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EXISTENCE AND DECAY OF SOLUTIONS TO A VISCOELASTIC PLATE EQUATION

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Abstract. In this article we study the fourth-order viscoelastic plate equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau = 0$$

in the bounded domain $\Omega = (0,\pi)\times(-\ell,\ell) \subset \mathbb{R}^2$ with non traditional boundary conditions. We establish the well-posedness and a decay result.

1. Introduction

This article is devoted to the well-posedness and the decay rate of the energy functional for the fourth-order viscoelastic plate problem

$$\begin{gathered}
u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau = 0, \quad \Omega\times(0,T) \\
u(0,y,t) = u_{xx}(0,y,t) = 0, \quad \text{for } (y,t)\in(-\ell,\ell)\times(0,T) \\
u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \quad \text{for } (y,t)\in(-\ell,\ell)\times(0,T) \\
u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, \quad \text{for } (x,t)\in(0,\pi)\times(0,T) \\
u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) = 0, \quad \text{for } (x,t)\in(0,\pi)\times(0,T) \\
u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), \quad \text{in } \Omega
\end{gathered} \tag{1.1}$$

where $\Omega = (0,\pi)\times(-\ell,\ell)$, $0 < \sigma < \frac{1}{2}$ and g is a positive and nonincreasing function. This type of problems models the motion of a viscoelastic plate. The fundamental work of Ferrero and Gazzola [14] in 2013, where they modeled a suspension bridge as a rectangular plate with the same boundary conditions as (1.1), suggests the investigation of the viscoelastic material used in construction. Al-Gwaiz et al [1] also investigated the bending and stretching energies of the rectangular plate model suggested in [14]. Contributions on the analysis of a suspension bridge have also come from Mckenna and Walter [24], Mckenna et al [15], Ma and Zhong [23] and Bochicchio et al [6].

The existence, decay and blow up properties of viscoelastic problems has attracted a lot of attention since the pioneer work by Dafermos [12, 13] in 1970. Hence, a considerable number of results for models similar to (1.1), for both second

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and fourth order have been established. We begin with the result of Messaoudi [25], where he considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0, \quad \text{in } \Omega \times (0,+\infty) \tag{1.2}$$

with general conditions on the relaxation function g, and proved a general decay result that is not necessarily of exponential or polynomial type. His result generalized and improved many results in literature such as [4, 5, 2, 3, 10]. Rivera et al [27] considered the fourth-order equation

$$u_{tt} + \gamma\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau = 0, \quad \text{in } \Omega \times (0,T) \tag{1.3}$$

together with initial and dynamical boundary conditions and proved that the sum of the first and second energies decays exponentially (polynomially) if the kernel g decays exponentially (polynomially). Mustafa and Ghassan [28] considered the plate equation

$$u_{tt} + \Delta^2 u = 0, \quad \text{in } \Omega \times (0,+\infty) \tag{1.4}$$

with viscoelastic damping localized on a part of the boundary and established a decay result. For more results related to the plate equation, we refer the reader to Messaoudi [26], Kang [17], Santos and Junior [29], Lagnese [19], Horn and Lasiecka [16], Lasiecka [20], and Lasiecka et al [21], Cabanillas et al [8] Lasiecka et al [9].

The aim of this work is to take advantage of the techniques used in [25] and the new model in [14] to establish a global existence and general decay results for problem (1.1). We organize this work as follows. In section 2, we present some important and fundamental materials to be used in establishing our main results. In section 3, we state and prove the global existence result. Finally, in section 4 we state and prove the general decay result.

2. Preliminaries

In this section, we present some fundamental materials needed for the proof of our main results. For this, we assume the following conditions on the relaxation function g.

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)ds = l_0 > 0. \tag{2.1}$$

(A2) There exists a differentiable function ξ satisfying

$$\begin{gathered} g'(t) \le -\xi(t)g(t), \quad t \ge 0 \\ \xi(t) > 0, \quad \xi'(t) \le 0, \quad \forall t > 0, \quad \int_0^{+\infty} \xi(s)ds = +\infty \end{gathered} \tag{2.2}$$

The following three functions satisfy (A1)–(A2).

$$g_1(t) = \frac{ae^{-t}}{(1+t)}, \quad a > 0,$$

$$g_2(t) = \frac{a}{(1+t)^p}, \quad p > 1,\ a > 0, g_3(t) = ae^{-b(1+t)^p}, \quad 0 < p \le 1,\ a,\ b > 0.$$

We introduce the space

$$H^2_*(\Omega) = \{w \in H^2(\Omega) : w = 0 \text{ on } \{0, \pi\} \times (-\ell, \ell)\}, \tag{2.3}$$

with the inner product

$$(u, v)_{H^2_*} = \int_\Omega [(\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]\, dx\, dy, \tag{2.4}$$

and set $\mathcal{H}(\Omega)$ as the dual of $H^2_*(\Omega)$. For completeness, we state some results from Ferrero and Gazzola [14].

Lemma 2.1 ([14])**.** *Assume $0 < \sigma < 1/2$. Then the norm $\|\cdot\|_{H^2_*(\Omega)}$ given by $\|\cdot\|^2_{H^2_*(\Omega)} = (\cdot,\cdot)_{H^2_*}$ is equivalent to the usual $H^2(\Omega)$-norm. Moreover, $H^2_*(\Omega)$ is a Hilbert space when endowed with the scalar product $(u, v)_{H^2_*}$.*

Theorem 2.2 ([14])**.** *Assume $0 < \sigma < 1/2$ and let $f \in L^2(\Omega)$. Then there exists a unique function $u \in H^2_*(\Omega)$ such that*

$$\int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]\, dx\, dy = \int_\Omega fv, \tag{2.5}$$

for all $v \in H^2_*(\Omega)$.

The function $u \in H^2_*(\Omega)$ satisfying (2.5) is called the weak solution of the stationary problem

$$\begin{gathered} \Delta^2 u = f, \\ u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0, \\ u_{yy}(x, \pm l) + \sigma u_{xx}(x, \pm l) = u_{yyy}(x, \pm l) + (2-\sigma)u_{xxy}(x, \pm l) = 0. \end{gathered} \tag{2.6}$$

Lemma 2.3 ([30])**.** *Let $u \in H^2_*(\Omega)$ and assume $1 \le p < +\infty$. Then, there exists a positive constant $C_e = C_e(\Omega, p) > 0$ such that*

$$\|u\|_p^p \le C_e \|u\|^p_{H^2_*(\Omega)}.$$

Let us also introduce the energy functional associated to problem (1.1),

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2}\int_0^t g(t-\tau)\|u(t) - u(\tau)\|^2_{H^2_*(\Omega)} d\tau. \tag{2.7}$$

3. Well-posedness

In this section, we show that problem (1.1) has a unique global weak solution.

Definition 3.1. A function

$$u \in C([0, T), H^2_*(\Omega)) \cap C^1([0, T), L^2(\Omega)) \cap C^2([0, T), \mathcal{H}(\Omega)) \tag{3.1}$$

is called a weak solution of (1.1) if

$$\begin{gathered} \int_\Omega u_{tt}w + (u, w)_{H^2_*(\Omega)} - \int_0^t g(t-\tau)(u(\tau), w)_{H^2_*(\Omega)} d\tau = 0, \quad \forall w \in H^2_*(\Omega), \\ u(0) = u_0, \quad u_t(0) = u_1. \end{gathered} \tag{3.2}$$

Theorem 3.2. *Let $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$. Assume that* (A1), (A2) *hold. Then problem* (1.1) *has a unique weak global solution*

$$u \in C([0, T), H^2_*(\Omega)), \quad u_t \in C([0, T), L^2(\Omega)), \quad u_{tt} \in C([0, T), \mathcal{H}(\Omega)) \tag{3.3}$$

Proof. We use the Galerkin approximation method. Let $\{w_j\}_{j=1}^{\infty}$ be a basis of the separable space $H_*^2(\Omega)$ and $V_m = \operatorname{span}\{w_1, w_2, \dots, w_m\}$ be a finite subspace of $H_*^2(\Omega)$ spanned by the first m vectors. Let

$$u_0^m(x,y) = \sum_{j=1}^{m} a_j w_j(x,y) \quad \text{and} \quad u_1^m(x,y) = \sum_{j=1}^{m} b_j w_j(x,y)$$

be sequences in $H_*^2(\Omega)$ and $L^2(\Omega)$ respectively, such that

$$u_0^m \to u_0 \text{ in } H_*^2(\Omega), \quad u_1^m \to u_1 \text{ in } L^2(\Omega). \tag{3.4}$$

We seek a solution of the form

$$u^m(x,y,t) = \sum_{j=1}^{m} c_j(t) w_j(x,y),$$

which satisfies the approximate problem

$$\begin{gathered}
\int_\Omega u_{tt}^m(x,y,t) w_j + (u^m(x,y,t), w_j)_{H_*^2(\Omega)} \\
- \int_0^t g(t-\tau)(u^m(x,y,\tau), w_j)_{H_*^2(\Omega)} d\tau = 0, \quad \forall w_j \in V_m,\ j = 1,2,\dots,m. \\
u^m(0) = u_0^m, \quad u_t^m(0) = u_1^m\,.
\end{gathered} \tag{3.5}$$

We note that (3.5) leads to system of ODEs with m unknown functions c_j, $j = 1, 2, \dots, m$. Thus, using ODE theory (see [11]), we obtain functions

$$c_j : [0, t_m) \to \mathbb{R},\ j = 1, 2, \dots, m,$$

which satisfy (3.5) for almost every $t \in (0, t_m),\ 0 < t_m < T$. Therefore, we obtain a local solution u^m of (3.5) in a maximal interval $[0, t_m), t_m \in (0, T]$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this, we multiply (3.5) by $c_j'(t)$ and sum over $j = 1, 2, \dots, m$, to obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}\|u_t^m\|_{L^2(\Omega)}^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|u^m\|_{H_*^2(\Omega)}^2 \\
&\quad + \frac{1}{2}\int_0^t g(t-\tau)\|u^m(t) - u^m(\tau)\|_{H_*^2(\Omega)}^2 d\tau\Big] \\
&= \frac{1}{2}\int_0^t g'(t-\tau)\|u^m(t) - u^m(\tau)\|_{H_*^2(\Omega)}^2 d\tau - \frac{1}{2} g(t)\|u^m\|_{H_*^2(\Omega)}^2
\end{aligned}$$

It follows from (2.7) that

$$\frac{d}{dt}E^m(t) = \frac{1}{2}\int_0^t g'(t-\tau)\|u^m(t) - u^m(\tau)\|_{H_*^2(\Omega)}^2 d\tau - \frac{1}{2} g(t)\|u^m\|_{H_*^2(\Omega)}^2 \le 0, \tag{3.6}$$

by assumptions (A1) and (A2). Integrating (3.6) over $(0,t)$, $t \in (0, t_m)$ and noting that (u_0^m) and (u_1^m) are bounded in $H_*^2(\Omega)$ and $L^2(\Omega)$ respectively (as convergent sequences (3.4)), we obtain

$$E^m(t) \le E^m(0) = \frac{1}{2}\|u_1^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u_0^m\|_{H_*^2(\Omega)}^2 \le C \tag{3.7}$$

where C is a positive constant independent of m and t. Therefore,

$$\frac{1}{2}\|u_t^m\|_{L^2(\Omega)}^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|u^m\|_{H_*^2(\Omega)}^2$$

$$+\frac{1}{2}\int_0^t g(t-\tau)\|u^m(t)-u^m(\tau)\|^2_{H^2_*(\Omega)}d\tau \le C.$$

This implies

$$\frac{1}{2}\sup_{t\in(0,t_m)}\|u^m_t\|^2_{L^2(\Omega)}+\frac{l_0}{2}\sup_{t\in(0,t_m)}\|u^m\|^2_{H^2_*(\Omega)}\le C. \tag{3.8}$$

So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T. Moreover, we obtain from (3.8) that

$$\begin{aligned} &(u^m)\text{is a bounded sequence in } L^\infty((0,T),H^2_*(\Omega)),\\ &(u^m_t) \text{ is a bounded sequence in } L^\infty((0,T),L^2(\Omega)). \end{aligned} \tag{3.9}$$

Thus, there exists a subsequence (u^k) of (u^m) such that

$$\begin{aligned} &u^k \rightharpoonup u \text{ weakly star in } L^\infty((0,T),H^2_*(\Omega)) \text{ and weakly in } L^2((0,T),H^2_*(\Omega))\\ &u^k_t \rightharpoonup u_t \text{ weakly star in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^2((0,T),L^2(\Omega)) \end{aligned} \tag{3.10}$$

Using that $H^2_*(\Omega)$ is compactly embedded in $L^2(\Omega)$ (remember that Ω is bounded and $H^2_*(\Omega)\subset H^2(\Omega)$), we can extract a subsequence (u^l) of (u^k) such that

$$\begin{gathered} u^l \to u \text{ strongly in } L^2((0,T),L^2(\Omega)),\\ u^l \to u \text{ a.e in } \Omega\times(0,T). \end{gathered}$$

Now, replacing (u^m) by (u^l) in (3.5) and integrating over $(0,t)$ we obtain

$$\begin{aligned} &\int_\Omega u^l_t w_j + \int_0^t (u^l,w_j)_{H^2_*(\Omega)}dt - \int_0^t\int_0^s g(s-\tau)(u^l(\tau),w_j)_{H^2_*(\Omega)}d\tau ds\\ &= \int_\Omega u^l_1 w_j, \quad \forall j\le l. \end{aligned} \tag{3.11}$$

Letting $l\to+\infty$, we obtain

$$\begin{aligned} &\int_\Omega u_t w_j + \int_0^t (u,w_j)_{H^2_*(\Omega)}dt - \int_0^t\int_0^s g(s-\tau)(u(\tau),w_j)_{H^2_*(\Omega)}d\tau ds\\ &= \int_\Omega u_1 w_j, \quad \forall j\ge 1. \end{aligned} \tag{3.12}$$

This implies

$$\begin{aligned} \int_\Omega u_t w = &-\int_0^t (u,w)_{H^2_*(\Omega)}dt + \int_0^t\int_0^s g(s-\tau)(u(\tau),w)_{H^2_*(\Omega)}d\tau ds\\ &+\int_\Omega u_1 w, \quad \forall w\in H^2_*(\Omega). \end{aligned} \tag{3.13}$$

Now, observe that the terms in the right-hand side of (3.13) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$, hence differentiable almost everywhere. Thus, differentiating (3.13), we obtain that for a.e $t\in(0,T)$,

$$\int_\Omega u_{tt}w + (u,w)_{H^2_*(\Omega)} - \int_0^t g(t-\tau)(u(\tau),w)_{H^2_*(\Omega)}d\tau = 0 \tag{3.14}$$

for all $w\in L^2((0,T),H^2_*(\Omega))$. To handle the initial conditions, we note that

$$\begin{aligned} &u^l \rightharpoonup u \text{ weakly in } L^2((0,T),H^2_*(\Omega))\\ &u^l_t \rightharpoonup u_t \text{ weakly in } L^2((0,T),L^2(\Omega)) \end{aligned} \tag{3.15}$$

Thus, using Lions' Lemma [22], we obtain
$$u^l \to u \quad \text{in } C([0,T), L^2(\Omega)). \tag{3.16}$$
Therefore, $u^l(x,y,0)$ makes sense and $u^l(x,y,0) \to u(x,y,0)$ in $L^2(\Omega)$. Also we have that
$$u^l(x,y,0) = u_0^l(x,y) \to u_0(x,y) \quad \text{in } H_*^2(\Omega).$$
Hence
$$u(x,y,0) = u_0(x,y). \tag{3.17}$$
As in [14, 18], let $\phi \in C_0^\infty(0,T)$ and replacing (u^m) by (u^l), we obtain from (3.5) and for any $j \le l$ that
$$\begin{aligned} &-\int_0^T (u_t^l(t), w_j)_{L^2(\Omega)}\phi'(t)dt \\ &= -\int_0^T (u^l(t), w_j)_{H_*^2(\Omega)}\phi(t)dt + \int_0^T\int_0^t g(t-\tau)(u^l(\tau), w_j)_{H_*^2(\Omega)}\phi(t)d\tau dt. \end{aligned} \tag{3.18}$$
As $l \to +\infty$, we obtain that
$$\begin{aligned} &-\int_0^T (u_t(t), w_j)_{L^2(\Omega)}\phi'(t)dt \\ &= -\int_0^T (u(t), w_j)_{H_*^2(\Omega)}\phi(t)dt + \int_0^T\int_0^t g(t-\tau)(u(\tau), w_j)_{H_*^2(\Omega)}\phi(t)d\tau dt, \end{aligned}$$
for all $j \ge 1$. This implies
$$\begin{aligned} &-\int_0^T (u_t(t), w)_{L^2(\Omega)}\phi'(t)dt \\ &= -\int_0^T (u(t), w)_{H_*^2(\Omega)}\phi(t)dt + \int_0^T\int_0^t g(t-\tau)(u(\tau), w)_{H_*^2(\Omega)}\phi(t)d\tau dt, \end{aligned}$$
for all $w \in H_*^2(\Omega)$. This means $u_{tt} \in L^2([0,T), \mathcal{H}(\Omega))$. Thus,
$$u_t \in L^2([0,T), L^2(\Omega)), \quad u_{tt} \in L^2([0,T), \mathcal{H}(\Omega)) \Longrightarrow u_t \in C([0,T), \mathcal{H}(\Omega)). \tag{3.19}$$
So, $u_t^l(x,y,0)$ makes sense (see [18, p.116]). It follows that
$$u_t^l(x,y,0) \to u_t(x,y,0) \quad \text{in } \mathcal{H}(\Omega).$$
But
$$u_t^l(x,y,0) = u_1^l(x,y) \to u_1(x,y) \quad \text{in } L^2(\Omega).$$
Hence
$$u_t(x,y,0) = u_1(x,y). \tag{3.20}$$
For the uniqueness, suppose u and $\bar{u}$ satisfy (3.14), (3.17) and (3.20). Then $v = u - \bar{u}$ satisfies
$$\begin{gathered} \int_\Omega v_{tt}w + (v,w)_{H_*^2(\Omega)} - \int_0^t g(t-\tau)(v(\tau), w)_{H_*^2(\Omega)}d\tau = 0, \\ \forall w \in L^2((0,T), H_*^2(\Omega)), \\ v(0) = v_t(0) = 0. \end{gathered} \tag{3.21}$$
Replacing w by v_t in (3.21), we obtain
$$\frac{d}{dt}\Big[\frac{1}{2}\int_\Omega v_t^2 + \frac{1}{2}\|v\|_{H_*^2(\Omega)}^2\Big] - \int_0^t g(t-\tau)(v(\tau), v_t(t))_{H_*^2(\Omega)}d\tau = 0. \tag{3.22}$$

We have that

$$
\begin{aligned}
J_1 &= -\int_0^t g(t-\tau)(v(\tau), v_t(t))_{H_*^2(\Omega)} d\tau \\
&= \int_0^t g(t-\tau)(v_t(t), v(t)-v(\tau))_{H_*^2(\Omega)} d\tau - \int_0^t g(s)ds (v_t(t), v(t))_{H_*^2(\Omega)} \\
&= \int_0^t g(t-\tau)\frac{d}{dt}\frac{1}{2}\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)} d\tau - \int_0^t g(s)ds \frac{d}{dt}\frac{1}{2}\|v(t)\|^2_{H_*^2(\Omega)} \\
&= \frac{d}{dt}\frac{1}{2}\int_0^t g(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)} d\tau \\
&\quad - \frac{1}{2}\int_0^t g'(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)} d\tau \\
&\quad - \frac{d}{dt}\frac{1}{2}\int_0^t g(s)ds \|v(t)\|^2_{H_*^2(\Omega)} + \frac{1}{2} g(t)\|v(t)\|^2_{H_*^2(\Omega)}.
\end{aligned} \tag{3.23}
$$

Inserting (3.23) into (3.22) and taking note of (2.7), we obtain

$$
\frac{d\tilde{E}(t)}{dt} = \frac{1}{2}\int_0^t g'(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)} d\tau - \frac{1}{2} g(t)\|v(t)\|^2_{H_*^2(\Omega)} \leq 0, \tag{3.24}
$$

by (A1) and (A2). Integrating (3.24) over $(0,t)$, we obtain

$$
\tilde{E}(t) \leq \tilde{E}(0) = 0. \tag{3.25}
$$

This implies

$$
\|v_t\|^2_{L^2(\Omega)} + \|v\|^2_{H_*^2(\Omega)} = 0.
$$

Therefore, $u = \bar{u}$. The proof is complete. □

4. Decay of solutions

In this section, we discuss the stability of solution of problem (1.1). Let us begin by defining the Lyapunov functional

$$
F(t) = E(t) + \epsilon_1 \Psi(t) + \epsilon_2 \chi(t), \tag{4.1}
$$

where ϵ_1 and ϵ_2 are positive constants to be specified later and

$$
\begin{gathered}
\Psi(t) = \int_\Omega u u_t, \\
\chi(t) = -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau)) d\tau \, dx \, dy.
\end{gathered} \tag{4.2}
$$

Lemma 4.1. *Assume* (A1), (A2) *hold. Then the energy functional, defined in* (2.7)*, satisfies*

$$
\frac{dE(t)}{dt} = \frac{1}{2}\int_0^t g'(t-\tau)\|u(t)-u(\tau)\|^2_{H_*^2(\Omega)} d\tau - \frac{1}{2} g(t)\|u\|^2_{H_*^2(\Omega)} \leq 0. \tag{4.3}
$$

Proof. By using (3.14) and the density of $H_*^2(\Omega)$ in $L^2(\Omega)$ we obtain

$$
\int_\Omega u_{tt} w + (u, w)_{H_*^2(\Omega)} - \int_0^t g(t-\tau)(u(\tau), w)_{H_*^2(\Omega)} d\tau = 0 \tag{4.4}
$$

for all $w \in L^2([0,T), L^2(\Omega))$. Repeating exactly the same arguments as in (3.22)-(3.24), we obtain the result. □

Lemma 4.2. *For every $u \in H_*^2(\Omega)$, we have*

$$\begin{aligned} &\int_\Omega \Big(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \Big)^2 \,dx\,dy \\ &\leq C_e(1-l_0) \int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H_*^2(\Omega)} d\tau, \end{aligned} \tag{4.5}$$

where $C_e > 0$ is the embedding constant introduced in Lemma 2.3.

Proof. Since g is positive, we have

$$\int_\Omega \Big(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \Big)^2 = \int_\Omega \Big(\int_0^t \sqrt{g(t-\tau)}\sqrt{g(t-\tau)}(u(t)-u(\tau))d\tau \Big)^2$$

By applying Cauchy-Schwarz, (A1) and Lemma 2.3, we obtain

$$\begin{aligned} &\int_\Omega (\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau)^2 \\ &\leq \int_\Omega (\int_0^t g(s)ds)(\int_0^t g(t-\tau)(u(t)-u(\tau))^2 d\tau) \\ &\leq C_e(1-l_0) \int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H_*^2(\Omega)} d\tau. \end{aligned}$$

□

Lemma 4.3. *For ϵ_1 and ϵ_2 small enough, there exists two positive constants α_1 and α_2 such that*

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t) \tag{4.6}$$

The proof of the above lemma uses similar techniques as in [25, Lemma 3.3]; we omit it here.

Lemma 4.4. *Under assumptions* (A1), (A2)*, the functional*

$$\Psi(t) = \int_\Omega u u_t$$

satisfies, along the solution of (1.1)*,*

$$\Psi'(t) \leq \int_\Omega u_t^2 - \frac{l_0}{2}\|u\|^2_{H_*^2(\Omega)} + \frac{1-l_0}{2l_0}\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H_*^2(\Omega)} d\tau. \tag{4.7}$$

Proof. By using (3.14) and replacing w by u, direct differentiations yield

$$\Psi'(t) = \int_\Omega u_t^2 - \|u\|^2_{H_*^2(\Omega)} + \int_0^t g(t-\tau)(u(t),u(\tau))_{H_*^2(\Omega)} d\tau. \tag{4.8}$$

By using Cauchy-Schwarz and Young's inequalities, we estimate the third term

$$J_2 = \int_0^t g(t-\tau)(u(t),u(\tau))_{H_*^2(\Omega)} d\tau,$$

for any $\eta > 0$, as follows

$$\begin{aligned} J_2 &\leq \int_0^t g(t-\tau)\|u(t)\|_{H_*^2(\Omega)}\|u(\tau)\|_{H_*^2(\Omega)} d\tau \\ &\leq \frac{1}{2}\|u(t)\|^2_{H_*^2(\Omega)} + \frac{1}{2}\Big(\int_0^t g(t-\tau)(\|u(t)-u(\tau)\|_{H_*^2(\Omega)} + \|u(t)\|_{H_*^2(\Omega)})d\tau \Big)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{2}\Big(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)}d\tau\Big)^2 \\
&\quad + \frac{1}{2}(\int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)})d\tau)^2 \\
&\quad + \Big(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)}d\tau\Big)\Big(\int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)}d\tau\Big).
\end{aligned}
$$

By using Lemma 4.2, we obtain

$$
\begin{aligned}
J_2 &\le \frac{1}{2}(1+(1-l_0)^2)\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2}(1-l_0)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\
&\quad + \frac{\eta}{2}(\int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)}d\tau)^2 + \frac{1}{2\eta}(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)}d\tau)^2 \\
&\le \frac{1}{2}(1+(1-l_0)^2(1+\eta))\|u\|^2_{H^2_*(\Omega)} \\
&\quad + \frac{1}{2}(1-l_0)(1+\frac{1}{\eta})\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau.
\end{aligned} \tag{4.9}
$$

Now, substituting (4.9) in (4.8), we obtain

$$
\begin{aligned}
\Psi'(t) &\le \int_\Omega u_t^2 + \frac{1}{2}((1-l_0)^2(1+\eta)-1)\|u\|^2_{H^2_*(\Omega)} \\
&\quad + \frac{1}{2}(1-l_0)(1+\frac{1}{\eta})\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau, \quad \forall \eta > 0.
\end{aligned} \tag{4.10}
$$

We choose $\eta = \frac{l_0}{1-l_0}$ and obtain the result. $\square$

Lemma 4.5. *Assume conditions (A1) and (A2) hold. Then the functional*

$$
\chi(t) = -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\,dx\,dy \tag{4.11}
$$

satisfies, along the solution of (1.1),

$$
\begin{aligned}
\chi'(t) &\le \Big(\frac{\delta}{2} - \int_0^t g(s)ds\Big)\int_\Omega u_t^2 + \frac{\delta}{2}(1+2(1-l_0)^2)\|u\|^2_{H^2_*(\Omega)} \\
&\quad - \frac{C_e g(0)}{2\delta}\int_0^t g'(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\
&\quad + (1-l_0)(\delta+\frac{1}{\delta})\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau, \quad \forall \delta > 0.
\end{aligned} \tag{4.12}
$$

Proof. By differentiating (4.11) and using (3.14), with u instead of w, we obtain

$$\begin{aligned}
\chi'(t) &= -\Big(\int_0^t g(s)ds\Big)\int_\Omega u_t^2 - \int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau\,dx\,dy \\
&\quad - \int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\,dx\,dy \\
&= -\Big(\int_0^t g(s)ds\Big)\int_\Omega u_t^2 - \int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau\,dx\,dy \\
&\quad + \Big(u(t), \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\Big)_{H_*^2(\Omega)} \\
&\quad - \int_0^t g(t-\tau)\Big(u(\tau), \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\Big)_{H_*^2(\Omega)} d\tau.
\end{aligned} \tag{4.13}$$

By using Cauchy-Schwarz inequality, Young's inequality and Lemma 4.2 for $-g'$ instead of g, we estimate the terms in the right-hand side of (4.13). Thus, for the term

$$J_3 = -\int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau\,dx\,dy,$$

we have that for any $\delta > 0$,

$$\begin{aligned}
J_3 &\le \frac{\delta}{2}\int_\Omega u_t^2 + \frac{1}{2\delta}\int_\Omega \Big(\int_0^t -g'(t-\tau)(u(t)-u(\tau))d\tau\Big)^2 dx\,dy \\
&\le \frac{\delta}{2}\int_\Omega u_t^2 + \frac{1}{2\delta}\int_\Omega \Big(\int_0^t -g'(s)ds\Big)\Big(\int_0^t -g'(t-\tau)(u(t)-u(\tau))^2 d\tau\Big)\,dx\,dy \\
&\le \frac{\delta}{2}\int_\Omega u_t^2 - \frac{C_e g(0)}{2\delta}\int_0^t g'(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}^2 d\tau.
\end{aligned} \tag{4.14}$$

For the term

$$J_4 = (u(t), \int_0^t g(t-\tau)(u(t)-u(\tau)d\tau)_{H_*^2(\Omega)},$$

we have

$$\begin{aligned}
J_4 &\le \|u(t)\|_{H_*^2(\Omega)}\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}d\tau \\
&\le \frac{\delta}{2}\|u(t)\|_{H_*^2(\Omega)}^2 + \frac{1}{2\delta}(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}^2 d\tau)^2 \\
&\le \frac{\delta}{2}\|u(t)\|_{H_*^2(\Omega)}^2 + \frac{(1-l_0)}{2\delta}\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}^2 d\tau.
\end{aligned} \tag{4.15}$$

Similarly, for the term

$$J_5 = -\int_0^t g(t-\tau)(u(\tau), \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau)_{H_*^2(\Omega)}d\tau,$$

we obtain

$$
\begin{aligned}
J_5 &\le \Big(\int_0^t g(t-\tau)\|u(\tau)\|_{H_*^2(\Omega)}d\tau\Big)\Big(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}d\tau\Big)\\
&\le \frac{\delta}{2}\Big(\int_0^t g(t-\tau)(\|u(t)-u(\tau)\|_{H_*^2(\Omega)}+\|u(t)\|_{H_*^2(\Omega)})d\tau\Big)^2\\
&\quad + \frac{1}{2\delta}\Big(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}d\tau\Big)^2\\
&\le \frac{\delta}{2}\Big(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}d\tau\Big)^2\\
&\quad + \frac{\delta}{2}\Big(\int_0^t g(t-\tau)\|u(t)\|_{H_*^2(\Omega)}d\tau\Big)^2\\
&\quad + \delta(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}d\tau)\Big(\int_0^t g(t-\tau)\|u(t)\|_{H_*^2(\Omega)}d\tau\Big)\\
&\quad + \frac{(1-l_0)}{2\delta}\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H_*^2(\Omega)}d\tau\\
&\le (\delta+\frac{1}{2\delta})(1-l_0)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H_*^2(\Omega)}d\tau\\
&\quad + \delta(1-l_0)^2\|u\|^2_{H_*^2(\Omega)}.
\end{aligned}
\tag{4.16}
$$

By substituting (4.14)–(4.16) in (4.13), we obtain (4.12), for any $\delta > 0$. □

Theorem 4.6. *Let* $(u_0,u_1)\in H_*^2(\Omega)\times L^2(\Omega)$. *Assume* g *and* ξ *satisfy* (A1) *and* (A2). *Then, for any* $t_0>0$, *there exist positive constants* K *and* λ *such that the solution of* (1.1) *satisfies*

$$
E(t)\le Ke^{-\lambda\int_{t_0}^t \xi(s)ds},\quad \forall t\ge t_0. \tag{4.17}
$$

Proof. Since g is positive, continuous, and $g(0)>0$, then for any $t\ge t_0$ we have

$$
\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds = g_0 > 0.
$$

Combination of (4.3), (4.7) and (4.12), gives that for any $t\ge t_0$,

$$
\begin{aligned}
&F'(t)\\
&\le -\Big(\epsilon_2(g_0-\frac{\delta}{2})-\epsilon_1\Big)\int_\Omega u_t^2 - \Big(\frac{\epsilon_1 l_0}{2}-\epsilon_2\frac{\delta}{2}(1+2(1-l_0)^2)\Big)\|u\|^2_{H_*^2(\Omega)}\\
&\quad + \Big(\frac{1}{2}-\epsilon_2\frac{C_e g(0)}{2\delta}\Big)\int_0^t g'(t-\tau)\|u(t)-u(\tau)\|^2_{H_*^2(\Omega)}d\tau\\
&\quad + \Big(\frac{\epsilon_1(1-l_0)}{2l_0}+\epsilon_2(\delta+\frac{1}{\delta})(1-l_0)\Big)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H_*^2(\Omega)}d\tau.
\end{aligned}
\tag{4.18}
$$

Now, we choose δ small enough such that

$$
g_0-\frac{\delta}{2}>\frac{g_0}{2},\quad \frac{4\delta}{l_0}(1+2(1-l_0)^2)<\frac{g_0}{4}. \tag{4.19}
$$

By using (4.19), we easily check that any ϵ_1 and ϵ_2, satisfying

$$
\frac{\epsilon_2 g_0}{16}<\epsilon_1<\frac{\epsilon_2 g_0}{2}, \tag{4.20}
$$

will make

$$\beta_1 = \Big(\epsilon_2(g_0 - \frac{\delta}{2}) - \epsilon_1\Big) > 0, \quad \beta_2 = \Big(\frac{\epsilon_1 l_0}{2} - \epsilon_2\frac{\delta}{2}(1 + 2(1 - l_0)^2)\Big) > 0.$$

Next, we pick ϵ_1 and ϵ_2 small enough such that (4.6) and (4.20) remain valid and further we have

$$\frac{1}{2} - \epsilon_2\frac{C_e g(0)}{2\delta} > 0, \quad \frac{\epsilon_1(1 - l_0)}{2l_0} + \epsilon_2(\delta + \frac{1}{\delta})(1 - l_0) > 0.$$

Thus, (4.18) becomes

$$\begin{aligned} F'(t) &\le -\beta_1\int_\Omega u_t^2 - \beta_2\|u\|^2_{H^2_*(\Omega)} + \tilde{C}\int_0^t g(t-\tau)\|u(t) - u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\ &\le -\beta E(t) + C\int_0^t g(t-\tau)\|u(t) - u(\tau)\|^2_{H^2_*(\Omega)}d\tau, \quad \forall t \ge t_0. \end{aligned} \tag{4.21}$$

Multiplying (4.21) by $\xi(t)$ and using the facts that ξ is decreasing and

$$g'(t) \le -\xi(t)g(t), \quad E'(t) \le \frac{1}{2}\int_0^t g'(t-\tau)\|u(t) - u(\tau)\|^2_{H^2_*(\Omega)}d\tau, \tag{4.22}$$

we arrive at

$$\begin{aligned} \xi(t)F'(t) &\le -\beta\xi(t)E(t) + C\xi(t)\int_0^t g(t-\tau)\|u(t) - u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\ &\le -\beta\xi(t)E(t) + C\int_0^t \xi(t-\tau)g(t-\tau)\|u(t) - u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\ &\le -\beta\xi(t)E(t) + C\int_0^t -g'(t-\tau)\|u(t) - u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\ &\le -\beta\xi(t)E(t) - CE'(t), \quad \forall t \ge t_0. \end{aligned}$$

This gives

$$(\xi(t)F(t) + CE(t))' - \xi'(t)F(t) \le -\beta\xi(t)E(t), \quad \forall t \ge t_0.$$

Consequently,

$$(\xi(t)F(t) + CE(t))' \le -\beta\xi(t)E(t), \quad \forall t \ge t_0. \tag{4.23}$$

Let

$$L = \xi F + CE \sim E, \tag{4.24}$$

since $F \sim E$ and $0 \le \xi(t) \le \xi(0)$. Then (4.23) and (4.24) lead to

$$L'(t) \le -\lambda\xi(t)L(t), \quad \forall t \ge t_0. \tag{4.25}$$

A simple integration in (t_0, t) yields

$$L(t) \le L(t_0)e^{-\lambda\int_{t_0}^t \xi(s)ds}, \quad \forall t \ge t_0. \tag{4.26}$$

Again, recalling (4.24), we obtain

$$E(t) \le Ke^{-\lambda\int_{t_0}^t \xi(s)ds}, \quad \forall t \ge t_0. \tag{4.27}$$

This completes the proof. $\square$

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