

**EXISTENCE AND UNIQUENESS OF SOLUTIONS TO
FIRST-ORDER SYSTEMS OF NONLINEAR IMPULSIVE
BOUNDARY-VALUE PROBLEMS WITH SUB-, SUPER-LINEAR
OR LINEAR GROWTH**

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ABSTRACT. In this work we present some new results concerning the existence and uniqueness of solutions to an impulsive first-order, nonlinear ordinary differential equation with “non-periodic” boundary conditions. These boundary conditions include, as a special case, so-called “anti-periodic” boundary conditions. Our methods to prove the existence and uniqueness of solutions involve new differential inequalities, the classical fixed-point theorem of Schaefer, and the Nonlinear Alternative. Our new results apply to systems of impulsive differential equations where the right-hand side of the equation may grow linearly, or sub- or super-linearly in its second argument.

1. INTRODUCTION

At certain points in time, many dynamic phenomena experience sudden, instantaneous, rapid change exhibited by a jump in their states. Such behaviour is seen in a range of physical problems from: mechanics; chemotherapy; population dynamics; optimal control; ecology; industrial robotics; biotechnology; spread of disease; harvesting; and physics. The reader is referred to [12, 14, 16, 20, 22, 23, 24, 27, 28, 32, 36, 38, 39, 40] and references therein for some nice examples and applications to the above areas.

The branch of modern, applied analysis known as “impulsive” differential equations furnishes a natural framework to mathematically describe the aforementioned jumping processes. Consequently, the area of impulsive differential equations has been developing at a rapid rate, with the wide applications significantly motivating a deeper theoretical study of the subject.

This paper considers the existence and uniqueness of solutions to the following first-order differential system:

$$x' = f(t, x), \quad t \in [0, N], \quad t \neq t_1; \quad (1.1)$$

$$Ax(0) + Bx(N) = \alpha, \quad 0 < N \in \mathbb{R}; \quad (1.2)$$

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where: $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous on $(t, p) \in ([0, N] \setminus \{t_1\}) \times \mathbb{R}^n$; $n \geq 1$; A and B are $n \times n$ matrices with real-valued elements; α is a constant vector in \mathbb{R}^n ; and the impulse at $t = t_1$ is given by a continuous function $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with

$$x(t_1^+) = x(t_1^-) + I_1(x(t_1)), \quad t_1 \in (0, 1), \quad t_1 \text{ fixed}; \quad (1.3)$$

using the notation $x(t_1^-) := \lim_{t \rightarrow t_1^-} x(t)$ and $x(t_1^+) := \lim_{t \rightarrow t_1^+} x(t)$.

Equations (1.1)–(1.3) are collectively known as an impulsive boundary value problem (BVP). Besides the natural physical applications of impulsive differential equations, significance of the study of the system (1.1)–(1.3) lies in the fact that most types of impulsive BVPs with linear boundary conditions can be written in the form (1.1)–(1.3). For example, through a simple substitution $x_i := x^{(i)}$, $i = 1, \dots, k$, impulsive BVPs with second- or higher-order derivatives may be reduced to the system (1.1)–(1.3). Thus the study of (1.1)–(1.3) can lead to a deeper understanding of a range of impulsive BVPs, including those of a higher-order.

In the case when: $A = B$ equals the identity matrix; and $\alpha = 0$ in (1.2), that is, (1.2) becomes the periodic boundary conditions

$$x(0) = x(N),$$

some recent and influential papers examining existence of solutions to (1.1)–(1.3) include [4, 8, 10, 15, 20, 21, 22, 23, 26].

Throughout this paper the condition $\det(A + B) \neq 0$ is assumed to hold, so in this sense, the boundary conditions (1.2) do not include the periodic conditions $x(0) - x(N) = 0$. Few papers, apart from [9, 18, 29, 30], have examined the existence and uniqueness of solutions to (1.1)–(1.3) under this “non-periodic” boundary condition, even though these types of boundary conditions appear in many applications, especially the case of “anti-periodic” boundary conditions

$$x(0) = -x(N),$$

for example, see [1, 2, 3, 5, 9, 11, 13, 19, 25, 31, 37].

This article is organised as follows. Section 2 presents some preliminary ideas associated with the impulsive BVP (1.1)–(1.3). Sections 3 and 4 contain the main results of the paper and are devoted to the existence and uniqueness of solutions to (1.1)–(1.3). There, new differential inequalities in the impulsive-setting are introduced, developed and applied, in conjunction with Schaefer’s theorem [17, Theorem 4.4.12] and the Nonlinear Alternative [7, Theorem 5.1, p.61], to prove the existence and uniqueness of solutions to (1.1)–(1.3). The main ideas rely on: novel differential inequalities; and *a priori* bounds on solutions to a certain family of integral operator equations, with the operator being compact.

The new results compliment and extend those of [4, 9, 10, 15, 18, 20, 21, 22, 26, 29, 30] in the sense that: our ideas permit super-linear growth of $\|f(t, p)\|$ in $\|p\|$ in (1.1), whereas the theorems in [18] do not; our investigation tackles a wider range or a different class of boundary conditions than those in [15, 18, 20, 21, 22, 26, 29, 30]; and our results apply to systems of impulsive BVPs, unlike the papers [9, 10] which have concentrated on scalar-valued equations. This last point is of particular significance when dealing with large systems of equations, as traditional methods, like the method of upper and lower solutions, are rather cumbersome to apply to (1.1)–(1.3) when n is large.

Section 5 presents an example to illustrate how to apply some of the newly developed theoretical results. A particular example is constructed so that the theorems in [4, 9, 10, 15, 18, 20, 21, 22, 26, 29, 30] do not directly apply.

One could consider impulsive BVPs with a finite number of impulses I_i , so that (1.3) could take the form, for $i = 1, \dots, p$

$$x(t_i^+) = x(t_i^-) + I_i(x(t_i)), \quad \text{each } t_i \text{ in } (0, 1) \text{ and fixed.}$$

However, for clarity and brevity, attention is restricted to BVPs with one impulse. In addition, the difference between the theory of one or an arbitrary number of impulses is quite minimal.

Our new results were particularly motivated by the recent works [6], [15], [33], [34] and [35].

To understand the notation used above and the ideas in the remainder of the paper, some appropriate concepts connected with impulsive differential equations are now introduced. The following notation comes from [18] and further information can be found in the references therein.

Assume that

$$f(t_1^+, x) := \lim_{t \rightarrow t_1^+} f(t, x) \quad \text{and} \quad f(t_1^-, x) := \lim_{t \rightarrow t_1^-} f(t, x)$$

both exist with

$$f(t_1^-, x) = f(t_1, x).$$

Introduce and denote the Banach space $PC([0, N]; \mathbb{R}^n)$ by

$$PC([0, N]; \mathbb{R}^n) := \{u : [0, N] \rightarrow \mathbb{R}^n, u \in C([0, N] \setminus \{t_1\}; \mathbb{R}^n), u \text{ is left continuous at } t = t_1, \text{ the right hand limit } u(t_1^+) \text{ exists}\}$$

with the norm

$$\|u\|_{PC} := \sup_{t \in [0, N]} \|u(t)\|$$

where $\|\cdot\|$ is the usual Euclidean norm and $\langle \cdot, \cdot \rangle$ will be the Euclidean inner product.

Let $t_0 = 0$ and $t_2 = N$. In a similar fashion to the above, define and denote the Banach space $PC^1([0, N]; \mathbb{R}^n)$ by

$$PC^1([0, N]; \mathbb{R}^n) := \{u \in PC([0, N]; \mathbb{R}^n), u|_{(t_k, t_{k+1})} \in C^1((t_k, t_{k+1}); \mathbb{R}^n) \text{ for } k = 0, 1, \text{ and the limits } u'(t_1^+), u'(t_1^-) \text{ exist}\}$$

with the norm

$$\|u\|_{PC^1} := \max\{\|u(t)\|_{PC}, \|u'(t)\|_{PC}\}.$$

For an $n \times n$ matrix A with real-valued elements a_{ij} , $\|A\|$ will denote the norm of matrix A given by

$$\|A\| := \left(\sum_{j=1}^n [a_{1j}^2] + \dots + \sum_{j=1}^n [a_{nj}^2] \right)^{1/2}.$$

A solution to the impulsive BVP (1.1)–(1.3) is a function $x \in PC^1([0, N]; \mathbb{R}^n)$ that satisfies (1.1)–(1.3) for each $t \in [0, N]$.

2. OPERATOR FORMULATION

In this section the impulsive BVP (1.1)–(1.3) is reformulated as an appropriate integral equation so that potential solutions to the integral equation will be solutions to the impulsive BVP (1.1)–(1.3). The motivation behind this approach is to define a suitable integral operator, with fixed-points of the operator corresponding to solutions of the BVP (1.1)–(1.3).

The following results are included to keep the paper self-contained for the benefit of the reader. Recall that the Heaviside function is defined as $H(s) = 0$ if $s \leq 0$, and $H(s) = 1$ if $s > 0$,

Lemma 2.1. *Consider the impulsive BVP (1.1)–(1.3) with $\det(A + B) \neq 0$. Let $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ both be continuous.*

(i) *If $x \in PC^1([0, N]; \mathbb{R}^n)$ is a solution of (1.1)–(1.3) then*

$$\begin{aligned} x(t) &= (A + B)^{-1} \left[\alpha - B \left(\int_0^N f(s, x(s)) ds + I_1(x(t_1)) \right) \right] \\ &\quad + \int_0^t f(s, x(s)) ds + H(t - t_1) \cdot I_1(x(t_1)), \quad t \in [0, N]; \end{aligned} \quad (2.1)$$

(ii) *If $x \in PC([0, N]; \mathbb{R}^n)$ satisfies (2.1) then $x \in PC^1([0, N]; \mathbb{R}^n)$ and x is a solution of (1.1)–(1.3).*

Proof. (i) Let $x \in PC^1([0, N]; \mathbb{R}^n)$ be a solution to (1.1)–(1.3). Integrating (1.1) from 0 to $t < t_1$ we have

$$x(t) = x(0) + \int_0^t f(s, x(s)) ds,$$

and from t_1 to t with $t \in (t_1, N]$ we get

$$x(t) = x(t_1^+) + \int_{t_1}^t f(s, x(s)) ds.$$

A similar integration of (1.1) from 0 to t_1 shows that

$$x(t_1^-) = x(0) + \int_0^{t_1} f(s, x(s)) ds.$$

Hence combining the previous expressions we have for $t \in [0, t_1]$

$$x(t) = x(0) + \int_0^t f(s, x(s)) ds = x(0) + H(t - t_1) \cdot I_1(x(t_1)) + \int_0^t f(s, x(s)) ds,$$

and for each $t \in (t_1, N]$

$$\begin{aligned} x(t) &= x(0) + x(t_1^+) - x(t_1^-) + \int_0^t f(s, x(s)) ds \\ &= x(0) + I_1(x(t_1)) + \int_0^t f(s, x(s)) ds. \end{aligned} \quad (2.2)$$

Letting $t = N$ in (2.2) and using the boundary conditions (1.2) we obtain

$$\begin{aligned} Bx(N) &= B \left[x(0) + I_1(x(t_1)) + \int_0^N f(t, x(t)) dt \right] \\ &= \alpha - Ax(0). \end{aligned}$$

A rearrangement in the previous expression then gives

$$x(0) = (A + B)^{-1} \left[\alpha - B \left(\int_0^N f(t, x(t)) dt + I_1(x(t_1)) \right) \right]$$

which is substituted into (2.2) and a rearrangement leads to (2.1).

(ii) Let $x \in PC([0, N]; \mathbb{R}^n)$ be a solution to (2.1). Since f is continuous it is easy to see that $x \in PC^1([0, N]; \mathbb{R}^n)$. To verify that x also satisfies the impulsive BVP (1.1)–(1.3) just differentiate (2.1) to obtain (1.1) and also show that (1.2) and (1.3) hold by direct substitution. \square

In view of Lemma 2.1 a useful operator will now be introduced so that fixed-points of the operator will be solutions of the impulsive BVP (1.1)–(1.3).

Lemma 2.2. *Consider the impulsive BVP (1.1)–(1.3) with $\det(A + B) \neq 0$. Let $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ both be continuous. Consider the mapping $T : PC([0, N]; \mathbb{R}^n) \rightarrow PC([0, N]; \mathbb{R}^n)$ defined by*

$$\begin{aligned} (t) := & (A + B)^{-1} \left[\alpha - B \left(\int_0^N f(s, x(s)) ds + I_1(x(t_1)) \right) \right] \\ & + \int_0^t f(s, x(s)) ds + H(t - t_1) \cdot I_1(x(t_1)), \quad t \in [0, N]. \end{aligned} \quad (2.3)$$

If T has a fixed-point q , that is $Tq = q$ for some $q \in PC([0, N]; \mathbb{R}^n)$, then this fixed-point q is also a solution to the impulsive BVP (1.1)–(1.3).

The above lemma follows from Lemma 2.1.

The topologically-inspired fixed point theorems that will be used to guarantee the existence of at least one fixed-point of T requires that T be a “compact” map [17, pp.54-55].

Recall that a mapping between normed spaces is compact if it is continuous and carries bounded sets into relatively compact sets.

Lemma 2.3. *Consider (2.3) with $\det(A + B) \neq 0$. Let $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ both be continuous. Then $T : PC([0, N]; \mathbb{R}^n) \rightarrow PC([0, N]; \mathbb{R}^n)$ is a compact map.*

Proof. This follows in a standard step-by-step process and so is omitted. \square

The following two well-known fixed-point theorems will be of use in the sections to follow. In particular, the Nonlinear Alternative [7, Theorem 5.1, p.61] and Schaefer’s Theorem [17, Theorem 4.4.12] will be employed.

Theorem 2.4 (Nonlinear Alternative). *Let X be a normed space with C a convex subset of X . Let U be an open subset of C with $0 \in C$ and consider a compact map $H : \bar{U} \rightarrow C$. If*

$$u \neq \lambda Hu \quad \text{for all } u \in \partial U \text{ and for all } \lambda \in [0, 1]$$

then H has at least one fixed-point.

Theorem 2.5 (Schaefer). *Let X be a normed space with $H : X \rightarrow X$ a compact mapping. If the set*

$$S := \{u \in X : u = \lambda Hu \text{ for some } \lambda \in [0, 1]\}$$

is bounded then H has at least one fixed-point.

3. EXISTENCE: HOMOGENEOUS CASE

This section presents some new existence results for solutions to the following “homogenous” problem ($\alpha = 0$)

$$x' = f(t, x), \quad t \in [0, N], \quad t \neq t_1; \quad (3.1)$$

$$Ax(0) + Bx(N) = 0, \quad 0 < N \in \mathbb{R}; \quad (3.2)$$

$$x(t_1^+) = x(t_1^-) + I_1(x(t_1)), \quad t_1 \in (0, 1), \quad t_1 \text{ fixed.} \quad (3.3)$$

The ideas use novel differential inequalities in the impulsive equation setting and standard fixed–point methods of integral operators. In particular, the Nonlinear Alternative [7, Theorem 5.1, p.61] and Schaefer’s Theorem [17, Theorem 4.4.12] will be employed.

The following existence result involves sublinear growth of $\|f(t, p)\|$ in $\|p\|$.

Theorem 3.1. *Consider the impulsive BVP (3.1)–(3.3) with $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ both being continuous and $\det(A + B) \neq 0$. Let ρ and σ be non-negative constants and let $\psi : [0, \infty) \rightarrow (0, \infty)$ be a continuous, non-decreasing function such that*

$$\|f(t, p)\| \leq \rho\psi(\|p\|), \quad \text{for all } (t, p) \in [0, N] \setminus \{t_1\} \times \mathbb{R}^n; \quad (3.4)$$

$$\|I_1(q)\| \leq \sigma\|q\|, \quad \text{for all } q \in \mathbb{R}^n; \quad (3.5)$$

$$\sup_{c \geq 0} \frac{c}{\psi(c)} > K_1 := \frac{\rho N (\|(A + B)^{-1}B\| + 1)}{1 - \sigma (\|(A + B)^{-1}B\| + 1)}; \quad (3.6)$$

$$\sigma (\|(A + B)^{-1}B\| + 1) < 1; \quad (3.7)$$

then the impulsive BVP (3.1)–(3.3) has at least one solution.

Proof. We will use the Nonlinear Alternative. From (3.6) there exists a constant $Q > 0$ such that

$$\frac{Q}{\psi(Q)} > K_1. \quad (3.8)$$

Consider the mapping $T_1 : PC([0, N]; \mathbb{R}^n) \rightarrow PC([0, N]; \mathbb{R}^n)$

$$\begin{aligned} T_1 x(t) := & (A + B)^{-1} \left[-B \left(\int_0^N f(s, x(s)) ds + I_1(x(t_1)) \right) \right] \\ & + \int_0^t f(s, x(s)) ds + H(t - t_1) \cdot I_1(x(t_1)), \quad t \in [0, N]. \end{aligned}$$

By Lemma 2.3, T_1 is a compact mapping. Let

$$\bar{\Omega} := \{x \in PC([0, N]; \mathbb{R}^n) : \|x\|_{PC} < Q\}.$$

We consider $T_1 : \bar{\Omega} \rightarrow PC([0, N]; \mathbb{R}^n)$ and the family of problems

$$x = \lambda T_1 x, \quad \lambda \in [0, 1]. \quad (3.9)$$

Let x be a solution to (3.9) with $x \in \bar{\Omega}$. We show that $x \notin \partial\Omega$. From (3.9) and (2.3) we have, for each $t \in [0, N]$,

$$\begin{aligned} \|x(t)\| &= \|\lambda T_1 x(t)\| \\ &\leq (1 + \|(A + B)^{-1}B\|) \left[\int_0^N \|f(t, x(t))\| dt + \|I_1(x(t_1))\| \right] \\ &\leq (1 + \|(A + B)^{-1}B\|) \left[\int_0^N \rho\psi(\|x(t)\|) dt + \sigma\|x(t_1)\| \right] \\ &\leq (1 + \|(A + B)^{-1}B\|) \left[\rho N\psi\left(\sup_{t \in [0, N]} \|x(t)\|\right) + \sigma \sup_{t \in [0, N]} \|x(t)\| \right]. \end{aligned}$$

Hence we have

$$\sup_{t \in [0, N]} \|x(t)\| \leq (1 + \|(A + B)^{-1}B\|) \left[\rho N\psi\left(\sup_{t \in [0, N]} \|x(t)\|\right) + \sigma \sup_{t \in [0, N]} \|x(t)\| \right]$$

so that a rearrangement in the previous line gives

$$\sup_{t \in [0, N]} \|x(t)\| \leq K_1\psi\left(\sup_{t \in [0, N]} \|x(t)\|\right)$$

where K_1 is defined in (3.6). Hence, by (3.8) we must have $\sup_{t \in [0, N]} \|x(t)\| \neq Q$, that is $\|x\|_{PC} \neq Q$. The Nonlinear Alternative is applicable and thus the existence of at least one solution follows. \square

The following result allows $\|f(t, p)\|$ to grow more than linearly in $\|p\|$.

Theorem 3.2. *Consider the impulsive BVP (3.1)–(3.3) with $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ both being continuous and $\det(A + B) \neq 0$. If there exist non-negative constants a, b, β, L such that:*

$$\|f(t, p)\| \leq 2a\langle p, f(t, p) \rangle + b, \quad \text{for all } (t, p) \in [0, N] \setminus \{t_1\} \times \mathbb{R}^n; \tag{3.10}$$

$$\|I_1(q)\| \leq \beta\|q\| + L, \quad \text{for all } q \in \mathbb{R}^n; \tag{3.11}$$

$$\|B^{-1}A\| \leq 1; \tag{3.12}$$

$$\beta(\|(A + B)^{-1}B\| + 1) < 1; \tag{3.13}$$

then the impulsive BVP (3.1)–(3.3) has at least one solution.

Proof. Consider the mapping $T_1 : PC([0, N]; \mathbb{R}^n) \rightarrow PC([0, N]; \mathbb{R}^n)$

$$\begin{aligned} T_1 x(t) &:= (A + B)^{-1} \left[-B \left(\int_0^N f(s, x(s)) ds + I_1(x(t_1)) \right) \right] \\ &\quad + \int_0^t f(s, x(s)) ds + H(t - t_1) \cdot I_1(x(t_1)), \quad t \in [0, N]. \end{aligned}$$

By Lemma 2.3, T_1 is a compact mapping. Consider the equation

$$x = T_1 x. \tag{3.14}$$

To show that T_1 has at least one fixed point, we apply Schaefer’s Theorem by showing that all potential solutions to

$$x = \lambda T_1 x, \quad \lambda \in [0, 1]; \tag{3.15}$$

are bounded *a priori*, with the bound being independent of λ . With this in mind, let x be a solution to (3.15). Note that x is also a solution to

$$\begin{aligned}x' &= \lambda f(t, x), \quad t \in [0, N], \quad t \neq t_1; \\Ax(0) + Bx(N) &= 0; \\x(t_1^+) &= x(t_1^-) + \lambda I_1(x(t_1)).\end{aligned}$$

Note that (3.12) and (1.2) imply

$$\|x(N)\| \leq \|B^{-1}Ax(0)\| \leq \|B^{-1}A\|\|x(0)\| \leq \|x(0)\|.$$

We also have, for each $t \in [0, N]$,

$$\begin{aligned}\|x(t)\| &= \lambda \|Tx(t)\| \\&\leq (1 + \|(A+B)^{-1}B\|) \left[\int_0^N \|\lambda f(t, x(t))\| dt + \|\lambda I_1(x(t_1))\| \right] \\&\leq (1 + \|(A+B)^{-1}B\|) \left[\int_0^N 2a\langle x(s), \lambda f(s, x(s)) \rangle + \lambda b ds + \beta \|x(t_1)\| + L \right] \\&\leq (1 + \|(A+B)^{-1}B\|) \left[\int_0^N 2a\langle x(s), x'(s) \rangle + b ds + \beta \|x\|_{PC} + L \right] \\&= (1 + \|(A+B)^{-1}B\|) \left[\int_0^N a \frac{d}{ds} (\|x(s)\|^2) + b ds + \beta \|x\|_{PC} + L \right] \\&= (1 + \|(A+B)^{-1}B\|) [a(\|x(N)\|^2 - \|x(0)\|^2) + bN + \beta \|x\|_{PC} + L] \\&\leq (1 + \|(A+B)^{-1}B\|) [bN + \beta \|x\|_{PC} + L]\end{aligned}$$

Thus, taking the supremum above and rearranging we obtain

$$\|x\|_{PC} = \sup_{t \in [0, N]} \|x(t)\| \leq \frac{[bN + L][1 + \|(A+B)^{-1}B\|]}{1 - (1 + \|(A+B)^{-1}B\|)\beta}.$$

Thus we see that the bound on all possible solutions to (3.15) is independent of λ and Schaefer's Theorem applies, yielding the existence of at least one fixed-point to T_1 and thus (3.1)–(3.3) has at least one solution. \square

Theorem 3.2 may be suitably modified to include an alternate class of f as follows.

Theorem 3.3. *Consider the impulsive BVP (1.1)–(1.3) with $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ both being continuous. Let the conditions of Theorem 3.2 hold with (3.10) and (3.12) respectively replaced by*

$$\|f(t, p)\| \leq -2a\langle p, f(t, p) \rangle + b, \quad \text{for all } (t, p) \in [0, N] \setminus \{t_1\} \times \mathbb{R}^n. \quad (3.16)$$

$$\|A^{-1}B\| \leq 1; \quad (3.17)$$

Then the impulsive BVP (1.1)–(1.3) has at least one solution.

Proof. The proof is a minor variation to that of Theorem 3.2 and so is not discussed. \square

Although the proofs of Theorems 3.2 and 3.3 are similar, the two results differ in sense that Theorem 3.2 may apply to certain problems, whereas Theorem 3.3 may not apply, and vice-versa. For example, in the scalar case,

$$f(t, p) := -p^3 - t, \quad t \in [0, 1];$$

satisfies (3.16) for the choices $a = 1/2$ and $b = 100$, but the above f cannot satisfy (3.10) for any choice of non-negative a and b .

4. EXISTENCE AND UNIQUENESS: INHOMOGENEOUS CASE

This section presents existence and uniqueness results for solutions to the general impulsive BVP (1.1)–(1.3) where α may be non-zero.

The following general existence result allows linear growth of $\|f(t, p)\|$ in $\|p\|$.

Theorem 4.1. *Consider the impulsive BVP (1.1)–(1.3) with $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ both being continuous and $\det(A + B) \neq 0$. Let u, v, w, z be non-negative constants such that*

$$\|f(t, p)\| \leq u\|p\| + v, \quad \text{for all } (t, p) \in [0, N] \setminus \{t_1\} \times \mathbb{R}^n; \quad (4.1)$$

$$\|I_1(q)\| \leq w\|q\| + z, \quad \text{for all } q \in \mathbb{R}^n; \quad (4.2)$$

$$(\|(A + B)^{-1}B\| + 1)[Nu + w] < 1. \quad (4.3)$$

Then the impulsive BVP (1.1)–(1.3) has at least one solution.

Proof. We use Schaefer's Theorem. Consider the mapping $T : PC([0, N]; \mathbb{R}^n) \rightarrow PC([0, N]; \mathbb{R}^n)$,

$$\begin{aligned} Tx(t) := & (A + B)^{-1} \left[\alpha - B \left(\int_0^N f(s, x(s)) ds + I_1(x(t_1)) \right) \right] \\ & + \int_0^t f(s, x(s)) ds + H(t - t_1) \cdot I_1(x(t_1)), \quad t \in [0, N]. \end{aligned}$$

By Lemma 2.3, T is a compact mapping. Consider the equation

$$x = Tx. \quad (4.4)$$

In order to show that T has at least one fixed point, we apply Schaefer's Theorem by showing that all potential solutions to

$$x = \lambda Tx, \quad \lambda \in [0, 1]; \quad (4.5)$$

are bounded *a priori*, with the bound being independent of λ . With this in mind, let x be a solution to (4.5). Note that x is also a solution to

$$x' = \lambda f(t, x), \quad t \in [0, N], \quad t \neq t_1, \quad (4.6)$$

$$Ax(0) + Bx(N) = \lambda \alpha; \quad (4.7)$$

$$x(t_1^+) = x(t_1^-) + \lambda I_1(x(t_1)). \quad (4.8)$$

We then have, for each $t \in [0, N]$,

$$\begin{aligned} & \|x(t)\| \\ &= \|\lambda T x(t)\| \\ &\leq \|(A+B)^{-1}\alpha\| + (1 + \|(A+B)^{-1}B\|) \left[\int_0^N \|\lambda f(t, x(t))\| dt + \|\lambda I_1(x(t_1))\| \right] \\ &\leq \|(A+B)^{-1}\alpha\| + (1 + \|(A+B)^{-1}B\|) \left[\int_0^N u\|x(t)\| + v dt + w\|x(t_1)\| + z \right] \\ &\leq \|(A+B)^{-1}\alpha\| + (1 + \|(A+B)^{-1}B\|) \left[N(u\|x\|_{PC} + v) + w\|x\|_{PC} + z \right] \end{aligned}$$

Thus, taking the supremum and rearranging we obtain

$$\|x\|_{PC} = \sup_{t \in [0, N]} \|x(t)\| \leq \frac{\|(A+B)^{-1}\alpha\| + [1 + \|(A+B)^{-1}B\|](Nv + z)}{1 - (\|1 + (A+B)^{-1}B\|)[Nu + w]}.$$

Thus we see that the bound on all possible solutions to (3.15) is independent of λ and Schaefer's Theorem applies, yielding the existence of at least one fixed-point to T and thus (1.1)–(1.3) has at least one solution. \square

The following uniqueness result for solutions (1.1)–(1.3) is now obtained with the help of Theorem 4.1.

Theorem 4.2. *Consider the impulsive BVP (1.1)–(1.3) with $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ both being continuous and $\det(A+B) \neq 0$. Let u_1 and w_1 be non-negative constants such that*

$$\begin{aligned} \|f(t, p) - f(t, q)\| &\leq u_1\|p - q\|, \quad \text{for all } (t, p, q) \in [0, N] \setminus \{t_1\} \times \mathbb{R}^{2n}; \\ \|I_1(p) - I_1(q)\| &\leq w_1\|p - q\|, \quad \text{for all } (p, q) \in \mathbb{R}^{2n}; \\ (\|(A+B)^{-1}B\| + 1)[Nu_1 + w_1] &< 1. \end{aligned}$$

Then the impulsive BVP (1.1)–(1.3) has a unique solution.

Proof. The conditions of the theorem imply that

$$\begin{aligned} \|f(t, p) - f(t, 0)\| &\leq u_1\|p - 0\|, \quad \text{for all } (t, p) \in [0, N] \setminus \{t_1\} \times \mathbb{R}^n; \\ \|I_1(p) - I_1(0)\| &\leq w_1\|p - 0\|, \quad \text{for all } p \in \mathbb{R}^n. \end{aligned}$$

A rearrangement of the above two inequalities leads to (4.1) and (4.2) for

$$v = \sup_{t \in [0, N]} \|f(t, 0)\|,$$

$u = u_1$, $w = w_1$ and $z = \|I_1(0)\|$. Thus, since (4.3) holds, all of the conditions of Theorem 4.1 are satisfied and the existence of at least one solution to (1.1)–(1.3) follows.

$$\begin{aligned}
& \text{Now let } x \text{ and } y \text{ be two solutions to (1.1)–(1.3). We have, for each } t \in [0, N] \\
& \|x(t) - y(t)\| \\
& \leq (1 + \|(A + B)^{-1}B\|) \left[\int_0^N \|f(t, x(t)) - f(t, y(t))\| dt + \|I_1(x(t_1)) - I_1(y(t_1))\| \right] \\
& \leq (1 + \|(A + B)^{-1}B\|) \left[u_1 \int_0^N \|x(t) - y(t)\| dt + w_1 \|x(t_1) - y(t_1)\| \right] \\
& \leq (1 + \|(A + B)^{-1}B\|) [u_1 N \|x - y\|_{PC} + w_1 \|x - y\|_{PC}].
\end{aligned}$$

Hence rearranging and taking the supremum above we have

$$[1 - [1 + \|(A + B)^{-1}B\|][Nu_1 + w_1]] \|x - y\|_{PC} \leq 0$$

and (4.2) ensures $x = y$. Thus, the solutions are unique. \square

The following corollary is a special case of Theorem 4.1 and involves global bounds on the functions f and I_1 .

Corollary 4.3. *Consider the impulsive BVP (1.1)–(1.3) with $f : [0, N] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $I_1 : \mathbb{R}^n \times \mathbb{R}^n$ both being continuous and $\det(A + B) \neq 0$. Let v and z be non-negative constants such that*

$$\begin{aligned}
\|f(t, p)\| &\leq v, \quad \text{for all } (t, p) \in [0, N] \setminus \{t_1\} \times \mathbb{R}^n; \\
\|I_1(q)\| &\leq z, \quad \text{for all } q \in \mathbb{R}^n.
\end{aligned}$$

Then the impulsive BVP (1.1)–(1.3) has at least one solution.

Proof. The proof involves taking $u = 0 = w$ so that all of the conditions of Theorem 4.1 are satisfied. \square

5. EXAMPLES

In this section some examples are presented to highlight the theory. We firstly consider the following scalar-valued differential equation case.

Example 5.1. Consider the impulsive BVP given by

$$x' = x^3 + x + t, \quad t \in [0, 1], t \neq t_1; \quad (5.1)$$

$$x(0) = 2x(1); \quad (5.2)$$

$$x(t_1^+) = x(t_1^-) + x(t_1)/2 \quad (5.3)$$

where x is scalar-valued ($n = 1$). The above impulsive BVP has at least one solution.

Proof. Let $f(t, p) = p^3 + p + t$ and see that $|f(t, p)| \leq |p^3| + |p| + 1$ for $(t, p) \in [0, 1] \times \mathbb{R}$. For a and b to be chosen below, see that

$$\begin{aligned}
2apf(t, p) + b &= 2a(p^4 + p^2 + pt) + b \\
&= (p^4 + 1) + [p^2 + pt + 40.25], \quad \text{for the choices } a = 1/2, b = 41.25 \\
&= (p^4 + 1) + [(p + t/2)^2 + 40.25 - t^2/4] \\
&\geq (|p^3|) + [|p| + 1] \\
&\geq |f(t, p)| \quad \text{for all } (t, p) \in [0, 1] \times \mathbb{R}
\end{aligned}$$

and thus (3.10) holds. It is easy to see that (3.11), (3.12) and (3.13) hold for $\beta = 1/2$, $L = 0$, $N = 1$, $A = 1$, $B = -2$. Thus, all of the conditions of Theorem

3.2 hold and the solvability follows. The theorems of [18, 29, 30], for example, do not apply to the above because of the wider class of boundary conditions. \square

We now consider an example involving a system of differential equations.

Example 5.2. Consider (1.1)–(1.3) with $n = 2$ and f given by

$$\begin{aligned} f(t, p) &= (h(t, y, z), j(t, y, z)), \quad t \in [0, 1], \\ &:= ((t+1)y^3 + ye^{-z^2} + 1, (t+1)z^3 + ze^{-y^2}). \end{aligned} \quad (5.4)$$

and

$$(y(0), z(0)) - 2(y(1), z(1)) = (0, 0)$$

with

$$(y(t_1^+), z(t_1^+)) = (y(t_1^-), z(t_1^-)) + (y(t_1)/2, z(t_1)/2).$$

The above impulsive BVP has at least one solution.

Proof. We show that above f satisfies the conditions of Theorem 3.2. Note that for all $(t, p) \in [0, 1] \times \mathbb{R}^2$ we have

$$\begin{aligned} \|f(t, p)\| &\leq |h(t, y, z)| + |j(t, y, z)| \\ &\leq 2|y|^3 + |y|e^{-z^2} + 2|z|^3 + |z|e^{-y^2} + 1. \end{aligned}$$

Below, we will need the following simple inequalities:

$$\begin{aligned} w^4 &\geq |w|^3 - 1, \quad w^4 + w \geq |w|^3 - 10, \quad \text{for all } w \in \mathbb{R}, \\ d^2e^{-c^2} &\geq |d|e^{-c^2} - 1, \quad \text{for all } (c, d) \in \mathbb{R}^2. \end{aligned}$$

For $a \geq 0$ and $b \geq 0$ to be chosen below, consider for $(t, p) \in [0, 1] \times \mathbb{R}^2$,

$$\begin{aligned} 2a\langle p, f(t, p) \rangle + b &\geq 2a \left[y^4 + y + y^2e^{-z^2} + z^4 + z^2e^{-y^2} \right] + b \\ &\geq 2a \left[|y|^3 - 10 + |y|e^{-z^2} - 1 + |z|^3 - 1 + |z|e^{-y^2} - 1 \right] + b \\ &\geq 2|y|^3 + |y|e^{-z^2} + 2|z|^3 + |z|e^{-y^2} + 1, \quad \text{for } a = 1, b = 27 \\ &\geq \|f(t, p)\| \end{aligned}$$

Thus f satisfies the conditions of Theorem 3.2 for the choices $a = 1$ and $b = 27$.

It is not difficult to verify that the remaining conditions of Theorem 3.2 hold with $\beta = 1/2$, $N = 1$ and $L = 0$. Thus we conclude that our problem has at least one solution. \square

The theorems in [4, 9, 10, 15, 18, 20, 21, 22, 26, 29, 30] do not directly apply to the previous example as: the growth of $\|f(t, p)\|$ in $\|p\|$ is super-linear; the boundary conditions are a wider range or a different class; and the problem involves a system of impulsive BVPs.

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