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EXISTENCE AND ASYMPTOTIC EXPANSION OF SOLUTIONS TO A NONLINEAR WAVE EQUATION WITH A MEMORY CONDITION AT THE BOUNDARY

NGUYEN THANH LONG, LE XUAN TRUONG

Abstract. We study the initial-boundary value problem for the nonlinear wave equation

$$\begin{gathered} u_{tt} - \frac{\partial}{\partial x}(\mu(x,t)u_x) + K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t = f(x,t), \\ u(0,t) = 0 \\ -\mu(1,t)u_x(1,t) = Q(t), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{gathered}$$

where $p \ge 2$, $q \ge 2$, K, λ are given constants and u_0, u_1, f, μ are given functions. The unknown function $u(x,t)$ and the unknown boundary value $Q(t)$ satisfy the linear integral equation

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u_t(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds,$$

where K_1, λ_1, g, k are given functions satisfying some properties stated in the next section. This paper consists of two main sections. First, we prove the existence and uniqueness for the solutions in a suitable function space. Then, for the case $K_1(t) = K_1 \ge 0$, we find the asymptotic expansion in K, λ, K_1 of the solutions, up to order $N+1$.

1. Introduction

In this paper, we consider the following problem: Find a pair of functions (u, Q) satisfying

$$u_{tt} - \frac{\partial}{\partial x}(\mu(x,t)u_x) + F(u,u_t) = f(x,t), \quad 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

$$u(0,t) = 0, \tag{1.2}$$

$$-\mu(1,t)u_x(1,t) = Q(t), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.4}$$

where $F(u,u_t) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$, with $p, q \ge 2$, K, λ are given constants and u_0, u_1, f, μ are given functions satisfying conditions specified later; the unknown

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function $u(x,t)$ and the unknown boundary value $Q(t)$ satisfy the integral equation

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u_t(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds, \tag{1.5}$$

where g, k, K_1, λ_1 are given functions. Santos [10] studied the asymptotic behavior of solution of problem (1.1), (1.2) and (1.4) associated with a boundary condition of memory type at $x = 1$ as follows

$$u(1,t) + \int_0^t g(t-s)\mu(1,s)u_x(1,s)ds = 0, \quad t > 0. \tag{1.6}$$

To make such a difficult condition simpler, Santos transformed (1.6) into (1.3), (1.5) with $K_1(t) = \frac{g'(0)}{g(0)}$, and $\lambda_1(t) = \frac{1}{g(0)}$ positive constants.

In the case $\lambda_1(t) \equiv 0$, $K_1(t) = h \geq 0$, $\mu(x,t) \equiv 1$, the problem (1.1)–(1.5) is formed from the problem (1.1)–(1.4) wherein, the unknown function $u(x,t)$ and the unknown boundary value $Q(t)$ satisfy the following Cauchy problem for ordinary differential equations

$$\begin{gathered} Q''(t) + \omega^2 Q(t) = hu_{tt}(1,t), \quad 0 < t < T, \\ Q(0) = Q_0, \quad Q'(0) = Q_1, \end{gathered} \tag{1.7}$$

where $h \geq 0$, $\omega > 0$, Q_0, Q_1 are given constants [6].

An and Trieu [1] studied a special case of problem (1.1)–(1.4) and (1.7) with $u_0 = u_1 = Q_0 = 0$ and $F(u,u_t) = Ku + \lambda u_t$, with $K \geq 0$, $\lambda \geq 0$ are given constants. In the later case the problem (1.1)–(1.4) and (1.7) is a mathematical model describing the shock of a rigid body and a linear viscoelastic bar resting on a rigid base [1].

From (1.7) we represent $Q(t)$ in terms of Q_0, Q_1, ω, h, $u_{tt}(1,t)$ and then by integrating by parts, we have

$$Q(t) = hu(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds, \tag{1.8}$$

where

$$g(t) = -(Q_0 - hu_0(1))\cos\omega t - \frac{1}{\omega}(Q_1 - hu_1(1))\sin\omega t, \tag{1.9}$$

$$k(t) = h\omega\sin\omega t. \tag{1.10}$$

Bergounioux, Long and Dinh [2] studied problem (1.1), (1.4) with the mixed boundary conditions (1.2), (1.3) standing for

$$u_x(0,t) = hu(0,t) + g(t) - \int_0^t k(t-s)u(0,s)ds, \tag{1.11}$$

$$u_x(1,t) + K_1u(1,t) + \lambda_1u_t(1,t) = 0, \tag{1.12}$$

where

$$g(t) = (Q_0 - hu_0(0))\cos\omega t + \frac{1}{\omega}(Q_1 - hu_1(0))\sin\omega t, \tag{1.13}$$

$$k(t) = h\omega\sin\omega t. \tag{1.14}$$

where $h \geq 0$, $\omega > 0$, Q_0, Q_1, K, λ, K_1, λ_1 are given constants.

Long, Dinh and Diem [7] obtained the unique existence, regularity and asymptotic behavior of the problem (1.1), (1.4) in the case of $\mu(x,t) \equiv 1$, $Q(t) =$

$K_1u(1,t) + \lambda u_t(1,t)$, $u_x(0,t) = P(t)$ where $P(t)$ satisfies (1.7) with $u_{tt}(1,t)$ is replaced by $u_{tt}(0,t)$.

Long, Ut and Truc [9] gave the unique existence, stability, regularity in time variable and asymptotic behavior for the solution of problem (1.1)–(1.5) when $F(u,u_t) = Ku + \lambda u_t$. In this case, the problem (1.1)–(1.5) is the mathematical model describing a shock problem involving a linear viscoelastic bar.

The present paper consists of two main parts. In Part 1 we prove a theorem of global existence and uniqueness of weak solutions (u,Q) of problem (1.1) - (1.5). The proof is based on a Galerkin type approximation associated to various energy estimates-type bounds, weak-convergence and compactness arguments. The main difficulties encountered here are the boundary condition at $x=1$ and with the advent of the nonlinear term of $F(u,u_t)$. In order to solve these particular difficulties, stronger assumptions on the initial conditions u_0, u_1 and parameters K, λ will be modified. We remark that the linearization method in the papers [3, 7] cannot be used in [2, 5, 6]. In addition, in the case of $K_1(t) \equiv K_1 \geq 0$, we receive a theorem related to the asymptotic expansion of the solutions with respect to K, λ, K_1 up to order $N+1$. The results obtained here may be considered as the generalizations of those in An and Trieu [1] and in Long, Dinh, Ut and Truc [2, 3], [5-10].

2. The existence and uniqueness theorem of solution

Put $\Omega = (0,1)$, $Q_T = \Omega\times(0,T)$, $T>0$. We omit the definitions of usual function spaces: $C^m(\overline{\Omega})$, $L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0,1,\dots$ The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or pair of dual scalar product of continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u:(0,T)\to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p dt\Big)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p=\infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, and $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, and $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. We put

$$V = \{v \in H^1(0,1) : v(0) = 0\}, \tag{2.1}$$

$$a(u,v) = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx. \tag{2.2}$$

The set V is a closed subspace of H^1 and on V, $\|v\|_{H^1}$ and $\|v\|_V = \sqrt{a(v,v)} = \|v_x\|$ are two equivalent norms. Then we have the following result.

Lemma 2.1. *The imbedding $V \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \leq \|v\|_V, \text{ for all } v \in V. \tag{2.3}$$

The proof is straightforward and we omit the details. We make the following assumptions:

(H1) $K, \lambda \geq 0$,

(H2) $u_0 \in V \cap H^2$, $u_1 \in H^1$,
(H3) $g, K_1, \lambda_1 \in H^1(0,T)$, $\lambda_1(t) \geq \lambda_0 > 0$, $K_1(t) \geq 0$,
(H4) $k \in H^1(0,T)$,
(H5) $\mu \in C^1(\overline{Q_T})$, $\mu_{tt} \in L^1(0,T;L^\infty)$, $\mu(x,t) \geq \mu_0 > 0$, for all $(x,t) \in \overline{Q_T}$,
(H6) $f, f_t \in L^2(Q_T)$.

Then we have the following theorem.

Theorem 2.2. *Let (H1)–(H6) hold. Then, for every $T > 0$, there exists a unique weak solution (u,Q) of problem* (1.1)–(1.5) *such that*

$$\begin{gathered} u \in L^\infty(0,T;V \cap H^2), \\ u_t \in L^\infty(0,T;V), \quad u_{tt} \in L^\infty(0,T;L^2), \\ u(1,\cdot) \in H^2(0,T), \quad Q \in H^1(0,T). \end{gathered} \tag{2.4}$$

Remark 2.3. (i) Noting that with the regularity obtained by (2.4), it follows that the component u in the weak solution (u,Q) of problem (1.1)–(1.5) satisfies

$$\begin{gathered} u \in L^\infty(0,T;V \cap H^2) \cap C^0(0,T;V) \cap C^1(0,T;L^2), \\ u_t \in L^\infty(0,T;V), u_{tt} \in L^\infty(0,T;L^2), \quad u(1,\cdot) \in H^2(0,T). \end{gathered} \tag{2.5}$$

(ii) From (2.4) we can see that u, u_x, u_t, u_{xx}, u_{xt}, $u_{tt} \in L^\infty(0,T;L^2) \subset L^2(Q_T)$. Also if $(u_0,u_1) \in (V \cap H^2) \times H^1$, then the component u in the weak solution (u,Q) of problem (1.1)–(1.5) belongs to $H^2(Q_T) \cap L^\infty(0,T;V \cap H^2) \cap C^0(0,T;V) \cap C^1(0,T;L^2)$. So the solution is almost classical which is rather natural since the initial data u_0 and u_1 do not belong necessarily to $V \cap C^2(\overline{\Omega})$ and $C^1(\overline{\Omega})$, respectively.

Proof of the Theorem 2.2. The proof consists of Steps four steps.
Step 1. The Galerkin approximation. Let $\{w_j\}$ be a denumerable base of $V \cap H^2$. We find the approximate solution of problem (1.1)- (1.5) in the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{2.6}$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations as follows

$$\begin{aligned} &\langle u_m''(t), w_j\rangle + \langle \mu(t) u_{mx}(t), w_{jx}\rangle + Q_m(t) w_j(1) + \langle F(u_m(t), u_m'(t)), w_j\rangle \\ &= \langle f(t), w_j\rangle, 1 \leq j \leq m, \end{aligned} \tag{2.7}$$

$$Q_m(t) = K_1(t) u_m(1,t) + \lambda_1(t) u_m'(1,t) - \int_0^t k(t-s) u_m(1,s) ds - g(t), \tag{2.8}$$

$$\begin{gathered} u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to u_0 \quad \text{strongly in } V \cap H^2, \\ u_m'(0) = u_{1m} = \sum_{j=1}^{m} \beta_{mj} w_j \to u_1 \quad \text{strongly in } H^1. \end{gathered} \tag{2.9}$$

From the assumptions of Theorem 2.2, system (2.7)–(2.9) has solution (u_m, Q_m) on an interval $[0, T_m]$. The following estimates allow one to take $T_m = T$ for all m.
Step 2. A priori estimates: *A priori estimates I.* Substituting (2.8) into (2.7), then multiplying the j^{th} equation of (2.7) by $c_{mj}'(t)$, summing up with respect to j and

afterwards integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned} S_m(t) &= S_m(0) + \int_0^t ds \int_0^1 \mu'(x,s) u_{mx}^2(x,s)dx + \int_0^t K_1'(s) u_m^2(1,s)ds \\ &\quad + 2\int_0^t g(s) u_m'(1,s)ds + 2\int_0^t u_m'(1,s)\Big(\int_0^s k(s-\tau)u_m(1,\tau)d\tau\Big)ds \\ &\quad + 2\int_0^t \langle f(s), u_m'(s)\rangle ds, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} S_m(t) &= \|u_m'(t)\|^2 + \|\sqrt{\mu(t)}u_{mx}(t)\|^2 + K_1(t)u_m^2(1,t) + \frac{2K}{p}\|u_m(t)\|_{L^p}^p \\ &\quad + 2\lambda\int_0^t \|u_m'(s)\|_{L^q}^q ds + 2\int_0^t \lambda_1(s)|u_m'(1,s)|^2 ds. \end{aligned} \tag{2.11}$$

Using the inequality

$$2ab \le \beta a^2 + \frac{1}{\beta}b^2, \quad \forall a,b\in\mathbb{R}, \forall\beta>0, \tag{2.12}$$

and the following inequalities

$$S_m(t) \ge \|u_m'(t)\|^2 + \mu_0\|u_{mx}(t)\|^2 + 2\lambda_0\int_0^t |u_m'(1,s)|^2 ds, \tag{2.13}$$

$$|u_m(1,t)| \le \|u_m(t)\|_{C^0(\overline{\Omega})} \le \|u_{mx}(t)\| \le \sqrt{\frac{S_m(t)}{\mu_0}}, \tag{2.14}$$

we shall estimate respectively the following terms on the right-hand side of (2.10) as follows

$$\int_0^t ds\int_0^1 \mu'(x,s)u_{mx}^2(x,s)dx \le \frac{1}{\mu_0}\|\mu'\|_{C^0(\overline{Q_T})}\int_0^t S_m(s)ds, \tag{2.15}$$

$$\int_0^t K_1'(s)u_m^2(1,s)ds \le \frac{1}{\mu_0}\int_0^t |K_1'(s)|S_m(s)ds, \tag{2.16}$$

$$2\int_0^t g(s)u_m'(1,s)ds \le \frac{1}{\beta}\|g\|_{L^2(0,T)}^2 + \frac{\beta}{2\lambda_0}S_m(t), \tag{2.17}$$

$$\begin{aligned} &2\int_0^t u_m'(1,s)\Big(\int_0^s k(s-\tau)u_m(1,\tau)d\tau\Big)ds \\ &\le \frac{\beta}{2\lambda_0}S_m(t) + \frac{1}{\beta\mu_0}T\|k\|_{L^2(0,T)}^2\int_0^t S_m(s)ds, \end{aligned} \tag{2.18}$$

$$2\int_0^t \langle f(s), u_m'(s)\rangle ds \le \|f\|_{L^2(Q_T)}^2 + \int_0^t S_m(s)ds. \tag{2.19}$$

In addition, from the assumptions (H1), (H2), (H5) and the embedding $H^1(0,1)\hookrightarrow L^p(0,1)$, $p>1$, there exists a positive constant C_1 such that for all m,

$$S_m(0) = \|u_{1m}\|^2 + \|\sqrt{\mu(0)}u_{0mx}\|^2 + K_1(0)u_{0m}^2(1) + \frac{2K}{p}\|u_{0m}\|_{L^p}^p \le C_1 \tag{2.20}$$

Combining (2.10), (2.11), (2.15)–(2.20) and choosing $\beta = \frac{\lambda_0}{2}$, we obtain

$$S_m(t) \le M_T^{(1)} + \int_0^t N_T^{(1)}(s) S_m(s) ds, \tag{2.21}$$

where

$$\begin{gathered} M_T^{(1)} = 2C_1 + \frac{4}{\lambda_0}\|g\|^2_{L^2(0,T)} + 2\|f\|^2_{L^2(Q_T)}, \\ N_T^{(1)}(s) = 2[1 + \frac{2}{\lambda_0\mu_0} T\|k\|^2_{L^2(0,T)} + \frac{1}{\mu_0}\|\mu'\|_{C^0(\overline{Q_T})} + \frac{1}{\mu_0}|K_1'(s)|], \\ N_T^{(1)} \in L^1(0,T). \end{gathered} \tag{2.22}$$

By Gronwall's lemma, we deduce from (2.21), (2.22), that

$$S_m(t) \le M_T^{(1)} \exp(\int_0^t N_T^{(1)}(s) ds) \le C_T, \quad \text{for all } t \in [0,T]. \tag{2.23}$$

A priori estimates II. Now differentiating (2.7) with respect to t , we have

$$\begin{aligned} &\langle u_m'''(t), w_j\rangle + \langle \mu(t) u_{mx}'(t) + \mu'(t) u_{mx}(t), w_{jx}\rangle + Q_m'(t) w_j(1) \\ &+ K(p-1)\langle |u_m|^{p-2} u_m', w_j\rangle + \lambda(q-1)\langle |u_m'|^{q-2} u_m'', w_j\rangle \\ &= \langle f'(t), w_j\rangle, \end{aligned} \tag{2.24}$$

for all $1 \le j \le m$. Multiplying the j^{th} equation of (2.24) by $c_{mj}''(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t, we have after some rearrangements

$$\begin{aligned} X_m(t) &= X_m(0) + 2\langle \mu'(0) u_{0mx}, u_{1mx}\rangle - 2\langle \mu'(t) u_{mx}(t), u_{mx}'(t)\rangle \\ &\quad + 2\int_0^t \langle \mu''(s) u_{mx}(s), u_{mx}'(s)\rangle ds + 3\int_0^t ds \int_0^1 \mu'(x,s) |u_{mx}'(x,s)|^2 dx \\ &\quad - 2\int_0^t \big(K_1'(s) - k(0)\big) u_m(1,s) u_m''(1,s) ds \\ &\quad - 2\int_0^t \big(K_1(s) + \lambda_1'(s)\big) u_m'(1,s) u_m''(1,s) ds \\ &\quad + 2\int_0^t u_m''(1,s)\big(g'(s) + \int_0^s k'(s-\tau) u_m(1,\tau) d\tau\big) ds \\ &\quad - 2(p-1)K\int_0^t \langle |u_m(s)|^{p-2} u_m'(s), u_m''(s)\rangle ds + 2\int_0^t \langle f'(s), u_m''(s)\rangle ds, \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} X_m(t) &= \|u_m''(t)\|^2 + \|\sqrt{\mu(t)} u_{mx}'(t)\|^2 + 2\int_0^t \lambda_1(s) |u_m''(1,s)|^2 ds \\ &\quad + \frac{8}{q^2}(q-1)\lambda \int_0^t \|\frac{\partial}{\partial s}\big(|u_m'(s)|^{\frac{q-2}{2}} u_m'(s)\big)\|^2 ds. \end{aligned} \tag{2.26}$$

From the assumptions (H1), (H2) , (H5), (H6) and the imbedding $H^1(0,1) \hookrightarrow L^p(0,1)$, $p > 1$, there exists positive constant $\widetilde{D}_1$ depending on μ, u_0, u_1, K, λ, p,

q, f such that

$$\begin{aligned} &X_m(0) + 2\langle \mu'(0)u_{0mx}, u_{1mx}\rangle \\ &= \|u''_m(0)\|^2 + \|\sqrt{\mu(0)}u_{1mx}\|^2 + 2\langle \mu'(0)u_{0mx}, u_{1mx}\rangle \\ &\le \|\mu(0)u_{0mxx} + \mu_x(0)u_{0mx} - K|u_{0m}|^{p-2}u_{0m} - \lambda|u_{1m}|^{q-2}u_{1m} + f(0)\|^2 \\ &\quad + \|\sqrt{\mu(0)}u_{1mx}\|^2 + 2\|\mu'(0)\|_{L^\infty(\Omega)}\|u_{0mx}\|\|u_{1mx}\| \le \widetilde{D}_1, \end{aligned} \tag{2.27}$$

for all m. Using the inequality (2.12) where β is replaced by β_1 and the following inequalities

$$X_m(t) \ge \|u''_m(t)\|^2 + \mu_0\|u'_{mx}(t)\|^2 + 2\lambda_0\int_0^t |u''_m(1,s)|^2 ds, \tag{2.28}$$

$$|u_m(1,t)| \le \|u_m(t)\|_{C^0(\overline{\Omega})} \le \|u_{mx}(t)\| \le \sqrt{\frac{S_m(t)}{\mu_0}} \le \sqrt{\frac{C_T}{\mu_0}}, \tag{2.29}$$

$$|u'_m(1,t)| \le \|u'_m(t)\|_{C^0(\overline{\Omega})} \le \|u'_{mx}(t)\| \le \sqrt{\frac{X_m(t)}{\mu_0}}, \tag{2.30}$$

we estimate, without difficulty the following terms in the right-hand side of (2.25) as follows

$$-2\langle \mu'(t)u_{mx}(t), u'_{mx}(t)\rangle \le \beta_1 X_m(t) + \frac{1}{\beta_1\mu_0}C_T\|\mu'\|^2_{C^0(\overline{Q_T})}, \tag{2.31}$$

$$\begin{aligned} &2\int_0^t \langle \mu''(s)u_{mx}(s), u'_{mx}(s)\rangle ds \\ &\le 2\int_0^t \|\mu''(s)\|_{L^\infty}\|u_{mx}(s)\|\|u'_{mx}(s)\| ds \\ &\le \beta_1\frac{1}{\mu_0}\int_0^t \|\mu''(s)\|_{L^\infty}\|u_{mx}(s)\|^2 ds + \beta_1\mu_0\int_0^t \|\mu''(s)\|_{L^\infty}\|u'_{mx}(s)\|^2 ds \\ &\le \beta_1\int_0^t \|\mu''(s)\|_{L^\infty}X_m(s)ds + \frac{C_T}{\beta_1\mu_0}\|\mu''\|_{L^1(0,T;L^\infty)}, \end{aligned} \tag{2.32}$$

$$3\int_0^t ds\int_0^1 \mu'(x,s)|u'_{mx}(x,s)|^2 dx \le \frac{3}{\mu_0}\|\mu'\|_{C^0(\overline{Q_T})}\int_0^t X_m(s)ds, \tag{2.33}$$

$$-2\int_0^t (K'_1(s) - k(0))u_m(1,s)u''_m(1,s)ds \le \frac{\beta_1}{2\lambda_0}X_m(t) + \frac{C_T}{\beta_1\mu_0}\|K'_1 - k(0)\|^2_{L^2(0,T)}, \tag{2.34}$$

$$\begin{aligned} &-2\int_0^t (K_1(s) + \lambda'_1(s))u'_m(1,s)u''_m(1,s)ds \\ &\le \frac{2}{\beta_1\mu_0}\int_0^t (|K_1(s)|^2 + |\lambda'_1(s)|^2)X_m(s)ds + \frac{\beta_1}{2\lambda_0}X_m(t), \end{aligned} \tag{2.35}$$

$$
\begin{aligned}
&2\int_0^t u_m''(1,s)(g'(s)+\int_0^s k'(s-\tau)u_m(1,\tau)d\tau)ds \\
&\le \frac{\beta_1}{2\lambda_0}X_m(t)+\frac{2}{\beta_1}[\|g'\|^2_{L^2(0,T)}+\frac{C_T}{\mu_0}T\|k'\|^2_{L^1(0,T)}],
\end{aligned}
\tag{2.36}
$$

$$
-2(p-1)K\int_0^t \langle |u_m(s)|^{p-2}u_m'(s), u_m''(s)\rangle ds \le 2\frac{p-1}{\sqrt{\mu_0}}K(\frac{C_T}{\mu_0})^{\frac{p-2}{2}}\int_0^t X_m(s)ds, \tag{2.37}
$$

$$
2\int_0^t \langle f^{/}(s), u_m''(s)\rangle ds \le \beta_1\int_0^t X_m(s)ds+\frac{1}{\beta_1}\|f^{\,\prime}\|^2_{L^2(Q_T)}. \tag{2.38}
$$

In terms of (2.25), (2.27), (2.31)–(2.38) and by the choice of $\beta_1>0$ such that

$$
\beta_1(1+\frac{3}{2\lambda_0})\le\frac{1}{2},
$$

we obtain

$$
X_m(t)\le \widetilde{M}_T^{(2)}+\int_0^t N_T^{(2)}(s)X_m(s)ds, \tag{2.39}
$$

where

$$
\begin{aligned}
\widetilde{M}_T^{(2)} &= 2\widetilde{D}_1+\frac{2C_T}{\beta_1\mu_0}[\|\mu'\|^2_{C^0(\overline{Q_T})}+\|\mu''\|_{L^1(0,T;L^\infty)}+\|K_1'-k(0)\|^2_{L^2(0,T)}] \\
&\quad+\frac{2}{\beta_1}[2\|g'\|^2_{L^2(0,T)}+\frac{2C_T}{\mu_0}T\|k'\|^2_{L^1(0,T)}+\|f^{\,\prime}\|^2_{L^2(Q_T)}], \\
N_T^{(2)}(s) &= 2\beta_1+4\frac{p-1}{\sqrt{\mu_0}}K(\frac{C_T}{\mu_0})^{\frac{p-2}{2}}+\frac{6}{\mu_0}\|\mu'\|_{C^0(\overline{Q_T})}+2\beta_1\|\mu''(s)\|_{L^\infty} \\
&\quad+\frac{4}{\beta_1\mu_0}(|K_1(s)|^2+|\lambda_1'(s)|^2), \\
&N_T^{(2)}\in L^1(0,T).
\end{aligned}
\tag{2.40}
$$

From (2.39)–(2.40) and applying Gronwall's inequality, we obtain that

$$
X_m(t)\le M_T^{(2)}\exp\big(\textstyle\int_0^t N_T^{(2)}(s)ds\big)\le C_T \quad \text{for all } t\in[0,T]. \tag{2.41}
$$

On the other hand, we deduce from (2.8), (2.11), (2.23), (2.26) and (2.41), that

$$
\begin{aligned}
\|Q_m'\|^2_{L^2(0,T)} &\le \frac{5D_T}{2\lambda_0}\|\lambda_1\|^2_\infty+\frac{5T^2C_T}{\mu_0}\|k'\|^2_{L^2(0,T)}+5\|g'\|^2_{L^2(0,T)} \\
&\quad+\frac{5D_T}{\mu_0}(\|K_1+\lambda_1'\|^2_{L^2(0,T)}+\|K_1'-k(0)\|^2_{L^2(0,T)}),
\end{aligned}
\tag{2.42}
$$

where $\|\lambda_1\|_\infty=\|\lambda_1\|_{L^\infty(0,T)}$. From the assumptions (H3) and (H4), we deduce from (2.42), that

$$
\|Q_m\|_{H^1(0,T)}\le C_T \quad \text{for all } m, \tag{2.43}
$$

where C_T is a positive constant depending only on T.

Step 3. Limiting process. From (2.11), (2.23), (2.26), (2.41) and (2.43), we deduce the existence of a subsequence of $\{(u_m,\,Q_m)\}$ still also so denoted, such that

$$\begin{gathered} u_m \to u \quad \text{in } L^\infty(0,T;V) \quad \text{weak*}, \\ u_m' \to u' \quad \text{in } L^\infty(0,T;V) \quad \text{weak*}, \\ u_m'' \to u'' \quad \text{in } L^\infty(0,T;L^2) \quad \text{weak*}, \\ u_m(1,\cdot) \to u(1,\cdot) \quad \text{in } H^2(0,T) \quad \text{weakly}, \\ Q_m \to \widetilde{Q} \quad \text{in } H^1(0,T) \quad \text{weakly}. \end{gathered} \tag{2.44}$$

By the compactness lemma in Lions [4: p.57] and the imbedding $H^2(0,T) \hookrightarrow C^1([0,T])$, we can deduce from $(2.44)_{1,2,3,4,5}$ the existence of a subsequence still denoted by $\{(u_m, Q_m)\}$ such that

$$\begin{gathered} u_m \to u \quad \text{strongly in } L^2(Q_T), \\ u_m' \to u' \quad \text{strongly in } L^2(Q_T), \\ u_m(1,\cdot) \to u(1,\cdot) \quad \text{strongly in } C^1([0,T]), \\ Q_m \to \widetilde{Q} \quad \text{strongly in } C^0([0,T]). \end{gathered} \tag{2.45}$$

From (2.8) and $(2.45)_3$ we have that

$$Q_m(t) \to K_1(t)u(1,t) + \lambda_1(t)u'(1,t) - g(t) - \int_0^t k(t-s)\,u(1,s)ds \equiv Q(t) \tag{2.46}$$

strongly in $C^0([0,T])$.

Combining $(2.45)_4$ and (2.46), we conclude that

$$Q(t) = \widetilde{Q}(t). \tag{2.47}$$

By means of the inequality

$$\big||x|^{\delta-2}x - |y|^{\delta-2}y\big| \le (\delta-1)R^{\delta-2}|x-y| quad \forall x,y \in [-R;R], \tag{2.48}$$

for all $R>0$, $\delta \ge 2$, it follows from (2.39), that

$$||u_m|^{p-2}u_m - |u|^{p-2}u| \le (p-1)R^{p-2}|u_m-u| \quad \text{with } R = \sqrt{\frac{C_T}{\mu_0}}. \tag{2.49}$$

Hence, it follows from $(2.45)_1$ and (2.49), that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{strongly in } L^2(Q_T). \tag{2.50}$$

By the same way, we deduce from (2.48), with $R = \sqrt{\frac{C_T}{\mu_0}}$ and $(2.44)_3$, $(2.45)_2$, that

$$|u_m'|^{q-2}u_m' \to |u'|^{q-2}u' \quad \text{strongly in } L^2(Q_T). \tag{2.51}$$

Passing to the limit in (2.7)–(2.9) by $(2.44)_{1,5}$, (2.46), (2.47), (2.50) and (2.51) we have (u,Q) satisfying

$$\begin{aligned} &\langle u''(t), v\rangle + \langle \mu(t)u_x(t), v_x\rangle + Q(t)v(1) + \langle K|u|^{p-2}u + \lambda|u'|^{q-2}u', v\rangle \\ &= \langle f(t), v\rangle, \quad \forall v \in V, \end{aligned} \tag{2.52}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{2.53}$$

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u_t(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds, \tag{2.54}$$

On the other hand, from $(2.44)_5$, (2.52) and assumptions (H5)-(H6) we have

$$u_{xx} = \frac{1}{\mu(x,t)}(u'' - \mu_x u_x + K|u|^{p-2}u + \lambda|u'|^{q-2}u' - f) \in L^\infty(0,T;L^2). \tag{2.55}$$

Thus $u \in L^\infty(0,T;V \cap H^2)$ and the existence of the theorem is proved completely.
Step 4. Uniqueness of the solution. Let (u_1, Q_1), (u_2, Q_2) be two weak solutions of problem (1.1)–(1.5), such that

$$\begin{gathered} u_i \in L^\infty(0,T;V\cap H^2), \quad u_i' \in L^\infty(0,T;H^1), \quad u_i'' \in L^\infty(0,T;L^2), \\ u_i(1,\cdot) \in H^2(0,T), \quad Q_i \in H^1(0,T), \quad i=1,2. \end{gathered} \tag{2.56}$$

Then (u, Q) with $u = u_1 - u_2$ and $Q = Q_1 - Q_2$ satisfy the variational problem

$$\begin{gathered} \langle u''(t), v\rangle + \langle \mu(t) u_x(t), v_x\rangle + Q(t)v(1) + K\langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2 \ , v\rangle \\ +\lambda\langle |u_1'|^{q-2}u_1' - |u_2'|^{q-2}u_2', v\rangle = 0 \quad \forall v \in V, \\ u(0) = \quad u'(0) = 0, \end{gathered} \tag{2.57}$$

and

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u'(1,t) - \int_0^t k(t-s)\, u(1,s)ds. \tag{2.58}$$

We take $v = u'$ in $(2.57)_1$, and integrating with respect to t, we obtain

$$\begin{aligned} \sigma(t) \le &\int_0^t \|\sqrt{|\mu'(s)|}u_x(s)\|^2 ds + \int_0^t K_1'(s)u^2(1,s)ds \\ &+ 2\int_0^t u'(1,s)ds \int_0^s k(s-\tau)\, u(1,\tau)d\tau \\ &- 2K\int_0^t \langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, u'\rangle ds, \end{aligned} \tag{2.59}$$

where

$$\sigma(t) = \|u'(t)\|^2 + \|\sqrt{\mu(t)}u_x(t)\|^2 + K_1(t)u^2(1,t) + 2\int_0^t \lambda_1(s)|u'(1,s)|^2 ds. \tag{2.60}$$

Noting that

$$\sigma(t) \ge \|u'(t)\|^2 + \mu_0\|u_x(t)\|^2 + 2\lambda_0 \int_0^t |u'(1,s)|^2 ds, \tag{2.61}$$

$$|u(1,t)| \le \|u(t)\|_{C^0(\overline{\Omega})} \le \|u_x(t)\| \le \sqrt{\frac{\sigma(t)}{\mu_0}}. \tag{2.62}$$

We again use inequalities (2.12) and (2.48) with $\delta = p$, $R = \max_{i=1,2}\|u_i\|_{L^\infty(0,T;V)}$, then, it follows from (2.59)–(2.62), that

$$\begin{aligned} \sigma(t) \le &\frac{1}{\mu_0}\int_0^t (\|\mu'\|_{C^0(\overline{Q_T})} + |K_1'(s)|)\sigma(s)ds + \frac{\beta}{2\lambda_0}\sigma(t) \\ &+ \frac{T}{\beta\mu_0}\|k\|^2_{L^2(0,T)}\int_0^t \sigma(\tau)d\tau + \frac{1}{\sqrt{\mu_0}}(p-1)KR^{p-2}\int_0^t \sigma(s)ds. \end{aligned} \tag{2.63}$$

Choosing $\beta > 0$, such that $\beta\frac{1}{2\lambda_0} \le 1/2$, we obtain from (2.63), that

$$\sigma(t) \le \int_0^t q_1(s)\sigma(s)ds, \tag{2.64}$$

where

$$
\begin{gathered}
q_1(s)=\frac{2}{\mu_0}(\|\mu'\|_{C^0(\overline{Q_T})}+|K_1'(s)|)+\frac{2T}{\beta\mu_0}\|k\|^2_{L^2(0,T)}+\frac{2}{\sqrt{\mu_0}}(p-1)KR^{p-2},\\
q_1\in L^2(0,T).
\end{gathered}\tag{2.65}
$$

By Gronwall's lemma, we deduce that $\sigma\equiv 0$ and Theorem 2.2 is completely proved. $\square$

Remark 2.4. In the case p, $q>2$, $K<0$, and $\lambda<0$, the question of existence for the solutions of problem (1.1)–(1.5) is still open. However we have also obtained the answer of problem (1.1)–(1.5) when $p=q=2$ and K, $\lambda\in\mathbb{R}$ published in [9].

3. Asymptotic expansion of the solution

In this part, we consider two given functions u_0, u_1 as $\widetilde{u}_0$, $\widetilde{u}_1$, respectively. Then we assume that $K_1(t)=K_1$ is a nonnegative constant and $(\widetilde{u}_0,\ \widetilde{u}_1,\ f,\ \mu,\ g,\ k,\ \lambda_1)$ satisfy the assumptions (H2)-(H6). Let $(K,\lambda,K_1)\in\mathbb{R}^3_+$. By Theorem 2.2, the problem (1.1)–(1.5) has a unique weak solution (u,Q) depending on (K,λ,K_1):

$$
u=u(K,\lambda,K_1),\quad Q=Q(K,\lambda,K_1).
$$

We consider the following perturbed problem, where K, λ, K_1 are small parameters such that, $0\le K\le K_*$, $0\le\lambda\le\lambda_*$, $0\le K_1\le K_{1*}$:

$$
\begin{gathered}
Au\equiv u_{tt}-\frac{\partial}{\partial x}(\mu(x,t)u_x)=-KF(u)-\lambda G(u_t)+f(x,t),\quad 0<x<1,0<t<T,\\
u(0,t)=0,\\
Bu\equiv-\mu(1,t)u_x(1,t)=Q(t),\\
u(x,0)=\widetilde{u}_0(x),\quad u_t(x,0)=\widetilde{u}_1(x),\\
Q(t)=K_1u(1,t)+\lambda_1(t)u_t(1,t)-g(t)-\int_0^t k(t-s)u(1,s)ds,
\end{gathered}\tag{3.1}
$$

where $F(u)=|u|^{p-2}u$, $G(u_t)=|u_t|^{q-2}u_t$, $p>N\ge 2$, $q>N\ge 2$. We shall study the asymptotic expansion of the solution of problem (P_{K,λ,K_1}) with respect to (K, λ, K_1). We use the following notation. For a multi-index $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\mathbb{Z}^3_+$ and $\overrightarrow{K}=(K,\lambda,K_1)\in\mathbb{R}^3_+$, we put

$$
\begin{gathered}
|\gamma|=\gamma_1+\gamma_2+\gamma_3,\quad \gamma!=\gamma_1!\gamma_2!\gamma_3!,\\
\|\overrightarrow{K}\|=\sqrt{K^2+\lambda^2+K_1^2},\quad \overrightarrow{K}^\gamma=K^{\gamma_1}\lambda^{\gamma_2}K_1^{\gamma_3},\\
\alpha,\beta\in\mathbb{Z}^3_+,\quad \beta\le\alpha\Longleftrightarrow\beta_i\le\alpha_i\quad\forall i=1,2,3.
\end{gathered}
$$

First, we shall need the following Lemma.

Lemma 3.1. *Let m, $N\in\mathbb{N}$ and $v_\alpha\in\mathbb{R}$, $\alpha\in\mathbb{Z}^3_+$, $1\le|\alpha|\le N$. Then*

$$
\Big(\sum_{1\le|\alpha|\le N}v_\alpha\overrightarrow{K}^\alpha\Big)^m=\sum_{m\le|\alpha|\le mN}T^{(m)}[v]_\alpha\overrightarrow{K}^\alpha,\tag{3.2}
$$

where the coefficients $T^{(m)}[v]_\alpha$, $m \le |\alpha| \le mN$ depending on $v = (v_\alpha)$, $\alpha \in \mathbb{Z}_+^3$, $1 \le |\alpha| \le N$ are defined by the recurrence formulas

$$\begin{gathered} T^{(1)}[v]_\alpha = v_\alpha, \quad 1 \le |\alpha| \le N, \\ T^{(m)}[v]_\alpha = \sum_{\beta \in A_\alpha^{(m)}} v_{\alpha-\beta} T^{(m-1)}[v]_\beta, \quad m \le |\alpha| \le mN, m \ge 2, \\ A_\alpha^{(m)} = \{\beta \in \mathbb{Z}_+^3 : \beta \le \alpha, 1 \le |\alpha - \beta| \le N, m-1 \le |\beta| \le (m-1)N\}. \end{gathered} \tag{3.3}$$

The proof of the above lemma can be found in [11]. Let $(u_0,\ Q_0) \equiv (u_{0,0,0},\ Q_{0,0,0})$ be a unique weak solution of the following problem (as in Theorem 2.2) corresponding to $(K, \lambda, K_1) = (0,0,0)$; i.e.,

$$\begin{gathered} Au_0 = P_{0,0,0} \equiv f(x,t), \quad 0 < x < 1, 0 < t < T, \\ u_0(0,t) = 0, \quad Bu_0 = Q_0(t), \\ u_0(x,0) = \widetilde{u}_0(x), \quad u_0'(x,0) = \widetilde{u}_1(x), \\ Q_0(t) = -g(t) + \lambda_1(t) u_0'(1,t) - \int_0^t k(t-s) u_0(1,s)ds, \\ u_0 \in C^0(0,T;V) \cap C^1(0,T;L^2) \cap L^\infty(0,T; V \cap H^2), \\ u_0' \in L^\infty(0,T;H^1), \quad u_0'' \in L^\infty(0,T;L^2), \\ u_0(1,\cdot) \in H^2(0,T), \quad Q_0 \in H^1(0,T). \end{gathered}$$

Let us consider the sequence of weak solutions (u_γ, Q_γ), $\gamma \in \mathbb{Z}_+^3$, $1 \le |\gamma| \le N$, defined by the following problems $(\widetilde{P}_\gamma)$:

$$\begin{gathered} Au_\gamma = P_\gamma, \quad 0 < x < 1,\ 0 < t < T, \\ u_\gamma(0,t) = 0, \quad Bu_\gamma = Q_\gamma(t), \\ u_\gamma(x,0) = u_\gamma'(x,0) = 0, \\ Q_\gamma(t) = \widehat{Q}_\gamma(t) + \lambda_1(t) u_\gamma'(1,t) - \int_0^t k(t-s) u_\gamma(1,s)ds, \\ u_\gamma \in C^0(0,T;V) \cap C^1(0,T;L^2) \cap L^\infty(0,T; V \cap H^2), \\ u_\gamma' \in L^\infty(0,T;H^1), \quad u_\gamma'' \in L^\infty(0,T;L^2), \\ u_\gamma(1,\cdot) \in H^2(0,T), \quad Q_\gamma \in H^1(0,T), \end{gathered} \tag{3.4}$$

where P_γ, $\widehat{Q}_\gamma$, $|\gamma| \le N$ are defined by the recurrence formula

$$\begin{gathered} \widehat{Q}_\gamma(t) = 0, \quad 1 \le |\gamma| \le N, \quad \gamma_3 = 0, \\ \widehat{Q}_\gamma(t) = u_{\gamma_1,\gamma_2,\gamma_3-1}(1,t), \quad 1 \le |\gamma| \le N, \quad \gamma_3 \ge 1, \end{gathered} \tag{3.5}$$

and

$$
\begin{gathered}
P_{1,0,0}=-F(u_0),\quad P_{0,1,0}=-G(u_0'),\quad P_{0,0,1}=0,\\
P_{0,0,\gamma_3}=0,\quad 2\le\gamma_3\le N,\\
P_{0,\gamma_2,\gamma_3}=-\sum_{m=1}^{|\gamma|-1}\frac{1}{m!}G^{(m)}(u_0')T^{(m)}[u']_{0,\gamma_2-1,\gamma_3},\quad 2\le\gamma_2+\gamma_3\le N,\gamma_2\ge 1,\\
P_{\gamma_1,0,\gamma_3}=-\sum_{m=1}^{|\gamma|-1}\frac{1}{m!}F^{(m)}(u_0)T^{(m)}[u]_{\gamma_1-1,0,\gamma_3},\quad 2\le\gamma_1+\gamma_3\le N,\gamma_1\ge 1,\\
P_{\gamma}=-\sum_{m=1}^{|\gamma|-1}\frac{1}{m!}[F^{(m)}(u_0)T^{(m)}[u]_{\gamma_1-1,\gamma_2,\gamma_3}+G^{(m)}(u_0')T^{(m)}[u']_{\gamma_1,\gamma_2-1,\gamma_3}],\\
2\le|\gamma|\le N,\gamma_1\ge 1,\gamma_2\ge 1,
\end{gathered}
\tag{3.6}
$$

here we have used the notation $u=(u_\gamma)$, $\gamma\in\mathbb{Z}_+^3$, $|\gamma|\le N$. Let $(u,Q)=(u_{K,\lambda,K_1},Q_{K,\lambda,K_1})$ be a unique weak solution of problem (3.1). Then (v,R), with

$$
v=u-\sum_{|\gamma|\le N}u_\gamma\overrightarrow{K}^\gamma\equiv u-h,\quad R=Q-\sum_{|\gamma|\le N}Q_\gamma\overrightarrow{K}^\gamma,
$$

satisfies the problem

$$
\begin{gathered}
\begin{aligned}
Av\equiv v_{tt}-\frac{\partial}{\partial x}(\mu(x,t)v_x)&\\
=-K\big[F(v+h)-F(h)\big]-\lambda\big[G(v_t+h_t)-G(h_t)\big]&\\
+\widetilde{E}_N(\overrightarrow{K}),\quad 0<x<1,\ 0<t<T,&
\end{aligned}\\
v(0,t)=0,\quad Bv\equiv-\mu(1,t)v_x(1,t)=R(t),\\
R(t)=K_1v(1,t)+\lambda_1(t)v_t(1,t)+\widetilde{G}_N(\overrightarrow{K})-\int_0^t k(t-s)v(1,s)ds,\\
v(x,0)=v_t(x,0)=0,\\
v\in C^0(0,T;V)\cap C^1(0,T;L^2)\cap L^\infty(0,T;V\cap H^2),\\
v'\in L^\infty(0,T;H^1),\quad v''\in L^\infty(0,T;L^2),\\
v(1,\cdot)\in H^2(0,T),\quad R\in H^1(0,T),
\end{gathered}
\tag{3.7}
$$

where

$$
\widetilde{E}_N(\overrightarrow{K})=f(x,t)-KF(h)-\lambda G(h_t)-\sum_{|\gamma|\le N}P_\gamma\overrightarrow{K}^\gamma,\tag{3.8}
$$

$$
\widetilde{G}_N(\overrightarrow{K})=\sum_{|\gamma|=N+1,\gamma_3\ge 1}u_{\gamma_1,\gamma_2,\gamma_3-1}(1,t)\overrightarrow{K}^\gamma.\tag{3.9}
$$

Then, we have the following lemma.

Lemma 3.2. *Let (H2)–(H6) hold. Then*

$$
\|\widetilde{E}_N(\overrightarrow{K})\|_{L^\infty(0,T;V)}\le\widetilde{C}_{1N}\|\overrightarrow{K}\|^{N+1},\tag{3.10}
$$

$$
\|\widetilde{G}_N(\overrightarrow{K})\|_{H^2(0,T)}\le\widetilde{C}_{2N}\|\overrightarrow{K}\|^{N+1},\tag{3.11}
$$

for all $\overrightarrow{K} = (K,\ \lambda,\ K_1) \in \mathbb{R}^3_+$, $\|\overrightarrow{K}\| \le \|\overrightarrow{K_*}\|$ *with* $\overrightarrow{K_*} = (K_*, \lambda_*, K_{1*})$, *where* $\widetilde{C}_{1N}$, $\widetilde{C}_{2N}$ *are positive constants depending only on the constants* $\|\overrightarrow{K_*}\|$, $\|u_\gamma\|_{L^\infty(0,T;V)}$, $\|u'_\gamma\|_{L^\infty(0,T;V)}$, $(|\gamma| \le N)$, $\|u_{\gamma_1,\gamma_2,\gamma_3-1}(1,\cdot)\|_{H^2(0,T)}$, $(|\gamma| = N+1,\ \gamma_3 \ge 1)$.

Proof. In the case of $N = 1$, the proof of Lemma 3.2 is easy, hence we omit the details, which we only prove with $N \ge 2$. Put

$$h = u_0 + h_1, h_1 = \sum_{1\le|\gamma|\le N} u_\gamma \overrightarrow{K}^\gamma. \tag{3.12}$$

By using Taylor's expansion of the function $F(h) = F(u_0 + h_1)$ around the point u_0 up to order $N-1$, we obtain

$$F(h) = F(u_0) + \sum_{m=1}^{N-1} \frac{1}{m!} F^{(m)}(u_0) h_1^m + \frac{1}{N!} F^{(N)}(u_0 + \theta_1 h_1) h_1^N, \tag{3.13}$$

where $0 < \theta_1 < 1$. By Lemma 3.1, we obtain from (3.13), after some rearrangements in order to of $\overrightarrow{K}^\gamma$, that

$$\begin{aligned} KF(h) &= KF(u_0) \\ &\quad + \sum_{2\le|\gamma|\le N,\quad \gamma_1\ge 1} \sum_{m=1}^{|\gamma|-1} \frac{1}{m!} F^{(m)}(u_0) T^{(m)}[u]_{\gamma_1-1,\gamma_2,\gamma_3} \overrightarrow{K}^\gamma + R^{(1)}(F, \overrightarrow{K}), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} &R^{(1)}(F, \overrightarrow{K}) \\ &= K \sum_{m=1}^{N-1} \frac{1}{m!} F^{(m)}(u_0) \sum_{N\le|\gamma|\le mN} T^{(m)}[u]_\gamma \overrightarrow{K}^\gamma + \frac{1}{N!} F^{(N)}(u_0 + \theta_1 h_1) K h_1^N, \end{aligned} \tag{3.15}$$

Similarly, we use Taylor's expansion of the function $G(h_t) = G(u'_0 + h'_1)$ around the point u'_0 up to order $N-1$, we obtain

$$\begin{aligned} \lambda G(h_t) = \lambda G(u'_0) &+ \sum_{2\le|\gamma|\le N,\quad \gamma_2\ge 1} \sum_{m=1}^{|\gamma|-1} \frac{1}{m!} G^{(m)}(u'_0) T^{(m)}[u']_{\gamma_1,\gamma_2-1,\gamma_3} \overrightarrow{K}^\gamma \\ &+ R^{(2)}(G, \overrightarrow{K}), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} &R^{(2)}(G, \overrightarrow{K}) \\ &= \lambda \sum_{m=1}^{N-1} \frac{1}{m!} G^{(m)}(u'_0) \sum_{N\le|\gamma|\le mN} T^{(m)}[u']_\gamma \overrightarrow{K}^\gamma + \lambda \frac{1}{N!} G^{(N)}(u'_0 + \theta_2 h'_1)(h'_1)^N, \end{aligned} \tag{3.17}$$

and $0 < \theta_2 < 1$. Combining (3.6), (3.8), (3.14)–(3.17), we then obtain

$$\begin{aligned}
\widetilde{E}_N(\overrightarrow{K}) &= f(x,t) - KF(u_0) - \lambda G(u_0') \\
&\quad - \sum_{2\le|\gamma|\le N,\ \gamma_1\ge 1} \sum_{m=1}^{|\gamma|-1} \frac{1}{m!} F^{(m)}(u_0) T^{(m)}[u]_{\gamma_1-1,\gamma_2,\gamma_3} \overrightarrow{K}^{\gamma} \\
&\quad - \sum_{2\le|\gamma|\le N,\ \gamma_2\ge 1} \sum_{m=1}^{|\gamma|-1} \frac{1}{m!} G^{(m)}(u_0') T^{(m)}[u']_{\gamma_1,\gamma_2-1,\gamma_3} \overrightarrow{K}^{\gamma} \\
&\quad - \sum_{|\gamma|\le N} P_\gamma \overrightarrow{K}^{\gamma} - R^{(1)}(F,\overrightarrow{K}) - R^{(2)}(G,\overrightarrow{K}) \\
&= -R^{(1)}(F,\overrightarrow{K}) - R^{(2)}(G,\overrightarrow{K}).
\end{aligned} \tag{3.18}$$

We shall estimate respectively the following terms on the right-hand side of (3.18).
Estimate for $R^{(1)}(F,\overrightarrow{K})$. By the boundedness of the functions u_γ, $\gamma \in \mathbb{Z}_+^3$, $|\gamma| \le N$ in the function space $L^\infty(0,T;H^1)$, we obtain from (3.13), that

$$\begin{aligned}
&\|R^{(1)}(F,\overrightarrow{K})\|_{L^\infty(0,T;L^2)} \\
&\le |K| \sum_{m=1}^{N-1} \sum_{N\le|\gamma|\le mN} \frac{1}{m!} \|F^{(m)}(u_0)\|_{L^\infty(0,T;V)} \|T^{(m)}[u]_\gamma\|_{L^\infty(0,T;L^2)} |\overrightarrow{K}^{\gamma}| \\
&\quad + \frac{1}{N!} K \|F^{(N)}(u_0+\theta_1 h_1)\|_{L^\infty(0,T;V)} \|h_1\|^N_{L^\infty(0,T;V)}.
\end{aligned} \tag{3.19}$$

Using the inequality

$$|\overrightarrow{K}^{\gamma}| \le \|\overrightarrow{K}\|^{|\gamma|}, \text{ for all } \gamma \in \mathbb{Z}_+^3, |\gamma| \le N, \tag{3.20}$$

it follows from (3.19) and (3.20) that

$$\|R^{(1)}(F,\overrightarrow{K})\|_{L^\infty(0,T;L^2)} \le \widetilde{C}_{1N}^{(1)} \|\overrightarrow{K}\|^{N+1}, \quad \|\overrightarrow{K}\| \le \|\overrightarrow{K_*}\|, \tag{3.21}$$

where

$$\begin{aligned}
\widetilde{C}_{1N}^{(1)} &= \sum_{m=1}^{N-1} \sum_{N\le|\gamma|\le mN} C_{p-1}^m \|u_0\|^{p-m-1}_{L^\infty(0,T;V)} \|T^{(m)}[\widehat{u}]_\gamma\|_{L^\infty(0,T;L^2)} \|\overrightarrow{K_*}\|^{|\gamma|-N} \\
&\quad + C_{p-1}^N \|\overrightarrow{K_*}\|^{-N} \Big(\sum_{|\gamma|\le N} \|u_\gamma\|_{L^\infty(0,T;V)} \|\overrightarrow{K_*}\|^{|\gamma|} \Big)^{p-1},
\end{aligned} \tag{3.22}$$

$\overrightarrow{K_*} = (K_*, \lambda_*, K_{1*})$, and $C_{p-1}^m = \frac{(p-1)(p-2)\dots(p-m)}{m!}$.
Estimate for $R^{(2)}(G,\overrightarrow{K})$. From (3.17) We obtain in a similar manner corresponding to the above part, that

$$\|R^{(2)}(G,\overrightarrow{K})\|_{L^\infty(0,T;L^2)} \le \widetilde{C}_{1N}^{(2)} \|\overrightarrow{K}\|^{N+1}, \quad \|\overrightarrow{K}\| \le \|\overrightarrow{K_*}\|, \tag{3.23}$$

where

$$\begin{aligned}
\widetilde{C}_{1N}^{(2)} &= \sum_{m=1}^{N-1} \sum_{N\le|\gamma|\le mN} C_{q-1}^m \|u_0'\|^{q-m-1}_{L^\infty(0,T;V)} \|T^{(m)}[\widehat{u}']_\gamma\|_{L^\infty(0,T;L^2)} \|\overrightarrow{K_*}\|^{|\gamma|-N} \\
&\quad + C_{q-1}^N \|\overrightarrow{K_*}\|^{-N} \Big(\sum_{|\gamma|\le N} \|u_\gamma'\|_{L^\infty(0,T;V)} \|\overrightarrow{K_*}\|^{|\gamma|} \Big)^{q-1}.
\end{aligned} \tag{3.24}$$

Therefore, it follows from (3.18), (3.21)–(3.24) that

$$\begin{gathered}\|\widetilde{E}_N(\overrightarrow{K})\|_{L^\infty(0,T;L^2)} \leq (\widetilde{C}_{1N}^{(1)} + \widetilde{C}_{1N}^{(2)})\|\overrightarrow{K}\|^{N+1} \equiv \widetilde{C}_{1N}\|\overrightarrow{K}\|^{N+1}, \\ \|\overrightarrow{K}\| \leq \|\overrightarrow{K_*}\|.\end{gathered} \tag{3.25}$$

Hence, the first part of Lemma 3.2 is proved.

With $\widetilde{G}_N(\overrightarrow{K})$, then, we obtain from (3.9) in a similar manner to the above part, that

$$\|\widetilde{G}_N(\overrightarrow{K})\|_{H^2(0,T)} \leq \widetilde{C}_{2N}\|\overrightarrow{K}\|^{N+1}, \tag{3.26}$$

where

$$\widetilde{C}_{2N} = \sum_{|\gamma|=N+1,\gamma_3\geq 1} \|u_{\gamma_1,\gamma_2,\gamma_3-1}(1,\cdot)\|_{H^2(0,T)}. \tag{3.27}$$

The proof of Lemma 3.2 is complete. □

Theorem 3.3. *Let (H2)–(H6) hold. Then, for every $\overrightarrow{K} \in \mathbb{R}^3_+$, with $0 \leq K \leq K_*$, $0 \leq \lambda \leq \lambda_*$, $0 \leq K_1 \leq K_{1*}$, problem* (3.1) *has a unique weak solution $(u, Q) = (u_{K,\lambda,K_1},\ Q_{K,\lambda,K_1})$ satisfying the asymptotic estimations up to order $N + 1$ as follows*

$$\begin{aligned}&\|u' - \sum_{|\gamma|\leq N} u'_\gamma \overrightarrow{K}^\gamma\|_{L^\infty(0,T;L^2)} + \|u - \sum_{|\gamma|\leq N} u_\gamma \overrightarrow{K}^\gamma\|_{L^\infty(0,T;V)} \\ &+ \|u'(1,\cdot) - \sum_{|\gamma|\leq N} u'_\gamma(1,\cdot)\overrightarrow{K}^\gamma\|_{L^2(0,T)} \\ &\leq \widetilde{D}^*_N\|\overrightarrow{K}\|^{N+1},\end{aligned} \tag{3.28}$$

and

$$\|Q - \sum_{|\gamma|\leq N} Q_\gamma \overrightarrow{K}^\gamma\|_{L^2(0,T)} \leq \widetilde{D}^{**}_N\|\overrightarrow{K}\|^{N+1}, \tag{3.29}$$

for all $\overrightarrow{K} \in \mathbb{R}^3_+$, $\|\overrightarrow{K}\| \leq \|\overrightarrow{K_*}\|$, $\widetilde{D}^*_N$ and $\widetilde{D}^{**}_N$ are positive constants independent of $\overrightarrow{K}$, the functions $(u_\gamma,\ Q_\gamma)$ are the weak solutions of problems (3.4)*, $\gamma \in \mathbb{Z}^3_+$, $|\gamma| \leq N$.*

Remark 3.4. In [9], as in this special case for problem (1.1)–(1.5), Long, Ut and Truc have obtained a result about the asymptotic expansion of the solutions with respect to two parameters (K, λ) up to order $N + 1$.

Proof of Theorem 3.3. First, we note that, if the data $\overrightarrow{K}$ satisfy

$$0 \leq K \leq K_*, \quad 0 \leq \lambda \leq \lambda_*, \quad 0 \leq K_1 \leq K_{1*}, \tag{3.30}$$

where K_*, λ_*, K_{1*} are fixed positive constants. Therefore, the a priori estimates of the sequences $\{u_m\}$ and $\{Q_m\}$ in the proof of theorem 2.2 satisfy

$$\|u'_m(t)\|^2 + \mu_0\|u_{mx}(t)\|^2 + 2\lambda_0 \int_0^t |u'_m(1, s)|^2 ds \leq M_T, \forall t \in [0, T], \tag{3.31}$$

$$\|u''_m(t)\|^2 + \mu_0\|u'_{mx}(t)\|^2 + 2\lambda_0 \int_0^t |u''_m(1, s)|^2 ds \leq M_T, \forall t \in [0, T], \tag{3.32}$$

$$\|Q_m\|_{H^1(0,T)} \leq M_T, \tag{3.33}$$

where M_T is a constant depending only on T, $\widetilde{u}_0$, $\widetilde{u}_1$, λ_0, μ_0, f, g, k, μ, λ_1, K_*, λ_*, K_{1*} (independent of $\overrightarrow{K}$). Hence, the limit (u, Q) in suitable function spaces of the sequence $\{(u_m,\, Q_m)\}$ defined by (2.7)– (2.9) is a weak solution of the problem (1.1)–(1.5) satisfying the a priori estimates (3.31)–(3.33).

Multiplying the two sides of $(3.7)_1$ with v', and integrating in t, we find without difficulty from Lemma 3.2 that

$$\begin{aligned}\sigma(t) \le\ & 2T(\frac{2}{\lambda_0}\widetilde{C}_{2N}^2 + \widetilde{C}_{1N}^2)\|\overrightarrow{K}\|^{2N+2} \\ & + 2[1 + K + \frac{1}{\mu_0}\|\mu'\|_{C^0(\overline{Q_T})} + \frac{2}{\lambda_0\mu_0}T\|k\|^2_{L^2(0,T)}]\int_0^t \sigma(s)ds \\ & + 2K\int_0^t \|F(v+h) - F(h)\|^2 ds,\end{aligned} \tag{3.34}$$

where

$$\sigma(t) = \|v'(t)\|^2 + \|\sqrt{\mu(t)}v_x(t)\|^2 + K_1 v^2(1,t) + 2\int_0^t \lambda_1(s)|v'(1,s)|^2 ds. \tag{3.35}$$

By using the same arguments as in the above part we can show that the component u of the weak solution (u, Q) of problem (P_{K,λ,K_1}) satisfies

$$\|u'(t)\|^2 + \mu_0\|u_x(t)\|^2 + 2\lambda_0\int_0^t |u'(1,s)|^2 ds \le M_T, \forall t \in [0,T], \tag{3.36}$$

where M_T is a constant independent of K, λ, K_1. On the other hand,

$$\|h\|_{L^\infty(0,T;V)} \le \sum_{|\gamma|\le N} \|u_\gamma\|_{L^\infty(0,T;V)}\|\overrightarrow{K_*}\|^{|\gamma|} \equiv R_1. \tag{3.37}$$

We again use inequality (2.48) with $\delta = p$, $R = \max\{R_1, \sqrt{\frac{M_T}{\mu_0}}\}$, then, it follows from (3.35)–(3.37), that

$$\int_0^t \|F(v+h) - F(h)\|^2 ds \le \frac{1}{\mu_0}(p-1)^2 R^{2p-4}\int_0^t \sigma(s)ds. \tag{3.38}$$

Combining (3.34) and (3.38), we then obtain

$$\sigma(t) \le 2T(\frac{2}{\lambda_0}\widetilde{C}_{2N}^2 + \widetilde{C}_{1N}^2)\|\overrightarrow{K}\|^{2N+2} + \sigma_{1T}\int_0^t \sigma(s)ds, \tag{3.39}$$

for all $t \in [0,T]$, where

$$\sigma_{1T} = 2\big[1 + K_* + \frac{1}{\mu_0}\|\mu'\|_{C^0(\overline{Q_T})} + \frac{2}{\lambda_0\mu_0}T\|k\|^2_{L^2(0,T)} + \frac{1}{\mu_0}(p-1)^2 R^{2p-4}K_*\big], \tag{3.40}$$

By Gronwall's lemma, we obtain from (3.39) that

$$\sigma(t) \le 2T(\frac{2}{\lambda_0}\widetilde{C}_{2N}^2 + \widetilde{C}_{1N}^2)\|\overrightarrow{K}\|^{2N+2}\exp(T\sigma_{1T}) \equiv \widetilde{D}_T^{(1)}\|\overrightarrow{K}\|^{2N+2}, \tag{3.41}$$

for all $t \in [0,T]$ and all $\overrightarrow{K} \in \mathbb{R}^3_+$, $\|\overrightarrow{K}\| \le \|\overrightarrow{K_*}\|$. It follows that

$$\|v'(t)\|^2 + \mu_0\|v_x(t)\|^2 + 2\lambda_0\int_0^t |v'(1,s)|^2 ds \le \sigma(t) \le \widetilde{D}_T^{(1)}\|\overrightarrow{K}\|^{2N+2}. \tag{3.42}$$

Hence

$$\|v'\|_{L^\infty(0,T;L^2)} + \|v\|_{L^\infty(0,T;V)} + \|v'(1,\cdot)\|_{L^2(0,T)} \le \widetilde{D}_N^*\|\overrightarrow{K}\|^{N+1}, \tag{3.43}$$

or

$$\begin{aligned} &\|u' - \sum_{|\gamma|\le N} u'_\gamma \overrightarrow{K}^\gamma\|_{L^\infty(0,T;L^2)} + \|u - \sum_{|\gamma|\le N} u_\gamma \overrightarrow{K}^\gamma\|_{L^\infty(0,T;V)} \\ &+ \|u'(1,\cdot) - \sum_{|\gamma|\le N} u'_\gamma(1,\cdot)\overrightarrow{K}^\gamma\|_{L^2(0,T)} \\ &\le \widetilde{D}^*_N \|\overrightarrow{K}\|^{N+1}, \end{aligned} \tag{3.44}$$

for all $\overrightarrow{K} \in \mathbb{R}^3_+$, $\|\overrightarrow{K}\| \le \|\overrightarrow{K_*}\|$, where $\widetilde{D}^*_N$ is a constant independent of $\overrightarrow{K}$. On the other hand, it follows from (3.11), (3.43), that

$$\begin{aligned} \|R\|_{L^2(0,T)} &\le K_1\|v\|_{L^\infty(0,T;V)} + \|\lambda_1\|_\infty \|v'(1,\cdot)\|_{L^2(0,T)} + \|\widetilde{G}_N(\overrightarrow{K})\|_{L^2(0,T)} \\ &\quad + \sqrt{\frac{1}{\mu_0} T}\|k\|_{L^2(0,T)} \Big(\int_0^T \sigma(s)ds\Big)^{1/2} \\ &\le \widetilde{D}^{**}_N \|\overrightarrow{K}\|^{N+1}, \end{aligned} \tag{3.45}$$

hence,

$$\|Q - \sum_{|\gamma|\le N} Q_\gamma \overrightarrow{K}^\gamma\|_{L^2(0,T)} \le \widetilde{D}^{**}_N \|\overrightarrow{K}\|^{N+1}, \tag{3.46}$$

where $\widetilde{D}^{**}_N$ is a constant independent of $\overrightarrow{K}$. The proof of Theorem 3.3 is complete. □

Remark 3.5. For the case $(K, \lambda, K_1) \in \mathbb{R}^2 \times \mathbb{R}_+$, but $p = q = 2$, we have received a theorem of the asymptotic expansion for the weak solution (u, Q) of problem (1.1)–(1.5) with respect to three mentioned parameters; however, the detailes of proof have been omitted.

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Nguyen Thanh Long
Department of Mathematics, Hochiminh City National University, 227 Nguyen Van Cu, Q5, HoChiMinh City, Vietnam
E-mail address: `longnt@hcmc.netnam.vn, longnt2@gmail.com`

Le Xuan Truong
Department of Mathematics, Faculty of General Science, University of Technical Education in HoChiMinh City, 01 Vo Van Ngan Str., Thu Duc Dist., HoChiMinh City, Vietnam
E-mail address: `lxuantruong@gmail.com`