractions knowledge can have consequences in the second stage of learning fractions and even upper-level mathematics classes in middle school, high school, and college (Reeder, 2017). Therefore, it is more likely to show instructional coherence when teaching the topic of introduction of fractions.

Overall, the purpose of this study is to acquire a more in-depth knowledge of how teachers in a particular province in China introduce fractions and the coherent manner in which they teach. The result of this study will offer teachers in the U.S. an international perspective in addition to a new perspective on ways to plan and implement the first stage of teaching fractions.

Research Questions

My study addressed the following research questions:

1. How does different teachers' instruction display coherence in the classroom when teaching the unit on the topic of introducing fractions? In particular,

a. with respect to conceptual instructional coherence; and

b. with respect to procedural instructional coherence.

2. How does teachers' use of different instructional resources influence the instructional coherence in their instruction?

Definition of Terms

The definitions of the terms used in this study are presented here:

Instructional coherence. Previous studies defined instructional coherence from different aspects. As Hiebert et al. (2003) focused on "the interrelation of all mathematics components of the lesson (p. 196)", Wang and Murphy (2004) further emphasized the critical role of "meaningful discourse" that linked the class activities. In this study, *instructional coherence* is defined as a characteristic of instruction that manifests while instructors present class activities in a particular sequence using well-designed discourse, which purposefully indicates the progressive and consistent relationships within mathematical knowledge. Specifically, my study will focus on two aspects of instructional coherence: conceptual coherence and procedural coherence. Conceptual coherence is the connection among the mathematics themes while procedural coherence is procedural connection reflected by the routine of the teachers' teaching.

Instructional resources. Instructional resources are the resources that are accessible to the teachers to help them with their teaching, which include textbook, curriculum standards, and teaching guides. It may also contain the non-physical resources such as support or assistance from other teachers or the program.

Introduction of fractions. According to the Common Core Standards (CCSSM, 2010), students' learning of fractions contains two main stages (Wu, 2001), the first stage is from the grade 3 to part of grade 4 when students begin to learn the basic idea of fractions with a variety of representations and by using simple analogies and intuitive reasoning to make simple computations. The Chinese Curriculum Standard (中华人民共和国教育部制定, 2011)) has a similar grade band that students learn fractions mainly in grade 3 and grade 5. The learning in grade 3 could be seen as the introduction of fractions. In this research, the introduction of fractions is defined as the content that students learn in this initial stage. To better fit the definition for the research, I will be more specific and define *introduction of fractions* as a series of lessons that start from introducing the concept of fractions as a brand-new topic, which begins with the notation and representation of fractions, and evolves to performing some simple computations and applications. It is only after completion of this first stage that students learn operations with fractions.

Delimitations

The main participants of my study are three teachers from an elementary school from a city in Shandong province, China. Considering the geographical area and large population of China, there may be regional disparities regarding cultural and educational environments. The way teachers implement their instruction in one small region is far from representing the whole of Chinese teachers' instruction. Thus, there should be consideration given to what extent the data can be representative of general Chinese teacher instruction when analyzing the data and reported findings.

Summary

Previous research (Hiebert et al., 2003; Gonzales, & Stigler, 2003; Shimizu, 2007; Wang & Murphy, 2004) have revealed that instructional coherence plays a vital role in

Chinese classrooms and is beneficial to students' learning. In light of these findings, researchers have examined teachers' instructional coherence by analyzing their discourse from the content and procedural perspective. However, most of the researches have focused on experienced teachers or model classes. To have a lens on general teachers' teaching, this study will be conducted to gain more insight into the instructional coherence achieved by not only experienced teachers but also by novice teachers.

2. LITERATURE REVIEW

This chapter presents the theoretical framework for the study and literature review from two main aspects: instructional coherence, and teaching and learning fractions. The review of previous research about instructional coherence as a broad term of both individual instructional coherence and instructional program coherence will respectively expound how they were informed by curriculum coherence, and later, will address the conceptual framework for this study. The section of teaching and learning of fractions will present the pertaining literature regarding teaching and learning of fractions.

Theoretical Framework

The theoretical framework that I use for this study is the zone of proximal development (ZPD) which was first presented by Vygotsky (1978). This theory not only explains how people learn but also guides the instructional method which leads to the theory of scaffolding. Vygotsky (1978) defined ZPD as:

> The distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers (p.86).

The *actual* development is what students are already able to do when they work on a problem or task independently and includes the knowledge and the ability they possess to solve a problem. The *potentia*l development is what students may be able to do later but could be accomplished now with the teachers' or more competent peers' assistance. Based on this theory, learning is the process of improving the students' actual development level by achieving their potential development level in a social environment like a classroom. The potential development level could be the learning goal that teachers help students to achieve. For example, as is shown in Figure 1, students come to the daily class with the archived knowledge and the ability that represent their actual development level. However, with teachers' instructional guidance and the interaction with participants in the class, students gain new knowledge and skills based on the archived knowledge. Then students' actual development has progressed into a new level and for the next day of class (the darker color shows the progression of students' actual development), what is being taught in today's class will be part of the new achieved knowledge.

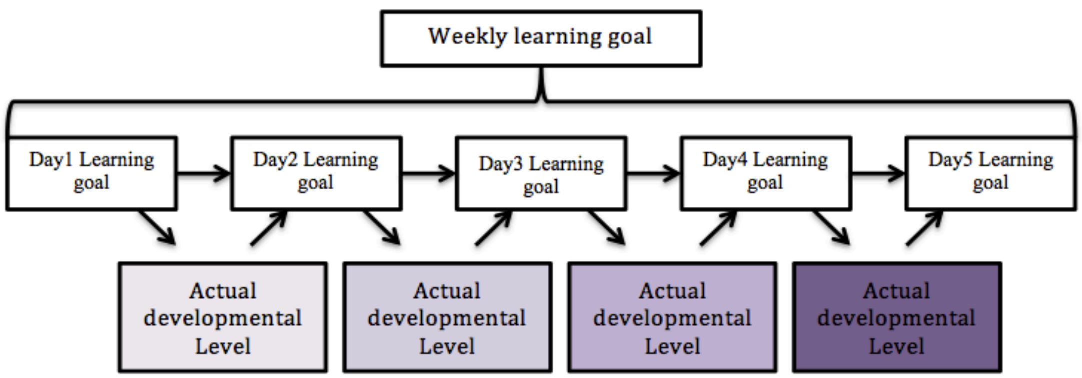

Figure 1. The process of knowledge building among lessons

This process is described by Vygotsky as "what is in the zone of proximal development today will be the actual developmental level tomorrow- that is, what a child can do with assistance today she will be able to do by herself tomorrow" (Vygotsky, 1978, p.87), as shown in Figure 1. It is noteworthy that learning in the setting of classrooms happens not only when students are directly guided by teachers, it also occurs when teachers interact with other students. For example, when a teacher corrects one student in front of the class or answers one student's question, the process may also help to correct other students' misunderstanding or eliminate others' confusion. During the learning in school settings, the learning goal can be broken down hierarchically as yearly learning goal, weekly learning goal, daily learning goal, and episode learning goal. As is shown in Figure 2, one class contains different episodes within which the teacher sets up a learning goal for the students. It could be an activity like solving a problem or a group discussion. Moreover, when the class gradually accomplishes each episode learning goal, the daily goal will be achieved.

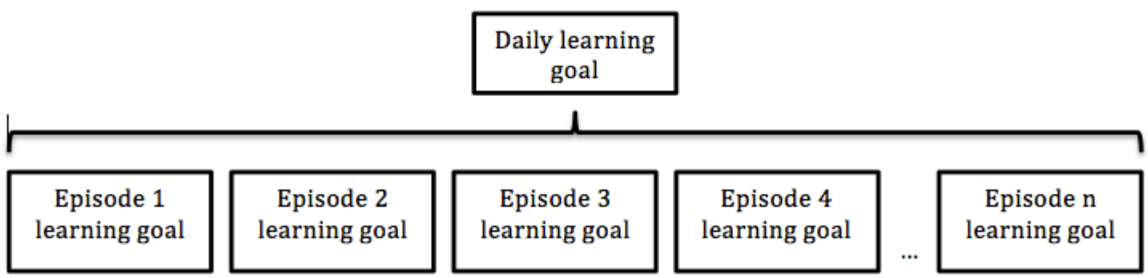

Figure 2. The process of knowledge building within a lesson

Considering the complex interpretation of ZPD, Stott (2016) presented an organizing framework as a structure to review ZPD in mathematics education research. The organizing framework is comprised of five elements and presented as five questions. By answering these five questions, the Table 1 below clarifies my theoretical perspective and illustrates how the theory of ZPD is embedded in this study.

Table 1.

Organizing framework of ZPD

Who learns/ develops in the ZPD?	Chinese students in the grade 3 mathematics classrooms
With whom does learning/ development take place?	Mainly the teacher, but also classmates and parents.
What is learned/developed?	Students' understanding of fractions. More specifically, the learning goal stated in the teaching and learning standards.
How is it learned/ developed?	By engaging in the class activity, which is mainly guided by the teacher; also, homework and assistance from classmates and parents.
Where does learning/development take place?	Mainly in the classroom and also where students work with the mathematics materials.

In addition to explaining the learning process, the theory of ZPD also indicates the critical role that teachers play in the students' learning process. Informed by the ZPD, researchers use the word "scaffolding" as a metaphor to express the support teachers provide students (Pea, 2004). Woods, Bruner, and Ross (1976) first proposed the term *scaffolding* and defined it as "the process that enables a child or novice to solve a problem, carry out a task, or achieve a goal which would be beyond his unassisted efforts'' (p. 90). In this study, students as the novice of working with fractions achieve a learning goal of understanding fractions with teachers' scaffolding.

Hammond and Gibbons (2005) further explored scaffolding in an enacted curriculum where they presented a model for scaffolding with two tiers: macro design-in scaffolding and micro contingent scaffolding. *Design-in scaffolding* pointed to the design of a class, which included the selection of tasks and the sequence of the instruction. *Contingent scaffolding* happened as teachers react to students' unpredictable activity like questioning and confusion. The spontaneous interactions with students cannot be precisely prepared when teachers design their class but instead must depend on their experience and knowledge to make the decisions on how to react and respond. This model will inform the analysis of my study. In particular, the design-in scaffolding informs how the conceptual and procedural coherence for the instruction should be analyzed, and the contingent scaffolding for analyzing informs how the teachers' discourse is indispensable for supporting students' learning.

Instructional Coherence

Hiebert et al. (2003) defined class coherence as " the interrelation of all mathematics components of the lesson (p. 196)." Regarding the definition, we could see a class covering several mutual unrelated topics and activities as a lack of coherence in contrast with a class with a clear theme and in which the activities are well designed to address the theme of the class progressively. Wang and Murphy (2004) further defined coherence with more emphasis on teachers' instructional discourse as the "unity or connectedness" of class activities informed by a series of purposeful discourse (p. 107). Although there is no unanimity on the definition of instructional coherence, researchers have agreed upon the attribution of instructional coherence as a feature of effective teaching (Stigler & Hiebert, 1999; Cai, 2014). Informed by the previous studies and to additionally address the goal of instructional coherence, it could be seen as a characteristic of instruction that manifests while instructors present class activities in a particular sequence using well-designed discourse, which purposefully indicates the progressive and consistent relationships within mathematical knowledge.

Previous studies have shown the positive effect of instructional coherence on students' learning as it helps to promote students' understanding and engagement (Okazaki, Kimura, & Watanabe, 2004). Stigler and Hiebert (1999) used an analogy to explain a well-formed story and to compare it with a coherent lesson; just as well-formed stories are easier to follow than ill-formed ones, coherent lessons are more accessible for students since the components of the lessons are well related. Also, knowing that a story-formed lesson would be more attractive for children (Okazaki, Kimura, & Watanabe, 2014), a coherent story-formed lesson would promote students' engagement.

The influence of teachers' knowledge on instructional coherence

Although instructional coherence is manifested when teachers are teaching, the construction of it requires effort in every regard. Therefore, it is important to explore the contributing factors of the instructional coherence. Informed by previous literature, the coherence of teachers' knowledge and the curriculum coherence could be seen as the factors that underlie the instructional coherence.

In a study comparing Chinese and US elementary teachers' knowledge, Ma (2000) discerned a predominant feature of Chinese teachers which is their understanding of mathematical knowledge as a whole. From the view of the Chinese teachers in Ma's study, mathematics topics are not independent; one topic could be a basis for another, and each topic connects with each other to some extent. Not only the teachers' views, but their performance in solving the problems given by Ma also showed that their knowledge is indeed interconnected and coherent. Ma (2000) indicated the essence of fundamental mathematics knowledge as "depth, breadth, and thoroughness (p. 122)" and teachers who comprehend the essence of mathematics knowledge tend to apply coherence in their teaching; teachers who apply coherence in their teaching tie back to previous knowledge to promote students' understanding and also consciously prepare students for their future learning by consolidating students' knowledge foundation. This result fully supports the idea that instructional coherence would entail the coherence of teacher knowledge.

Cai (2014) further supported the idea by interviewing 20 Chinese teachers from 13 different provinces and 16 U.S. teachers from 14 different states who were recognized as excellent teachers (experienced teachers that had been rewarded a prize in teaching) about their interpretation of instructional coherence. Cai (2014) reported teachers' view from both conceptual and procedural perspectives. Content-wise, Chinese teachers attended more to the logical interconnection of mathematical ideas and the essence of the mathematical knowledge, in comparison with U.S. teachers whose responses indicated the order and sequence of the knowledge. Procedurally, Chinese teachers distinguished real instructional coherence from a superficial interpretation by describing the real instructional coherence as not restricted to a smooth flow of teaching. Conversely, emergent teaching event like challenging students by asking questions which promote students' thinking and learning should be welcome.

Curriculum coherence

In the *Principles and Standards for School Mathematics*, the National Council of Teaching of Mathematics (NCTM) considered curriculum as a guideline for both teachers' teaching and students' learning (NCTM, 2000). Also, in the section of Curriculum Principle, NCTM strikingly emphasized the significance of curriculum coherence which "effectively organizes and integrates important mathematical ideas so that students can see how ideas build on or connect with, other ideas, thus enabling them to develop new understandings and skills (p.14)." Besides indicating the interrelationship between each mathematical topic, NCTM also discussed the curriculum coherence concerning classroom teaching, which directly pointed to the instructional coherence. Schmidt, Houang, & Cogan (2002) considered a curriculum to be coherent “if they are articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives (p.9).”

However, an analysis of the data from the Third International Math and Science Study (TIMSS, 1997) revealed a lack of coherence in the U.S. curriculum, which caused a negative influence on American students and teachers (Schmidt, Houang & Cogan, 2002). In order to have a better knowledge of the curricula in the top achieving TIMSS countries, Schmidt et al. (2002) investigated the top achieving TIMSS countries' curricula from grade one to grade eight in mathematics and developed a composite set of the topics that at least two-thirds of the top achieving countries included in their curricula. The results showed that those curricula are strongly coherent based on the researchers' definition of the coherent curriculum. Moreover, in contrast to TIMSS's top-achieving countries and regions, the pattern of curriculum guide in China exhibited a similar coherent pattern to those high achieving countries but in a more concentrated way (Wang et al., 2012). Also, in order to ensure the quality of implementation of the curriculum in China, the issued curriculum guide or curriculum standard is always accompanied by a policy document referred to as the curriculum plan. This document elaborates the policies about curriculum and instruction in detail for all the subjects, such as which subjects should be taught in each grade and the teaching time assigned for every subject per week (Zhu, 2002).

Moreover, Schmidt, Wang, & McKnight (2005) further argued that the content standards of the day which inform the curricula is insufficient to lead to high-quality instruction and achievement. Considering the deficiency of American curriculum revealed in the previous researchers, the Common Core State Standard for Mathematics (CCSSM) was released in order to improve U.S. curriculum (CCSSM, 2010). To examine the coherence shown in the CCSSM, Schmidt and Houang (2012) compared the CCSSM with a model standard designed by mathematicians that was based on the high-achieving countries' curricula. This involved measuring the overlap on a graphical distribution of the topics covered in each grade. The result showed that CCSSM displayed a significant coherence, as it quite resembles the model standard. The research also suggests a possible relationship between 50 states' standards and students' achievement on the 2009 National Assessment of Educational Progress (NAEP). In particular, students from the states with standards more like CCSSM showed better performance on the NAEP test.

Considering the benefits of instructional coherence on students' learning, I propose to explore what the instructional coherence looks like in classrooms. A review of previous studies about instructional coherence informed the investigation of three aspects: conceptual coherence, procedural coherence.

Conceptual coherence

Wang and Murphy (2004) stated that conceptual coherence could be examined both "within lessons and across lessons (p.108)". Chen and Li (2010) examined a Chinese teacher's instructional coherence by analyzing a sequence of four consecutive lessons on the topic of fraction division. The analysis of the study focused on the characteristics of instructional coherence both within and across individual lessons. The result showed that a central theme is embedded within an individual lesson while the difficulty of the four lessons is gradually increasing. Overall, the materials of the four lessons all lead to a common theme: fraction division.

Mok (2013) identified five strategies for coherence from four consecutive grade 7 lessons in a Chinese classroom taught by a Shanghai teacher on the topic of the system of equations. He addressed one strategy in the classroom as "the what-why-how in the thematic connection (p.124)" which indicated that students got to know and understand the algorithm before doing it. Ding and Anthorny (2011) conducted a case study to examine the instructional coherence across ten lessons on linear equations in a grade eight class in New Zealand. This class was one of two extension classes at the year nine level with 30 students. In this study, the researchers investigated how the teacher's pedagogical strategies associated with the selection and enactment of tasks and concluded that the action of 'sowing seeds' were key factors in establishing instructional coherence, and the "seeds" correspond to the multiple layers of new knowledge and methods embedded in the intended curriculum. By sowing the seeds, the instructor planned a logical sequence of knowledge construction that builds and links to students' existing and instant ideas.

Procedural coherence

Cazden and Beck (2003) reported that setting up a classroom routine that makes certain classroom activity familiar and the structure of the class as students expected can "free up students' mental energy to attend to higher order thinking (p. 179)", so that both teachers and students could be more engaging in a class. Previous studies explored how the class activity procedurally promotes the instructional coherence in a classroom. Chen and Li (2009) classified the classroom activities into three categories: "reviewing, introducing new content, and closure (p. 721)." By analyzing the procedural connection across the four lessons, they found a class routine where the teacher reviewed the material covered in the previous lesson at the beginning of each class and then summarized the knowledge at the end of each class. Mok (2013) also emphasized the contribution of teachers' review on the coherence of instruction since review is the process of reflecting on previous knowledge that sets up a foundation for future learning.

Class discourse

The term classroom discourse refers to the language that teachers and students use to communicate with each other in the classroom. Teachers used discourse to impart knowledge and instruction in the classroom (Mohr, 1998). The research showed that the development of students' mathematical understandings is enabled and constrained by the classroom interactions and the discourse in which students and the instructor participate (Cobb, Boufi, McClain, & Whitenack, 1997). Therefore, discourse, as the most direct way for teachers to express and convey information, plays a pivotal role in a classroom. A teacher's discourse used as a classroom move including speaking and writing has an intended purpose. It could be in different forms which, has corresponding consequences (Krussel, Springer & Edwards, 2004). Cazden and Beck (2003) addressed a shift of pattern on teacher's discourse in the last few decades. The shift went from the traditional pattern of classroom talk, in which teachers ask test-like questions and students give short test-like answers, to teachers leading discussions that stimulate and support higher order thinking.

Previous research has pointed out the impact of teachers' discourse towards students' learning and the necessity of deliberately constructing teachers' discourse (Nystrand, 2006). Teachers construct the instructional coherence through their discourse. In the study conducted by Wang and Murphy (2004), they examined the coherence of a model Chinese mathematics classroom in a Shanghai elementary school that was recognized by the local education institute. The topic of the class was on finding the area of a triangle with equal base and equal height. By analyzing the discourse throughout the six activities conducted in the class, Wang and Murphy interpreted the transitional statements the teacher made between different activities and concluded that the teacher used language to connect the well-structured activity explicitly and highlighted the connection between the old knowledge and new knowledge.

Su (2013) also illustrated how instructional coherence is achieved by giving an example of a Chinese mathematics lesson and analyzing the coherence of the class by following a teacher's discourse moves based on the framework "the teachers' discourse move" proposed by Kusse et al. (2004). The lesson Su (2013) analyzed was a grade 7 mathematics class in China on the topic of "the property of equality" that was taught by an experienced teacher who was identified as above average by the author. The teacher spent the 43 minutes of the lesson on different activities that were worthwhile and coherently organized. Su (2013) evaluated the lesson as being content-rich, purpose-oriented, topic-focused, and carefully planned, in which the tasks were closely tied to the topic of the lesson and all linked to each other to construct the coherence of the whole lesson to attain a coherent instruction.

Wang, Cai, & Hwang (2015) proposed a framework for examining the instructional coherence by applying classroom discourse theory. Cai (2014) identified three discourse management strategies adopted from Tomlin et al. (1997): rhetorical, referential, and focus. In this study, the researchers explored discourse strategies the teacher used to achieve instructional coherence through a fine-grained analysis of a videotaped lesson about the topic on circles in a grade 6 classroom in China. In this class taught by an experienced teacher, not only were the mathematical themes in the different stages within the instructional phase all closely connected to the new content taught in the stage of teaching new content, but there were also connections within the stage of teaching new content. The teacher used strategies such as writing on the board, comparing and contrasting themes, recurring themes, and using the oral response to make a discourse coherent so that students could focus on themes which are most central to the discourse.

However, those studies mainly focused on one teacher's instruction (e.g., Chen & Li, 2010; Ding & Anthorny, 2011; Mik, 2013; Su, 2013; Wang & Murphy, 2004; Wang, Cai, & Hwang, 2015), which then relies too much on the teacher's individual quality. Since there is no absolute standard for teachers' instructions, it is unclear if different instructions can result in coherence in different ways. One way to explore how different teachers construct the instructional coherence is by contrasting different teachers' instructions to examine their approach to instructional coherence. The study conducted by Okazaki, Kimura, and Watanabe (2014) examined more than one teachers' instructional coherence, however, before they collected data of each teachers' teaching, the researchers also provided teachers with some direction on their teaching. Therefore, the focus of the study is, not a comparison of different teachers' instructional coherence.

Nearly all the above researchers chose to observe experienced teachers or the classes observed and analyzed were recognized as model classes. These studies informed further questions like: are teachers with less experience able to present coherent instruction or are only the teachers with more experience able to do so? Moreover, do the regular lessons presented to students in their daily class resemble the instructional coherence in a model class? To further address these questions, this study will investigate the level of the instructional coherence of several Chinese math teachers in grade 3. By contrasting different ways of instruction over a common topic, I will describe how different teachers, both novice and experienced, construct instructional coherence. To be more specific, I will present literature related to the Chinese and the U.S. learning and teaching of fractions in the following sections.

Instructional Program Coherence

In contrast to instructional coherence, which focuses on an individual teacher's instruction, Instructional program coherence occurs at the school level, and the goal is to improve the efficiency of the entire program. As the instructional coherence is informed by the vertical coherence of curriculum, the idea of instructional program coherence aligns with the horizontal coherence of curriculum. Horizontal coherence refers to students' learning in different classes within the same grade and discipline aligning with each other even when taught by different teachers (Hidden curriculum, 2014).

Considering the gap between the intended curriculum or policy and the instruction implemented in the classroom, educators and researchers sought a more comprehensive description of the coherence of the education system. Researchers noticed that "when curriculum, instructional materials, and assessments are all focused on the same goal that is when the policy systems that frame education are coherent- the prospects for educational improvement are enhanced" (Koppich & Knapp, 1998, p. 2). Therefore, building on the curriculum coherence and several other criteria, Newmann, Smith, Allensworth and Bryk (2001a) introduced the concept of the instructional program coherence as "a set of interrelated programs for students and staff that are guided by a common framework for curriculum, instruction, assessment, the learning climate, and is pursued over a sustained period (p.299)."

Researchers (e.g., Newmann et al., 2001a; Oxley, 2008) have revealed several benefits of instructional program coherence both regarding students' learning and engagement and assisting teachers' instruction. Instructional program coherence could be seen as an extension of instructional coherence. As discussed above, instructional coherence benefits students' understanding and engagement by receiving a coherent instruction. However, researchers tended to perceive instructional coherence as an individual teacher's short-term instruction, like teaching in one class or a series of classes. Therefore, instructional coherence could be beneficial to a small number of students in a certain teacher's class for a limited period. Even those students' well-established learning states may be interrupted once students are assigned a new teacher or transferred to another school. The new teacher without knowing students' previous knowledge could repeat the materials some students already learned, or the new teacher could bring too advanced knowledge to the students.

In order to benefit more students and to address the issues discussed above, schools should consider applying the instructional program coherence because it helps maintain content coherence across different classes and grades. Specifically, the school level instructional program coherence requires teachers teaching the same grade to apply the same instructional strategies and assessment and to use common instructional resources (Newmann et al., 2001a). With the instructional program coherence implemented, students who start learning in a new grade are more likely to possess a common knowledge base, which would make the teaching more effective (Hirsch, 1999).

Schmidt, Houang and Cogan (2002) made an analogy of two countries' agricultural practices where one country disseminates a practical guideline that contains best farmers' ideas to all farmers. The other country disseminates only a list of ideas to which everyone contributed. The result is that most of the farmers in the first country strive toward a more common outcome in comparison to the other country where the farmers' performance remains diverse. This analogy appropriately addressed the situation in the field of education. We could see the ideas of good farming as the various policies. However, as stated by Finley (2000), "policy is just policy until it is incorporated into teaching practice," that is why the second country did not outperform even with a list of ideas.

Similarly, even with the substantial amount of policies, a practical guideline which supports teachers' implementation of those ideas is indispensable. Instructional program coherence serves as a practical guideline for schools and teachers and would affect teachers' instruction by supporting them. Also, instructional program coherence facilitates teachers' cooperation with each other (Newmann et al., 2001a). Efforts are centralized for creating common instructional strategies or assessment, which makes the preparation of instruction more efficient and allows teachers to receive support from one another.

In order to explore the influence of instructional program coherence on students' performance, Newmann et al., (2001a) gave surveys to about five thousand teachers from 222 Chicago elementary schools in 1994 and 1997 and used the surveys to measure the extent of each school's program coherence. By statistically analyzing the relationship between schools' instructional program coherence and their students' academic achievement, researchers found that the coherence is significantly related to students' achievement in a positive way. Oxley (2008) further showed the appearance of instructional program coherence in high schools by presenting the examples of Atlanta's Southside High School and Oregon's South Salem High School, in which teachers worked collaboratively on a common set of expectations for students' learning and applied the expectations into their instruction. With the implementation of instructional program coherence, Atlanta's Southside High School gained a higher graduation and attendance rate. In this study, the instructional program coherence will serve as a reference to inform the analysis of teachers' use of resources.

Teaching and Learning Fractions

In this section, I will describe both the National Council of Teachers of Mathematics (NCTM) standard and Common Core State Standard (CCSSM) on the topic of fractions which reflect a path for students' learning. Also, I will address some important aspects of fractions learning including the teachers' teaching, students' prior knowledge and how to improve by reviewing the previous literature.

The Number and Operations Standard of the *Principles and Standards for School Mathematics* documented by NCTM indicated the significance of "a deep and fundamental understanding of, and proficiency with, counting, numbers, and arithmetic, as well as an understanding of number systems and their structures (NCTM, 2000, p. 32)." Further, the expectations for students' learning of fractions in different grades automatically separate the fractions learning into two periods. To be specific, students in grades 3 through 5 are expected to interpret fractions as parts of a whole, a number and also a division of whole numbers. Moreover, students are expected to use different models to compare the magnitude of fractions and distinguish equivalent fractions (p.148). Following that, students in grades 6 through 8 are expected to gain proficiency in using fractions with an understanding of the meaning of arithmetic operations (p.214). From the NCTM standards, we can see the continuous learning of fractions through the different levels and stages of students' knowledge and learning ability.

The learning of fractions as a vital topic runs through the grade 3 to grade 7 according to the CCSSM (2010). The CCSSM clearly shows a learning path for students in each grade: The goal of learning fractions in grade 3 is using models to understand unit fractions 1/b as a number and interpret other fractions as multiples of unit fractions. In addition, students learn to compare fractions with the same denominator or numerator. Learning fractions in grade 4 stays at the visual level where students compare the area of the fraction model to reasoning about equivalent fractions. Adding or subtracting fractions with the same denominator is also a part of grade 4 but with the help of visual models. Multiplication will be introduced in grade 4 although it only refers to multiplication of fractions by a whole number. In grade 5, students begin to interpret fractions as division, for example, *a*/*b* could be interpreted as *a* divided by *b*. With this interpretation, students learn how to add, subtract fractions and also multiply fractions by fractions. In grade 6, students learn division of fractions by fractions, and grade 7 examines division of fractions more intensively. Wu (2013) analyzed the CCSSM and concluded two stages of students' learning of fractions: the initial stage is in grade 3 and part of grade 4 while the second stage is called "formal mathematical development of fractions (Wu, 2013, p.2)", and starts around grade 4 and develops until grade 7.

Steffe and Olive (2010) presented a progression of fraction schema (Norton & Wilkins, 2009) for U.S. students. A more recent study conducted by Norton, Wilkins, and Xu (2018) indicated a common progression concerning the fraction schema between Chinese and U.S students despite the different curriculum. Both U.S. and Chinese students tend to construct part–whole schemes (PWS: producing *m/n* by partitioning a whole into *n* pieces and extracting *m* of those pieces) prior to constructing schemas that support measurement conceptions including Partitive unit fractions (PUFS: determining the size of a unit fraction relative to a given unpartitioned whole by iterating the unit fraction to produce a continuous, partitioned whole), Partitive fraction scheme (PFS: determining the size of a proper fraction relative to a given unpartitioned whole by partitioning the proper fraction to produce a unit fraction and iterating the unit fraction to reproduce the proper fraction and the whole) and Reversible partitive fraction scheme (RPFS: producing an implicit whole from a proper fraction of the whole by splitting the fraction to produce a unit fraction and iterating that fraction the appropriate number of times). Moreover, students tend to construct the measurement schema for unit fractions (PUFS) before constructing the measurement schema for non-unit fractions (PFS). The finding of the study suggests a common cognitive core in students' development of fraction knowledge, which provides a resource for educators and the curriculum makers.

Blyth (2006) gave two reasons for knowing fractions. One reason is that if students do not understand and master the ideas of fractions, then they will find learning algebra difficult because the techniques learned for numerical fractions are the same as those needed for parts of algebra involving fractional expressions. Also, Bailey's (2010) study showed that measures of fluency with computational fractions in grade 6 significantly predicted grade 7 mathematics achievement, more so than the influence of fluency in computational whole number arithmetic, performance on number fluency and number line tasks, central executive span, and intelligence. It demonstrates the impact that the understanding of fractions can have on students' subsequent learning in middle school, which suggests that a goal of learning should be the fluency in working with fractions at the elementary grades. Fraction knowledge is related to algebra readiness, more so than general number magnitude knowledge. Students' magnitude knowledge of unit fractions (i.e., those with a numerator of 1) appears particularly important (Booth, Newton, & Kristie, 2012; Mousley, & Kelly, 2017). Moreover, the learning of fractions also predicted students' success in high school, as Siegler (2013) showed in his research findings; students' understanding of fractions and long division are unique predictors of their later success in high school algebra and their overall mathematical achievement.

Despite the importance of fractions for students, it is generally considered a difficult topic for both children and adults. National tests showed nearly half of eighthgraders were not able to order three fractions by size (Sue, 2013). Ndalichako (2013) analyzed the problems related to fractions in the Primary School Leaving Examination (PSLE), which is a national examination in Singapore taken by students at the end of their sixth year in primary school. Their research indicated that students' performance on this test showed the part involving fractions to be weak. For instance, in questions involving the addition of fractions, they were treating numerators and denominators as separate entities. Johanning (2008) also revealed that students did not simply take the concepts and skills learned in formal fractions units and use them in their other mathematical content areas. Students' understanding of how to use fractions was related to the situations in which it was being used. Even for some college students, fractions are still a complicated topic. Foley (2003) conducted a study to examine community college students' understanding of fractions. The researcher gave 23 students from three community colleges 15 computational and contextual problems to solve. Students' performances in solving problems indicate that most students only possessed a rudimentary understanding of fractions (e.g., fractional representation and proportional reasoning for contextual problems).

Students' difficulty in the computation of or in the arithmetic of fractions is widely acknowledged (Lortie-Forgues, Tian & Siegler, 2015). Although researchers noticed the lack of understanding of fractions, there are substantial researches ascribed towards the second stage of learning fractions. However, the first-stage learning of fractions which mainly works on students' conceptual understanding of fractions is the basis for the second stage of learning that focuses on sophisticated and procedural operations. In another words, students' deficiency of conceptual understanding may influence their proficiency in applying fraction operations (Stigler et al., 2010). Therefore, it is indispensable for teachers to carefully guide students in their first-stage learning, for instance, to precisely define the whole when working with a model (Wu, 2013).

Previous researchers have widely studied the teaching of fractions as it is most related to students' learning of fractions. However, fractions as a challenging topic are not only difficult for students to learn but also for teachers to teach (Ball, 1993; Harvey, 2012; Lamon, 2012; Newton, 2008). Firstly, it is reasonable to assume that teachers teaching fractions should possess a reasonable knowledge base on fractions. However, previous research has revealed a limitation of some teachers' content knowledge of fractions. Van (2014) conducted a study on 290 pre-service teachers with 184 first-year and 106 final-years about their knowledge of fractions by giving them a test. The pre-service teachers were from two teacher education institutions and enrolled in a three-year professional bachelor degree in Flanders. By analyzing pre-services teachers' test results which reflect their conceptual and procedural knowledge, Van (2014) addressed a limitation of the participants' general knowledge of fractions with a better procedural knowledge than conceptual knowledge. In addition, the study showed that the final-year pre-service teachers did not perform better on fraction operations than the first-year pre-service teachers, even though the final-year pre-service teachers received more mathematics education courses than the first-year pre-service teachers. In the study conducted by Ma (1999), the U.S. teachers' knowledge of division by fractions was noticeably weaker than their knowledge of other topics. In contrast, Chinese teachers' performance on the task for division by fractions was not noticeably different from other topics. Also, a higher percentage of Chinese teachers in the study succeeded in computation and generating appropriate representations. Ma (1999) raised a point that the Chinese teachers acquired the knowledge of fractions from elementary schooling, and their later learning as teachers reinforced their understanding of the topic from the teachers' perspective.

Mathematics education researchers and educators have conducted extensive research regarding different aspects to improve the teaching and learning of fractions. Dunham (2008) examined the impact of different instructional strategies teachers used when teaching grade 6 students in a control group and a treatment group. The results suggest that the manner in which students are taught how to compare fractions directly impacts their accompanying explanations on fraction comparison problems. The research also emphasized the critical role of instruction focused on students' conceptual understanding while it suggested that the traditional focus on procedural method should be modified to include conceptual strategies at early grade levels.

In order to develop effective instruction of fractions for kindergarten to grade 8, the U.S. Organization Institute of Education (IES) published a practice guide (Stigler et al., 2010). The panel who wrote the IES practice guide analyzed a substantial amount of literature, combined their expertise, and came up with a series of evidence-based recommendations for teachers to improve students' understanding of fractions from kindergarten to grade 8. The recommendation fully indicated that instruction should build on students' previous knowledge on understanding fractions and emphasized the interpretation of fractions as numbers. Also, it suggests making the procedure of operations meaningful is beneficial to students' learning of fractions (p.8).

Lortie-Forgues, Tian & Siegler (2015) also addressed the impact of students' prior knowledge on fractions learning that "the limited knowledge of whole number arithmetic and of the magnitudes of individual fractions, as well as limited general processing abilities (p. 218)" would cause difficulty in learning fractions. Moreover, considering the issue of curriculum's lack of coherence addressed by Schmidt, Houang, & Cogan (2002), so many topics need to be covered within a limited time. Therefore, each item cannot be deeply explored and as a result, the time students spend on each topic is limited. However, it takes time and practice for students to assimilate the knowledge, not only in understanding the material but also in consolidating the knowledge. In other words, learning mathematics, especially fractions, takes practice and time. Saxe, Gearhart, & Seltzer (1999) conducted a study to investigate the relationship between classroom practices and student learning in the domain of fractions. By analyzing the classroom observation and pre- and post-instruction achievement in 19 upper elementary classrooms, the results of the study indicated that for students who began with a rudimentary understanding of fractions, there exists a linear relationship between the measures of classroom practice and problem-solving.

Summary

In this chapter, I presented Vygotsky's (1987) Zone of Proximal Development (ZPD) as the theoretical framework for this study, which informed the mechanism of students' learning with teachers' support and addressed the necessity of coherence in teachers' teaching. In addition, the review of previous literature in terms of instructional coherence exposed a gap in the research that more attention should be paid to the novice teachers' instructional coherence in the daily lessons.

3. METHODOLOGY

The review of the literature in the last chapter provides a theoretical foundation for my study. The research suggests that instructional coherence as related to my study can be examined by analyzing teachers' and students' discourse; through conceptual and procedural perspectives; and the ways and challenges of teaching and learning of fractions. Previous research on instructional coherence, however, has mainly focused on the instruction of experienced teachers, with only a limited understanding of how novice teachers construct instructional coherence in their teaching. In my research study, I expand upon this foundation by providing a multi-case study, including within case and cross-case analysis.

The purpose of my study is to examine and contrast the instructional coherence of different teachers' teaching in a sequence of classes in the region of Dongying, China. In particular, I observed and analyzed how both novice and experienced teachers implemented a sequence of lessons in which they introduced the topic of fractions. This was accomplished by exploring the instructional resources a sample of teachers used. In addition, I investigated how these resources that teachers brought to the classroom impacted the instructional coherence in their respective classrooms.

Research Questions

The study examined the following research questions:

1. How do different teachers' instruction display coherence in the classroom when teaching the unit on the topic of introducing fractions? In particular,
    a. with respect to conceptual instructional coherence; and
    b. with respect to procedural instructional coherence.

2. How does the teachers' use of different instructional resources influence the instructional coherence in their instruction?

Because the purpose of the study is to examine each of these teachers teaching a set of lessons in depth within the real-world context (two Chinese elementary schools), a case study research methodology is used. Yin (2013) defined case study research methodology as "an empirical inquiry that investigates a contemporary phenomenon (the "case") in depth and within the real-world context (p, 16). My study contains four cases, which are the four Chinese teachers.

In this study, employing the purposeful sampling strategy (Creswell, 2013) for conducting a case study, four Chinese teachers with different experiences teaching grade 3 mathematics are the participants, and the multiple sources of information including observation and interviews are used for the data analysis. Upon applying my purposeful sampling, in the observation stage, I realized one case (one teacher), did not meet two of the criteria. Specifically, as I observed and videotaped the four teachers teaching a series of classes that introduces the topic of fractions, one teacher from the Donger elementary school field site, produced minimal evidence of: (1) classroom peer interactions, nor (2) a student-centered classroom. Where my study focused on teachers, my theoretical framework derived from a social constructivism paradigm. While my intent was to have four participants from two field sites, my purposeful selection criteria necessitated attrition after the observation phase. I denoted attrition with an asterisk in Table 2. I interviewed three teachers about their perspective on instructional coherence.

Sites and Participants

I collected the data in two elementary schools in Dongying, Shandong, China. Dongying is a prefecture-level city, on the northern coast of Shandong province, which has two million residents. There were two purposeful and convenient reasons to research in Dongying. Firstly, Dongying has a relatively high college entrance rate in Shandong province, which to some extent reflects the high quality of its education. Also, it was more convenient for me to access data resources since Dongying is my hometown where I received my education. One of the schools where I collected data is the one that I attended. There are 113 elementary schools in Dongying from which I could choose my potential candidates. Considering the large size of eligible candidates, I applied a two-phase approach (Yin, 2014) to the screening procedure of my case study. In the first stage, I consulted the experts from the local education bureau for the education quality. Using this information along with the consideration of the location and availability of the schools, I selected two elementary schools from the 113 elementary schools to be the site of my case study and their pseudonyms are Shiyi elementary school and Donger elementary school. Shiyi elementary was built in 1988, which is the first elementary school in Dongying. Donger elementary school is much newer than Shiyi elementary school, and it was established in 2011.

In the second stage of the case selection, I contacted the principals of the two elementary schools I selected and stated that I would like to observe and interview teachers who have different years of teaching experience. Based on my preference, the principals recommended two teachers from their respective schools considering the teachers' availability and schedule (Yin, 2014). Thus, two mathematics teachers teaching grade three mathematics from each of these two schools are the participants for the study. In the case of most Chinese elementary schools, instead of one teacher teaching all subjects like some schools in the U.S., each teacher only teaches one subject. Both of these two teachers were in charge of teaching two grade 3 mathematics classes. The four teachers all have different teaching experiences giving me access to examine their instructional coherence ranging from the experienced teacher to the novice teacher. I chose the grade 3 mathematics class because this is usually the first stage for students to learn fractions in China. Teachers start the topic by introducing the concept of fractions as part-whole and continue with a sequence of lessons, after which, students are expected to gain a rudimentary understanding of fractions, such as how to solve some basic application problems using the part-whole interpretation of fractions. The topic lasts about two weeks with one class each day. The first lesson is on Tuesday for the first week, and the last lesson is on Thursday of the second week. Therefore, I obtained a total of eight class recordings for each teacher. The Table 2 below reports the background of the participants including teachers' years of experience and the kinds of teaching experiences each teacher has had. For example, Ms. Liu has been a teacher for 21 years, and she has taught all different grades at her elementary school. In contrast, Ms. Yang has been teaching for only three years and has taught grade 3 for only one year.

Table 2.

Participant Teachers' Background

School	Teacher	Years of experience	Teaching experience
Shiyi school	Ms. Liu	21 years	All different grades
	Ms. Xu*	9 years	Mainly grade1 to 3, taught grade 4 once
Donger school	Ms. Shi	5 years	Taught grade 1 and 2 before and this is the first time teaching grade 3
	Ms. Yang	3 years	Taught grade 1 twice, this is the first time teaching grade 3.

Analytic Tools and Procedure

The data collection process lasted about two weeks. The data collection tools that I used in the study were observations and interviews. During the two weeks, I videotaped a series of eight classes per teacher on the topic of introducing fractions, for a total of 32 videotaped lessons. During the observations, I wrote field notes and gathered questions for the post-lesson interviews based on my observations that I used later to gather feedback from the teachers. At the same time, I videotaped the entire class with a video camera.

I conducted three interviews for each of the four teacher participants during the data collection process. The first interview was conducted before the observation started. During the first interview, I gathered information about the teachers' backgrounds and their previous experiences of teaching. Also, I asked them about their goals for teaching the unit "introduction of fractions." gathering evidence about the teachers' expectations of their teaching and students' learning. The second interview was post-observation-stimulated, which was given during the two-week observation period. This interview was semi-structured with only part of the questions determined, and the other part of the questions produced during the observation of the class. The last interview was given after the entire fractions unit was taught. This interview contained more general questions that explored the teachers' teaching philosophies. For example, I asked teachers about their perspective on instructional coherence and their opinion about how to construct instructional coherence. The term “instructional coherence” was translated as 教学的连贯性 (Cai, 2014). Table 3 shows some examples of the interview questions.

Table 3.

Interview Protocol

Interview 1	1. Please introduce yourself regarding your teaching experience. For example, how long have you been a teacher and which grades have you ever taught. 2. What would be your expectation for students’ learning today?
Interview 2	1. I noticed that the model that we used in the class is a paper circle. Why do you choose it to be the model? 2. During the class, you repeatedly indicated the word “evenly divide”. And also use the same sentence pattern when describing the fractions. Why? And how could this help student understand fractions.
Interview 3	1. When people say a lesson is very coherent, what does the word “coherent” mean to you? What are the characteristics of a coherent lesson? 2. What previous knowledge had been taught as the preparation for the learning of fractions? How does that influence the instructional coherence?

Data Analyses

Through the preliminary analysis, it appeared that the learning environment created by Ms. Xu’s is more teacher-centered, students’ learning from the interaction with the teacher and students is not clearly shown in Ms. Xu’s class. Considering that Ms. Xu’s teaching failed to conform to the theoretical frameworks of this study, which indicated that learning happened through the interaction with the teacher and peers. Therefore, the case of Ms. Xu has not been included in the study.

The data sources I use in the study are (a) 24 videotaped lessons (eight from each teacher), (b) curriculum resources, including the textbook, teachers' guide, the works or exercises assigned to students, and (c) 9 interviews of the three teachers. The videotaped lessons are the main data source. I analyzed the video transcripts to answer the first research questions, which illustrated how these three teachers' instruction display coherence for both individual lessons and a sequence of lessons. An analysis of teaching resources including textbooks, teachers' guiding books, and teachers' interviews helped with answering the second research question about the use of different instructional resources impacting the instructional coherence. In addition, the exploration of the instructional program coherence in the school level provided further evidence of how the resources impact teachers' teaching.

I transcribed the videotaped lessons and interviews verbatim in Chinese and then translated to English. The English transcript was used in the final data analysis. Also, I analyzed the curriculum resources in their original Chinese and then translated into English when needed.

Informed by previous research (Chen, & Li, 2010; Mik, 2013), my first research question examined the instructional coherence in terms of two elements: the conceptual coherence and procedural coherence. Then I considered how both conceptual coherence and procedural coherence are evident in the classroom discourse. At this point, this study was analyzed using discourse as an analytic tool. Table 4 summarizes the elements to be examined, data sources, and analytic tools for each research question.

Table 4.

Data Analysis Table

Research Question	Element to be examined (Chen, & Li, 2010; Mik, 2013)	Data Source	Analytic tool	Operationalization
RQ1. How does four different teachers' instruction display coherence in the classroom when teaching the unit on the topic of introducing fractions? In particular, a. with respect to conceptual instructional coherence; and b. with respect to procedural instructional coherence.	Conceptual coherence & Procedural coherence	32 Videoed lessons, 8 per each teacher.	In-class Discourse analysis	Discourse (informal and formal language by the teacher or students) that displays instructional coherence The relationship of each class activities defined by the actions taken by the teachers and students.
RQ2. How does the teachers' use of different instructional resources impact the instructional coherence in their instruction?	The impact of the Resources available for teachers on teachers' instructional coherence, such as the coherence of the resources providing more clarity and precision for teachers' instruction.	1.Concrete resources such as: the curriculum standard, the textbook, and the teaching guide. 2.9 teachers' interview.	1. Intended curriculum analysis 2. Interview discourse analysis	Exploring the scope and sequence of the curriculum and compare with the teachers' instruction.

For the data analysis plan, firstly, it is imperative to distinguish the domain of conceptual coherence and procedural coherence. The examination of conceptual coherence is attending to the content of the lessons. Specifically, the goal was to explore the interrelationship of each topic and how the teachers were guiding students to build their knowledge on prior knowledge. Chen and Li (2010) presented an example that when teaching the division of fractions, the teacher drew on students' prior knowledge of multiplication of fraction that 1/2 multiply by 2/3 is 1/3 and using the pictorial representation to show that 2/3 divided by 2 is also 1/3, therefore students could naturally accept that these two operations lead to the same result.

In contrast, the procedural coherence reflects the connections between each classroom activity and the structure of the lessons, which is determined by the actions taken by the participants in their classes. For example, Chinese teachers tend to start each class with a review and elicit new knowledge from prior knowledge (Mik, 2013). Moreover, Shimizu (2007) highlighted teachers' summary of knowledge as a feature of instructional coherence in the Japanese classroom.

Chen and Li (2010) analyzed the instructional coherence from two layers: individual class and a series of classes. However, informed by the theoretical framework, individual classes are causally sequenced to devote to a common theme of a sequence of classes. Just like a TV show, the audience needs to follow every single episode to understand the development of the storyline. Considering "the coherence and sequential nature of mathematics (NMAP, 2008, p. 18)", I depicted the instructional coherence of the eight lessons as a big picture. So that the data analysis commenced on the content and procedure of each class based on the sequence presented in class to investigate the conceptual and procedural coherence as well as the transition between each class.

Conceptual coherence. While my study focused on teachers, it still derived from my Zone of Proximal Development (ZPD) framework (Vygotsky, 1978). Per ZPD, students' learning is enacted in the social context of the classroom through the interaction with the teacher and knowledgeable peers. Students come to the class with actual developmental levels, which reflect students' acquired knowledge. It is through the interaction with the teacher and their peers that they achieve the learning goal that is set up by the teacher. While my focus is the teachers, classroom interaction became a criterion of my purposeful sampling.

The unit I observed, made up of the eight lessons that were examined for this study, had a goal of enlightening students with a rudimentary cognition of fraction. To achieve this unit goal, the teachers organized and sequenced the eight lessons so that the knowledge was presented in an orderly way. Furthermore, to explore the conceptual coherence in the unit of eight classes, it is necessary to discern the content covered in each class and how they are connected. I focused on the shift of the themes in a sequence of classes to see how the topics of each lesson transits between those lessons. In addition, I examined the transition and connection between each class and how the goal of each class was devoted to the goal of the unit of lessons.

Specifically, each lesson I observed could also be seen as a string of several episodes, the learning goal of each episode will finally contribute to the learning goal of the lesson. Regarding that, I first explored the goal of the lesson, and then divided the lesson into several episodes based on the content covered in the class. An episode began as a new topic or concept appeared. For example, when the teacher posed the concept of fractions for the first time, it referred to a new episode. By exploring the goal of each episode and the previous knowledge the teachers assumed students to know, I explored the connections that teachers made between prior knowledge and new knowledge, in addition, examined how each episode was devoted to the theme of the lesson.

Procedural coherence. I further investigated the procedural aspect of the lessons. In particular, I examined procedural coherence as it was embodied in the class routine as reflected in the structure of the lesson. Adapted from Chen and Li (2010), I defined the class activity based on the actions made in the classroom. A class activity can be categorized as “Preparation for the class”, “Review”, “Lecturing”, “Group discussion”, “Students’ individual work”, “Student presenting”, or “Summary”.

By examining the pattern of enactment of class activities in each class, I found the routine of each class reflected by the sequence of the class activities and specified the role of the structure of the lesson. Then I classified each class as new-material instruction class and practicing classes. The goal of the new-material instructional class was to impart new material, and the goal of the practicing class was to review previous knowledge by practicing. Table 5 describes the flow of the class activities and their corresponding specifications that aid the analysis of the data.

Table 5.

Class Activity Instruction

Class activity	Specification
Preparation for the class	Teachers make command before class start. For example: put your pencil into your pencil box.
Review	Teacher go over the previous knowledge. It usually starts with “let us think about what we talked about in yesterday’s lesson”.
Lecturing	Teachers present to the class.
Group discussion	Students discuss as groups.
Students’ individual work	Students work on the task by themselves.
Student presenting	Students answer questions standing on their seats or in front of the class.
Summary	Teachers repeat or summarize on the statements made by the students or teachers summarize the knowledge. For example: today, we talked about fractions. Let us summarize the knowledge we have learned today.

In the second research question, I explored how teachers use the resources that were available to them to help them construct the instructional coherence. As mentioned in the Chapter 2, instructional program coherence is an important way to provide teachers with resources and support for their teaching. Newmann, Smith, Allensworth & Bryk (2001) developed a rubric to assess the extent of coherence of schools’ instructional program. Informed by this rubric, I investigated whether the teachers use the same resources and if they pursue any form of professional development.

To answer this question, I drew on teachers’ interviews to discern available resources for the teachers’ teaching. Furthermore, based on teachers’ responses, the resources were categorized into the use of the intended curriculum of concrete sources such as the textbook, teaching guide, and the curriculum standard, or non-physical resources such as the support from other teachers, professional development and other possible sources that the teachers report using. As mentioned by Wang & Murphy (2004), Chinese teachers' instructional coherence benefits from a supportive teaching program. In contrast to the teaching in the U.S., which is often viewed as a personal activity, Chinese teachers work like a group in which everyone contributes to it or gains valuable sources from it.

For the data analysis, I first explored the scope and sequence of the curriculum on the topic of fractions based on the Chinese curriculum standards and the textbooks used by the teachers to discern the teachers' fidelity to the textbook. Then I contrasted teachers' teaching with the teaching guide to explore how teachers' teaching followed the strategies as suggested in the teaching guide. In addition, in order to extend the understanding of how previous content prepared the learning of fractions and how the unit of content foreshadowed what was coming, I also read and analyzed the textbook from grade 2 and grade 5. The teachers' interviews further provided information about the non-physical resources that were available for them and how the resources impacted their teaching. For example, Ms. Shi stated in an interview that "Ms. Wang is the group leader of teaching in our grade (3rd), she organizes a weekly meeting for all the teachers in our grade to discuss the teaching plan for the following week." In addition, as mentioned in Chapter 2, the instructional program coherence facilitated the cooperation between teachers and pointed out approaches to support teachers by providing them with multiple resources. I also explored the following questions informed by the rubric presented by Newmann, Smith, Allensworth & Bryk (2001), such as: (1) whether the teachers use the same resources, like textbook or assessment? and, (2) is there any professional development activity for teachers and how were they helpful?

Validation

Creswell (2013) viewed the validation in qualitative research as “an attempt to assess the ‘accuracy’ of the finding (p.249).” As recommended by Creswell (2013), researchers need to use at least two procedures to ensure the validation of qualitative research. Therefore, I employed three procedures: triangulation, member checking, and peer review in my study. Because the use of multiple resources of evidence is significant and necessary in case study methodology, more so than other research methods (Yin, 2014), I used the triangulation procedure, specifically data triangulation (Yin, 2014), for which I attend to two main data resources: the interviews and the observations. These two resources are related and reflective. For example, the answer to some of the interview questions from the teachers could also be found from the observation of teachers’ teaching, and the teachers’ teaching also informed some of the interview questions. Therefore, the findings of my study were supported by more than one resource (Yin, 2014). Moreover, I applied the member checking procedure in the data analysis, which involved asking teachers for their views of the research finding or on whether I made a reasonable interpretation of the data. Thirdly, I underwent a peer review process by which I had peers ask me questions about the method I was implementing, including the meaning of my study which encouraged me to think deeply about my research.

Summary

Based on the design of the study, the two research questions were answered by analyzing the data collected from class observations and interviews, and the resources such as textbooks and teaching guides. Informed by previous studies mentioned in chapter 2, my study provided a comprehensive investigation of the instructional coherence comprised of conceptual coherence and procedural coherence and was conducted by discourse analysis. Furthermore, the teachers' interviews presented teachers' perspectives of instructional coherence and also offered a deeper look at the instructional program coherence.

4. RESULTS AND FINDINGS

The goal of this study is to have a deep understanding of different Chinese teachers' instructional coherence on the topic of introducing fractions. To achieve this goal, the study used several sources of data including recorded classroom lessons, teacher interviews, as well as the instructional resources such as a textbook and a teacher' guide. This chapter presents the findings of this study addressing the following research questions:

1. How do different teachers' instruction display coherence in the classroom when teaching the unit on the topic of introducing fractions? In particular,
    a. with respect to conceptual instructional coherence; and
    b. with respect to procedural instructional coherence.
2. How does the teachers' use of different instructional resources influence the instructional coherence in their instruction?

This chapter contains two main sections. The first section describes the finding for the first research question through three case reports. Each report contains three parts: conceptual coherence within each lesson, conceptual coherence among the lessons, and procedural coherence. The second section presents the result in terms of the second research question in three parts: textbook, teachers' guide, and other resources.

The participants of this study are three Chinese mathematics teachers from two different elementary schools. Each of the three teachers gave eight lessons on the topic of introducing fractions within two weeks. Based on the textbook and the teaching guide, the expected goal of the eight lessons is to build students' conceptual knowledge of fractions as well as some basic rules for computation. After analyzing all the lessons for each teacher, it is evident that the designs of the eight lessons are slightly different in the two elementary schools. Yet they both have lessons where students revisit previous concepts embedded in the lessons when new concepts are introduced. In this study, the analysis focused on the lessons where new concepts were introduced by the teachers. In the meanwhile, the goal of the practice lesson was to review and reinforce the knowledge learned in the previous lesson.

Within Case Result

Case 1: Ms. Liu

Ms. Liu is a mathematics teacher from Shiyi elementary school in Dongying, Shandong, China, she has been teaching elementary mathematics since 1996, right after she graduated from Dongying Normal Institute.

Conceptual coherence within each lesson

Based on the definition, conceptual coherence manifests when teachers guide students with their previous knowledge and use tools such as contextual examples and models to learn new knowledge. The following section presents how Ms. Liu made the connection between students' previous knowledge and the new knowledge and how Ms. Liu used the tools as a bridge to promote students' learning. A flow chart is shown in each lesson as a map of the knowledge building process. In the flow chart, each rectangular box represents the knowledge that will be used in the following activity. Specifically, dotted frames indicate the knowledge discussed in the previous lessons while solid frames indicate the knowledge gained from the previous activity in the current lesson. Each of the activities along with the instructional tools used in the activity such as models or contextual examples are displayed on the top and bottom of the arrows which form a flow path of the lesson. Furthermore, the dotted arrows point out the connection between the knowledge across the entire lesson.

Lesson 1: unit fractions

As the first lesson in a series of eight lessons on the topic of introducing fractions, Lesson 1 commenced with the learning of the unit fractions. In this lesson, Ms. Liu's instruction delivered two main goals: (1) to interpret unit fractions as part whole; and (2) to compare magnitude comparison of unit fractions. To achieve the first goal, Ms. Liu implemented a series of activities beginning with the introduction of specific fractions 1/2 and ¼, then generalized to any unit fractions 1/n. Assuming students have gained conceptual understanding of unit fractions, Ms. Liu then asked students to explore a method for comparing the magnitude of unit fractions using a model.

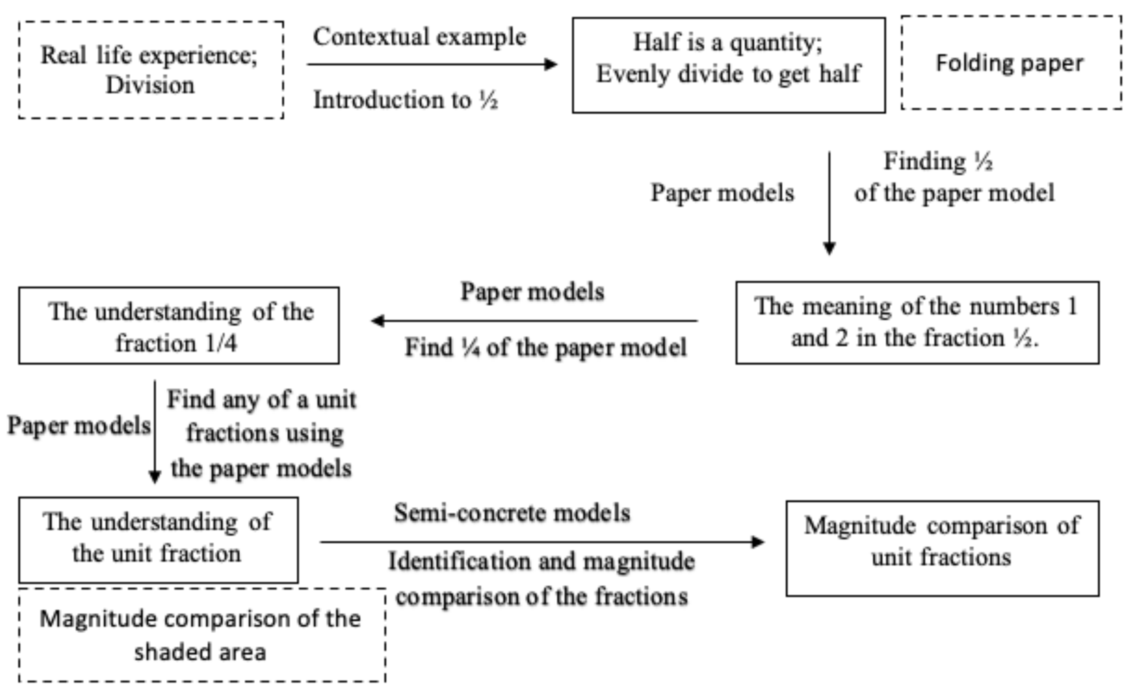

Figure 3. Flow chart of Lesson 1

Introduction to 1/2 with a contextual example. Ms. Liu started the lesson with a contextual problem. She brought to the class four apples, two oranges, and a mooncake and displayed them in front of the class. Then Ms. Liu said she would like to give the food to two students and asked all the students how two students can share the food equally. Students divided the set of four apples and the set of two oranges into two groups, each containing two apples and one orange. When sharing one mooncake, students answered that each student would get half of the mooncake, using their daily language “half”. In Chinese, there are two ways to present the numeral 1/2, one half(一半) and one-out-of- two. While one half is the daily language used to represent 1/2, one-out-of-two is how the numeral 1/2 read mathematically.

Ms. Liu also took move to reveal to the students that half is a quantity. She listed students’ response of two apples, one orange, and half mooncake on the blackboard shown in the following figure, from which she implied that two, one, and half are all in the same category of numeral.

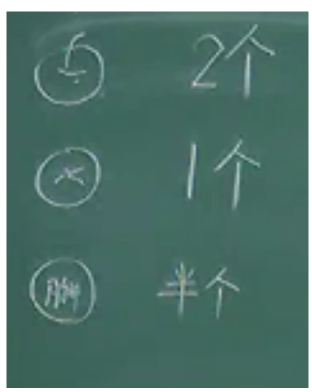

Figure 4. Ms. Liu listed the result of the contextual example on board

Ms. Liu: At the beginning of the lesson, you guys helped me share the four apples to two students, each student got two apples, share the two oranges to two persons, each one got one orange, how about the mooncake? you?

Student1: Cut in the middle, give half to one student, and another half to another student.

Ms. Liu: all right, so I cut it from the middle like this [Ms. Liu made an unfair cut on a circular paper]

Students: No, no.

Ms. Liu: Give this to one student, it that ok? [called a student name]

Student2: Their [referring to the two parts after the cut] volumes are different.

Ms. Liu: Why not work then? What do you think? [called another student]

Student3: Because if this is big and that is small, then it is not one half.

Ms. Liu: Why not? [called a student's name]

Student4: Because it is not evenly divided.

Ms. Liu: He used a word.

Student4: It is unlikable.

Ms. Liu: Ok, unlikable, that's because it is not fair, right?

Multiple students: Yes.

Ms. Liu: So, if I share something, I have to make sure……?

Students: Evenly divide.

Ms. Liu: Evenly divide, so that it will be fair. That means we should not share the mooncake randomly, so how should we do it?

Students: From the middle.

Ms. Liu: From the middle, it actually is …evenly divide.

In the discussion of how to get half a mooncake, it appears that students applied their life experience of sharing food reasoning that it should be fair to everyone, which means each of the two parts that everyone gets is congruent. With Ms. Liu's guide, students defined the sharing process using the archived concept "evenly divide", which, according to Ms. Liu, was used in division of whole numbers by whole numbers.

Finding ½ of the paper shape models. After the discussion of how to get 1/2, Ms. Liu continued to let students explore 1/2 using the shape models. Ms. Liu asked each student to take one piece of paper in any shape they like, then find the 1/2 of the shape by folding and drawing on their shape. Also, students were encouraged to discuss the process of finding 1/2 with each other. After students finished, Ms. Liu collected several students' works and let them explain the process of finding 1/2 of their paper models, including a circle model, a square model and a rectangle model. Ms. Liu then asked students why all the different shapes can be used to find 1/2, several students took turns to state their opinions and Ms. Liu summarized their responses at the end.

In the previous activity of introducing 1/2, Ms. Liu guided students to know 1/2 as a quantity and had students discuss the concept of "evenly divide", which provided students the knowledge base for the activity of finding 1/2 of the paper shape models. In this activity, Ms. Liu aimed to deepen students' part-whole understanding of 1/2 with the use of different varieties of paper shape models. Firstly, as students presented their paper models, Ms. Liu guided them to be clear about the reference unit by asking students the question: which shape is the one out of two of?

Ms. Liu: Then what does the one out of two represent?

Student1: Half of the shape.

Ms. Liu: Half is….?

Student1: One out of two.

Ms. Liu: Which shape is the one out of two of?

Student1: The square.

Student2: I fold the rectangle in half, then unfolded it. So, the left and right sides are both 1/2.

Ms. Liu: He said both left and right sides. What is the one out of two on the left side of?

Student2: The shape.

Ms. Liu: One out of two of the rectangles. How about the right side?

Student2: Also, one out of two.

Ms. Liu: Of what? Complete the sentence.

Student2: It is also one out of two of the rectangles.

Also, Ms. Liu encouraged students to use the shapes of their liking to find 1/2, which increased the variety of shapes. Ms. Liu purposefully chose different shaped models when she collected students' work. In a later class discussion, Ms. Liu posed a question based on the different shaped models that students used and she displayed the different shaped models on the board. This presented students with different whole or reference units to consider.

Figure 5. Students' work displayed on board

Ms. Liu: Some of you used the circular paper, some of you used rectangle or square paper, you all showed 1/2 of the shape you used. But I noticed that those papers had different sizes and they are in different shapes. Why are you able to use all of them to get 1/2?

Student1: As long as the shape is symmetric, it contains half of it no matter what. No matter what the shape is. No matter it is rectangle or square, they could be shared into two if they are symmetric.

Ms. Liu: Someone just said something about symmetric, what does that mean? [calls another student]

Student2: The two parts are exactly the same.

Ms. Liu: The two parts are exactly the same. Why is that? [calls another student]

Student3: Because only in this way, it is evenly divided.

Ms. Liu: After evenly dividing, the two shares are congruent. Then we just make sure that the half we got is one part of the two equal parts, then we can use one out of two to represent all of them. Therefore, this is one out of two of the rectangle, this is…? [pointing to the circle model]

Students (answering together): Half of the circular paper

Ms. Liu: This half is…? [pointing to the square model]

Students (answering together): One out of two of the square.

From concrete model to representative model. With the previous understanding of 1/2, Ms. Liu showed students four shapes with part of each shaded and asked students to determine if the shaded area of each shape can be represented by 1/2. By implementing this activity, Ms. Liu expected students to apply their conceptual understanding of 1/2 into the problem solving. Also, Ms. Lui is transitioning from concrete models (fruits and paper) into semi-concrete models (drawings) as illustrated in Figure 6.

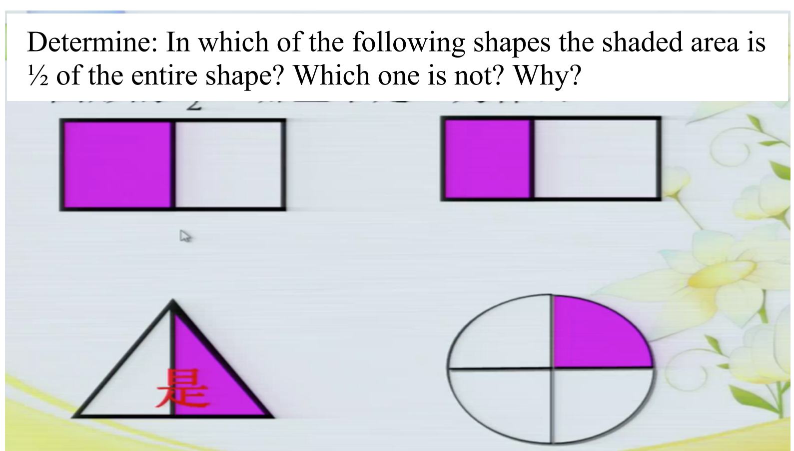

Figure 6. Problem shown to the students

Right after the practice, Ms. Liu guided students in a discussion about what the numbers 1 and 2 in the fraction 1/2 represents. Given the previous discussion about 1/2, Ms. Lui's expectation seemed to be that students may see connections between the '2' with their noticing of "two congruent parts" from dividing the figure, and the '1' as "one part" of the two congruent parts.

From 1/2 to 1/4. After the exploration of 1/2 using paper shape models, Ms. Liu guided the students to explore 1/4 following the same process as exploring 1/2. Similarly, when the students finished with finding 1/4 using different paper shape models, Ms. Liu collected several students' work including a circular model, a square model, and a rectangle model and let students present how they get 1/4 in each model. After that, Ms. Liu asked why all the different shapes can be used to find 1/4. Ms. Liu continued to ask students if different ways of folding can produce 1/4.

Although the activities of finding 1/2 and finding 1/4 are similar and follow the same process, the previous knowledge used in the two activities are different. To find 1/2 in a paper model, the students can use their previous experience of half to get half of a paper shape model, which is also 1/2. In comparison, the activity of 1/4 is conducted after they discussed the part-whole interpretation of 1/2 and also the meaning of each part of a fraction. The students are therefore able to understand 1/4 as one part out of four parts, and as the fractions 1/4.

Ms. Liu continued the discussion of different wholes when finding 1/4 of the model, and Ms. Liu extended the discussion by asking students if different ways of folding can get 1/4. The discussion guided students back to the key of getting a fraction "evenly divided" that as long as one object is evenly divided, no matter how you fold it, no matter what the whole is, one can always get 1/4.

Generalization of any unit fraction. After the exploration of specific fractions 1/2 and 1/4, Ms. Liu asked the students to create any unit fractions they wanted by folding and coloring their shape models. After two-minutes of individual work, Ms. Liu let students take turns talking about the fractions they created and how they came up with them. At the same time, Ms. Liu recorded the students' fractions on the board.

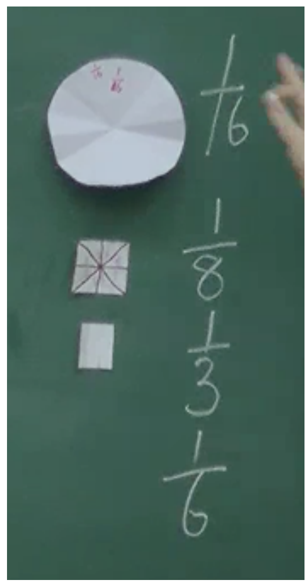

Figure 7. Ms. Liu listed the unit fractions students got on board

In the previous instruction, Ms. Liu took 1/2 and 1/4 as examples to illustrate the part-whole interpretation of fractions. She also guided the students in making a connection between the meaning of a fraction and the fractional notation, which prepared the students to generalize their knowledge to any unit fractions.

Magnitude comparison of unit fractions. Ms. Liu first showed the students two congruent rectangles and let the students identify the fractions represented by the shaded area in the two shape models. After the students distinguished the two fractions represented in the two graphs as 1/4 and 1/6, Ms. Liu asked the students to determine the magnitude of the two fractions and encouraged the students to present their ideas. Two students answered the question with their reasoning stated, and then concluded that as you divide the whole into more shares, each part will be smaller.

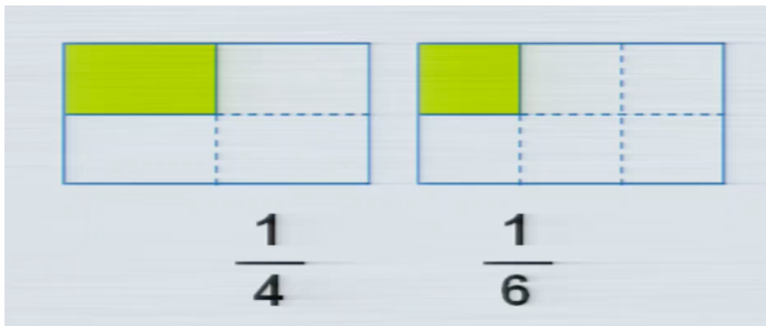

Figure 8. Two fractions represented by diagrams

To guide students to compare the magnitude of unit fractions, Ms. Liu used the rectangular models to visualize the two fractions 1/4 and 1/6 with the shaded areas of the two models. Through the comparison of the shaded area of the two rectangles, Ms. Liu helped the students to accomplish the magnitude comparison of the unit fractions.

Ms. Liu: Do you know which one is greater and which one is smaller?

Students: One sixth is smaller, one fourth is greater.

Ms. Liu: Can you talk about the reason? [called a student]

Student1: Because it was folded into small pieces, so that it has more grids. But if it has less grid, then the shape would be greater.

Ms. Liu: How about you? [called another student]

Student2: One-fourth is…divide it, divide a paper… The papers have the same size, if you divide it into four parts, then it is one-fourth, point twenty-five of the circle. The second is that divide it into six parts.

Ms. Liu: That means, the parts you have…

Student2: more parts, it will be smaller.

Ms. Liu: Then one part will be…

Student2: Smaller.

Ms. Liu: Is that what you mean? Good. Sit down please. Then one-fourth is greater than one-sixth. When the numerators are both one, smaller denominator means it has less parts, so that the fraction is greater. If the denominator is greater, it will have more parts, then the fraction is smaller.

To compare the magnitude of the two fractions 1/4 and 1/6, first required students to identify the two fractions that each shaded area represented in the two rectangles. In the previous activity of determining if the shaded area of each shape can be represented by 1/2, Ms. Liu had shown students the representative models. With knowledge of unit fractions covered, the students appeared capable of identifying the two fractions 1/4 and 1/6. Then, with the use of the model, magnitude comparison was more accessible for students by observing the shaded area in the two rectangles. The model seemed to help the students with the reasoning of why the larger the denominator, the smaller the unit fraction.

Lesson 2: non-unit fractions

Ms. Liu delivered Lesson 2 on the day after Lesson 1 was completed. In Lesson 2, Ms. Liu continued to build students' knowledge of non-unit fractions. Specifically, Lesson 2 was observed to have two learning objectives: 1) to interpret non-unit fractions, and 2) to explore the method of the magnitude comparison of like fractions.

From the observation, Ms. Liu introduced two ways to interpret the non-unit fractions. She first guided the students to explore the part-whole interpretation of non-unit fractions, then she demonstrated to the students that a non-unit fraction consists of multiple unit fractions. Once Ms. Liu introduced the conceptual knowledge of non-unit fractions, she had students compare the magnitude of like fractions (Fractions with same denominator).

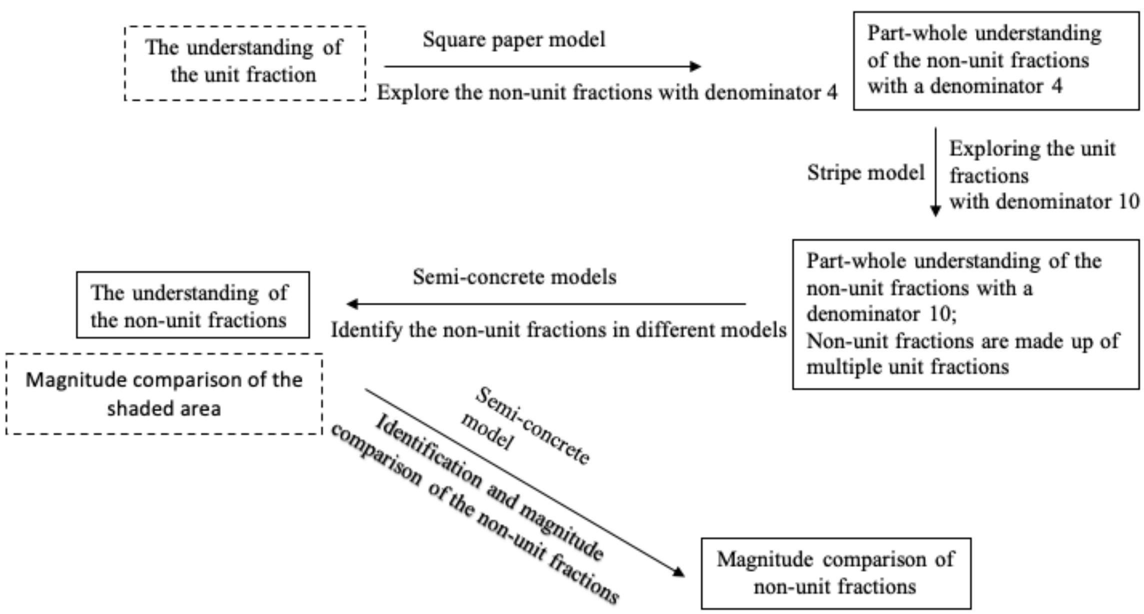

Figure 9. Flow chart for Lesson 2

Review of unit fractions. Ms. Liu started the lesson with a review of the content learned in Lesson 1. She showed students a slide with four different shapes with part of their areas shaded. Then she asked students if the shaded part in each shape could be represented by the fractions written below the shape. As students determined the correctness of each indicated fraction, Ms. Liu asked students to explain their reasoning and emphasized that even division is an important requirement when working with fractions.

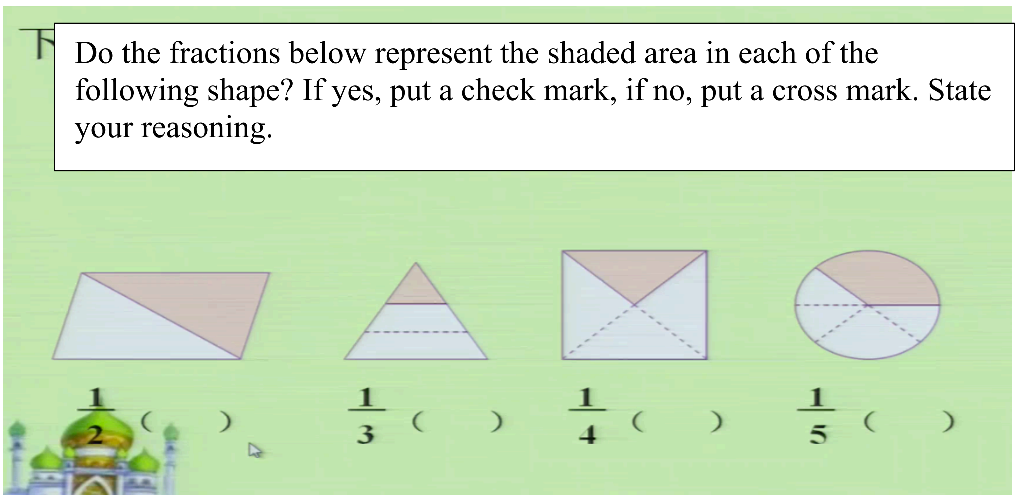

Figure 10. Problems shown to the students

Ms. Liu: Ok, the last one. [called a student name]

Student1: The last one is no, because the shaded part is larger than any other parts.

Ms. Liu: Oh, for this one, the shade part is….

Student 1: Greater than all others.

Ms. Liu: Greater than all others, therefore it is not …

Students (answered together): Evenly divided.

Ms. Liu: Evenly divided. Then can we use one-fifth to represent it?

Student 2 (answered loudly): No.

The problem provided students with an opportunity to reinforce their previous knowledge on the unit fractions. In Ms. Liu' words, the problem served as "an assessment to see if the students grasp the previous knowledge". Also, as shown in the vignette, through the discussion on the problem, Ms. Liu brought to the forefront the concept "evenly divided" that would be applied to the learning of non-unit fractions.

The exploration of the non-unit fractions with denominator 4. With the previous knowledge reviewed, Ms. Liu continued the lesson with the activity of exploring the non-unit fractions that have denominator 4. She asked the students to evenly divide the square shape model into four parts. Next, she asked them to color one or several parts out of the four parts and represent the colored area by a fraction. After five minutes of individual work, Ms. Liu asked the students to present their ideas of how they got their fractions.

> Student1: I folded it (the square model) into a big triangle, then folded it into the middle triangle, then unfolded it, this is one-fourth. (pointing to one part of the square model).
>
> Ms. Liu: One part out of them is one-fourth. You mentioned that you also found two-fourths. Can you talk about where the two-fourths is?
>
> Student1: Two parts out of it (the square model) is two-fourths.
>
> Ms. Liu: Keep going.

Student1: Three-fourths. Three parts out of it (the square model) is three-fourths. And four-fourths, the one square is four-fourths.

Ms. Liu: How many parts did you select?

Student1: four part like this.

To guide students in exploring the non-unit fractions, the students were encouraged to color one or more parts out of four. With the previous knowledge of the meaning of numerator and denominator discussed in Lesson 1, Ms. Liu attempted to guide students to come up with other fractions with denominator 4 by applying the part-whole interpretation of fractions.

The exploration of non-unit fractions with denominator 10. After the exploration of the non-unit fractions with denominator 4, Ms. Liu led the class in working on an activity from the textbook with the use of the paper strip model. First she asked students to evenly divide a one-decimeter paper strip into ten parts and checked with the class that one part out of the ten parts is 1/10. Then Ms. Liu gave the students about two minutes to figure out how to fill out the blanks by themselves. After most students finished, Ms. Liu assigned several students to show their work to the class.

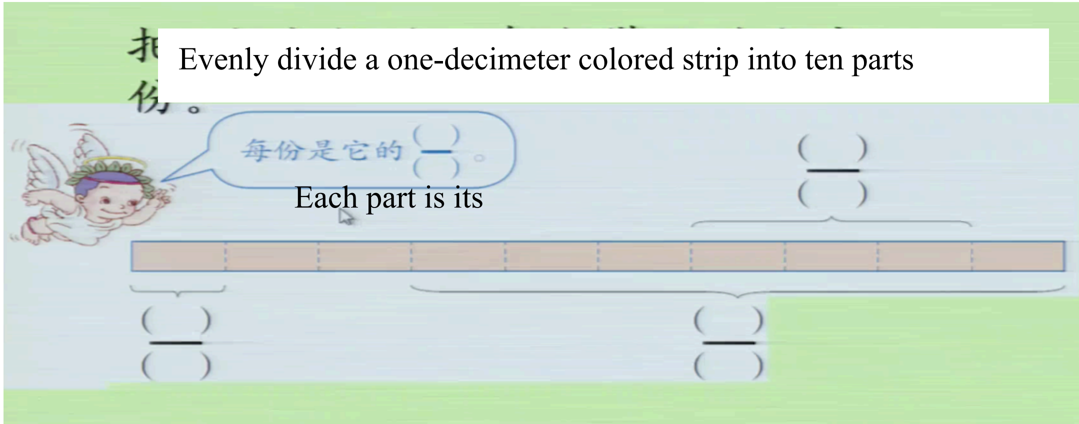

Figure 11. Activity of identifying fractions from textbook

When students presented their work, it seemed that Ms. Liu selected two students' work on purpose because they contained errors in order to check the students' knowledge.

Ms. Liu: Then let's check out this. This student wrote something like this. [project one student's work on board]. Can anyone talk about this? Seven-sevenths or seven-tenths, which one is right? [Called a student name]

Student 1: Seven-tenths is correct.

Ms. Liu: Why?

Student 1: Because the paper is evenly divided into ten parts not seven parts.

Ms. Liu: Where did we get the seven parts? We got the seven parts from the ten parts, so that the denominator should be…? [talking to the entire class]

Several students (answered together): It should be ten…ten.

From the observation, Ms. Liu helped students as they presented their work, to reinforce their part-whole understanding of non-unit fractions. After the students presented their work, Ms. Liu guided students to interpret the non-unit fractions in an alternate way when she summarized the problem.

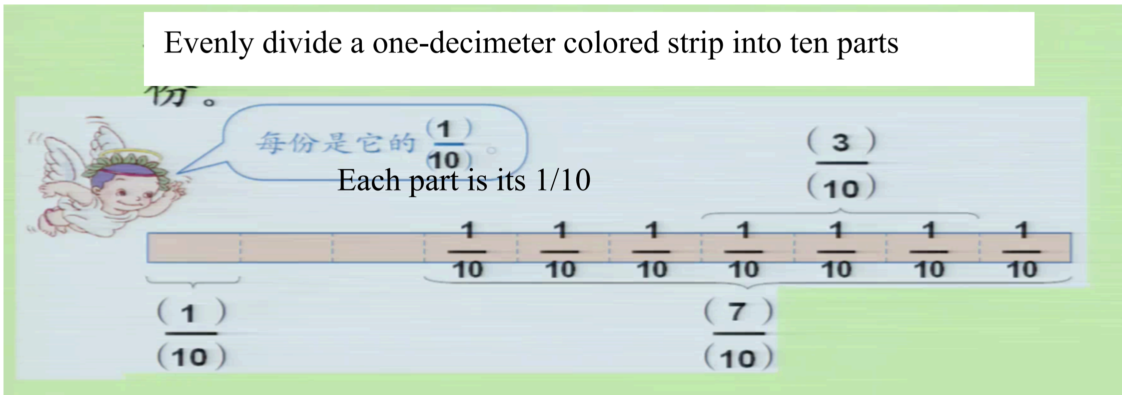

Figure 12. The example of identifying fractions by evenly dividing the paper stripe

Ms. Liu: All right, let’s look at the screen together. Well, each part is one-tenth out of it (the strip), therefore, by counting them one by one, how many one-tenths for the three parts?

Students: Three(-tenth).

Ms. Liu: Three. It contains three one-tenth. The last one?

Students: Seven-tenths.

Ms. Liu: Seven-tenth. Count it carefully.

Students: One, two, three, four, five, six, seven.

Ms. Liu: How many one-tenth?

Students: Seven.

Ms. Liu: So that is…? How many tenths?

Students: Seven-tenths.

In the conversation, Ms. Liu implicitly addressed another interpretation of non-unit fractions beside the part-whole. She first restated that one part out of the strip can be represented by one-tenth. Then she took one-tenth as a unit and asked the students to count the number of one-tenths in three-tenths, from which Ms. Liu made the connection that three out of ten can also be interpreted as three one-tenths, therefore three of the one-tenths is also three-tenths.

After the exploration on 3/10 and 7/10, Ms. Liu let the class discover other fractions using the paper strip model and provided the students with the opportunity to practice their understanding of non-unit fractions. The students came up with the fractions 8/10, 9/10, 2/10, 1/10, 10/10, 5/10, 6/10 and recorded them on the board. Then Ms. Liu went over the fractions shown on the board with the class.

Fractions in real life. After the introduction of non-unit fractions, Ms. Liu related the knowledge to real life by displaying several slides with examples of the using fractions to represent real life object. With the text shown in the picture, the activity was designed to consolidate students' knowledge on the part-whole interpretation of fraction.

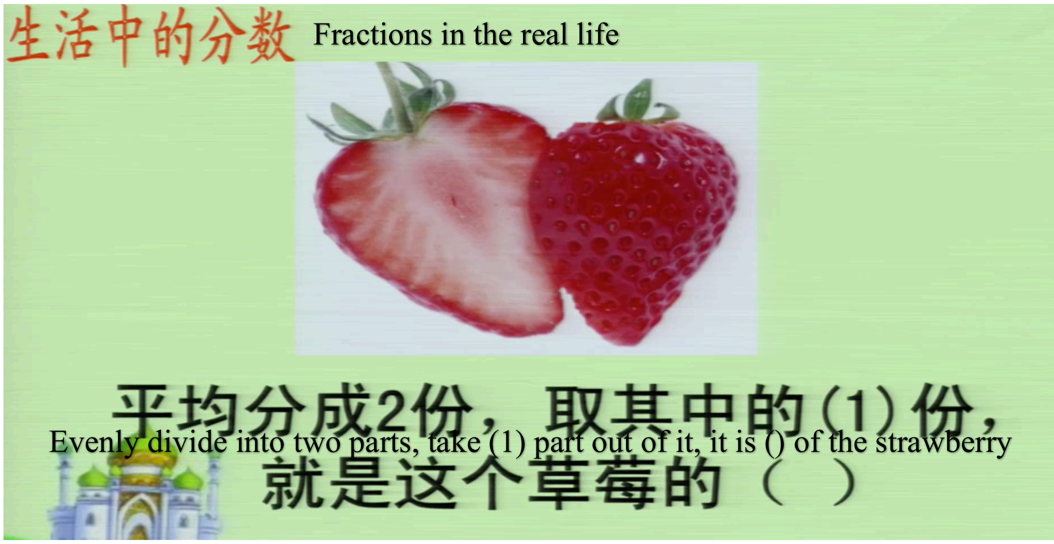

Figure 13. Example of applying fractions to represent the real life object.

Identify the non-unit fractions in different models. The activities of exploring non-unit fractions with denominator four and ten, respectively, led the students to interpret non-unit fractions in two ways using two different models. To put the students' knowledge into practice, Ms. Liu assigned students with two problems that corresponded to the models used in the previous activities.

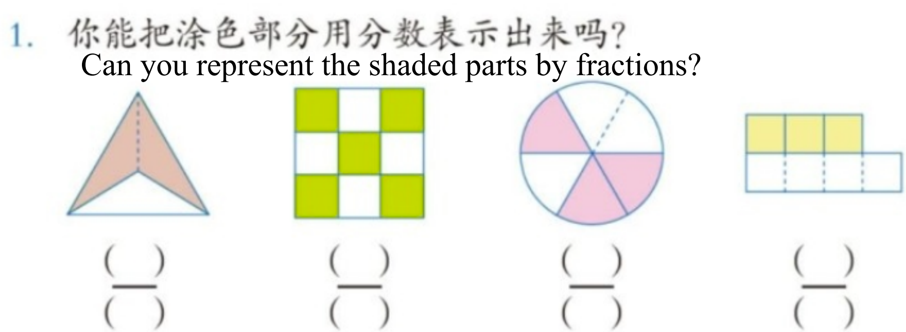

Figure 14. Problem of identifying fractions from textbook

The first problem was aligned with the activity of exploring non-unit fraction with the paper model. In this problem, students needed to identify the fractions represented by the shaded area of the representative shape model by applying the part-whole interpretation of fractions. The use of the representative model also laid the groundwork for the magnitude comparison of fractions.

Write out the fractions based on the picture

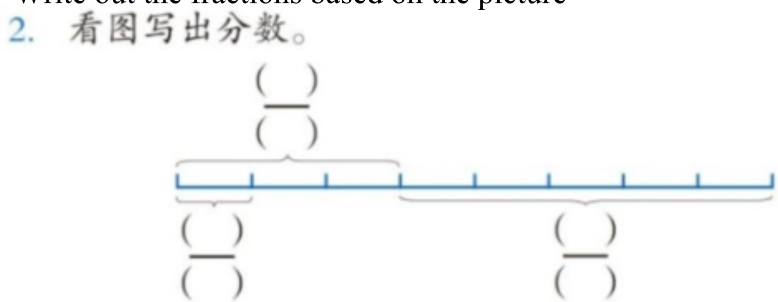

Figure 15. Problem of identifying fractions from textbook

The second problem is the same type of problem using a line segment model. This time, without the use of the concrete model, students directly apply their knowledge with the strategy learned in the previous activity.

Magnitude comparison of like fractions (fractions with equal denominators). Ms. Liu asked students to try the problem from the textbook shown in the Figure 16 to compare the magnitude of the fractions 2/5 and 3/5. Following that, Ms. Liu led students to discuss their ideas.

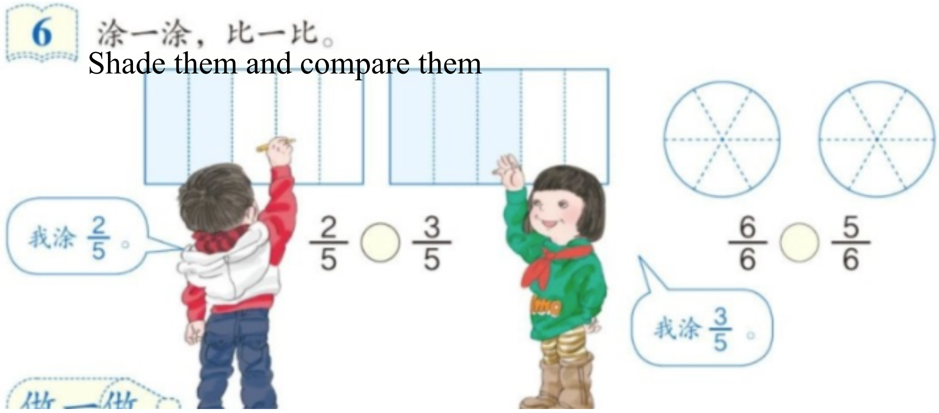

Figure 16. Activity of magnitude comparison from textbook

Ms. Liu: [called a student's name], talk about your thinking. Let's look at the first one, two-fifths and three-fifth, which is greater, and which is smaller? Talk about it. What do you think?

Student1: Three-fifths is greater, because the area shaded in three-fifths is more than two-fifths.

Ms. Liu: How much more?

Student1: One-fifth.

Ms. Liu: One-fifth more, that means one more part out of rectangle was taken. Well, two-fifths is getting from five parts and take….

Students: Two parts.

Ms. Liu: How about three-fifths?

Students: Three parts.

Ms. Liu: So that three-fifths is greater. Agree?

Students: Agree.

Lesson 3: simple computation of fractions

While in the first two lessons, Ms. Liu focused on the construction of students' conceptual understanding of fractions. In Lesson 3, Ms. Liu extended students' conceptual knowledge of fractions to the computation of fractions. Specifically, the learning objectives subsumed the conceptual understanding on the operation of the addition and subtraction of like-fractions and the application of the computation algorithm.

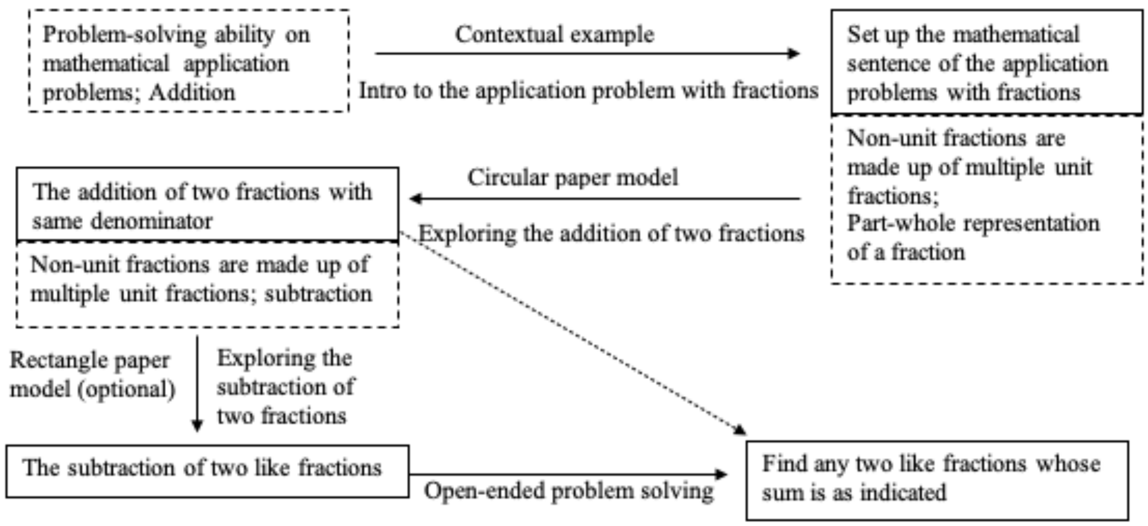

Figure 17. Flow chart of Lesson 3

Introduce simple computation of fractions. Ms. Liu introduced the addition and subtraction of like-fractions through a contextual example. She showed the class a picture of two children eating watermelon with the context that "there is one watermelon, the older brother eats two-eighths of it, and the younger brother eats one-eight".

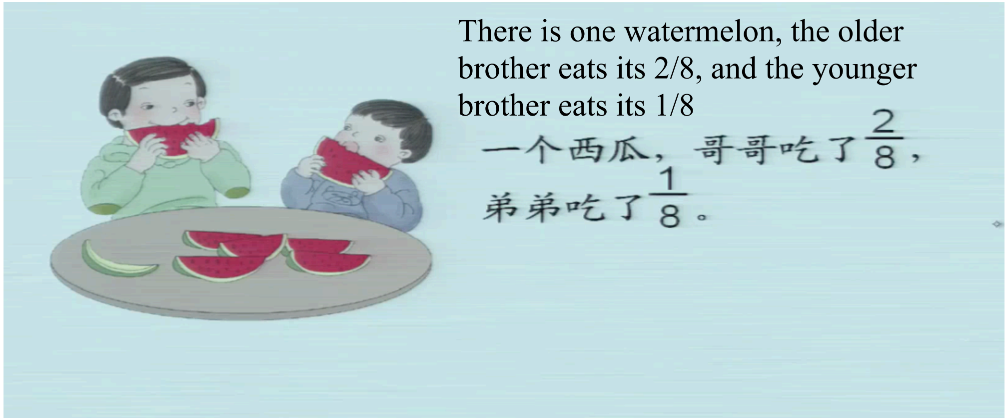

Figure 18. The contextual example shown to the class

Ms. Liu guided the students to pose mathematical questions based on the context and let the students set up the mathematical sentence based on the questions they made.

Ms. Liu: Ok. Two brothers are eating watermelon. What mathematical information can you find from your observation? [called a student's name]

Student 1: The older brother eats two-eighths, and the younger brother eats one-eight.

Ms. Liu: It would be better if you could say it louder. Anyone else would like to talk about this? [called another student's name].

Student 2: The older brother eats two-eighths, the younger brother eats one-eight.

Ms. Liu: Ok, we found two mathematical information. Based on this mathematical information, are you able to come up with some mathematical problems? [called a student's name]

Student 3: How much do they eat in total?

Ms. Liu: Ok. This one work, right? [called another student's name for other idea]

Student 4: How much less does the younger brother eat than the older brother?

Ms. Liu: Ok, sit down please. At first, Let's solve the first problem. Remind me of the first problem.

Students answered together: how much do they eat in total?

Ms. Liu: Who?

Student: The younger brother and the older brother.

Ms. Liu: How much of the watermelon do the younger brother and the older brother eat in total? How do we solve this problem?

Silence for several seconds.

Ms. Liu: Some students have got ideas. Who could talk about it? [called another student's name]

Student 5: Add two-eighths to one-eighth.

Ms. Liu: Ok. That is the mathematical sentence you got. Do you all have the same idea as him?

Students (answered together): Yes, we are the same.

Ms. Liu: Well, then what is the result of the addition of two-eighths and one-eighth?

In the vignette, Ms. Liu first asked the class to find the mathematical information in the context, then she had the students create their own problems by making up the questions based on the known mathematical information. The discussion on how to solve the first problem led the class to get the mathematical sentence of addition of two fractions. By guiding students with questions, Ms. Liu elicited the operation of addition and subtraction of like-fractions step-by-step starting by having students create possible problems involving the fractions. In this process, Ms. Liu also provided the students with a series of problem-solving strategies including extracting mathematical information and transitioning from a context to mathematical sentences with the students' previous knowledge about the word problem.

The exploration on the addition and subtraction of like- fractions After students came up with the mathematical sentence, Ms. Liu let students explore the answer for the sum by folding and coloring the circle model. To be clear on the direction of the activity, Ms. Liu suggested students use two circular papers to represent the amount of watermelon the two brothers eat respectively, then represent the amount of watermelon the brothers eat in total on one watermelon. As students worked individually, Ms. Liu selected several students' works and let the students explain their ideas. They had a discussion on why 1/8 + 2/8 is not equal to 3/16.

With the addition explored, Ms. Liu asked students to explore the result of 5/6 - 2/6 first by discussing with their classmates. If anyone had difficulty finding the result, students could use the rectangle model to help them. Ms. Liu let students solve the question made up by a student at the beginning of the lesson; how much more of the watermelon does the older brother eat than the younger brother?

Problem solving. At the end of the lesson, Ms. Liu showed students a slide with an open-ended question: $\frac{()}{5} + \frac{()}{5} = \frac{4}{5}$ to help the students gain algorithmic proficiency.

Lesson 4: simple application of fractions

In Lesson 4, Ms. Liu introduced the whole to multiple objects instead of one object. She guided the students in identifying the fractions when they take multiple objects as the whole, then solve the application problems with the part-whole interpretation of fractions. The progress of knowledge building is organized in the following diagram.

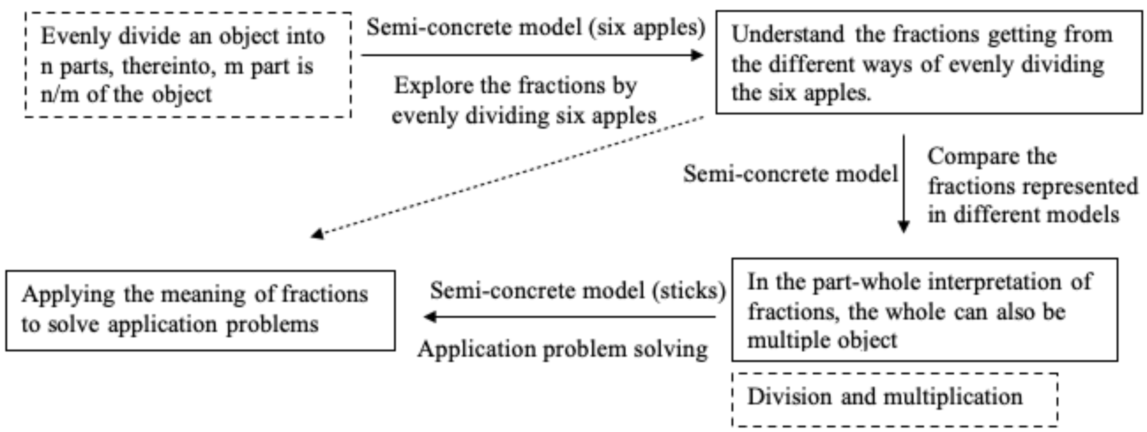

Figure 19. Flow chart of Lesson 4

Review of the previous knowledge. Ms. Liu started the lesson with a practice of identifying the fractions represented by the shaded area in the following shaped models.

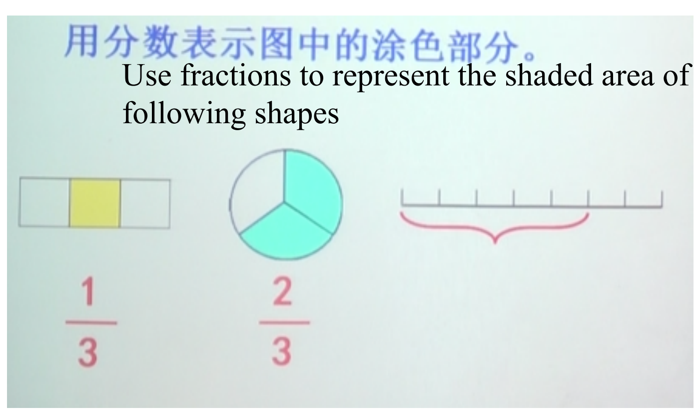

Figure 20. Problem of identifying fractions from textbook

Through the practice, Ms. Liu led the class in a review of the part-whole interpretation of fractions which was covered in the first two lessons. The models shown in the picture were used in the later part of the lesson when Ms. Liu attempted to show the difference between the fractions learned in the previous lessons and learned in Lesson 4.

Introduce multiple objects as whole. After the review, Ms. Liu immediately showed students a picture of six apples that had been evenly divided into three parts and asked students to identify the fraction that could represent one part out of the six apples. It was observed that Ms. Liu asked different students for the answer and the students came up with different fractions: three-sixths, two-sixths, one-third. Ms. Liu did not make any comments on the answers but asked the student who got one-third for his reasoning and let other students adjust their thoughts based on the student's response. Finally, Ms. Liu raised the question whether the fraction is determined based on the number of apples or the number of parts, which addressed the confusion that students had, that the fraction is determined by the number of parts but not the total number of the apples.

After the class discussion on the fractions obtained when six apples were evenly divided into three parts, Ms. Liu gave students a worksheet and had students figure out other ways to evenly divide the six apples. This activity allowed the students to figure out the relationship between the number of parts that was evenly divided into and the fraction used to represent each part of apples.

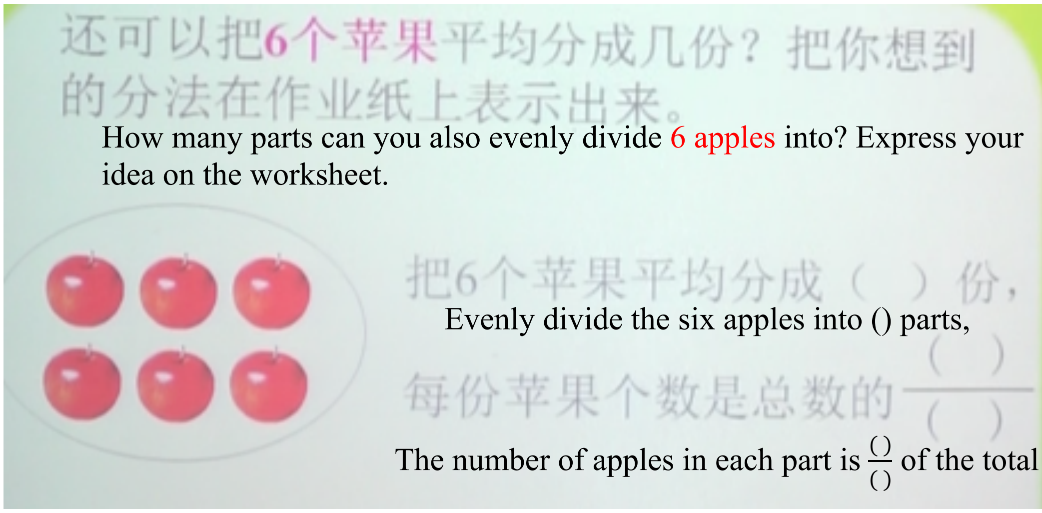

Figure 21. Activity of evenly dividing six apples and identifying the fractions

Discussion on the fractions obtained by evenly dividing the six apples. To deepen students' understanding, Ms. Liu raised two questions based on the activity of evenly dividing the six apples. She first showed the students a picture of three groups of six apples and asked the students to identify the fractions that could represent one part of the six apples. After students answered with one-third, she raised up the first question: why do we use different fractions to represent one part of all of them?

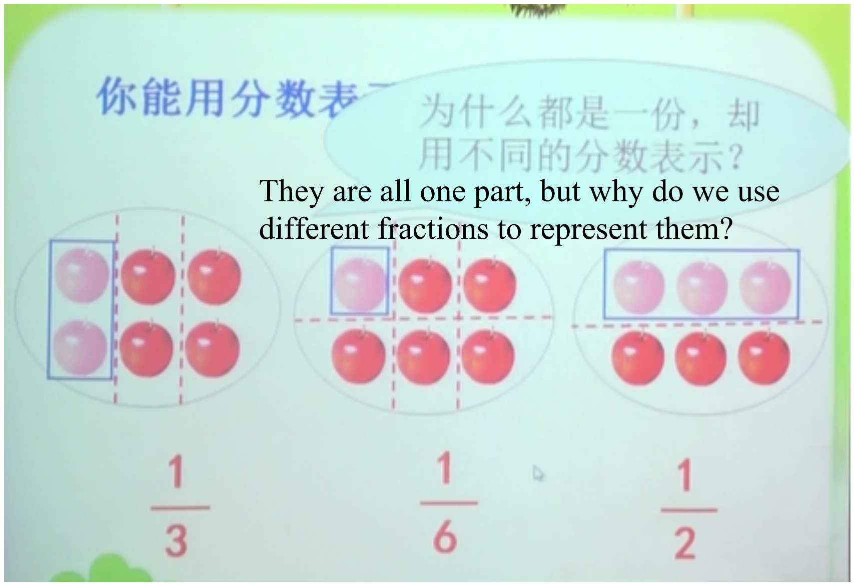

Figure 22. A slide shows different unit fractions indicated in the picture

The discussion on the first problem continued using another way to elicit students' thinking about how to determine the fractions and what matters in determining fractions. The fraction was not focused on the total number of the object, but Ms. Liu continued to show the class another picture with three groups of apples. This time the number of apples in each group was different. Still, she asked the class to identify fractions that represent the indicated parts of apples in each group. She then raised the second question: why can we use 1/3 to represent the circled part for all three groups. The question required the students to think about the relationship between fractions and the number of objects.

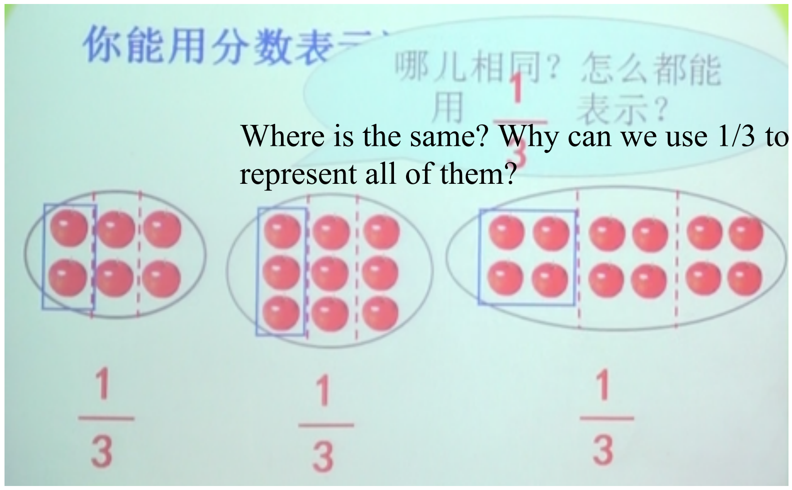

Figure 23. A slide shows the same fraction indicated in the picture

Contrast the fractions just learned with the fractions learned in the previous lesson. Next, Ms. Liu had students put their knowledge to practice. She showed students a slide with three groups of shapes, and asked students to identify the fractions represented by the shaded part.

After this practice, Ms. Liu attempted to summarize the new knowledge by showing students a picture on the screen and asking them to find the differences among the objects that were evenly divided. Finally, Ms. Liu concluded that, “evenly divide one object or multiple objects into a number of parts, we could use fractions to represent one or several parts out of them”.

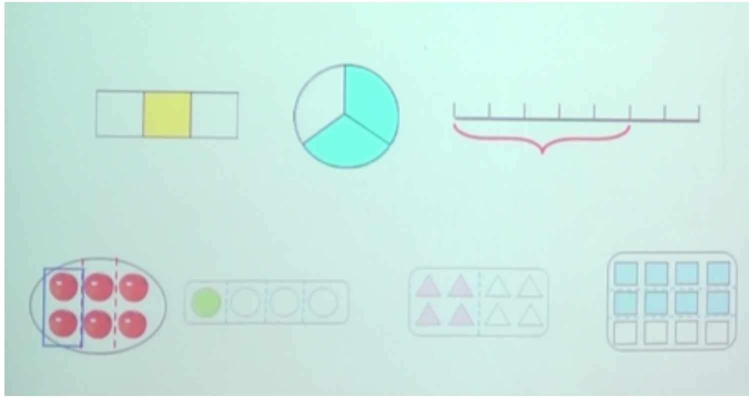

Figure 24. All the models used in the class

Application to problem solving. In the following part, Ms. Liu introduced the theme of the lesson: the simple application of fractions. Ms. Liu showed the class an application problem along with the instruction on the screen:

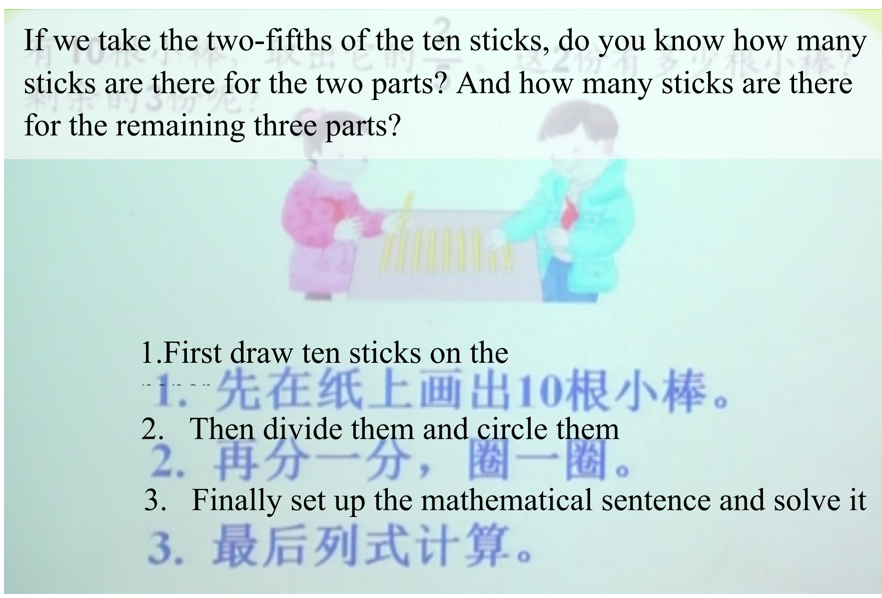

Figure 25. The application problem shown to the class

The instruction guided the students to approach the word problem with the representative model, Ms. Liu also encouraged the students to represent the sticks using short lines.

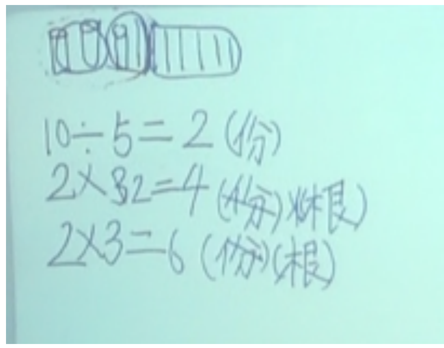

Figure 26. Student's work shown to the class

Conceptual coherence among each lesson

As mentioned in Chapter 3, the conceptual coherence among the lessons is examined by exploring the connection between the lessons. Based on my observations, Ms. Liu delivered four lessons to impart the new knowledge when teaching the unit of introduction of fractions. It appeared that Ms. Liu frequently referred to the knowledge covered in the previous lessons at the beginning or in the middle of each lesson. And the following flow chart shows the connection between Ms. Liu's lessons.

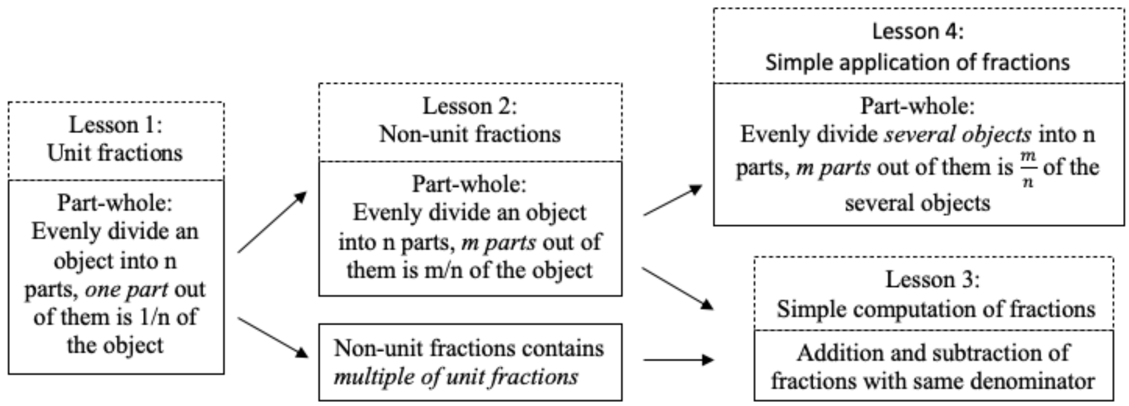

Figure 27. The conceptual coherence among each lesson

From Lesson 1 to Lesson 2. Based on the observation, Ms. Liu's teaching was devoted to a clear theme on the topic of unit fractions in Lesson 1. In Lesson 2, Ms. Liu introduced two ways to interpret the non-unit fractions which built on the knowledge of unit fractions. Firstly, Ms. Liu had students explore the part-whole understanding of non-unit fractions by folding the square paper model into four parts, which is similar to the activity of finding 1/4 of the square paper model implemented in Lesson 1. Instead of taking one part out of four parts, in order to explore non-unit fractions, Ms. Liu had students highlight more than one part out of four parts based on the meaning of denominator and numerator discussed in Lesson 1. Ms. Liu thus introduced to the students a part-whole understanding of non-unit fractions. Secondly, Ms. Liu interpreted non-unit fractions as multiple of unit fractions, which took unit fraction as a unit to measure non-unit fractions. Both ways of interpreting non-unit fractions are closely related to the knowledge of non-unit fractions.

From Lesson 2 to Lesson 3. In Lesson 3, Ms. Liu introduced the simple computation of fractions, which included the addition and subtraction of like fractions.

Ms. Liu provided students with opportunities to apply their previous knowledge covered in Lesson 1, namely the two ways to interpret non-unit fractions, to understand the operation of fractions.

From Lesson 2 to Lesson 4. In Lesson 4, Ms. Liu extended students' part-whole understanding of fractions by introducing that the whole could be multiple objects. This built on students' previous knowledge of the part-whole understanding of fractions.

Procedural coherence

In this study, procedural coherence is defined as the routines of the lessons as reflected in the structure within each lesson and among each lesson.

Procedural coherence within each lesson

As presented in Chapter 3, I will examine procedural coherence as it is embodied in the class routines. By examining the pattern of enactment of class activities in each class, I identify the routines of each class reflected by the sequence of the class activities and specify the role of the structure of the lesson. Table 6 describes the flow of the class activities which were adapted from Chen and Li (2010) and their corresponding specifications that will aid the analysis of the data for procedural coherence.

Table 6.

The Class Activity Instruction of Ms. Liu

Class activity	Specification
Preparation for the class	Teachers make command before class start. For example: put your pencil into your pencil box.
Review	Teacher go over the previous knowledge. It usually starts with ”let us think about what we talked about in yesterday’s lesson”.
Lecturing	Teachers present to the class.
Group discussion	Students discuss as groups.
Students’ individual work	Students work on the task by themselves.
Student presenting	Students answer questions standing on their seats or in front of the class.
Summary	Teachers repeat or summarize on the statements made by the students or teachers summarize the knowledge. For example: today, we talked about fractions. Let us summarize the knowledge we have learned today.

Preparation for the class. Ms. Liu was observed to help the students organize their stationary before the class started for all the four lessons. In Lesson 1, Ms. Liu was observed taking about five minutes guiding students to get ready for the lesson:

> Ms. Shi: Take out a practicing paper (a white paper), write down your class number, name. Put another A4 paper under your textbook. Pile up the manipulatives you have and put it on top of the textbook. Put your pencil box in front of you. The colored pen can be put on the right of your pencil box or in your drawer of your desk, you can take it whenever you need to use it. Ok. Check everything out, it is pretty warm in the room, if you feel uncomfortable, you can take off your coat and put it on the back of the classroom.

As Ms. Liu directed the students to organize their stuff on the desk, Ms. Liu walked around and checked if the students were ready. She also reminded one student to put aside her colored pen since it is so big.

Review. Based on the observation, Ms. Liu revisited the previous knowledge at the beginning of each lesson for all the four lessons expect for Lesson 1. Also, a routine was found that Ms. Liu would start each lesson with an introduction of the theme of each lesson, then she would lead the students to the review section.

Table 7.

The Time Ms. Liu Spent on The Review for Each Lesson

Review of	Lesson 2	Lesson 3	Lesson 4
Time spent	4min30sec	2min17sec	1min
Content reviewed	Lesson 1	Lesson 2	Lesson 2

Other activities. like class discussion, students' individual work, student presenting, and group discussion were also observed in Ms. Liu's lesson.

Procedural coherence between each lesson

To examine the procedural coherence between each lesson, I classified each class as either new-material instruction class or practicing classes. The goal of the new-material instructional class is to impart new material, and the goal of the practicing class is to review previous knowledge by practicing. The figure 28 below shows the flow of the eight lessons.

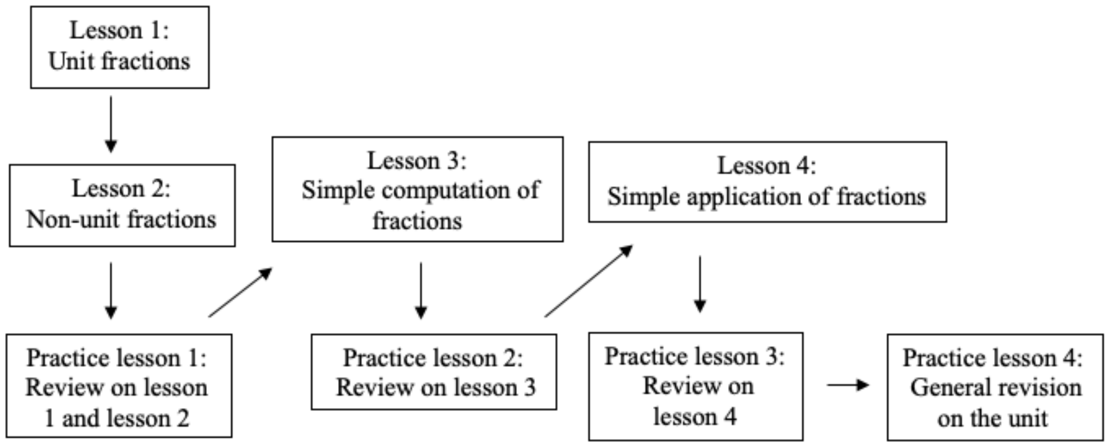

Figure 28. Flow chart among each lesson

Case 2: Ms. Yang

Ms. Yang is a mathematics teacher from Donger elementary school. At the time of data collection, she had three-years teaching experience including two years teaching grade 1 and this was her first time to teach grade 3 when Ms. Yang was observed.

Conceptual coherence within each lesson

During my time of observation, Ms. Yang delivered 8 lessons. While six of the lessons focused on the instruction of new content, the other three lessons were defined and introduced by the teacher at the beginning of the lesson as the practice lessons with the goal of reviewing or doing problems to reinforce the knowledge learned in the previous lesson. The theme for each lesson was explicitly shown by the teacher on her prepared slides. The themes aligned with the textbook.

Lesson 1: unit fractions

The theme of Lesson 1 was the conceptual understanding of unit fractions. Based on the observation from this lesson, Ms. Yang implemented a series of activities towards the goal of the lesson, which is shown in the flow chart in figure 29.

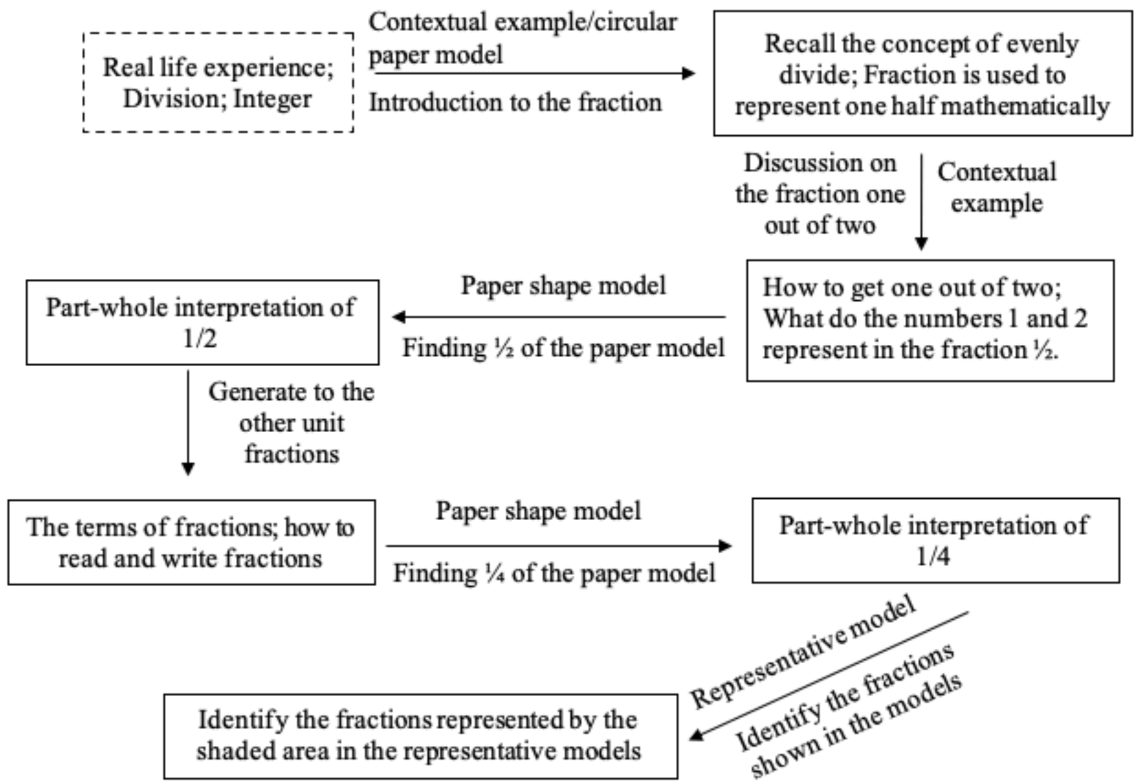

Figure 29. Flow chart for Lesson 1

Introduce the concept of fractions. At the beginning of the lesson, Ms. Yang used a problem with a real-life context which included a picture shown to the class:

> Zhang Zhiyong (a student in the class) told me that he was planning to go to the park with his sister. Let's check out the food he is going to bring. If Zhang Zhiyong and her sister would like to eat the food, how should they share the food?

Figure 30. A picture showing the context of sharing foods

Ms. Yang made Zhang Zhiyong, a student in the class, the leading role of the problem. In the process of exploring the contextual example, Ms. Yang first guided the class to recall the concept of ‘evenly divide’ when sharing the set of four apples, and the set of two bottles of water as illustrated in the vignette below:

Ms. Yang: Well, two bottles of water, each person gets one bottle, it was just evenly split. What do we call this?

Students answered together: Evenly divided.

Ms. Yang continued to ask about how to share the four apples and then the method of sharing one cake fairly:

Ms. Yang: Ok, that means there are four apples, you and your sister will take two apples for each of you. Then what? Is the way you share the apples fair?

Students answered together: It is fair

Ms. Yang: It is fair, then it is also…

Students answered together: Evenly division

Ms. Yang: Well, that means we are going to explore some knowledge about evenly division. Then besides water bottles and apples, can you evenly split the cake?

Students answered together: Yes.

As shown in the vignette, recalling the concept “evenly divided” is based on the students’ life experiences of sharing food evenly; two people need to get the same amount of food, or it is unfair. Also, as Ms. Yang mentioned in her interview, the concept of “evenly divided” was introduced when the students learned division. Thus, Ms. Yang transferred the concept of “evenly divided” from the operation of division and applied it to the concept of fractions.

After the discussion of how to share water bottles and apples, Ms. Yang asked another student how to share the cake. She then let the student demonstrate the splitting process using a circular paper model suggesting the student take the circular paper model as a cake to represent the student’s idea of sharing the cake.

After the student got one half by cutting the cake, Ms. Yang used an analogy with the integer one and two to reveal to the students that although one half is a quantity, one half is only a daily language but not mathematical language. Based on that, Ms. Yang introduced the concept of fraction which is a kind of number.

Ms. Yang: Yes, one bottle. Then what kind of number is “one” ? Do you know this number before?

Students answered together: Yes.

Ms. Yang: Ok, it is the number that we have learned before. Are you familiar with it?

Students answered together: Yes.

Ms. Yang: What kind of number is it? It is integer, Right? Evenly divide the four apples and each person gets two apples, then the two here is also… what kind of number is two?

Students (answered together): Integer.

Ms. Yang: It is also an integer. Then I have a question now. Ma Jiarui just said that after she evenly divided the cake from the middle, each person gets one half. What is one half? Is it a mathematical language?

Students answered together: No.

Ms. Yang: what if we want to use mathematical language to represent one half of the cake? How do we do that? Have a discussion with your desk mate.

In Chinese, there are two ways to present the numeral 1/2, one half(一半) and one-out-of- two. While one half is the daily language used to represent 1/2, one-out-of-two is how the numeral 1/2 read mathematically. That is why Ms. Yang pointed out that one half is not a mathematical language.

The conceptual understanding of one-out-of-two. After introducing the fraction one out of two as a number to represent one half, Ms. Yang recalled the process of sharing a cake and asked the students to explore the meaning of each part of the fraction one out of two.

Ms. Yang: Ok, let's first talk about the meaning of two. Where did you hear about the number two when we talk about sharing the cake? [called a student's name]

Student1: I heard two first, then one.

Ms. Yang: Ok, you heard two first, then one. Where did the two come from when we described the process of sharing the cake?

Student1: Divide the cake into two parts.

Ms. Yang: Yes! Divided into two parts, so what does the two mean?

Student1: Two should represent two parts.

Ms. Yang: It represents two…?

Student1: Two pieces of cakes.

Ms. Yang: Yes, two pieces of cake. Two pieces of cake is also two parts of one cake, right?

The interaction above illustrates how Ms. Yang emphasized the words of *one* and *two* when describing the process of sharing the cake to interpret half as part-whole concept.

After the discussion about using the words *one* and *two* in the context of sharing the cake and its relation with the word *half*, Ms. Liu asked students to find 1/2 of any paper shape model that the students would like to use based on their understanding of the fraction 1/2 as part-whole.

The conceptual understanding of 1/4. Ms. Yang displayed three paper shape models that respectively represent 1/2, 1/4 and 1/3 and made a conclusion that all of them are fractions.

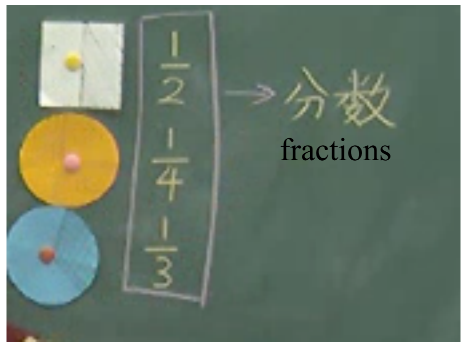

Figure 31. The paper models shown to the students

Ms. Yang then took the fraction 1/4 as an example to introduce the terms *denominator*, *numerator* and the *fractions bar* as well as their meaning. Ms. Yang also talked about how to read 1/4 and let students practice on reading the fractions. After that, Ms. Yang asked the students to represent 1/4 on a square paper individually, then Ms. Yang had several students present about how to represent 1/4 in front of class.

Ms. Yang drew the students' attention to the concept of the whole by showing the class various models with different sizes and shapes. She then discussed why 1/4 of the different models are in different sizes, reminding the student to be clear about the whole (or reference unit) when they refer to a fraction. Ms. Yang also demonstrated to the class that different ways of folding the papers can all illustrate 1/4 of the paper.

Identify the unit fractions within different models. Ms. Yang showed students pictures of real-life objects such as the national flags and let students distinguish the fractions from the picture. Ms. Yang continued to ask students for examples about fractions in real-life.

Ms. Yang showed students several figures and asked students to identify the fractions represented by the shaded area.

Students talked about what they learned for today's lesson and what they wanted to know in the future. Ms. Yang then concluded with the knowledge learned at the end.

Lesson 1 summary. In Lesson 1, Ms. Yang introduced unit fractions and its part-whole interpretation. She used contextual examples and real-life experiences to illustrate the concept of half and their part-whole interpretation to help students apply their previous knowledge on division as well as their real-life experiences to interpret the real-life concept one half as a fraction. Then Ms. Yang had the students explore the part-whole interpretation of unit fractions by finding 1/2 and 1/4 of a paper shape model.

Lesson 2: comparing magnitude of unit fractions

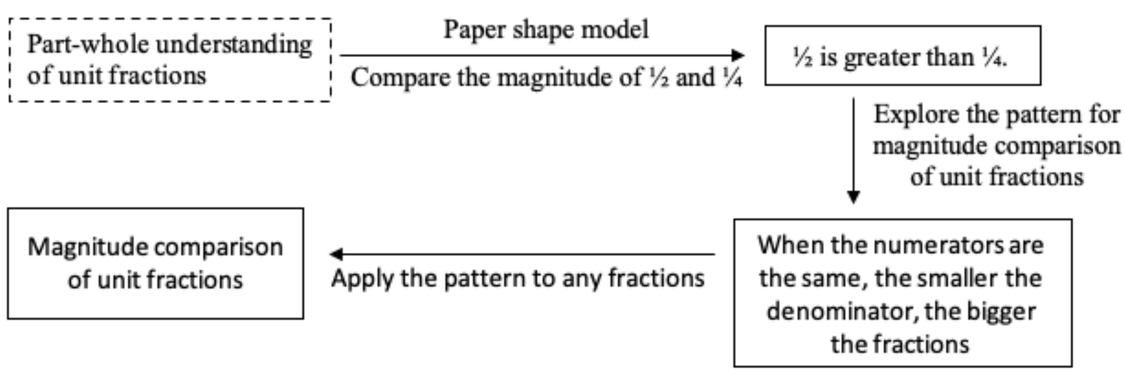

Figure 32. Flow chart for Lesson 2

Review of unit fractions. Ms. Yang started the lesson reviewing the topic of the previous lesson. She placed two square paper models on the board; one was evenly divided into 8 parts, and another paper was evenly divided into 4 parts. Then Ms. Yang let students describe the process of identifying the unit fractions 1/8 and 1/4 based on the two paper models.

Magnitude comparison of unit fractions. Noticing that there are many different unit fractions from the previous episode, Ms. Yang asked students what could be explored about fractions. She then pointed out the magnitude comparison of unit fractions as the theme of the lesson.

By showing students the different unit fractions while reviewing, Ms. Yang asked students to explore their differences, then transitioned to the topic of magnitude comparison of unit fractions.

Ms. Yang let students individually use their paper model to compare the magnitude of one out of two and one-fourth. After students finished working individually, then they discussed their ideas with their desk mate. After the discussion, Ms. Yang had students present their ideas.

Ms. Yang asked students to find a pattern for comparing the magnitude of fractions, Ms. Yang had students take turns presenting their ideas and making their conclusions at the end. She asked students to restate the pattern to their desk mates and prepare to use it later.

Ms. Yang asked one student to come up with two fractions, then assigned another student to compare the magnitude of the two fractions. They also discussed whether the denominator and numerator could be zero.

Ms. Yang and the students solved problems from the textbook together. They first read the problems together and then Ms. Yang had students come up with the answer.

Lesson 2 summary. In Lesson 2, Ms. Yang had the students explore a method for comparing the magnitude of the unit fractions. Using the topics of the previous lesson on unit fractions, Ms. Yang let the students compare the magnitude of 1/2 and 1/4 with the use of the same paper shape model, which visualized the magnitude of fractions as the area size of the shape. Ms. Yang then asked students to find a strategy to compare the magnitude of fractions based on their observations, from the concrete model to the application of the strategy to other pairs of fractions.

Lesson 3: non-unit fractions

The theme of Lesson 3 was non-unit fractions. Ms. Yang introduced two different ways to interpret non-unit fractions: 1) part-whole interpretation and 2) non-unit fractions as a composition of multiple unit fractions. Then Ms. Yang asked the students to compare the magnitude of fractions using the same denominator and different numerators.

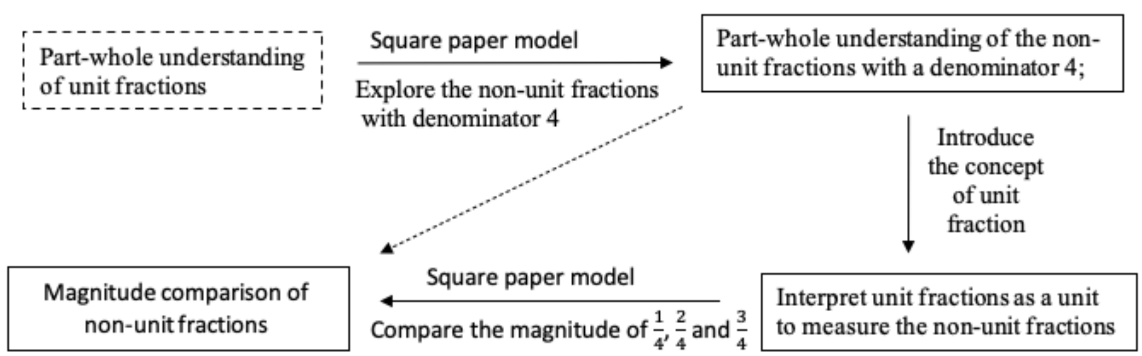

Figure 33. Flow chart for Lesson 3

Review of unit fractions. Ms. Yang started the lesson with a review of the concepts learned in Lesson 1. Ms. Yang showed students a shape model and asked them to describe the process of identifying a fraction based on the number of parts that the shape model is evenly divided into emphasizing the part-whole interpretation of fractions.

Non-unit fractions with denominator 4. Ms. Yang asked students to evenly divide the square model into four parts and shade as many parts as students wanted. Then Ms. Yang let students present their fractions.

The purpose of the activity is to have the students explore the part-whole interpretation of non-unit fractions. Similar to the activity of finding 1/4 of a square paper model that was implemented in Lesson 1, the students were asked to use the same square paper model evenly divided into four parts and to shade one or more parts instead of one part which extended the concept of the unit fraction 1/4 to other fractions with the same denominator but with numerator other than 1.

Introduce the concept of unit fractions. Ms. Yang asked students to find the relationship between the fractions obtained in their episode 2 activity and asked students to talk about one-fourth as the most important fraction among the other two. Then Ms. Yang introduced the concept of interpreting any non-unit fractions as several numbers of unit fractions. Ms. Yang attached a paper strip on the board and let students represent their fractions based on it.

Ms. Yang asked students to find the difference between the fractions learned in Lesson 1 (unit fractions) and the fractions they just learned (non-unit fractions). Students noticed that the numerators for those fractions were different.

Ms. Yang asked students to answer the questions and explain their reasoning.

Magnitude comparison of non-unit fractions. Ms. Yang asked students to choose any two fractions shown in the picture and compare their magnitude. The students then described how they made their comparisons.

Ms. Yang asked students to discuss and describe any strategy for comparing the magnitude of fractions with the same denominator in front of her class.

Ms. Yang showed students several problems on her slide, and asked different student to answer each one. She concluded her lesson with a summary of the knowledge that was covered in today's lesson.

Lesson 3 summary. In Lesson 3, Ms. Yang used previous concepts of non-unit fractions, which included the part-whole interpretation and magnitude comparison of non-unit fractions. Ms. Yang first had students explore the part-whole interpretation of non-unit fractions with the use of the paper shape model. Then she reintroduced unit fractions as a unit to measure non-unit fractions. With this, Ms. Yang guided students to interpret non-unit fractions as multiples of unit fractions. Using paper shape model and the two interpretations of non-unit fractions, Ms. Yang gave students the opportunity to create strategies for magnitude comparison of non-unit fractions.

Lesson 4: the simple computation of fractions

The theme of Lesson 4 was the addition and subtraction of fractions with equal denominators. Based on the observation, Ms. Yang's instruction followed the path which is shown in the flow chart.

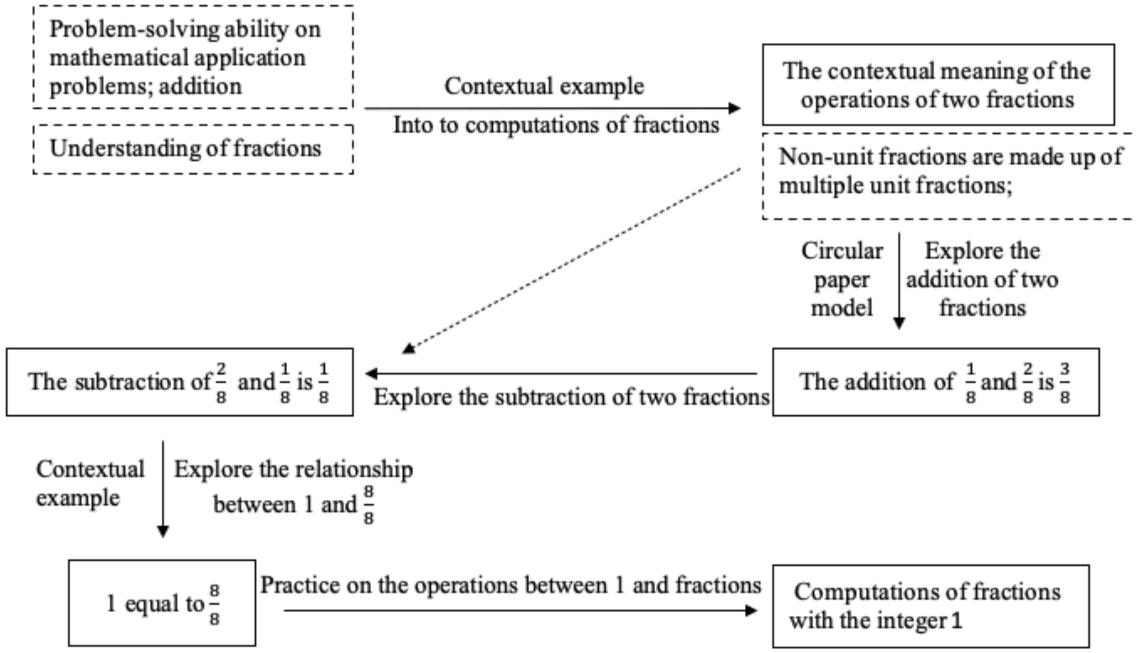

Figure 34. Flow chart for Lesson 3

Review of non-unit fractions. At the beginning of Lesson 4, Ms. Yang asked students to give examples of fractions, then interpret the meaning of the fraction as in previous lessons. In this process, Ms. Yang had students review the interpretation of fractions as part-whole and as multiple unit fractions.

Introduction to addition and subtraction of fractions. Ms. Yang showed students a picture of two children eating watermelon and asked students to explore the mathematical information given in the context. Then Ms. Yang asked students to create and pose mathematical questions based on the information they notice in the picture. Students came up with two questions: How much watermelon do the brothers eat altogether? How much more does the older brother eat than the younger brother?

Figure 35. A picture of two kids eating watermelon

Since no text or numerical information is shown in the picture, it required the students to interpret the situation shown in the picture and to think of a mathematical question. In order to determine that the older brother (the child on the left) eats 2/8 of the watermelon while the younger brother eats 1/8 of the watermelon, Ms. Yang guided the students to apply their part-whole understanding of fractions.

With Ms. Yang' s guidance, students associate the operation of addition and subtraction of like fractions with a real-life context, which can help students to make sense of the possible application of the computation.

Explore the addition and subtraction of fractions. Based on the question, the students came up with the mathematical sentence: $\frac{1}{8}+\frac{2}{8}=$. Ms. Yang let students explore how to add the two fractions using the circle model. She then asked students to present their reasoning. After the exploration of the addition of fractions, Ms. Yang guided students in solving the second question that the students proposed, the subtraction of fractions, by interpreting the fractions as a multiple of unit fractions. Ms. Yang asked students to summarize the strategies for adding and subtracting fractions.

Explore the relationship between 1 and 8/8. Ms. Yang asked students to explore the relationship between 1 and 8/8 using a circle model. First she let students think about a problem from the textbook by themselves, then asked them to represent their ideas. Ms. Yang gave students time to work on three questions on fraction computation with 1 in it and let students answer the questions in front of the class.

Ms. Yang showed students more questions and let them answer together.

Ms. Yang concluded the lesson with what knowledge was learned in today's lesson.

Lesson 4 summary. In Lesson 4, Ms. Yang introduced the notion of adding and subtracting fractions with equal denominator based on the concepts from previous lessons. Ms. Yang first elicited the computation of fractions using a contextual example, then Ms. Yang let the students explore a way to operate using a paper shape model using previous lessons' concepts. Ms. Yang also introduced a computational technique to the students, which is the transition from 1 to any fractions when adding or subtracting between any fractions and one. Finally, they observed patterns for the algorithm that may provide a connection from conceptual understanding to procedural understanding.

Lesson 5: application of fractions

In Lesson 5, Ms. Yang extended the part-whole interpretation of fraction by generating the whole from one object to multiple objects. In order to achieve the learning goal, Ms. Yang delivered the lesson in the following path.

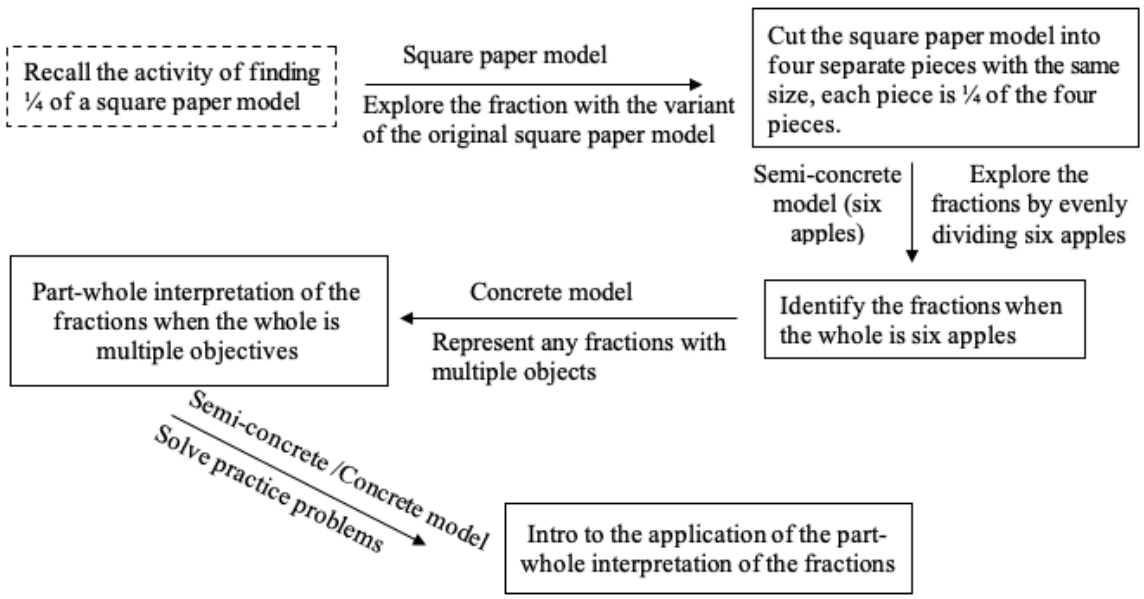

Figure 36. Flow chart for Lesson 5

Revisit the activity of finding 1/4 of a square paper model. Ms. Yang distributed square papers in different colors to the students at the beginning of the class. As the class started, Ms. Yang asked students to represent 1/4 on their square paper model. After the students finished, Ms. Yang collected three students' work and showed them to the class. In the first two student's works, the square paper was evenly divided into four small triangles, while in the third student's work, the square paper was divided into four squares. Then Ms. Yang asked students to state the process for how they got the fractions.

Ms. Yang had implemented the same activity of finding 1/4 in lesson one in order to have the students explore the unit fraction 1/4. In contrast, the current activity played a different role and appeared to lay the groundwork for the following activity.

Explore the fraction with the variant of the original square paper model. In the last activity, the students had evenly divided the square paper into four equal parts by folding the square paper model. In the following activity, with the same model used, the students were asked to tear or cut the square shape model along the folding lines to get four separate parts; some had four small triangles, the others had four small squares. Clearly, it all depended on how one folded the paper in the previous activity. After the students got four separate parts. Ms. Yang asked the students to piece the four small parts back together to form the original square paper and through her guidance, she intended to reveal to the students that the one part is still 1/4 of the combined four small parts after the square model was cut into the four smaller shapes.

Figure 37. Ms. Yang demonstrated the process of separating the four parts using the square paper model.

Ms. Yang asked a student to describe the fraction he got after he cut the square paper model into four triangles.

Ms. Yang: Before you cut it, this is a…?

Student1: Square paper.

Ms. Yang: Square paper. Then?

Student1: Then I cut it (square paper) along the folding lines and split it into four.

Ms. Yang: Ok. Which fraction can you represent one part when the whole is one object?

Student1: I can represent one-out-of-four.

Ms. Yang: Then? Right now?

Student1: Now I have cut it apart.

Ms. Yang: Ok. Then what did you get?

Student1: Four triangles.

Ms. Yang: Ok. You got four triangles. Then? Which fractions is one triangle out of the four triangles?

Student1: It is one-out-of-four of the four triangles.

In this activity, Ms. Yang first had student notice the part-whole relationship between one of the four parts transformed from the square model as one object among four objects.

Explore the fractions by evenly dividing six apples. Ms. Yang showed students a picture of six apples that was evenly divided into three parts. Ms. Yang asked students to find the fraction that could represent one part, two parts and three parts of the six apples.

Represent any fraction with multiple objects. Ms. Yang asked students to use the available manipulatives to represent any fraction they want and present the process in front of the class. Some students used sticks while others used coins.

Figure 38. A student demonstrated how to get the fraction

Problems solving. Ms. Yang asked students to work on the following problems in the textbook and present their solutions in front of the class.

Figure 39. Textbook problems

At the end of the class, Ms. Yang asked students to describe the process of getting fractions under a real-life context.

Lesson 5 summary. In Lesson 5, Ms. Yang provided students with opportunities to use the part-whole interpretation of fractions with a generalization from the whole being one object to multiple objects. First, she revisited the activity of finding 1/4 of a square paper model, then with the variant of the model, she showed the students that the whole has changed from one big square to four smaller squares or triangles. Then Ms. Yang showed the students a way to identify fractions by using the six apples. During the last part of the lesson, the students worked on problems by using representative models to connect to contextual problems.

Conceptual coherence between each lesson

Ms. Yang implemented a sequence of five lessons to introduce the concept of fractions, including addition and subtraction of fractions with equal denominators. She started from unit fractions, then moved to the non-unit fractions, and used the interpretation of these fractions to introduce addition and subtraction in application problems. Ms. Yang frequently started each lesson with a review of the knowledge covered in the previous lessons. Based on my observations, Ms. Yang's instruction demonstrated the conceptual coherence between each lesson, which is shown in the following diagram.

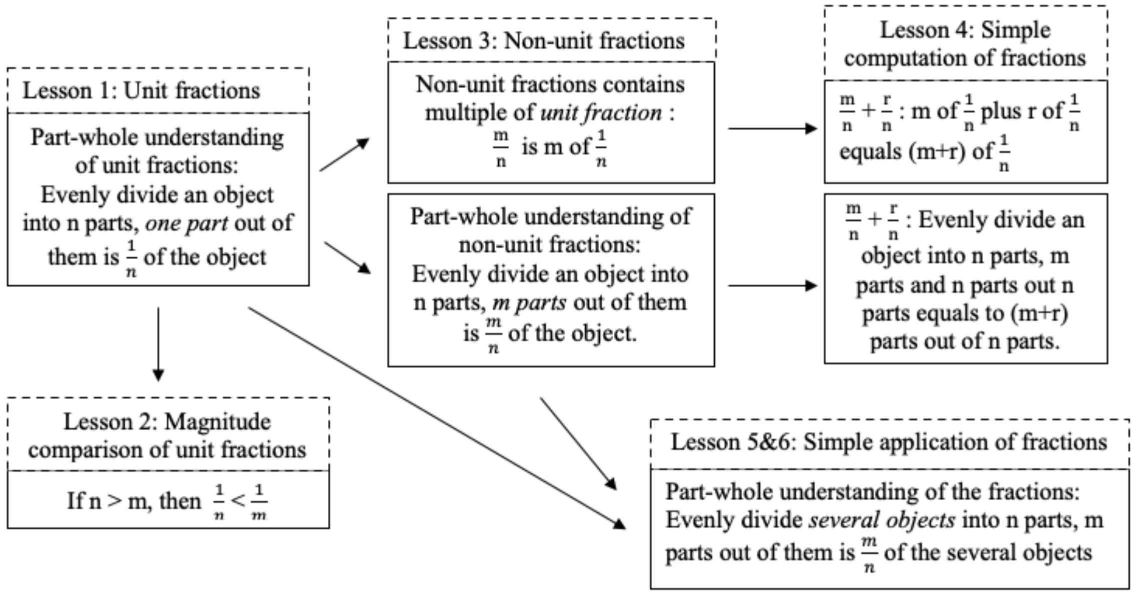

Figure 40. Flow chart among each lesson

From Lesson 1 to Lesson 2. At the beginning of Lesson 2, Ms. Yang recalled the students' part-whole interpretation of unit fractions 1/2 and 1/4 by having students state the meaning of the two fractions. In Lesson 1, the students had represented 1/2 and 1/4 of the paper shape model, following that, Ms. Yang let the students compare the magnitude of 1/2 and 1/4 with the use of the same paper shape model in Lesson 2. On one hand, the knowledge of the model used in lesson one appeared to help students visualize the magnitude of the two unit fractions so that the students could compare the magnitude of the fractions by observing the size of the shaded area of the model. On the other hand, the meaning of unit fractions provided a way for students to explain the reasoning behind the result, which is that the larger denominator corresponds to more parts that the shape was evenly divided into. Therefore, each part of the model which represents the unit fraction will be smaller.

From Lesson 1 to Lesson 3. Ms. Yang started Lesson 3 with a review of the part-whole interpretation of the unit fractions. In Lesson 3, Ms. Yang introduced two ways to interpret non-unit fractions, which built on previous lessons on unit fractions.

Ms. Yang first introduced the part-whole interpretation of non-unit fractions. While the unit fractions can be interpreted as one part out of n parts, the part-whole understanding of non-unit fractions is to take m parts instead of just one part out of the n parts. Therefore, Ms. Yang had the students explore non-unit fractions using the same square paper model that was used to explore the unit fraction1/4. In Lesson 3, Ms. Yang asked students to shade one or more parts to generate non-unit fractions all with denominator 4.

Ms. Yang also presented to students the non-unit fractions as multiple unit fractions by revisiting the concept of unit fraction. Thus, the learning of unit fraction served as a foundation for the learning of non-unit fractions.

From Lesson 3 to Lesson 4. In Lesson 3, Ms. Yang introduced two ways to interpret the non-unit fractions, which leads to the introduction of addition and subtraction of fractions with equal denominators in Lesson 4.

Ms. Yang first had the students explore the addition of the fractions 1/8 and 2/8 using the circle paper model which gave students the opportunity to use the part-whole interpretation of fractions and interpret the addition as taking two parts and one part out the eight parts, which is equal to taking three parts out of eight parts.

Ms. Yang then guided the students to make sense of the addition by taking 1/8 as a unit; therefore, one eighth plus two eighths is three one-out-of-eight, which is three-out-of-eight.

From Lesson 1 and Lesson 3 to Lesson 5. In Lesson 5, Ms. Yang mainly focused on using the part-whole interpretation of fractions and built from unit fractions to non-unit fractions as a process of generating the "part" from one part to several parts. Then in Lesson 5, students extended their part-whole interpretation with a change of the whole as an area unit representing one object to consisting of multiple discrete objects.

At the beginning of Lesson 5, Ms. Yang implemented the activity of finding 1/4 of a square paper model that was conducted in Lesson 1. However, the purposes of the activity in Lesson 1 and Lesson 5 appeared different. While the point of the activity was to have the students explore the unit fraction 1/4 in Lesson 1, the activity in Lesson 5 served as a reference that was used to demonstrate the possible variations of the "whole". With the previous knowledge that one part out of the four parts of the square paper is 1/4, Ms. Yang began to show the students that after cutting the square paper into the four triangles, which can still be spliced into the original square paper, one triangle, which is the same as one part of the original square paper, should still be 1/4 of the four triangles that was the same as the original one square paper.

In the later section, Ms. Yang guided the students to explore more fractions including the non-unit fractions. The understanding of the denominator and numerator as the number of parts instead of the number of objects in a whole enhanced students' ability to identify fractions when the whole contains multiple objects.

Procedural coherence

In this study, procedural coherence is defined as the routine of the lesson as reflected in the structure of the lesson. In the procedural coherence within each lesson and among each lesson.

Procedural coherence within each lesson

As presented in Chapter 3, I will examine procedural coherence as it is embodied in the class routine. By examining the pattern of enactment of class activities in each class, I will find the routine of each class reflected by the sequence of the class activities and specify the role of the structure of the lesson. Table 5 describes the flow of the class activities which were adapted from Chen and Li (2010) and their corresponding specifications that will aid the analysis of the data. The activity of Ms. Yang is highlighted in the following Table 8.

Table 8.

Class Activity Instruction of Ms. Yang

Class activity	Specification
Preparation for the class	Teachers make command before class start. For example: put your pencil into your pencil box.
Review	Teacher go over the previous knowledge. It usually starts with ”let us think about what we talked about in yesterday's lesson”.
Lecturing	Teachers present to the class.
Group discussion	Students discuss as groups.
Students' individual work	Students work on the task by themselves.
Student presenting	Students answer questions standing on their seats or in front of the class.
Summary	Teachers repeat or summarize on the statements made by the students or teachers summarize the knowledge. For example: today, we talked about fractions. Let us summarize the knowledge we have learned today.

Procedural coherence between each lesson

The procedural coherence between each lesson is presented in the Figure 41. There are six lessons focusing on the new knowledge implementation and two practice lessons focusing on the review on the previous knowledge.

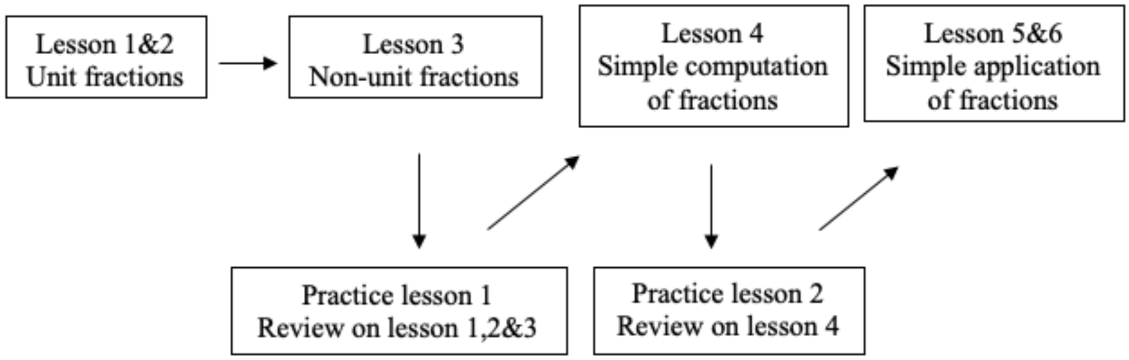

Figure 41. Flow chart of the eight lessons

Case 3: Ms. Shi

Ms. Shi is a mathematics teacher from Donger elementary school. At the time of data collection, she had five years teaching experience. She had taught grade 1 and grade 2 twice for the first four year and this was her first time teaching grade 3.

Conceptual coherence within each lesson

Lesson 1: unit fractions

In Lesson 1, Ms. Shi introduced the unit fractions to the students starting with how to interpret, read, and write the numeral 1/2 and then generalizing to other unit fractions. Recall that in Chinese there are two ways to talk about the fraction 1/2: 'one half'(一半) and 'one-out-of-two'（二分之一）. While 'one half' is the informal way to say it, 'one-out-of-two' is using the part-whole interpretation and how fractions numerals are read. This is true only for the 1/2 fraction; for all other fractions the way to say it is the part-whole format. The flow chart (Figure 42) depicts the different instructional goals or aims in terms of content or concepts (rectangles) and the instructional moves or activities (arrows connecting the rectangles). The dotted line rectangle represents previous content or concepts learned and used in the lesson to generate new content and the dotted arrows represent connections with content that are embedded into the activities. Ms. Shi provided opportunities for students to acquire the knowledge of fraction 1/2 from their previous knowledge on the process of "evenly divided" from division with whole numbers and their real-life experience, which can be interpreted as students' actual developmental level at the beginning of Lesson 1 and it is depicted in the diagram by the dotted-line rectangle (Vygotsky, 1978). Then, Ms. Shi continued giving opportunities for students to build the meaning of fraction using part-whole interpretation, mainly by using paper models first and then moving into diagrams – depicted in the flowchart as the arrows that link each area of content suggesting a sequence of pedagogical moves.

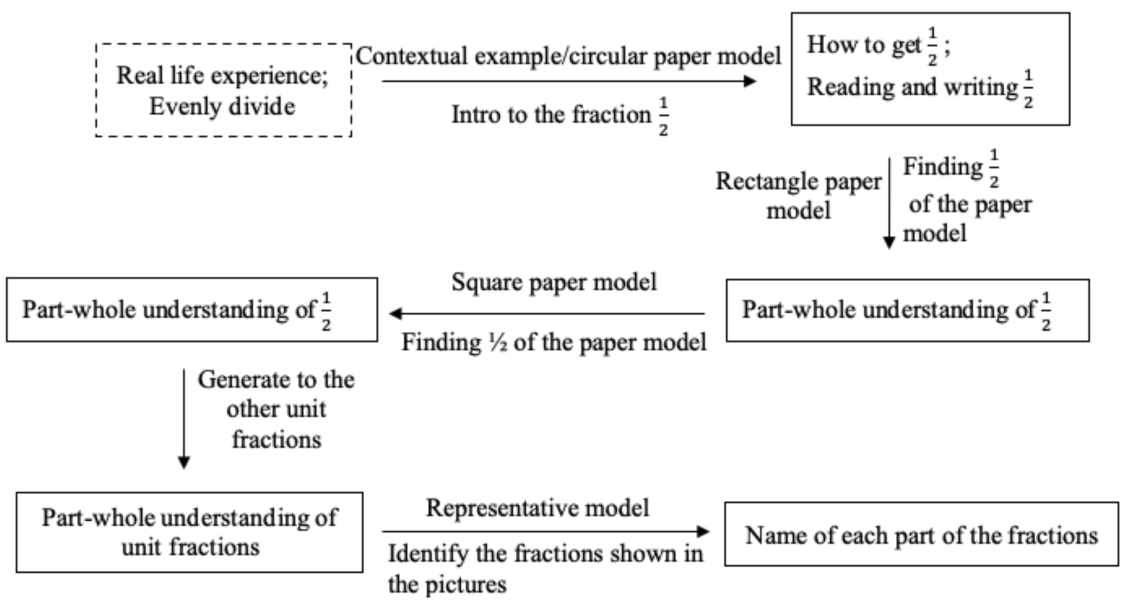

Figure 42. Flow chart for Lesson 1

The next subsections will be dedicated to providing the evidence for which the flow chart was constructed starting with the presentation of the real-life contextual example and ending with the identification of the part-whole interpretation of fractions using a model.

Introduction to 'one-out-of-two' with a contextual example. Ms. Shi started Lesson 1 with a story-based contextual example of Bear Da and Bear Er sharing mooncakes. Bear Da and Bear Er are two popular cartoon characters in China. The contextual example has a series of questions that are developmental in nature. The first situation (Figure 43) gives students a problem that they are expected to be familiar with (division of whole numbers with a whole number quotient, 4 divided by 2) but in the context of 'equal share' or 'evenly divided'.

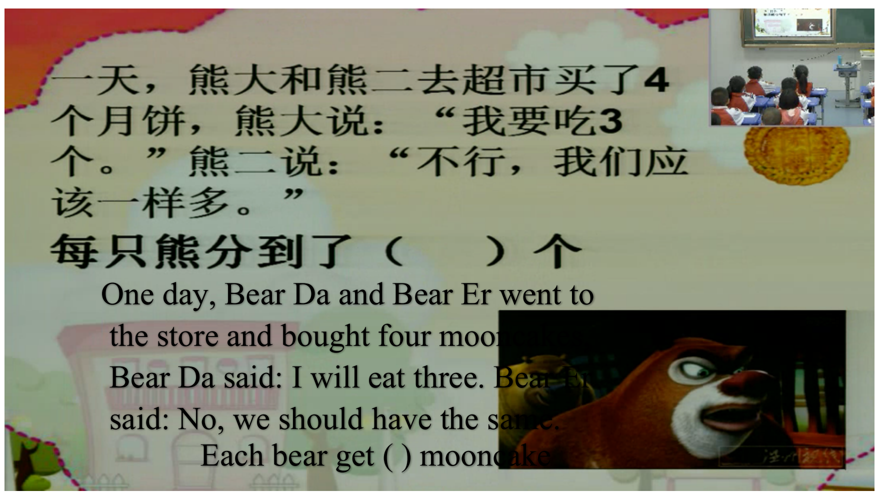

Figure 43. First situation of contextual example eliciting previous knowledge of division of whole numbers

A student answered the first question by saying that each bear should get two mooncakes, because they want to eat the same amount.

After a short discussion about the previous situation and concluding that each bear should get two mooncakes, Ms. Shi then presented a second situation with a corresponding second question (Figure 44). Although this second situation is also a division of whole numbers with a whole number as a quotient, it is following the sequence of 4 divided by 2 to 2 divided by 2, to eventually get to 1 divided by 2. This is evidence of providing students with a path in their zones of developmental proximities (Vygotsky, 1978).

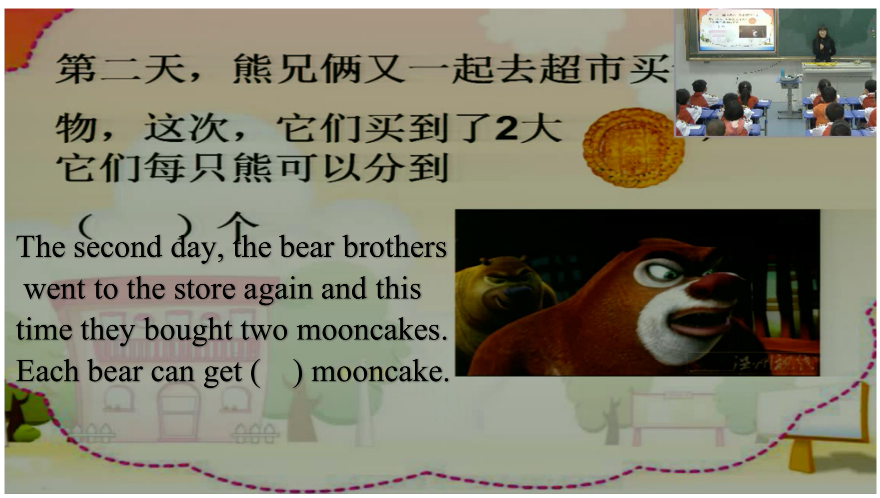

Figure 44. Second situation of contextual example eliciting previous knowledge of division of whole numbers

Below is a short vignette that illustrates the interaction between Ms. Shi and the students:

Ms. Shi: The second day, bear brothers went to buy mooncakes again. The mooncake is so delicious, how many did they buy this time? Two. How many mooncakes can each of them get this time?

Students answered together: One.

Ms. Shi: How did you get that? In mathematics, what do we call the method of dividing by the same amounts?

Students answered together: Evenly division.

Ms. Shi: Very good. How do we do the division?

Students answered together: By evenly dividing.

Ms. Shi: Yes. By evenly dividing. To ensure the fairness, we should use the method of evenly dividing. Therefore, each person gets one mooncake.

After revisiting the strategy, or method as students and teacher call it, “evenly dividing”, Ms. Shi presented the third situation with the corresponding question. She asked the class how the two bears should evenly divide one mooncake and had the students discuss with their desk mate about how to answer the question.

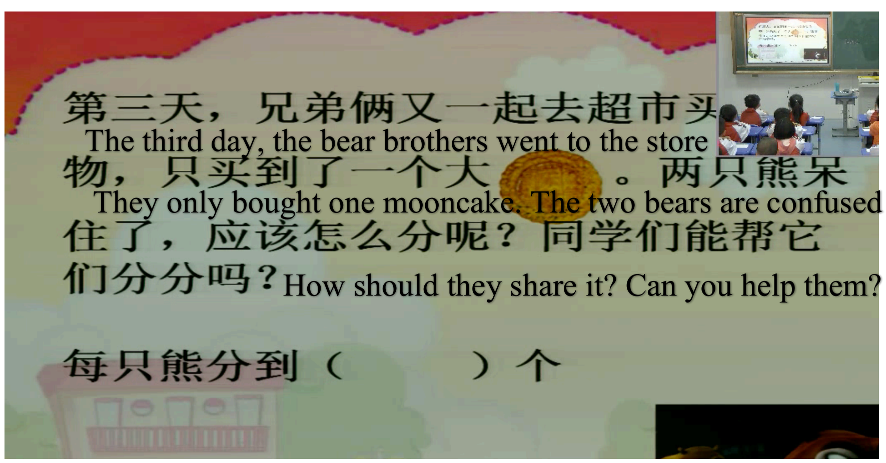

Figure 45. Third situation of context example eliciting division of whole numbers that does not have a whole number as a quotient

After students discussed the situation and problem in pairs, Ms. Shi asked a student to demonstrate the process of sharing the mooncake with a circular paper model:

> Ms. Shi: I would like to know how you evenly divided it. This is my mooncake [pointed to the circular paper model], how do you want to share it? Can anyone show it to us? [called a student' name].
>
> Student1: Split it into two parts.
>
> Ms. Shi: How to do it? This is yours. [Ms. Shi handed the circular paper model to the student]

The student folded the paper model from the middle. Ms. Shi continued to guide the student:

> Ms. Shi: I have a question now. I would like to ask you, how many mooncakes each bear can get right now?
>
> Student1: Half.
>
> Ms. Shi: Half. Do you all agree? [asked the class]
>
> Students answered together: Yes.
>
> Ms. Shi: Ok, go back to your seat please [talked to student 1]. Since he got half after dividing, we can also say that he got one half of the mooncake. Then can we use any integer that we have learned before to represent half?
>
> Students answered together: No.
>
> Ms. Shi: Well, we can not, right? Then today let's get to know the fractions.

Ms. Shi elicited the concept of fraction with a story-based contextual example at the beginning of the Lesson 1. Through the first two questions of sharing four mooncakes and two mooncakes, Ms. Shi evoked students' memory of the concept "evenly divided" that was used in division with students' awareness of being fair when sharing food. Then Ms. Shi applied it to the third question of how to evenly divide one mooncake.

When Ms. Shi asked a student to answer the third question, she let the student take the circular paper model as the mooncake, which gave the students an opportunity to make a connection with her real life experience. After the student answered that each bear would get half mooncake with the daily language used , Ms. Shi restated the answer as one-half to emphasize that half can be interpreted as a unit. And considering that 'one-half' cannot be represented with an integer anymore, it is essential to introduce fractions.

To introduce the fraction 1/2, Ms. Shi briefly summarized the process of how the student evenly divided the circular paper model and showed the students how to read and write the numeral '1/2' (see Figure 46).

> Ms. Shi: Using the method of evenly dividing, each person get one half of the mooncake, we can also use the daily language, half of the mooncake, right? Ok, then, how to write the fractions? Watch carefully! How many mooncakes do I have?
>
> Students answered together: One.
>
> Ms. Shi: How to divide it?
>
> Students answered together: Evenly divide.
>
> Ms. Shi: So we have a new number, do you know how to read it? [shown on the slide]

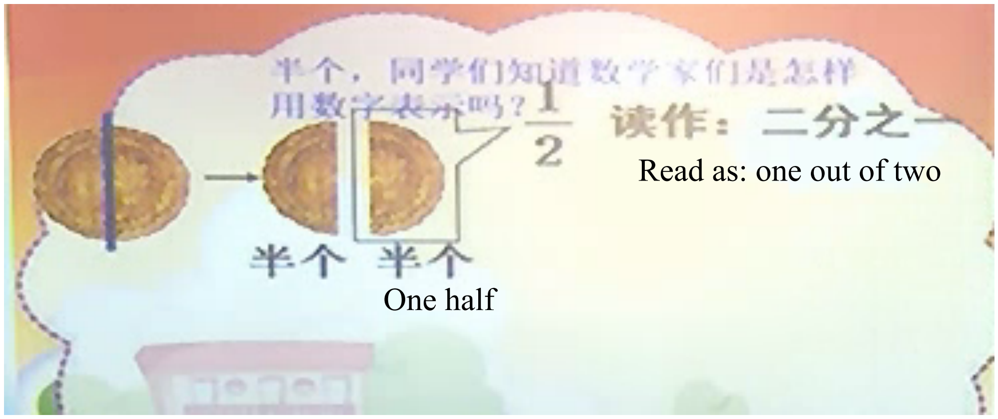

Figure 46. A slide showing how to read and write the numeral ‘1/2’

Finding 1/2 of the paper shape models. Ms. Shi asked the class to find 1/2 on a rectangular paper. Then she collected several students’ work and showed this to the class. Then she provided opportunities for understanding the process of identifying fractions and the meaning of each part of fraction, emphasizing the part-whole interpretation by repeating the process of getting 1/2 of rectangle:

> Ms. Shi: Then I will restate the process of getting fractions again, I evenly divide the rectangle paper into two parts. Any one part out of the two is out-out-of-two of the rectangle.

Ms. Shi also posed two questions for students to discuss:

1. Why are the areas that one-out-of-two represented in each model not the same?
2. Can we find one-out-of-two by folding the paper in different ways?

Generalizing other unit fractions. Ms. Shi asked students to evenly divide an object into any number of parts to create a unit fraction, then introduce their fractions with each other. Then Ms. Shi shared her own example, and let students present the fractions they created.

Student1: I evenly divided the circle into eighteen parts, each part is one-eighteenth of the circle.

Ms. Shi: Do you all agree?

Students answered together: Yes.

Lesson 2: comparing magnitude of unit fractions

With the part-whole interpretation of unit fractions covered in Lesson1, students' part-whole understanding of unit fractions turned to be the previous knowledge for the learning in Lesson 2 shown in the dotted-line rectangle in Figure 47. The arrows depict how the teachers provide ways for the students to achieve the desired new knowledge/understanding which in turn it is depicted in solid-line rectangles. Starting from the previous knowledge just mentioned, Ms. Shi provided opportunities for students to explore the magnitude comparison of unit fractions. The activities and models implemented Lesson 2 were organized in the following flow chart.

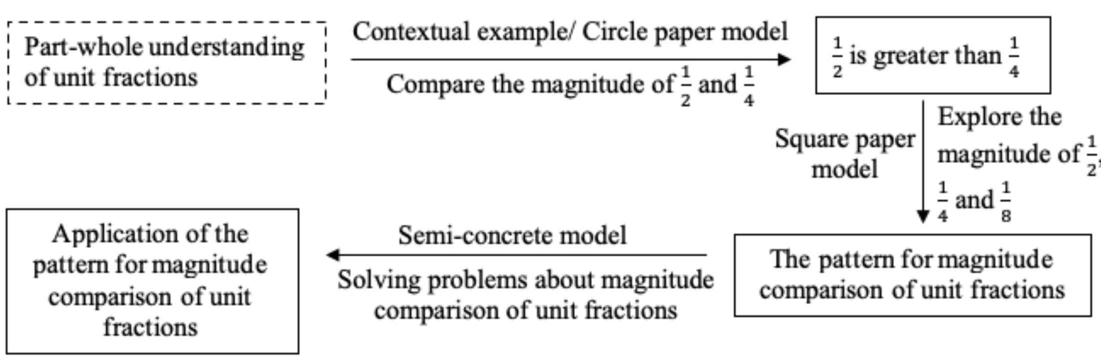

Figure 47. Flow chart for Lesson 2

Revisiting unit fractions. This refers to the dotted-line rectangle in the flow chart (Figure 47) At the beginning of Lesson 2, Ms. Shi took about eight and half minutes revisiting the knowledge of unit fractions covered in Lesson 1. She first led the class to recall the process of getting unit fractions and asked the students to interpret the several fractions shown on the slide: 1/2, 1/4, 1/7.

> Ms. Shi: Anyone want to try? [called a student's name]
>
> Student1: Evenly divide something into four parts, each part is one-fourth of it.
>
> Ms. Shi: Such a complete answer! Very good. maybe you can say evenly divide a shape.

Ms. Shi then showed the class some shapes on the slide and asked students to identify the fraction represented by the shaded area. Ms. Shi also presented the students with several statements and let the students find out the mistake in it. Given the statement that:

Divide a cake into 3 parts, each part is 1/3 of the cake.

Students pointed out that the statement is not correct, since it should say "evenly divide".

Comparing the magnitude of 1/2 and 1/4. This refers to the activity on the first arrow which direct from the rectangle of part whole understanding to the rectangle of 1/2 is greater than 1/4**.** Ms. Shi conducted the activity by showing students a contextual example about the story of Sun Wukong and Bajie, two well-known characters from one of the most famous Chinese classical novels "Journey to the West". While Sun Wukong was depicted to be smart and omnipotent, Bajie was considered to be a greedy and muddled figure. Based on that, Ms. Shi presented the contextual example:

Sun Wukong and Bajie were sharing a watermelon, Sun Wukong offered Bajie *one-out-of-two* of the watermelon. Bajie refused the offer and asked for *one-out-of-four* of the watermelon. But when he got it, he started to regret it. Why is that?

Ms. Shi went over the context with the students and asked the students using the square paper model to explore the reason why Bajie regretted taking 1/4 of the watermelon. Ms. Shi further addressed the question: Which one will you select for Bajie? 1/2 of the watermelon or 1/4 of the watermelon?

In the contextual example, Ms. Shi used a story of selecting a watermelon piece to elicit the topic of magnitude comparison. Based on the setting of the characters, Bajie would always select the bigger piece of the watermelon but he also made mistakes frequently. Bajie originally thought 1/4 is greater than 1/2, and that is why he regretted his decision once he got the watermelon.

Ms. Shi had students work as groups to explore the magnitude comparison of 1/2 and 1/4 using the square paper model. After that, Ms. Shi had students present their work in front of the class:

> Ms. Shi: Let's have a group present their work. [called a student's name], how about your group? Take your model. You can also have an assistant from your group.
>
> Student1: One-out-of-four is not greater than one-out-of-two. [student1 was standing on her seat]
>
> Ms. Shi: One-out-of-four is not greater than one-out-of-two. I feel that one-out-of-four is greater, four is greater than two.
>
> Student1: Since one-out-of-four is one divided into four small parts.

Ms. Shi: Ok, please demonstrate your idea to us. Anyone would like to show us?

Student2 from the same group stood up and came to the front of the classroom.

Student2: One-out-of-four is divided into four small squares, one-out-of-two is two big ones.

Ms. Shi: Any groups would like to add something? [a lot of students raised their hands]. Go back please (talk to student2), I understand what he means, but his expression is still not perfect, right? [called another student's name], your group. Come here to the front with your model. Go ahead.

Student2: Evenly divided the circle into four parts, each part is one-out-of-four, each part is pretty small.

Exploring strategies of magnitude comparison of unit fractions. Ms. Shi asked students to find 1/2, 1/4, 1/8 respectively on different square paper models and to explore the strategies for comparing magnitudes of unit fractions as groups. After students concluded that 1/2 > 1/4 > 1/8, Ms. Shi wrote the result on board (see Figure 48). And based on the result from the two activities, Ms. Shi asked students to find a strategy of magnitude comparison of unit fractions as groups.

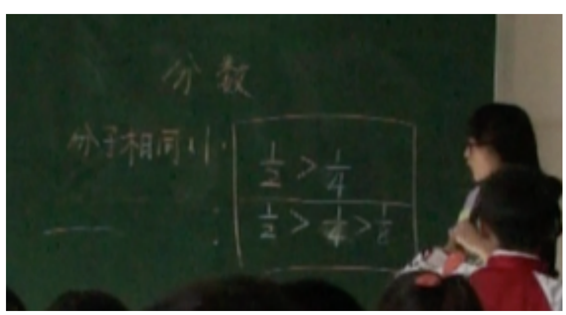

Figure 48. Ms. Shi listed the result of magnitude comparison on the board

Ms. Shi: This group stated that for fractions with numerator one, the greater the denominator, the greater the denominator, what does that represent? Divided into more or less parts?

Students answered together: More parts.

Ms. Shi: For instance, the fraction presented by [a student's name] has the denominator eight, which means it was evenly divided into… eight parts. What do you think about the eight parts? More or less?

Students answered together: More.

Ms. Shi: Therefore, each part, on the contrary, is smaller. Do you all agree?

Students answered together: Yes.

Lesson 3: non-unit fractions

While the theme of the first two lessons is the understanding of unit fractions, Lesson 3 presented the students with the non-unit fractions. Starting from students' actual development level developed in the first two lessons, Ms. Shi implemented several activities to introduce the meaning of non-unit fractions, which is shown in the flow chart (see Figure 49).

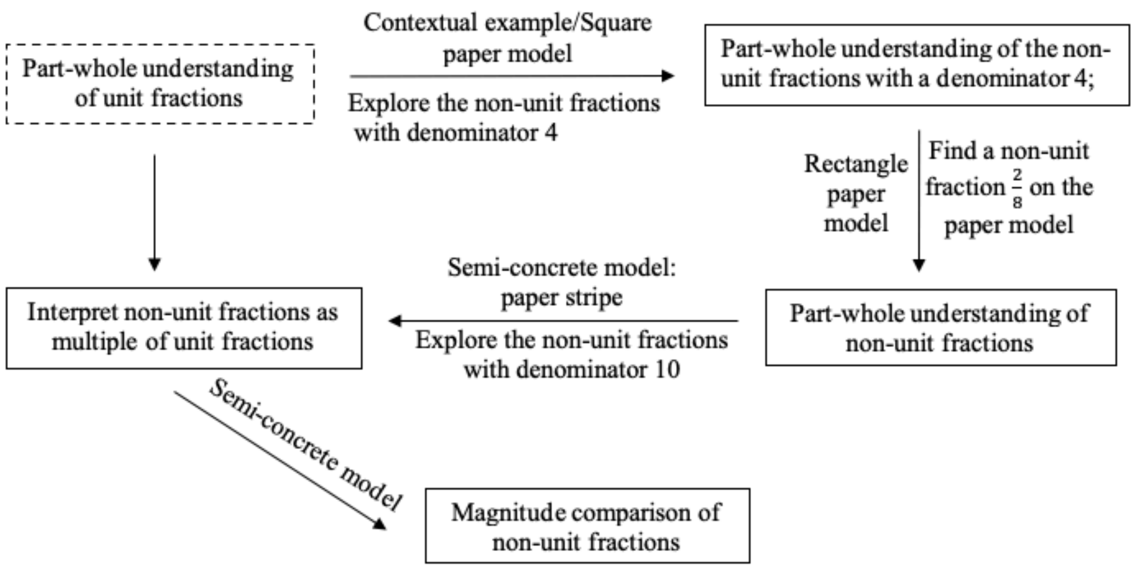

Figure 49. Flow chart for Lesson 3

Revisiting unit fractions. Ms. Shi started the lesson with a summary of the fractions learned in Lesson 1.

Ms. Shi: The fractions we learned have a common feature. [called a student's name]?

Student1: They are all one.

Ms. Shi: What do you mean by that?

Student1: Their numerators are all one.

Ms. Shi: Yes. Very good. Their numerators are all one. These kinds of fractions are classified as fractions with numerator 1(unit fractions).

Exploring the part-whole interpretation of non-unit fractions. Ms. Shi asked students to evenly divide a square paper model into four parts. Then she guided the student to generate fractions with denominator 4 by asking students to shade different numbers of parts as they were growing different plants on the ground:

Ms. Shi: Then next, let's play a game using this shape. I think a lot of you have heard about the game happy farm, did you?

Students answered together: Yes.

Ms. Shi: The game is about growing…

Students answered together: Vegetables.

Ms. Shi: Yes, vegetables. Right now, please grow two parts of strawberries on your square paper. How many parts are you going to grow?

Students answered together: Two.

Ms. Shi: You can plan it in any ways you want. The squares you have are different, you can do anything you like, drawing or shading, as long as you get two parts.

After students finished their drawing, Ms. Shi collected several students' work and projected them on board. Ms. Shi went over each of the students' work and checked if two parts were highlighted in each model. Then she asked students to discuss with each other about which fraction can be used to represent the two parts in the model and write the fraction on their square paper.

Figure 50. Students' work projected on board

Ms. Shi: I evenly divided the ground into four parts, and I take two parts to grow strawberry, it reminded me of a fraction, the fractions is…? Let's read the fraction together.

Students answered together: Two-out-of-four.

Ms. Shi: I would like to know the meaning of the fraction. What is the meaning of the fractions? Evenly divide into…?

Students answered together: Four parts.

Ms. Shi: How many parts do we take from them?

Students answered together: Two parts.

Ms. Shi: Do you still remember the name of this line?

Students answered together: Fraction bar.

Ms. Shi: What does it represent?

Students answered together: Even division.

Ms. Shi: How many parts?

Students answered together: Two parts…four parts.

Ms. Shi: This is the denominator. It was divided into four parts and take two parts out of them.

Exploring non-unit fractions as multiple of unit fractions. Ms. Shi showed the class a colored paper strip evenly divided into ten parts, then she asks students to identify the fractions with a certain number of parts selected.

Ms. Shi first guided the class to interpret the fraction as part-whole:

Ms. Shi: How many parts are there?

Students answered together: Ten parts.

Ms. Shi: Now I am going to select two parts, which fraction can you use to represent that?

Students answered together: Two-out-of-ten.

Ms. Shi: Then I select five parts.

Students answered together: Five-out-of-ten.

Then she guided students to interpret the non-unit fractions as multiple of unit fractions and using the unit fraction as a unit of measurement.

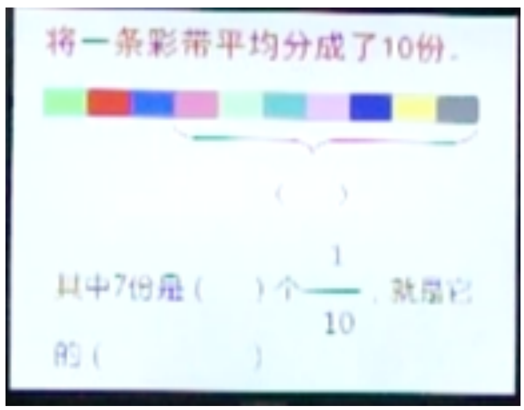

Figure 51. A projected slide showing the model of paper stripe

Ms. Shi: I will continue. One part is…?

Students answered together: One-out-of-ten.

Ms. Shi: Yes, that's what we learned last time, unit fraction. Ok, then I will randomly take (some parts of the paper stripe), tell me how many parts I've taken.

Students answered together: Five parts.

Ms. Shi: How many one-out-of-ten then?

Students answered together: Five.

Ms. Shi: Which fraction do we use to represent it?

Students answered together: Five-out-of-ten.

Ms. Shi: Five-out-of-ten. Five-out-of-ten will become five-tenths. Alright, I will continue to ask. I still evenly divided it into ten parts, how many parts did I take? [pointing to the slide]

Students answered together: Seven.

Ms. Shi: How many one-tenths? Which fraction can you think of?

Students answered together: Seven-out-of-ten.

Ms. Shi: Very good.

Comparing the magnitude of fractions with equal denominators. By interpreting 2/5 as two one-out-of-five, 3/5 as three one-out-of-five, along with a diagram, Ms. Shi asked the students to determine the magnitude of 2/5 and 3/5.

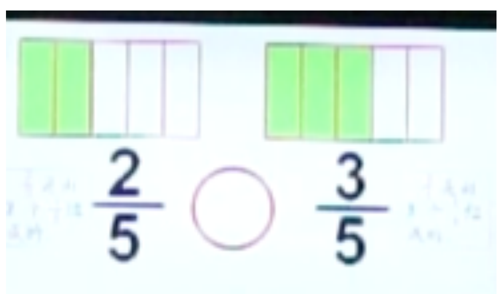

Figure 52. A diagram showing the magnitude of the two fractions

Lesson 4: simple computations of fractions (addition and subtraction of fractions with equal denominators)

Ms. Shi introduced the notion of fractions as a part-whole, including unit and non-unit fractions and their comparison during the first three lessons. Next, Ms. Shi introduced addition and subtraction of fractions with equal denominator in Lesson 4 (see

Figure 53 for flow chart). The interpretation of non-unit fractions as multiple unit fractions shown in dotted-line rectangle served as a starting point. Before introducing the notion of addition of fractions (solid-line rectangle in the lower right corner), the teacher introduced 1 as a fraction by engaging students in an exploration (solid arrow) and evoking previous learning experiences with addition in application problems (dotted line rectangle) The flowchart below displays the process of knowledge building along with the contextual examples and models used.

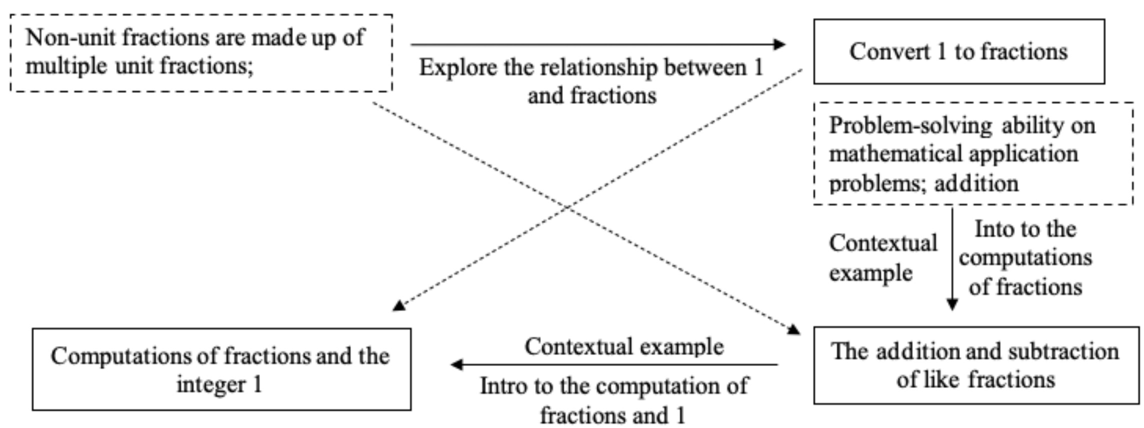

Figure 53. Flow chart for Lesson 4

Revisiting non-unit fractions. This section corresponds to description of the use of previous knowledge identified in the flow chart (Figure 53) by the dotted line rectangle in the upper left corner. Ms. Shi had students recall what they learned in the Lesson 1, by letting students identify the fractions represented by the shaded area (Figure 54) and interpret the non-unit fractions as a multiple of the unit fractions (Figure 55), they also talked about the relationship between 1 and fractions that have equal denominator and numerator.

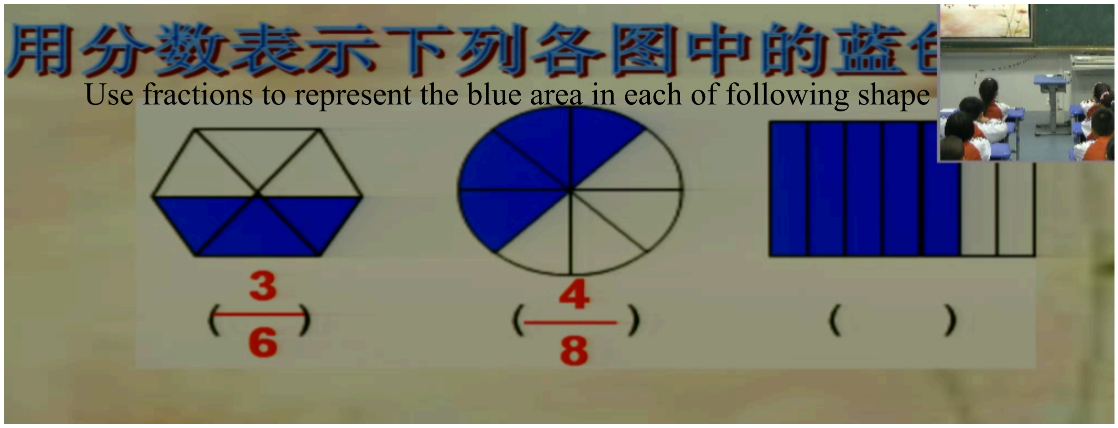

Figure 54. Problems shown to the students

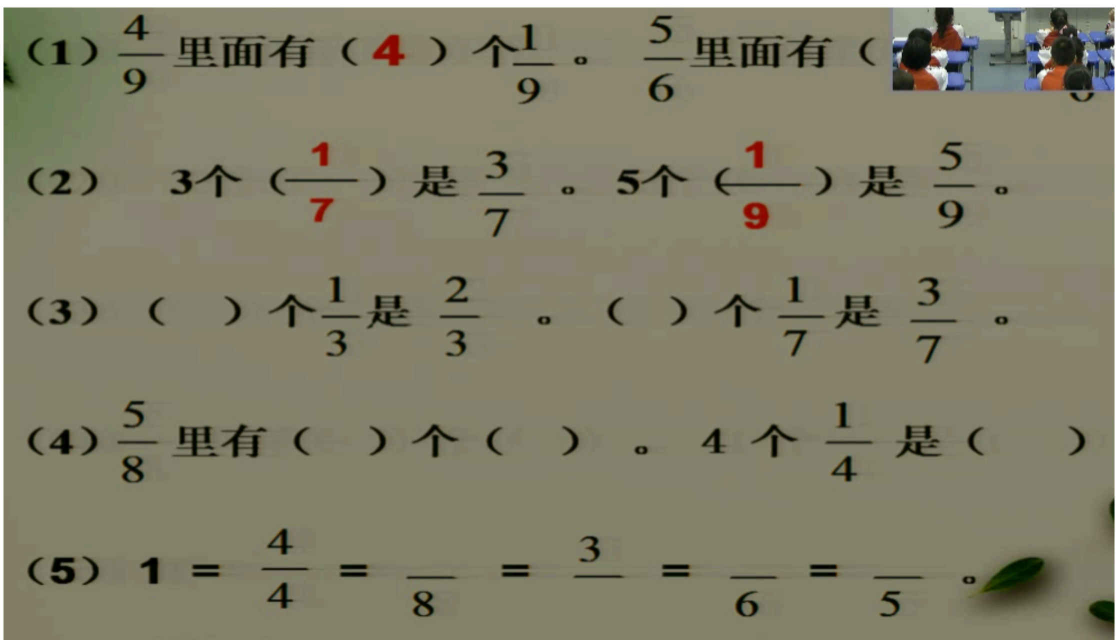

Figure 55. Practicing problems shown to the students

Exploring addition and subtraction of fractions. This section corresponds to the description of the way the teacher made the connection between the previous content and the notion of operating with fractions shown in the flow chart (Figure 53) by a solid arrow between rectangles. Ms. Shi showed students a slide of two children eating watermelon (see Figure 56 below). Ms. Shi asked students to create and pose questions using the mathematical information given in the context.

Figure 56. The context of brothers eating watermelon

Based on the context shown in the figure and after XX minutes, the students posed two questions: 1) how much watermelon do the brothers eat altogether? 2) how much watermelon does the older brother eat more than the younger brother?

With the previous knowledge on the addition and subtraction of whole numbers, the students then write the mathematical sentences based on the two questions raised: 1/8 + 2/8 = ; 2/8 – 1/8 = .

To perform the operations, Ms. Shi guided a student to apply his understanding of fractions as multiple of unit fractions to make sense of the addition of 1/8 and 2/8:

Ms. Shi: The thing is, I am curious about how [a student's name] did the calculation. How did you get three-out-of-eight?

Student1: Because one watermelon is evenly divided into eight parts, the older brother eats two parts, the younger brother eats one part.

Ms. Shi: They both eat watermelon from the eight parts. He eats (Ms. Shi points to 2/8)…

Student1: Two parts.

Ms. Shi: He eats (Ms. Shi points to 1/8)

Student1: One part.

Ms. Shi: How many one-out-of-eight he (the older brother) actually eats? ((Ms. Shi points to 2/8)

Student1: Two.

Ms. Shi: How many one-out-of-eight he (the younger brother) actually eats? ((Ms. Shi points to 1/8)

Student1: One.

Ms. Shi: Two eighths and one eighth altogether should be… how many eighths?

Student1: Three.

Ms. Shi: Three one-out-of-eights remind us the fraction…?

Student1: Three-out-of-eight.

Exploring the computation of fractions and 1. Ms. Shi asked students to consider another question: how much watermelon is left after the brothers eat their share. To answer the question, the students came up with sentence 8/8 – 3/8 and operated as 5/8.

Ms. Shi: I said the brothers eat three-out-of-eight of one watermelon and how much is left. Many students write eight-out-of-eight minus three-out-of-eight. Why?

Students answered together: Because the two brothers eat three-out-of-eight in total.

Ms. Shi: Then how did you get eight-out-of-eight

Students answered together: One watermelon was evenly divided into eight parts.

Ms. Shi: Ok, one watermelon was evenly divided into eight parts, one watermelon contains eight parts. eight-out-of-eight minus three-out-of-eight, by applying the method you concluded, which will stay the same?

Students answered together: The denominator stays the same.

Ms. Shi: Then?

Students answered together: Doing subtraction for the numerators.

Ms. Shi: Can you give me the result directly?

Students answered together: Five-out-of-eight.

Ms. Shi then summarized the strategy of rewriting 1 as a fraction that used in the computation of fractions involving 1:

Ms. Shi: When we face problems like this, check this out carefully. (Ms. Shi shows students the slide)

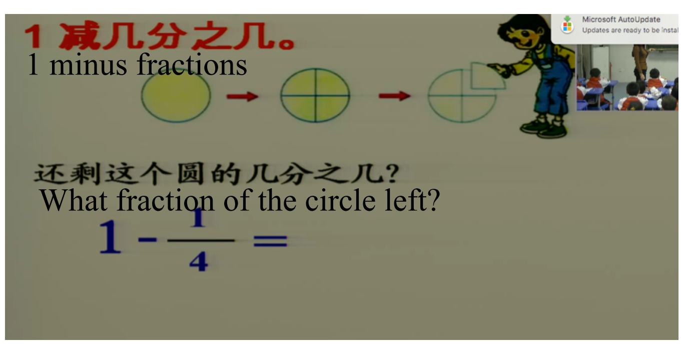

Figure 57. Introduction of the problem solving technique

Ms. Shi: Just like this, one subtracts one-out-of-four, what do you want to turn one into?

Students answered together: Four-out-of-four.

Lesson 5: simple application of fractions

In Lesson 5, Ms. Shi extended the meaning of whole in the part-whole understanding of fractions from one object into multiple objects. Starting with the activity of identifying the fractions represented by shaded area in one object, Ms. Shi demonstrated the shift of the whole with a series of activities and models used.

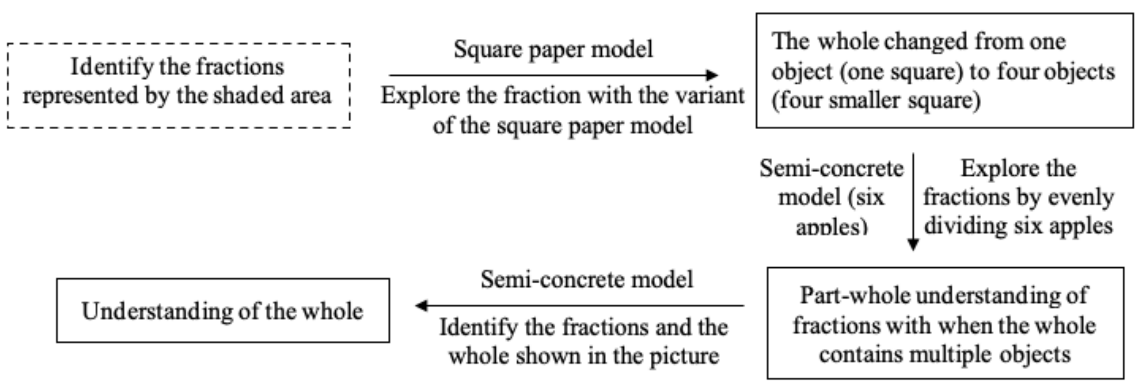

Figure 58. Flow chart for Lesson 5

Revisiting non-unit fractions. Ms. Shi asked students to recall what they had learned so far about fractions (dotted line rectangle in Figure 58). Then Ms. Shi showed students a slide of a square model and asked students to identify the shaded area with a fraction (solid arrow pointing to the next rectangle).

Explore the fractions with the variant of the square paper model. With the instructional goal of transitioning between a continuous and a discrete model (solid rectangle in the upper right corner), Ms. Shi cut a physical square paper model into four parts and guided the students to explore the change of the whole from one continuous area to four discrete objects (solid arrow pointing to the next rectangle). She did this by questioning:

Ms. Shi: What has been changed to the square shape after I cut it? Anyone would like to talk about this? Any changes? Mao Quan?

Mao Quan: It was divided into four parts.

Ms. Shi: What was divided into four parts?

Mao Quan: The square.

Ms. Shi: What is the original one?

Mao Quan: It was one, big one. Now it turns into four small ones.

Ms. Shi: Then I still have the same question, which fraction can represent the shaded small square out of the four squares.

Mao Quan: It does not change.

Ms. Shi: There is no change. Then what is the fraction?

Mao Quan: It is still one-fourth.

Ms. Shi: Still one-fourth. Do you all agree?

Students answered together: Yes.

Explore the fractions by evenly dividing six apples. With the instructional goal of understanding part-whole interpretation of fraction in a discrete setting (solid line rectangle in the lower right corner), Ms. Shi showed the class a picture of six apples, which were evenly divided into three parts (solid arrow between the two right rectangles). Then Ms. Shi asked students to find the fraction to represent one part out of the three parts, two part out three and also how to find the number that represents one-out-of-three of the set of six apples and two-out-of-three of the six apples.

Identify the fractions and the whole shown in the picture. Ms. Shi showed the class with several practicing problems on the slide and let students identify the whole for each shape.

Lesson 6: simple application of fractions continued

While Lesson 5 introduced the accessibility of having multiple objects being the whole (discrete model) when discussing the part-whole understanding of fractions, Lesson 6 focused on the application of the knowledge covered in Lesson 5 along with the previous knowledge on division. The flow chart below showed the development of application problem solving skills.

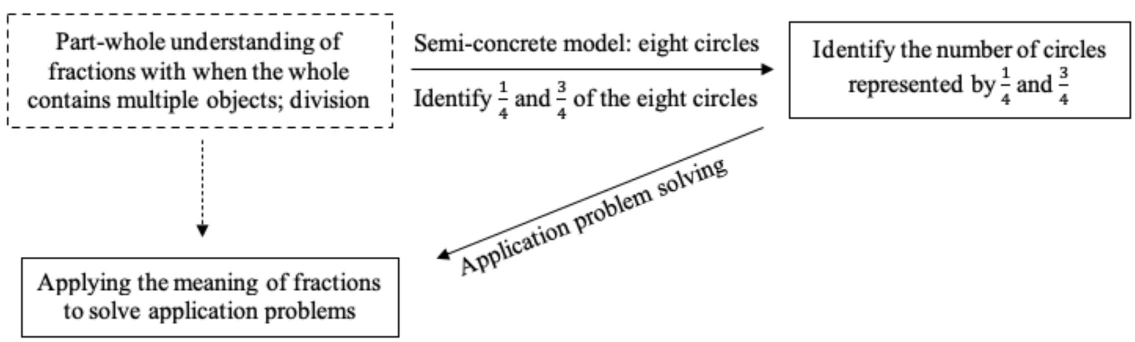

Figure 59. Flow chart for Lesson 6

Revisiting content of previous lessons. Ms. Shi revisited the fractions learned in the last lesson and compared it with the fractions learned earlier.

Identifying $\frac{1}{4}$ and $\frac{3}{4}$ of the set of eight circles. Ms. Shi let the students draw eight circles on their papers and find 1/4 and 3/4 of the set of eight circles. In the meantime, Ms. Shi had two students represent 1/4 and 3/4 on board.

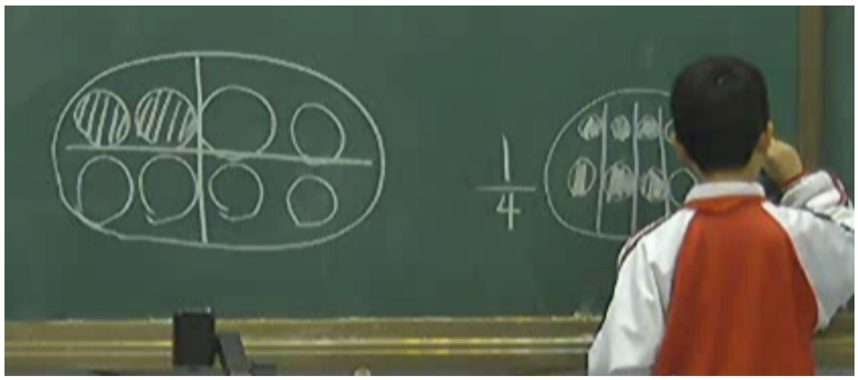

Figure 60. A student representing 1/4 of a set of eight circles on the board

After the students finished, Ms. Shi asked the students to find the number of circles that represent 1/4 and 3/4 of the set of eight circles. Also, she asked the students to justify their answer by showing a mathematical sentence.

Ms. Shi: Check this out, they all used different ways to divide the eight circles. Will the number of circles in one part be the same?

Students answered together: Same.

Ms. Shi: Same. Then how many?

Students answered together: Two.

Ms. Shi: How do we use mathematical sentences to get that? What should be the mathematical sentence? Anyone want to talk about that? [a student raised her hand and Ms. Shi called her name].

Student1: Two times four is eight. Eight divided by four is two.

Ms. Shi wrote down the $8 \div 4 = 2$ on board.

Ms. Shi: How did you come up with the mathematical sentence that eight divided by four is two?

Student1: Because there are eight circles, evenly divided into four parts, each part has two circles.

Application problem solving. After working with a discrete model of part-whole interpretation of fractions, Ms. Shi presented the class with the following application problems:

There are 12 mushrooms, 3/4 of them is ( ) mushroom? What is 3/4 of the 12 mushrooms?

There are 12 students playing games, while 1/3 out of them are female, 2/3 of them are male. How many male and female students are there?

Conceptual coherence between each lesson

Ms. Shi delivered four main topics through six lessons: Lesson 1 and Lesson 2 on the topic of unit fractions; Lesson 3 non-unit fractions; Lesson 4 simple computation of fractions (addition and subtraction of fractions with equal denominator); Lesson 5 and Lesson 6 simple application of fractions. Based on the observation, Ms. Shi started each lesson from the knowledge covered in the previous lessons, which provided the evidence for the conceptual coherence between each lesson, which is shown in the flowchart below.

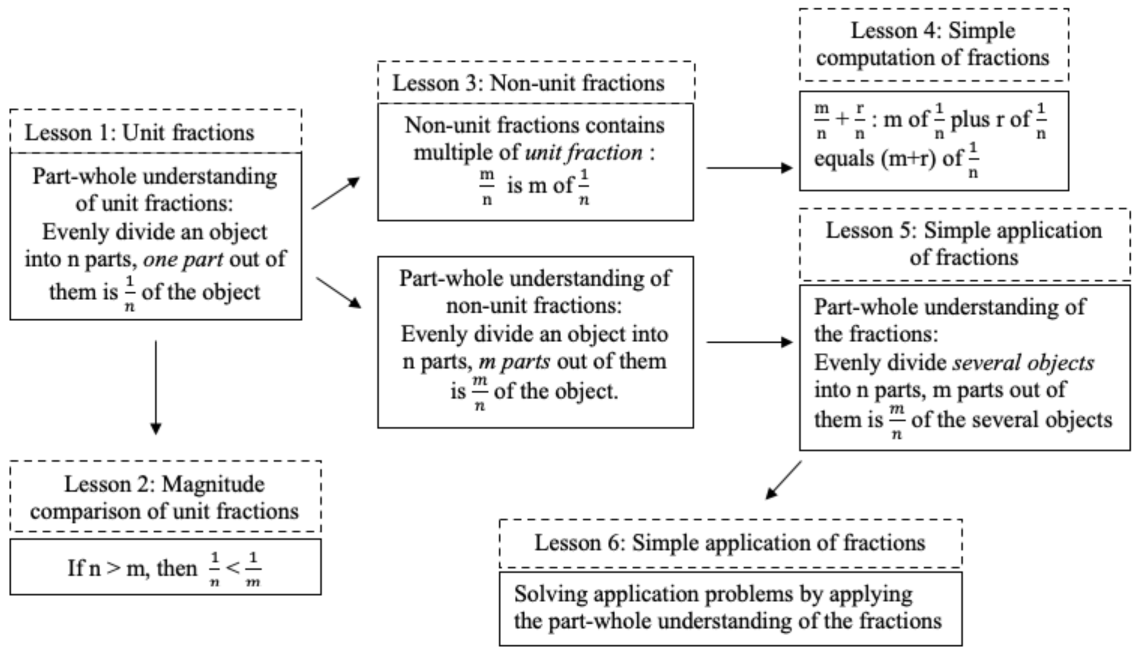

Figure 61. Flow chart among each lesson

From Lesson 1 to Lesson 2. Lesson 1 and Lesson 2 shared the same topic on unit fractions. While Lesson 1 introduced the part-whole understanding of unit fractions, Lesson 2 applied the knowledge of unit fractions covered in Lesson 1 to explore the magnitude comparison of unit fractions.

From Lesson 1 to Lesson 3 to Lesson 4. In Lesson 3, Ms. Shi introduced two ways to understand non-unit fractions, and the interpretation of non-unit fractions as multiple of unit fractions was applied to Lesson 4.

In Lesson 3, the concept of unit fractions covered in Lesson 1 was used as a unit to measure the non-unit fractions. Then in Lesson 4, when Ms. Shi guided the students to explore the addition of 1/8 and 2/8, she interpreted 1/8 as one one-out-of-eight, 2/8 as two one-out-of-eight, then 1/8 plus 2/8 is three one-out-of-eight, which is 3/8 or three-out-of-eight.

From Lesson 1 to Lesson 3 to Lesson 5. Another interpretation of the fractions that are embedded in almost all the five lessons is the part-whole understanding of fractions. In Lesson 1, Ms. Shi introduced the part-whole understanding of unit fractions, building on that, Ms. Shi had students use the same paper model but highlight more than one part out of four parts to construct the part-whole understanding of non-unit fractions. In Lesson 5, Ms. Shi continued building the part-whole understanding of fractions by shifting the whole from one object as a whole (area model) to a set of objects as a whole (discrete model).

From Lesson 5 to Lesson 6. Lesson 5 and Lesson 6 contributed to the same topic of the application of fractions. Lesson 5 introduced that the whole could be multiple objects, then Lesson 6 presented a type of application problems that need to be solved using the part-whole understanding of fractions when the whole is multiple objects.

Procedural coherence

In this study, procedural coherence is defined as the routine of the lesson as reflected in the structure of the lesson. In the procedural coherence within each lesson and among each lesson.

Procedural coherence within each lesson

The examination of procedural coherence within each lesson is based on the enactment of the class activities shown in the following Table 9 (Chen & Li, 2010).

Table 9.

Class Activity Instruction of Ms. Shi

Class activity	Specification
Preparation for the class	Teachers make command before class start. For example: put your pencil into your pencil box.
Review	Teacher go over the previous knowledge. It usually starts with ”let us think about what we talked about in yesterday’s lesson”.
Lecturing	Teachers present to the class.
Group discussion	Students discuss as groups.
Students’ individual work	Students work on the task by themselves.
Student presenting	Students answer questions standing on their seats or in front of the class.
Summary	Teachers repeat or summarize on the statements made by the students or teachers summarize the knowledge. For example: today, we talked about fractions. Let us summarize the knowledge we have learned today.

By examining the implementation of class activities in each lesson, Ms. Shi started each lesson with a review of what was learned the day before, and frequently led the class to summarize the main ideas learned at the end. In the middle of each lesson, besides presenting material to the whole class, Ms. Shi had students work individually, discuss as groups, and present their work or ideas.

Review. Ms. Shi began each lesson revisiting familiar concepts and ideas from the lesson the day before, and the time spent on the review and the content of the review is shown in Table 10. For example, the review of Lesson 2 was about the material of Lesson 1.

Table 10.

The Time Ms. Shi Spent on Review for Each Lesson

Review of	Lesson 2	Lesson 3	Lesson 4	Lesson 5	Lesson 6
Time spent	8min20sec	6min33sec	5min	3min30sec	50sec
Content reviewed	Lesson 1	Lesson 1	Lesson 3	Lesson 3	Lesson 5

As shown in the Table 10, for every lesson except for Lesson 1, Ms. Shi revisited the knowledge that was learned in the previous lessons that relates directly to the goal of the lesson, however the time spent varied from lesson to lesson decreasing as the unit was implemented, from 8 minutes and 20 seconds in Lesson 2 to 50 seconds in Lesson 6. On average, the amount of time spent revisiting the familiar content was 4min50sec. A possible explanation for this decreasing pattern is that as the unit unfolds, each lesson is more connected to the previous and therefore less time is needed to revisit the previous material. The content that was reviewed was related to the new knowledge, as analyzed in the flow charts of the lessons (Figure 61). For example, the topic of Lesson 3 non-unit fractions builds on the concept of unit fractions, yet not directly related to the magnitude comparison of unit fractions, which may be an explanation of why Ms. Shi did not revisited the comparison of fractions learned in Lesson 2 at the beginning of Lesson 3, but still revisited the content of lesson1.

Class discussion. For most of the time, Ms. Shi moved the lesson forward through class discussion. Ms. Shi would interact with the students by asking questions that required short answers. She frequently questioned students for the important words to engage students:

> Ms. Shi: look closely, how many mooncakes?
>
> Students answered together: One.
>
> Ms. Shi: How to divide?
>
> Students answered together: Evenly divided.
>
> Ms. Shi: After I divided the mooncake, I got its…
>
> Students answered together: half.

Ms. Shi: So, I created a new number, do you all know how to read it?

Students answered together: One out of two.

Other activities. Ms. Shi also provided students with opportunities to think independently and interact with their peers. Based on the observation, Ms. Shi would organize *group discussion* either in groups of four or groups of two after she posed a question. When exploring the model such as finding 1/2 of a paper model, Ms. Shi would have s*tudents working individually.* After the individual work or group discussion, Ms. Shi would also have students *presenting* their work or their thoughts about the question.

Summary. It was observed that, at the end of each lesson, Ms. Shi would summarize the knowledge learned, often by posing guiding questions that required short answers.

Ms. Shi: Alright, let's recall what we learned in this lesson. What did we get to know?

Students answered together: Fractions.

Ms. Shi: How did we get fractions? We have to make sure…?

Students answered together: Evenly divide.

Ms. Shi: Very good. It has to be evenly divided. Based on that, the number of parts you have divided into determines the unit fractions. Also, do you still remember how to read and write fractions? You should be fine with the reading. It is important to remember when you write fractions what you should write first, what is the line that represents 'evenly divide' called?

Students answered together: Fraction bar.

Procedural coherence between each lesson

To examine the procedural coherence between each lesson, I classified each class as new-material instruction class and practicing classes. The goal of the new-material instructional class is to impart new material, and the goal of the practicing class is to review previous knowledge by practicing. As suggested by the teaching guide (人民教育出版社课程教材研究所小学数学课程教材研究开发中心, 2012), it took eight lessons to implement the unit of introduction of fractions. The sequence of the eight lessons is shown in the Figure 62.

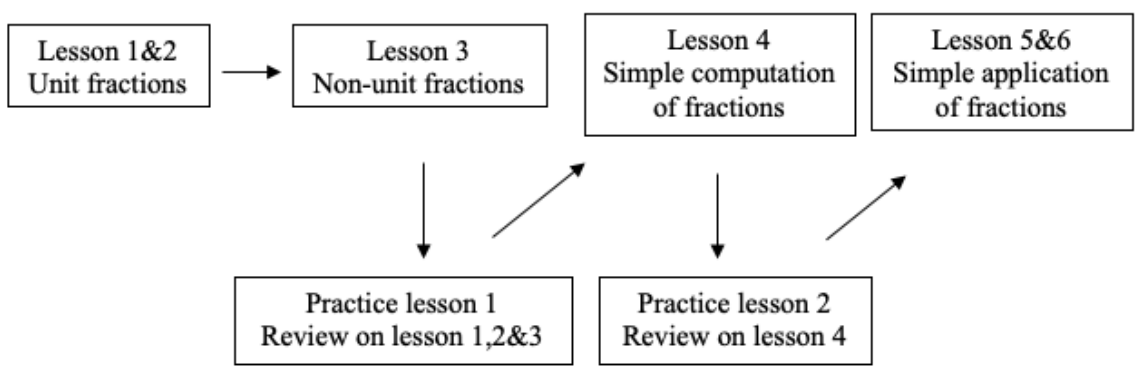

Figure 62. The flow chart of the eight lessons

Practicing lessons. As shown in the Figure 62, two practice lessons were embedded in the six main lessons. In the practicing Lesson 1, which was conducted after the Lesson 3, Ms. Shi revisited the material learned in the first three lessons. For the practice Lesson 1, Ms. Shi provided students with some practice problems and had students try the problem by themselves and then discuss together as a class. The goal of practice Lesson 1, addressed by Ms. Shi in the interview, is to consolidate students' knowledge on the part-whole interpretation of fractions and the magnitude comparison of fractions. The second practice lesson followed Lesson 4 on simple computation of fractions. While Lesson 4 focused on introducing and exploring the operations, the objective of practice Lesson 2 was to reinforce students' procedural understanding of the algorithms of adding and subtracting fractions with equal denominators.

Between Case Result

Instructional resources

Wang and Murphy (2004) stated the Chinese teachers in their studies were exposed to a supportive environment with plenty of instructional resources available for them to enhance their instructional coherence. They assert that, besides the physical materials including the curriculum standard, textbook, teacher's guide, the teachers are also able to gain access to other resources such as assistance from the leading teachers and professional development. However, this particular study did not provide evidence on how the resources support teachers' instructional coherence and although China has a centralized system of education and their conclusion may generalize to all teachers, there is variation in the use and access of instructional resources across the country. The evidence presented below relate to use of instruction resources by the three teachers in my study in order to support their goal of instructional coherence.

Curriculum standard

Mathematics curriculum standards in the period of Chinese compulsory education is created by the Ministry of Education of the People's Republic of China. It not only serves as a guide for the textbook developers, but also provides the teachers with principles and ideas for their teaching. The curriculum standard emphasizes the concept of mathematical way of thinking which is referenced by all the three teachers when they were interviewed. The curriculum standard also advocates for the development of the students' core accomplishment of ten aspects in mathematics including number sense, symbol awareness, spatial concept and so on. Ms. Shi expressed her teaching philosophy that turned to be consistent with the curriculum standard:

> Nowadays, we are encouraged to develop students' core accomplishments. My teaching not only focuses on the knowledge level, but also teaches students the mathematical way of thinking and mathematical methods. We guide students to transfer and analogize the knowledge.

Textbook

As the most frequently used instructional resources (Ding, Li, Li & Gu, 2012), textbooks were observed to be used by both teachers and the students during the class. The topic of introduction of fractions is first presented in the textbook (grade 3 volume one) from pages 90 to page 103. On those pages, three main parts (sections) were identified: Part 1: Preliminary understanding of fractions; Part 2: Simple computations of fractions; Part 3: Simple applications of fractions.

Table 11.

The Distribution of the Content in the Textbook

Section	Part 1 Preliminary understanding of fractions			Part 2 Simple computation of fractions		Part 3 Simple application of fraction	
Content	Unit fraction	Non-unit fraction	Practice	Addition and subtraction of like fractions	Practice	Simple application of fraction	Practice
Page	P90-91	P92-93	P94-95	P96-97	P98-99	P100-101	P102-103

The content in the textbook

The magnitude comparison of fractions. Although the magnitude comparison of fractions is usually perceived as one topic, it was split into two parts in the textbook; the magnitude comparison of unit fractions and fractions with equal denominators. Consistent with the textbook, all the three teachers were observed to deliver the two parts in two different lessons. When introducing the unit fractions, the teachers implemented the activity of comparing the magnitude of unit fractions which builds on the interpretation of part-whole of unit fractions. In Lesson 2 of non-unit fractions, with a focus on the changed numerators, the teacher showed the students the magnitude comparison of fractions with equal denominators.

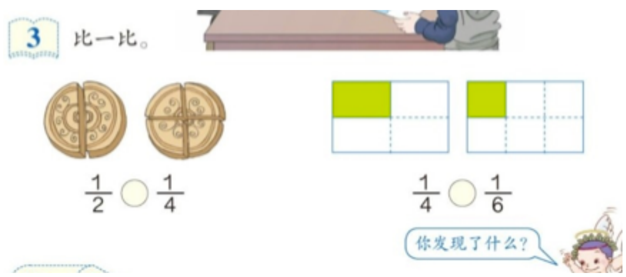

Figure 63. The activity of magnitude comparison of unit fractions in the textbook

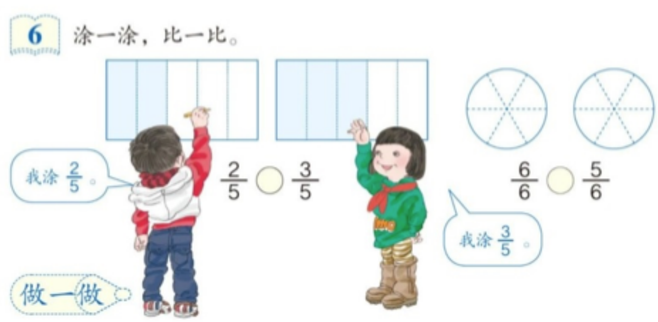

Figure 64. The activity of magnitude comparison of fractions with equal denominator in the textbook

Figure 65. The activity of finding non-unit fractions from the textbook

Simple computations of fractions. The textbook introduces the simple computation of fractions with the contextual example of eating a watermelon, which was used in all three teachers' instruction. To interpret the addition of 2/8 and 1/8, the textbook presents the process of the addition of 2/8 and 1/8 with a circular model, it also interprets the addition as two eighths plus one eighth, which is three eighths, or 3/8.

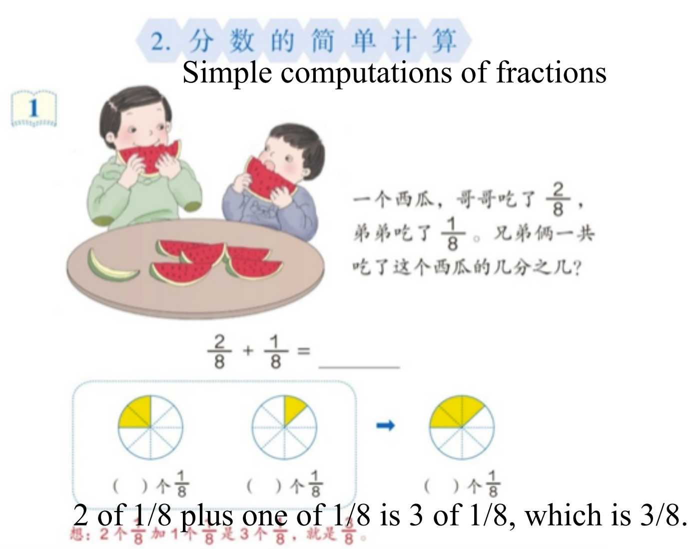

Figure 66. The activity of exploring addition of fractions from the textbook

Simple applications of fractions. In the section on simple applications of fractions, the textbook presented two examples. The first example is about exploring the fraction that represent the shaded area before and after the square model is cut, this activity was implemented by Ms. Shi and Ms. Yang to show the change of the whole from one object to multiple objects.

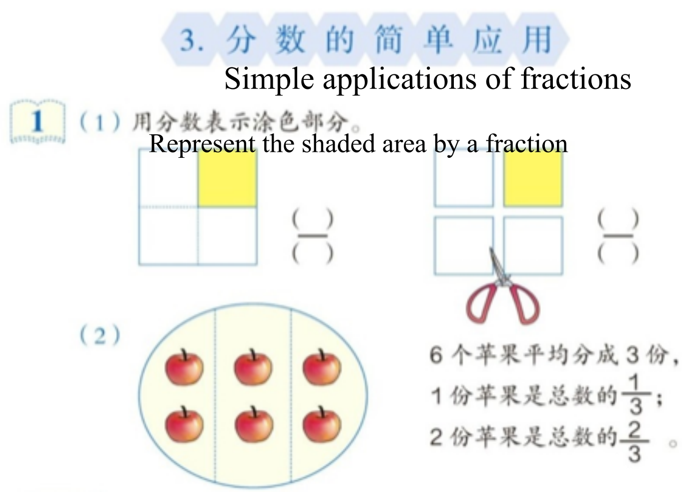

Figure 67. The activity of shifting the square paper model in the textbook

Summary. The three teachers' instruction demonstrated a strong fidelity to the textbook. It explained why the three teachers used similar activities and followed a similar path in their teaching.

Teacher's teaching guide

Another instructional resource available for teachers is the teachers' teaching guide, which is aligned with the textbook that the teachers used. As indicated by Wang & Murphy (2004), a detailed teacher's teaching guide plays an essential role in supporting teachers in preparing a coherent lesson.

The first part of the teaching guide generally illustrates the textbook (grade 3 volume one) in terms of its content and teaching objectives. In order to give the teachers a better understanding of the textbook, the first part of the teaching guide addresses the characteristics of the textbook from the perspective of the editor. It also suggests some manipulatives or models that may be used in the teaching as well as the number of lessons that teachers could spend on each topic.

The second part of the teaching guide is the detailed illustration for each unit of the textbook. It presents an explicit explanation of the textbook, page by page, regarding the intent of each activity and practicing problem. This is followed by the activity's intention. The teaching advice of how to facilitate the activities along with the practicing problems are given at the bottom of the page as shown in the figure.

编写意图 Activity intention

（1）通过两名学生平均分月饼的情境，引出分数的认识。小精灵话中的“平均分”和“它的”是关键词，明确指出了分数概念的基本要素。

（2）把一块月饼平均分成四份，通过类比推理，认识$\frac{1}{4}$。

（3）在实物模型的基础上，借助面积模型——圆和长方形，认识$\frac{1}{3}$、$\frac{1}{5}$。

（4）通过上述活动积累一定具体认识后，说明“像……这样的数，都是分数”。

（5）以$\frac{1}{3}$为例介绍分数各部分的名称，同时指导如何正确地读、写分数。

几分之一

1

这块月饼我们一人一半。

把一块月饼平均分成2份，每份是这块月饼的一半，也就是它的二分之一，写作$\frac{1}{2}$。

把一块月饼平均分成4份，每份是它的（ ）分之一，写作$\frac{1}{(\ )}$。

把一个圆平均分成3份，每份是它的（ ）分之（ ），写作$\frac{(\ )}{(\ )}$。

把一张长方形纸平均分成5份。指出它的五分之一，并涂上颜色。

像$\frac{1}{2}$、$\frac{1}{3}$、$\frac{1}{4}$、$\frac{1}{5}$这样的数，都是分数。

1 ……分子
—……分数线　读作：三分之一
3 ……分母

90

教学建议 Teaching advice

（1）丰富表象支撑，初步理解分数含义。

引出两个小朋友平均分月饼的问题后，借助学生已有的每人得到“半块”的生活经验，在如何表示“半块”的过程中引出分数，帮助他们理解一人一半就是每人分得这块月饼的$\frac{1}{2}$。随后，可以让学生想一想、说一说、折一折，体会$\frac{1}{2}$不仅可以表示半块月饼，还可以表示许多事物的“一半”，感受数学模型的作用。用学习$\frac{1}{2}$的经验做依托，可根据教材提供的直观图让学生用分数表示，通过画一画、折一折等活动使学生形成正确的表象，体会分数的具体含义。注意强调只有平均分才能得到它的几分之一，体会分的是“谁”就是“谁”的几分之一。

（2）联系具体例子，体会分数含义。

注意联系前面学习的具体例子，使学生明确$\frac{1}{2}$、$\frac{1}{3}$、$\frac{1}{4}$、$\frac{1}{5}$都表示把一个物体或图形平均分成几份，有这样的1份。认识分母、分子也要联系上面的例子，使学生弄清把一个物体或图形平均分成了几份，分母就是几，表示这样的1份，分子就是1。

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Figure 68. One page of the teaching guide of textbook illustration

Other resources

Other resources such as the assistant from the leading teachers, communication among peers, and professional development such as teaching competition.

Cross-Case Analysis

As suggested by the teaching guide, all three teachers implemented eight lessons to cover the knowledge on the introduction of fractions. Ms. Shi and Ms. Yang, from the same elementary school, shared the same lesson plan. This differed slightly from Ms. Liu, as noted in the figure below.

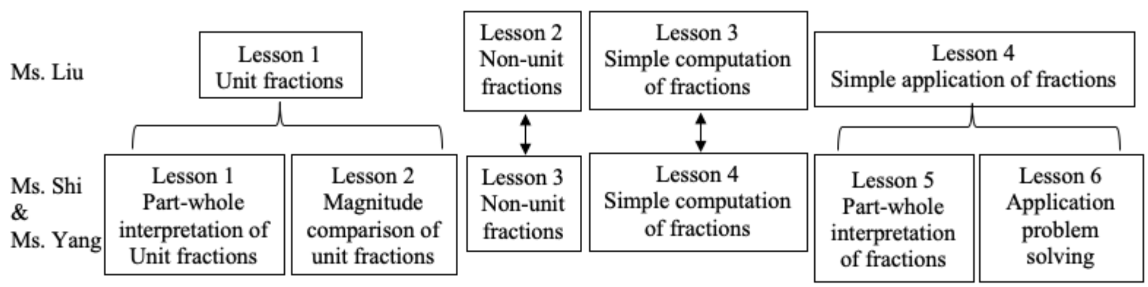

Figure 69. Comparison of the sequence of lessons delivered by teachers

As shown in figure 69, Ms. Liu delivered four lessons to introduce the new knowledge, with each lesson lasting about 40 to 45 minutes. In comparison, Ms. Shi and Ms. Yang had six lessons on the new knowledge implementation, and each lesson lasted about 35 to 40 minutes. Guided by the same instructional resources, including the textbook and teaching guide, all three teachers followed a similar path on the sequence of lesson implementation. They both start with an introduction to unit fractions, then non-unit fractions, following the simple computation of fractions and simple application of fractions. Yet, Ms. Liu used one lesson to cover the topic of unit fractions; in comparison, Ms. Yang and Ms. Shi split the two learning goals on the subject of unit fractions into two lessons. Similarly, Ms. Shi and Ms. Yang distributed the content covered in Ms.

Liu's Lesson 4 into Lesson 5 and Lesson 6. Other than that, Ms. Liu's Lesson 2 and Lesson 3 aligned with Ms. Yang and Ms. Shi' s Lesson 3 and Lesson 4.

From the observation, Ms. Liu categorized the eight lessons into four new knowledge implementation lessons and four practice lessons. At the same time, Ms. Shi and Ms. Yang embedded the practice and review into each lesson, with only two lessons specifically focused on the review and practice of the previous knowledge. Also, Ms. Shi and Ms. Yang took more time on the exploration on specific topics than Ms. Liu, for example, they had students explore the magnitude comparison of unit fraction using one entire lesson in Lesson 2.

5. DISCUSSION

This study is a detailed examination of instructional coherence cases of three grade 3 Chinese teachers from the region of Shandong on the topic of introduction to fractions. This study is guided by the following two research questions:

1. How does different teachers' instruction display coherence in the classroom when teaching the unit on the topic of introducing fractions? In particular,
    a. with respect to conceptual instructional coherence; and
    b. with respect to procedural instructional coherence.
2. How does the teachers' use of different instructional resources influence the instructional coherence in their instruction?

To answer the two research questions, I analyzed the data collected from Donger and Shiyi elementary schools in Shandong, China. Data sources include eight videotaped lessons for each teacher, semi-structured interviews, and instructional materials artifacts. With the findings presented in Chapter 4, this chapter presents the discussion of the findings.

Discussion of Findings

Instructional coherence

In this study, instructional coherence is defined as a characteristic of instruction that manifests while instructors present class activities in a particular sequence using well-designed discourse, which purposefully indicates the progressive and consistent relationships within mathematical knowledge (Wang & Murphy, 2004). Specifically, this study focuses on two aspects of instructional coherence: conceptual coherence and procedural coherence. While conceptual coherence is the connection among the mathematics themes, procedural coherence is procedural connections reflected by the routines of the teachers' teaching (Chen & Li, 2010).

Informed by the theoretical framework of this study, zone of proximal development (ZPD) learning theory (Vygotsky, 1978), learning happens through students' interaction with peers and instructors. Furthermore, the theory asserts that students' actual development level based on the students' previous knowledge and the way teachers guide the students from their actual development level to the level of potential development depicts the manifestation of the instructional coherence construct used in this study.

The role of the teachers in their classroom.

Based on the ZPD learning theory, the process of guiding students from their known state to unknown provides evidence of the level of instructional coherence in the teachers' teaching (Vygotsky, 1978). Thus, instances where students were given opportunities to actively learn, think, and communicate within and between lessons were the main focus of analysis for this study.

According to the Chinese elementary mathematics curriculum standard, which is mandated by the government as a national guide for all the Chinese mathematics educators, the role of the teacher is viewed as an organizer and facilitator (John, 2007) to create a student-centered environment in the classroom (中华人民共和国教育部制定, 2011). Therefore, as teachers were observed, instances of teachers playing this role were to some extent accounted as a possible explanation of the opportunities given to the students to interact and actively learn from each other.

Teachers' role as an organizer. Based on my analysis of the observations described in Chapter 4, all three teachers acted as organizers by assigning the students with different tasks that aligned well with the intended curriculum's proposed tasks (人民教育出版社课程教材研究所小学数学课程教材研究开发中心, 2012) and by providing the students with the opportunity to explore the concepts within each activity. Ms. Liu, as an experienced teacher with 21 years of experience teaching at the elementary level, showed her steady and calm manner during teaching. She barely presented any facts or comments, but instead, provided students with opportunities to explore the concepts by themselves through the activities or had different students express their ideas so that the students could learn through their communication with each other. In her class, there seemed to be a culture where students were comfortable expressing their ideas and commenting on others' responses. Often, students were able to adjust their ideas based on the information provided by their peers.

Similar to Ms. Liu, Ms. Yang and Ms. Shi played organizer roles in their teaching as well. They both offered opportunities for students to work individually, discuss in small groups, present their work or ideas in front of their classes. Yet, compared to Ms. Liu, Ms. Shi and Ms. Yang would provide more information to the students or sometimes have students observe her working on the models. Overall, the three teachers, to some degree, cultivated students' communication abilities, including the use of mathematical terms and having the students present their ideas using concise and complete mathematical language. This observed level of organizational skills with respect to trying to create a student-center environment provides some evidence that these classrooms have, to some extent, the necessary conditions for developmental learning from the ZDP perspective (Vygotsky, 1978) and that it provides evidence of procedural instructional coherence which is discussed in more detailed later in the chapter.

Teachers' role as facilitator. The three teachers also played the role of a facilitator in the classroom by posing questions as students worked on and presented activities, scaffolding discussion, and setting up exploration tasks. However, the enactment of these pedagogical activities varied among the three teachers. For example, to make the contextual example more closely related to the students, Ms. Yang assigned a student in the class a leading role. Ms. Shi used cartoon characters and a story-based contextual example to draw interest from students. Ms. Liu brought food to the classroom to make contextual examples more realistic. Although it is not clear the degree of effectiveness of these different approaches in terms of facilitating learning, they all gave students opportunities for engagement and motivation, and that in turn created a path in students' zone of proximal development and hence evidence of an intended instructional coherence from the part of the teachers which discussed in more detailed in the following section of conceptual coherence.

In terms of posing questions, Ms. Liu seemed to use questioning with the purpose of deepening students' understanding. In contrast, Ms. Yang and Ms. Shi used questioning to keep students focused since their questions required short answers in the form of completing sentences.

As they expressed it during their interviews, teachers held the belief that the goal of their teaching is to develop students' mathematical way of thinking by providing the appropriate guidance. However, they had different implementation techniques that impacted the level of students' interactions and therefore the level of guidance.

Conceptual coherence

As presented in Chapter 3, the examination of conceptual coherence is attending to the content of the lessons. Specifically, the goal is to explore the interrelationship of each topic and how the teachers were guiding students to build their knowledge on prior knowledge. Informed by the theoretical framework of ZPD (Vygotsky, 1978), the analytic framework for conceptual coherence is shown in Figure 70 and Figure 71.

As presented in Figure 70, each lesson could be seen as a string of several episodes, the learning goal of each episode will finally contribute to the learning goal of the lesson. By exploring the goal of each activity and the previous knowledge the teachers assumed students to know, I explored the connections that teachers made between prior knowledge and new knowledge, in addition, examine how each episode is devoted to the theme of the lesson.

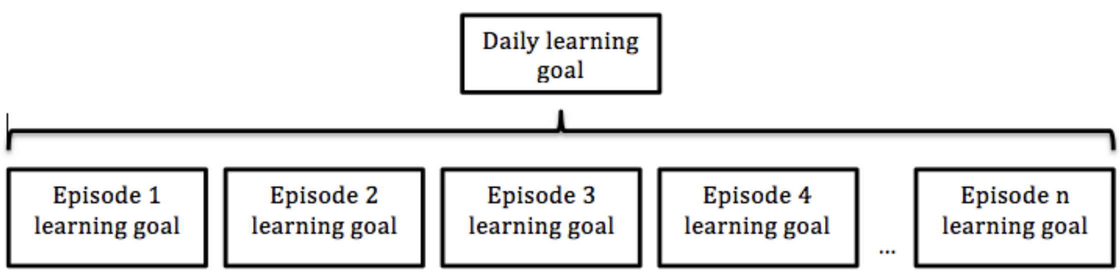

Figure 70. The process of knowledge building within a lesson

To explore the conceptual coherence among each lesson, it was necessary to discern the content covered in each class and how they are connected. The analysis focused on the shift of the themes in a sequence of classes to see how the topics of each lesson transits between those lessons.

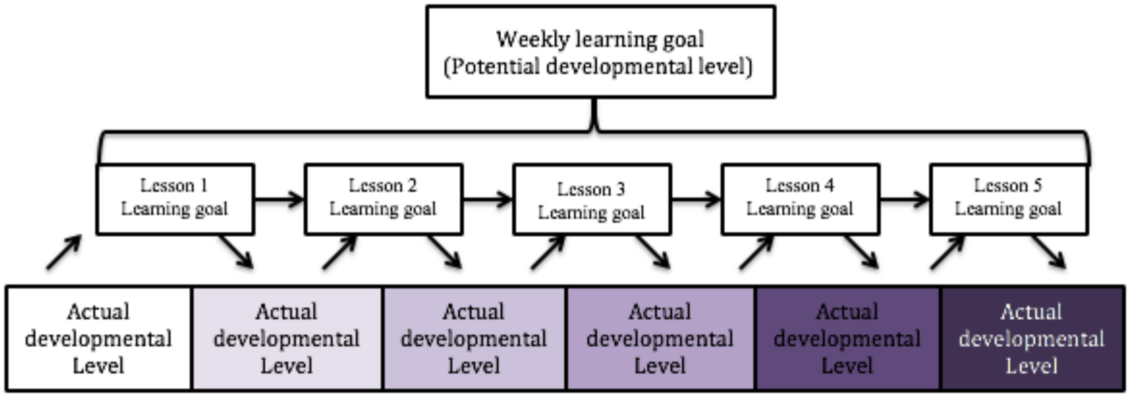

Figure 71. The process of knowledge building among lesson

The use of students' previous knowledge

Previous knowledge is a starting point that is necessary for students to learn, and it also determines the way the teachers guide students to their further learning (Stigler et al., 2010). For example, Ms. Yang addressed the critical role of students' solid background knowledge, which may contribute to the smooth flow when imparting new knowledge. As the analysis of the lessons such as the simple computation of fractions illustrates, the previous knowledge was not limited to the students' conceptual and procedural understanding of specific mathematical content, it also referred to the student's ability to deal with mathematics, in other words, the mathematical way of thinking.

The previous knowledge applied in the learning of fractions included the understanding of the binary operations of whole numbers such as division, addition, and subtraction. All three teachers solicited students' previous knowledge of division with whole numbers to make the connection with the key concept of dividing evenly or equal shares. Also, when introducing the computation of fractions, the teachers presented the topic to the students using a contextual example. This instructional decision gave opportunities for students to build their problem-solving skills through the teachers' guide.

Vygotsky (1978) notes that the knowledge learned in one day would turn to be the previous knowledge for the next day based on the zone of proximal development (ZPD), which emphasizes the strong relationship between the knowledge states. There is evidence that all three teachers in this study provided opportunities to achieve the connection between the knowledge in their lessons, which formed a virtuous circle. On one hand, teachers used developed knowledge as a starting point that gives opportunities to students to access the new knowledge. On the other hand, in the learning process, teachers referred to the knowledge learned in the previous lessons, which gave opportunities to consolidate students' previous knowledge. However, the effectiveness of these linkages remains less clear with respect to achieving instructional coherence, as discussed later in the chapter.

The instructional tools used by the teachers.

All three teachers used instructional tools such as concrete (paper), semi-concrete (diagrams) models, and contextual examples to connect students' previous knowledge and the new knowledge. Furthermore, these instructional tools seemed to provide an ideal space for posing questions, creating discussions, and engaging students' active learning (by folding the papers to create fractions).

The use of models. The models used by the three teachers included the concrete and semi-concrete models. Models consisted mainly of papers cut in regular polygons (for area models), strip paper models (for number line models), and the semi-concrete models were the pictorial form of the concrete models. Not only did the use of the models provide opportunities for connections between numerical and graphical representations, but it also served as a pedagogical tool to achieve conceptual coherence.

In many instances (see Chapter 4 for details), the concrete models used by the teachers were aligned with the semi-concrete models. For example, right after the exploration of non-unit fractions with the concrete paper strip model, the paper strip model turned toward a pictorial model shown in Figure 72.

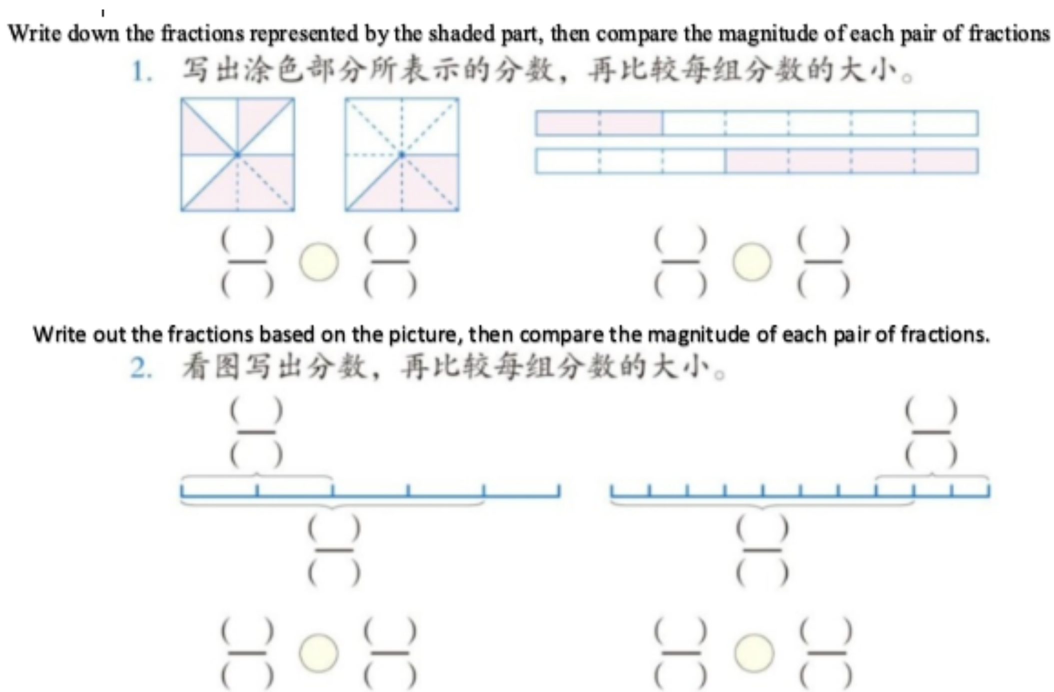

Figure 72. The problems with the semi-concrete models used from the textbook

The use of models following the sequence of concrete to pictorial to abstract (CPA) is an instructional strategy that has been shown to benefit students' learning of fractions (Purwadi, Sudiarta, & Suparta, 2019). Also, the activities presented in the textbook in each topic were in the sequence of CPA. In the three teachers' instruction, concrete models (paper) were used to explore the meaning of the fractions or the computation, while the pictorial models were commonly used when first introducing the practicing problems. After the students explored problem-solving strategies, such as the rule for the subtraction and addition of like fractions, the practicing problems were more abstract without the support of representative models. For example, when teaching the topic of magnitude comparison of unit fractions, Ms. Liu gave opportunities for students to explore the unit fractions with a concrete model (paper shape model) and had students compare the magnitude of unit fractions with the assistance of a pictorial model. Then Ms. Liu asked the students to explore the strategies for magnitude comparison of unit fractions. With a strategy explored, the students were expected to solve problems with abstract mathematics. Ms. Yang and Ms. Shi followed a similar path as well.

The three teachers were observed using similar models, such as the paper shape model, in most of their lessons. Through this process, students were given the opportunity to become familiar with the use of the models, which may influence the manifestation of the conceptual coherence of the lessons. For instance, Ms. Liu asked the students to explore the unit fraction ¼ with a square paper model in Lesson 1. In the next lesson, Ms. Liu asked the students to explore the non-unit fractions with denominator 4 using the same square paper model, which provided an opportunity for a learning process between the two lessons to be more efficient and connected. Ms. Liu's and other teachers' use of models could be traced back to their beliefs in teaching. For example, in her interview, Ms. Liu mentioned that the "use of models is one way to embody the mathematical way of thinking." From that, I gathered that the goal of education from the teachers' view is not only to help students understand the knowledge with the use of models but also to cultivate students with the idea and method about the use of models in solving mathematics problems.

The use of contextual examples. The contextual approach is a teaching method commonly used in the mathematics classroom, by which the teacher engages students with the real-life context, and that the students could make sense of mathematics using their life experience (Johnson, 2002). Informed by Selvianiresa & Prabawanto (2017), the contextual approach could also strengthen students' ability to make mathematical connections.

From the observation of the three teachers' teaching, all of them used a contextual example of sharing food when introducing the fraction one-out-of-two. Although the examples used by the three teachers were not exactly the same, they all provided familiar contexts to students from evenly dividing four objects, two objects, and even one object. As addressed by the teaching guide that accompanies the curriculum (人民教育出版社课程教材研究所小学数学课程教材研究开发中心, 2012), the purpose of the contextual example is to evoke students' life experience of sharing food and emphasize that the result after evenly dividing may not always result in an integer.

Figure 73. The example of introducing ½ in the textbook

All three teachers used the contextual example provided in the textbook of two brothers eating watermelon to elicit the operations of addition and subtraction of fractions. When implementing this example, students had the opportunity to consider the addition and subtraction of fractions under a real-life context and potentially relate the operations to their real-life.

Figure 74. The example of introducing one-out-of-two in the textbook

Only Ms. Shi used two other contextual examples when she introduced the magnitude comparison of unit fractions and the introduction of non-unit fractions. Ms. Shi's preference for contextual examples reveals that she paid a lot of attention to students' interests and their engagement as her contextual examples were story-based questions and cartoon characters that were popular among her students.

Instances with not enough evidence for conceptual coherence

While all the three teachers' teaching manifested the conceptual coherence in most of their teaching, there were also instances where there was not enough evidence to claim conceptual coherence.

The confusion made by the use of the models. When Ms. Liu guided the students to explore the addition of 1/8 and 2/8, she asked the students to take two circular paper models as the watermelon and respectively represent 1/8 on and 2/8 on the two circles. It went pretty well at the beginning of the activity since the two fractions 1/8 and 2/8 are clearly shown on two circles, but when the students added the two fractions together, it caused confusion about the reference unit for the addition problem. Based on the model, some students came up with the answer of 3/16, and the students who got the answer 3/8 also had a hard time explaining their reasoning since they feel like the denominator should not be added, but there are two models which is indeed 16 parts. In the students' discussion, Ms. Liu immediately noticed the students' confusion. She had different students explain their ideas and warned the students that there is still one watermelon in eight parts. She also provided the students with another way to interpret the addition, which is to interpret the fractions as multiple unit fractions.

In Ms. Shi's class, the students did not explore addition using the model. Instead, she led the students in applying the meaning of non-unit fractions as multiples of unit fractions to interpret the addition of 1/8 and 2/8 as one eighth plus two eighths.

Ms. Yang used the same paper model, but she asked the students to use only one circular paper model and represented both of the two fractions on one model. The students could, clearly see that when adding the two fractions, the denominator stayed the same since there is only one watermelon and avoided the kind of confusion Ms. Liu had.

The textbook provided two ways to interpret the addition of 2/8 and 1/8; the use of the model and the use of the meaning of the fractions as multiple of unit fractions. As suggested by the teaching guide (人民教育出版社课程教材研究所小学数学课程教材研究开发中心, 2012), teachers can help students understand the addition of fractions by using the meaning of the fractions with the assistance of the models. This suggestion served to combine the part-whole interpretation and the interpretation of fraction as multiple of unit fractions. As shown in Figure 75, the textbook used two models to represent the two different fractions, which may explain why Ms. Liu asked the students to use two circles. It is worth noticing that in the second example with the subtraction between 5/6 and 2/6, only one rectangle model was displayed in the picture. The model clearly shows the process of how to take away the two sixths from five-sixths.

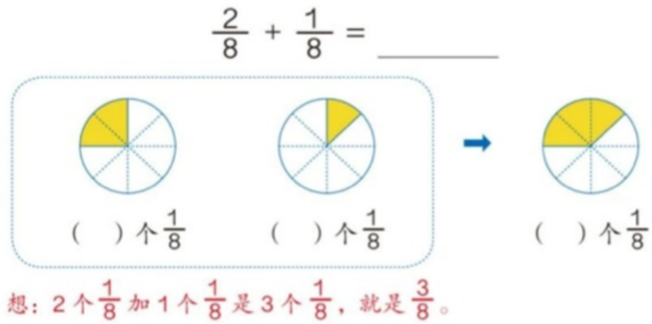

Figure 75. The interpretation of addition of fractions shown in the textbook

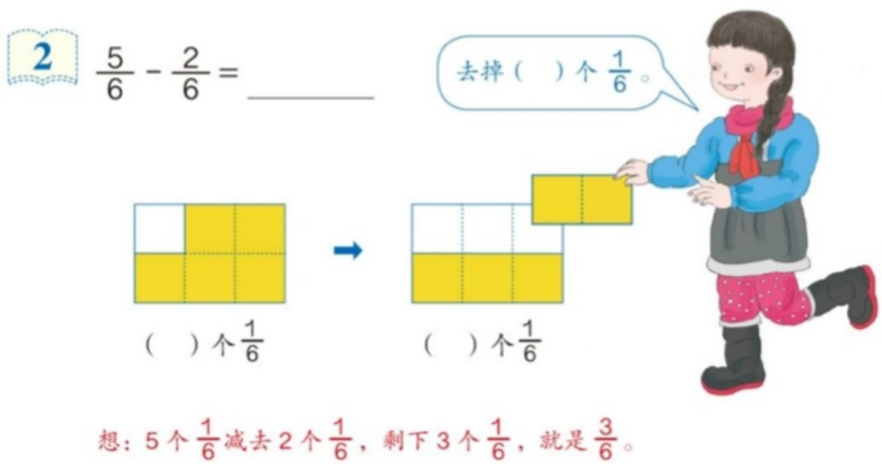

Figure 76. The interpretation of subtraction of fractions shown in the textbook

It is clear that the transition of instructional tools to models suggested by the curriculum materials may lead to a lesson that is not coherent with respect to representations and ultimately to the mathematical concepts. From the observations, the teachers seemed to be aware of this limitation and provided an opportunity for discussion and resolution of confusion. However, it is unclear whether the confusion was resolved since there was no explicit observation of students' final understanding.

Disconnection to the students' previous knowledge. When introducing the topic of a simple application of fractions, Ms. Liu started the lesson by showing the class a picture of a set of six apples, which was evenly divided into three equal subsets and asked students to represent one subset by a fraction. Students were familiar with how to identify fractions represented in the area models, but they had never identified fractions from a set of discrete objects. This unfamiliarity was apparent as the class fell silent after Ms. Liu posed the question. After several answers were presented by the students, it became apparent from my observation that the students were not prepared to answer this question. As I observed the lesson and positioned myself as a student, I felt confused because of disconnection to the previous knowledge. However, with Ms. Liu's guidance and with different students' input, the class gradually realized that the number of parts that the whole was divided into, and not the number of objects, determined the fraction.

Before showing a picture of the set of six apples, both Ms. Yang and Ms. Shi implemented an activity in their lessons to help make a connection between students' previous knowledge of the unit fractions to the new knowledge. The activity was presented in the textbook as shown in Figure 77. With the implementation of the first activity, students were able to apply their previous knowledge of identifying the fractions in the square paper model and naturally discovering the change of the whole from one object to a composition of multiple objects. It turned out that Ms. Liu did not apply the activity presented in the textbook, which may be an explanation of the gap in the students' learning.

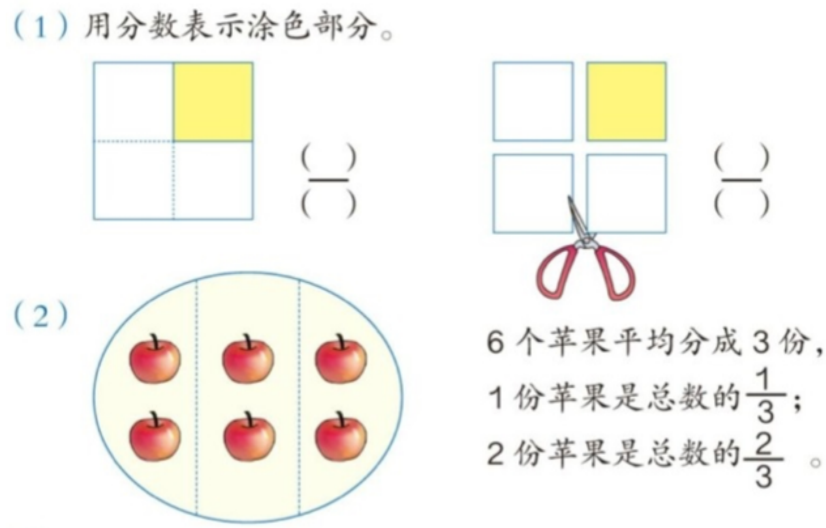

Figure 77. The activities in the textbook on the topic of a simple application of fractions

The confusion made by a question raised. At the end of Ms. Yang's Lesson 2, right after the class had finished a practice problem of comparing the magnitude of 1/4 and 1/5 on a line segment, she posed a problem asking if a line segment is 20 cm, how long is 1/5 of the line segment. Students appeared fine with the question and came up with the answer 4. Ms. Yang then tried to make a connection to the multiplication of fractions. She gave students the opportunity to think about how to apply multiplication to find 1/5 of 20. Students appeared to get confused and came up with answers such as 20/100 and 80/20. Then the class ended. This could have been an impromptu idea by Ms. Yang to guide students, and it could be seen as a tryout. Although that did not work out, the students appeared to enjoy the thinking process and actively sharing their answers. Also Ms. Yang pointed out to the students that this was intended as an extension for the students to think about, even if students became confused, but it would not influence their understanding of the knowledge that they were learning. Since the knowledge of multiplication of fractions will be covered in grade 5, it would be a challenge to guide students from their knowledge of unit fractions to the higher-level knowledge in such a short time in grade 3.

Procedural coherence

In this study, procedural coherence is defined as the class routine that was reflected in the structure of the lesson (Chen& Li, 2010). By examining the pattern of enactment of class activities in each class, I identified the general routine of each class reflected by the sequence of the class activities and specified the role of the structure of the lesson. For the procedural coherence among each lesson, I classify each class as new-material instruction class and practicing classes.

Getting ready to learn

Before the instruction started, the three teachers always communicated with their students as a routine. The teachers took this time to get the students ready to learn. They would ask the students to get their paper model ready, put away any materials that they won't need so that the students would not be distracted by the messy desk and could immediately reach whatever they needed in the lesson. The teachers would also suggest the students take off their thick coats in the warm classrooms to ensure their physical comfort Although the class preparation was not explicitly about materials related to learning mathematics, it played an essential part in getting the students into a learning mode and ensuring the lesson could be conducted efficiently and with minimal interruptions.

The use of the review

Previous research (Chen & Li, 2010; Mok, 2013) demonstrated the use of review and summary as characteristics of teachers' procedural coherence in Chinese mathematics teachers' teaching. In this study, the three teachers were all observed using the review and summary as part of trying to construct a coherent class.

The three teachers were observed starting each lesson with a review. It is worth noticing that the content reviewed at the beginning of each class was helpful for teachers in introducing the new knowledge or knowledge needed later. For example, at the beginning of the lesson on the topic of non-unit fractions, All the three teachers reviewed the part-whole understanding of unit-fractions that was learned in the previous lesson. Yet they did not go over the comparison of unit fractions that was not closely related to the learning of non-unit fractions even though that was also learned in the previous lesson.

The use of summary

At the end of each lesson, the three teachers would usually organize the class to summarize the knowledge learned on that day. In addition, all the teachers also provided brief summaries within the lessons. For example, after the introduction of non-unit fractions, Ms. Yang and Ms. Shi both guided students to a conclusion about the difference between non-unit fractions and unit fractions that they had learned in the previous lesson.

Procedural coherence among each lesson. Although all three teachers delivered eight lessons on the unit of introduction of fractions as suggested by the teaching guide, it was observed that the lesson plan followed by Ms. Shi and Ms. Yang was slightly different from Ms. Liu. While Ms. Shi and Ms. Yang mixed the practicing lesson with the new content lesson, Ms. Liu made a clearer distinction between new content lesson and practicing lessons.

The use of the instructional resources

The textbook and teaching guide. The contrast of the teachers' instruction as enacted and the primary instructional resources the teachers used, namely the textbook and teaching guide, revealed the three teachers' high fidelity to the textbook. This is not surprising since the curriculum is mandatory. This could be one of the explanations as to why the three teachers shared a lot of commonality in their design of lessons. It appeared that the coherence manifested in the teachers' instruction reflected the textbook and suggestions by the teaching guide. With an understanding of the intentions of the activities in the textbook, most teachers achieved their coherence in their instruction. The high fidelity to the textbook and the use of the teaching guide also informed teachers about students' previous knowledge level. Even if students came from different teachers' classes or moved from different schools, the content covered would still be the same.

Other resources. From the observation, Ms. Shi and Ms. Yang from Doner elementary school shared more similarity considering the class activities they enacted and the lesson plan they followed procedurally, while Ms. Liu from Shiyi elementary school had slightly different arrangements on each lesson. The non-physical resources of group lesson preparation accounts for the different implementation of instruction in the two different schools.

Summary

The evidence in this study provides some revelations on how Chinese teachers' teaching manifest a coherence on the topic of introduction of fractions, although the differences of teachers' teaching experience and teaching environments account for the variations in teachers' enactments of the class activities. There is no significant gap in terms of the instructional coherence, which could be explained by teachers' high fidelity to the textbook with a curriculum that is mandatory and suggest a sequence of activities to follow.

This study suggests that instructional coherence as a reflection of teachers' carefully crafted instructional skill provides students with better opportunities to access their learning process. Informed by the three cases, teacher's instructional coherence is not limited to the teachers' teaching experience. Moreover, the teachers' guide which addressed coherence in the curriculum and the transition teachers made from the intended curriculum to the actual instruction played a critical role in improving one's instructional coherence.

Adding to the Literature

Shown by this study, all three teachers applied the instructional techniques indicated in previous literature to improve students' understanding of fractions. For instance, the three teachers all combined the procedural knowledge with conceptual knowledge in their teaching (Dunhan, 2008) and provided students with opportunities to practice and solve problems (Seltzer, 1999), thus guiding students to apply their knowledge of fractions (Johanning, 2008). In addition, this study illustrates how teachers addressed the interrelation and development of conceptual and procedural knowledge.

While previous studies have indicated the critical role of curriculum coherence (Schmidt, Houang & Cogan, 2002), this study further reveals the close relationship between curriculum coherence and instructional coherence by providing evidence of how teachers use the instructional resources in their teaching. Moreover, by studying both novice and experienced teachers under a regular class setting, instead of focusing on the experienced teachers (Chen & Li, 2008), the study shows that novice teachers also manifested coherence in their teaching with the use of the instructional resources.

Limitation

The research study contains several limitations. By examining only three teachers from one Chinese province, the findings may be limited to these teachers. Another limitation is the failure to examine the instructional coherence from the students' perspective. Although the study shows that teachers' teaching manifested a coherence by providing evidence of teaching that provided students with the opportunity to apply their previous knowledge to learn the new knowledge, without any data on students' learning, it is not clear how the instructional coherence impacted students. This is an area for future study to examine how instructional coherence is implemented by teachers and its effectiveness in student learning. Additionally, informed by what we learn from this study, one could examine how instructional coherence is enacted in U.S. classrooms and other countries.

Implication for Future Research

Findings of this study suggest that teachers provided opportunities for students to learn fractions based on students' previous knowledge such as division. This informs us that teachers' instructional coherence is not only manifested within one topic but also across different mathematical topics. Also, the learning of fractions is embedded in the entire elementary level education that students will learn the knowledge of fractions more deeply in grade 5. Thus, the study of instructional coherence is not limited to within lesson or among each lesson analysis, but across different grades.

Future study could include the teaching and learning of fractions in grade 5 mathematics and working on the instructional coherence not only within or between each lesson but across different grades for a comprehensive understanding of the teaching and learning of fractions through the entire elementary levels. This would tie into better understanding learning trajectories. Another direction for future research is to compare and contrast the learning and teaching of fractions in China and the U.S to provide a reference for both of the educational systems.

APPENDIX SECTION

APPENDIX A

Interview protocol

Section one: Hello, thanks for being here, can I ask you some question about your background? What is your name? How do you become a math teacher?

Please introduce yourself regarding your teaching experience. For example, how long have you been a teacher and which grades have you ever taught?

What quality do you value most as an elementary mathematics teacher?

Have you ever experience any in-service training? If yes, please talk about your experience specifically.

Tell me something about how do you prepare for the class. What mathematics topics or ideas are you going to teach today? What do you think students need to know in order to work on this lesson? What would be your expectation for students' learning today? How did you prepare for today's class? What resources did you use for the preparation?

Section two: after class& class-related questions

From the class observation, I noticed that the model that we used in the class is a paper circle. Why do you choose it to be the model?

During the class, you repeatedly indicated the word "evenly divide". And also use the same sentence pattern when describing the fractions. Why? And how could this help student understand fractions.

How much practice do you think is necessary for students to do for each lesson? What is role of the practicing class?

Section three: Instructional coherence

When people say a lesson is very coherent, what does the word "coherent" mean to you? What are the characteristics of a coherent lesson?

If you mentor a new teacher, how would you guide the new teacher to achieve coherence in her or his teaching?

What are the factors that influence the instructional coherence?

What do you think of the coherence of the resources you used for example the textbook in the class? Does the sequence of the lessons matters? How does that influence your instructional coherence?

What previous knowledge had been taught as the preparation for the learning of fractions? How does that influence the instructional coherence.

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