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ON OLECK-OPIAL-BEESACK-TROY INTEGRO-DIFFERENTIAL INEQUALITIES

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ABSTRACT. We obtain necessary and sufficient conditions for the integro-differential inequality

$$\int_a^b \dot{x}^2(t)\,dt \geq \gamma \int_a^b q(t)\,|\dot{x}(t)x(t)|\,dt$$

to be valid with one of the three boundary conditions: $x(a) = 0$, or $x(b) = 0$, or $x(a) = x(b) = 0$. For a power functions q, the best constants γ are found.

INTRODUCTION

Beesack [2] found that the best constant is $\overline{\gamma} = \frac{2}{b-a}$, such that the inequality

$$\int_a^b \dot{x}^2(t)\,dt \geq \gamma \int_a^b |\dot{x}(t)x(t)|\,dt \tag{0.1}$$

holds for all $\gamma \leq \overline{\gamma}$ and for all continuously differentiable functions $x : [a, b] \to \mathbb{R}$ satisfied the condition $x(a) = 0$ (or $x(b) = 0$). For inequality (0.1) with two constrictions $x(a) = x(b) = 0$ the best constant is $\overline{\gamma} = \frac{4}{b-a}$ which was found by Olech and Opial in [8, 9]. Troy [10] obtained some estimates of the best constant γ for the inequality

$$\int_a^b \dot{x}^2(t)\,dt \geq \gamma \int_a^b (t-a)^p\,|\dot{x}(t)x(t)|\,dt \tag{0.2}$$

with the condition $x(a) = 0$ (and also with $x(a) = x(b) = 0$). But the question on the sharpness of these estimates remained open.

The aim of the paper is to answer this question and find the best constants for extended inequalities with different kinds of the boundary conditions. The Perm Seminar on Functional Differential Equations has developed methods for the investigation nonclassical variational problems (see, for example, [1, 5]). These methods allow to prove some known integral and integro-differential inequalities more

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effectively and obtain new inequalities, for example, for integrals with deviating arguments [3, 4]. Here we apply these methods to the integro-differential inequality

$$\int_a^b \dot{x}^2(t)\,dt \geq \gamma \int_a^b q(t)\,|\dot{x}(t)x(t)|\,dt \tag{0.3}$$

which generalizes the inequalities from [2, 8, 9, 10]. We obtain estimates of such constants γ for which (0.3) is valid. We also compute the best constants in some cases including inequality (0.2).

In particular, Troy have proved that (0.3) holds for all continuously differentiable functions $x : [a, b] \to \mathbb{R}$ such that $x(a) = 0$ if

$$\gamma \leq \frac{2\sqrt{p+1}}{(b-a)^{p+1}}, \quad p > -1. \tag{0.4}$$

We prove that (0.3) holds for all absolutely continuous functions $x : [a, b] \to \mathbb{R}$ such that $x(a) = 0$ if and only if

$$\gamma \leq \overline{\gamma}_p = \frac{(p+1)^2\eta_p^2}{2|p|(b-a)^{p+1}}, \quad p \neq 0, \quad p > -1,$$

where η_p is the smallest positive root of the Bessel function $J_{-\frac{1+2p}{1+p}}$ for $p < 0$ and the smallest positive root of the modified Bessel function $I_{-\frac{1+2p}{1+p}}$ for $p > 0$. In section 1.3.a some estimates for $\overline{\gamma}_p$, which are better than (0.4), are obtained. Moreover, in this section the best constants $\overline{\gamma}_p$ are evaluated explicitly for some cases. The singular case $p = -1$ is considered in section 1.3.b.

The method is based on the reduction of the integro-differential inequality to the problem of the minimization for the quadratic functional

$$\mathcal{J}(z) = \int_a^b \big(z(t) - (Kz)(t)\big)z(t)\,dt \to \min \tag{0.5}$$

in the space $\mathbf{L}_2$ of square integrable functions $z : [a, b] \to \mathbb{R}$ with the norm $\|z\| = \sqrt{\int_a^b z^2(t)\,dt}$. The linear operator $K : \mathbf{L}_2 \to \mathbf{L}_2$ in (0.5) is supposed to be self-adjoint and bounded.

It is clear that either a function $z \equiv 0$ is a point of the a minimum of $\mathcal{J}$ and problem (0.5) is solvable ($\min_{z\in\mathbf{L}_2} \mathcal{J}(z) = 0$) or $\mathcal{J}$ is unbounded below on $\mathbf{L}_2$. It's well known that problem (0.5) is solvable if and only if all points of the spectrum of the operator K is not greater than one. Moreover, the following statement is valid [6].

Theorem 0.1. *Suppose $\int_a^b (Kz)(t)z(t)\,dt \geq 0$ for all nonnegative functions $z \in \mathbf{L}_2$. Then problem (0.5) is solvable if and only if the norm (the spectral radius) of the operator K is less than or equal to one.*

So if the conditions of Theorem 0.1 are fulfilled, the question on the solvability of (0.5) is reduced to the computation or the estimation of $\|K\|$. The norm is equal to the spectral radius for self-adjoin operators in $\mathbf{L}_2$. If K is completely continuous, then the norm is equal to the largest eigenvalue of K.

Note that we can compute the norm (or the spectral radius) of the integral operator $Q : \mathbf{L}_2 \to \mathbf{L}_2$ exactly only in rare, extraordinary cases. We can estimate its norm, in particular, with the help of the known inequality:

$$\|Q\| \le \Big(\int_a^b \int_a^b Q_n^2(t,s)\,ds\,dt\Big)^{1/(2n)}, \tag{0.6}$$

$n = 1, 2, \dots$, where $Q_n(t,s)$ is the kernel of the integral operator Q^n. In addition, we use the following helpful result (see, for example, [7, c.406]).

Lemma 0.1. *Let $Q : \mathbf{L}_2 \to \mathbf{L}_2$ be a completely continuous integral operator with non-negative kernel. If there exists a positive function $v \in \mathbf{L}_2$ such that*

$$(Qv)(t) \le rv(t) \text{ almost everywhere on } [a,b], \tag{0.7}$$

then $\|Q\| \le r$.

1. Integro-differential inequalities with the condition $x(a) = 0$

By $\mathbf{W}_a$ denote the space of absolutely continuous functions $x : [a,b] \to \mathbb{R}$ such that $\dot{x} \in \mathbf{L}_2$ and $x(a) = 0$. Suppose the function $q : [a,b] \to \mathbb{R}$ is measurable, nonnegative on $[a,b]$, and $\int_a^b (t-a)q^2(t)\,dt < \infty$.

Our task is to find all γ such that for any function $x \in \mathbf{W}_a$ inequality (0.3) holds. Obviously, here and below we must consider only $\gamma > 0$, since for all $\gamma \le 0$ inequality (0.3) is valid. Note that the functional on $\mathbf{W}_a$

$$x \longmapsto \int_a^b \Big(\dot{x}^2(t) - \gamma q(t)|\dot{x}(t)x(t)|\Big)\,dt$$

defined by (0.3) is neither quadratic nor differentiable at zero. Therefore at first we reduce the question on the solvability of (0.3) to the minimization problem for a functional of form (0.5).

Lemma 1.1. *Inequality (0.3) holds for all $x \in \mathbf{W}_a$ if and only if the zero function is a solution of the variational problem*

$$\begin{gathered} J(x) = \int_a^b \Big(\dot{x}^2(t) - \gamma q(t)\dot{x}(t)x(t)\Big)\,dt \to \min, \\ x(a) = 0. \end{gathered} \tag{1.1}$$

Proof. If $x \equiv 0$ is not a solution to (1.1), then the functional J is negative for some function $x_0 \in \mathbf{W}_a$. Hence,

$$\int_a^b (\dot{x}_0^2(t) - \gamma q(t)|\dot{x}_0(t)x_0(t)|)\,dt \le \int_a^b (\dot{x}_0^2(t) - \gamma q(t)\dot{x}_0(t)x_0(t))\,dt < 0.$$

So inequality (0.3) does not hold for the function x_0.

If for a function $x_1 \in \mathbf{W}_a$ inequality (0.3) does not hold, then $J(x_2) < 0$ for the function $x_2(t) = \int_a^t |\dot{x}_1(s)|\,ds$. Indeed, we have $\dot{x}_2(t) = |\dot{x}_1(t)|$, $x_2(t) \ge |x_1(t)|$ on $[a,b]$. Therefore,

$$J(x_2) = \int_a^b \Big(\dot{x}_2^2(t) - \gamma q(t)|\dot{x}_2(t)x_2(t)|\Big)\,dt \le \int_a^b \Big(\dot{x}_1^2(t) - \gamma q(t)|\dot{x}_1(t)x_1(t)|\Big)\,dt < 0.$$

Thus inequality (0.3) holds if and only if variational problem (1.1) is solvable. $\square$

By $Q_b : \mathbf{L}_2 \to \mathbf{L}_2$ denote the linear integral operator

$$(Q_b z)(t) = \int_a^b Q_b(t,s)z(s)\,ds := q(t)\int_a^t z(s)\,ds + \int_t^b q(s)z(s)\,ds,$$

where

$$Q_b(t,s) = \begin{cases} q(t) \text{ if } a \le s < t \le b, \\ q(s) \text{ if } a \le t \le s \le b. \end{cases}$$

Since

$$\|Q_b\|^2 \le \int_a^b \int_a^b Q_b^2(t,s)\,dt\,ds = 2\int_a^b (t-a)q^2(t)\,dt < \infty,$$

the operator Q_b is completely continuous.

Our main result in this section is the following.

Theorem 1.1. *Inequality* (0.3) *holds for all* $x \in \mathbf{W}_a$ *if and only if*

$$\gamma \le \overline{\gamma} := 2/\|Q_b\|.$$

Proof. The substitution $x(t) = \int_a^t z(s)\,ds$ reduces problem (1.1) to the problem of the form (0.5):

$$\int_a^b \Big(z^2(t) - \gamma q(t) z(t) \int_a^t z(s)\,ds\Big) dt = \int_a^b \big(z(t) - (Kz)(t)\big) z(t)\,dt \to \min,$$

where the integral operator $(Kz)(t) = \int_a^b K(t,s)z(s)\,ds$ has the symmetric kernel $K(t,s) = \dfrac{\gamma}{2} Q_b(t,s)$. By Theorem 0.1, problem (1.1) is solvable if and only if $\|K\| = \dfrac{\gamma}{2}\|Q_b\| \le 1$. □

Estimating the norm of the operator Q_b by inequality (0.6) for $n = 1$, we obtain

Corollary 1.1. *If*

$$\gamma \le \frac{\sqrt{2}}{\sqrt{\int_a^b (t-a)q^2(t)\,dt}}, \tag{1.2}$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}_a$.

Letting $v = 1$ in inequality (0.7), we obtain

Corollary 1.2. *If*

$$\gamma \le \frac{2}{\operatorname*{ess\,sup}_{t\in[a,b]} \big(q(t)(t-a) + \int_t^b q(s)\,ds\big)}, \tag{1.3}$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}_a$.

Corollary 1.3. *If the function* q *is non-increasing and*

$$\gamma \le \frac{2}{\int_a^b q(t)\,dt},$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}_a$.

Proof. If the function q is non-increasing, then $Q_b(t,s) \le q(s)$ for all $t \in [a,b]$. Therefore, $\|Q_b\| \le \int_a^b q(s)\,ds$. □

From Lemma 0.1, we obtain

Corollary 1.4. *If*

$$\gamma \le \frac{2}{(b-a)\operatorname*{ess\,sup}_{t\in[a,b]} q(t)},$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}_a$.

The corollaries given above yield lower estimates for the best constant $\overline{\gamma}$. It is easy to obtain upper estimates. For example, putting $x(t) = t - a$ in (0.3) gives

$$\overline{\gamma} \le \frac{b-a}{\int_a^b (t-a)q(t)\,dt}.$$

Another way for finding the best constant $\overline{\gamma}$ is presented by the following result.

Theorem 1.2. *Inequality* (0.3) *holds for all* $x \in \mathbf{W}_a$ *if and only if* $\gamma \le \overline{\gamma}$, *where* $\overline{\gamma}$ *is smallest* $\gamma > 0$ *such that the Cuachy problem*

$$\begin{gathered} \dot{x}(t) = \frac{\gamma}{2}\Big(q(t)x(t) + \int_t^b q(s)\dot{x}(s)\,ds\Big), \\ x(a) = 0 \end{gathered} \tag{1.4}$$

has a nonzero solution.

Proof. Substituting $x(t) = \int_a^t z(s)\,ds$ in (1.4), we obtain $\gamma^{-1}z = \frac{1}{2}Q_b z$. So, $\overline{\gamma}^{-1}$ is the largest eigenvalue of the operator $\frac{1}{2}Q_b$. Therefore, $\gamma\overline{\gamma}^{-1} = \|K\|$ is the largest eigenvalue of the operator K. Then it follows from Theorem 0.1 that problem (0.5) is solvable if and only if $\gamma \le \overline{\gamma}$. ☐

If q is absolutely continuous on $[a,b]$, problem (1.4) can be written in the form

$$\begin{gathered} \dot{x}(t) = \frac{\gamma}{2}\Big(q(b)x(b) - \int_t^b \dot{q}(s)x(s)\,ds\Big), \\ x(a) = 0. \end{gathered} \tag{1.5}$$

Corollary 1.5. *If the function* q *is absolutely continuous and*

$$\gamma \le \frac{2}{(b-a)q(b) + \int_a^b (s-a)|\dot{q}(s)|\,ds}, \tag{1.6}$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}_a$.

Proof. Problem (1.5) is equivalent to the equation

$$x(t) = \frac{\gamma}{2}\Big(q(b)x(b)(t-a) - \int_a^t \int_s^b \dot{q}(\tau)x(\tau)\,d\tau ds\Big).$$

Let

$$\gamma < \gamma^* := \frac{2}{(b-a)q(b) + \int_a^b (s-a)|\dot{q}(s)|\,ds}.$$

If x is a solution to problem (1.5), then

$$\begin{aligned} \max_{t\in[a,b]} |x(t)| &\le \frac{\gamma}{2}\Big(q(b)|x(b)|(b-a) + \int_a^b \int_s^b |\dot{q}(\tau)|\,d\tau ds \max_{t\in[a,b]} |x(t)|\Big) \\ &\le \frac{\gamma}{2}\Big(q(b)(b-a) + \int_a^b (s-a)|\dot{q}(s)|\,ds\Big) \max_{t\in[a,b]} |x(t)| \le \frac{\gamma}{\gamma^*} \max_{t\in[a,b]} |x(t)|. \end{aligned}$$

Hence, $x = 0$. Thus we get $\gamma* \le \overline{\gamma}$. ☐

From representation (1.5), it is easy to obtain the following theorem.

Theorem 1.3. *Let the function q be absolutely continuous on $[a,b]$. Then inequality (0.3) holds for all $x \in \mathbf{W}_a$ if and only if $\gamma \le \overline{\gamma}$, where $\overline{\gamma}$ is the smallest $\gamma > 0$ such that the boundary value problem for the second order ordinary differential equation*

$$\ddot{x}(t) = \frac{\gamma}{2}\dot{q}(t)x(t), \tag{1.7}$$

$$x(a) = 0, \tag{1.8}$$

$$\dot{x}(b) = \frac{\gamma}{2}q(b)x(b) \tag{1.9}$$

has a nonzero solution.

If we can integrate equation (1.7), then it is possible to obtain the exact value of the best constant. In this paper, we confine ourselves to the case of power functions q. First of all, we will gather some information on differential equations and Bessel functions.

The general solution to the equation

$$\ddot{x}(t) = -\frac{1}{2}\gamma|p|t^{p-1}x(t), \tag{1.10}$$

can be written in the form

$$x(t) = c_1u^-(t) + c_2v^-(t), \tag{1.11}$$

where c_1, c_2 are arbitrary constants, and the functions u^-, v^- are defined with the help of the Bessel functions of the first and second kind respectively:

$$u^-(t) = \sqrt{t}J_{\frac{1}{p+1}}(\frac{\sqrt{2\gamma|p|}}{p+1}t^{\frac{p+1}{2}}), \quad v^-(t) = \sqrt{t}Y_{\frac{1}{p+1}}(\frac{\sqrt{2\gamma|p|}}{p+1}t^{\frac{p+1}{2}}).$$

Note that there exist the finite non-zero limits $\lim_{t\to 0}\dfrac{u^-(t)}{t}$ and $\lim_{t\to 0}v^-(t) = v(0)$.

The solution to (1.10) satisfies the equation

$$\dot{x}(t) + \frac{\operatorname{sgn}(p)}{2}\gamma t^p x(t) = c_1u_1^-(t) + c_2v_1^-(t), \tag{1.12}$$

where

$$\operatorname{sgn}(p) = \begin{cases} 1 & \text{if } p > 0, \\ 0 & \text{if } p = 0, \\ -1 & \text{if } p < 0, \end{cases}$$

$$u_1^-(t) = -\frac{\operatorname{sgn}(p)}{2}\gamma t^{p+\frac{1}{2}}J_{-\frac{2p+1}{p+1}}(\frac{\sqrt{2|p|\gamma}}{p+1}t^{\frac{p+1}{2}}),$$

$$v_1^-(t) = -\frac{\operatorname{sgn}(p)}{2}\gamma t^{p+\frac{1}{2}}Y_{-\frac{2p+1}{p+1}}(\frac{\sqrt{2|p|\gamma}}{p+1}t^{\frac{p+1}{2}})$$

(here we use the recurrent formulae for Bessel functions $\Theta'_\nu(z) = \Theta_{\nu-1}(z) - \frac{\nu}{z}\Theta_\nu(z)$) and $\Theta_{\nu-1}(z) = 2\frac{\nu}{z}\Theta_\nu(z) - \Theta_{\nu+1}(z)$). There exists the finite non-zero limits

$$\lim_{t\to 0}u_1^-(t) = u_1^-(0) \quad \text{and} \quad \lim_{t\to 0}v_1^-(t) = v^-(0).$$

If $\frac{1}{p+1}$ is half-integer (that is, for $p = \dots, -\frac{7}{5}, -\frac{5}{3}, -3, 1, -\frac{1}{3}, -\frac{3}{5}, -\frac{5}{7}, \dots$), then Bessel functions and solutions to equation (1.10) can be expressed by elementary functions (see Appendix 1).

Similarly, the general solution to the equation

$$\ddot{x}(t) = \frac{1}{2}\gamma|p|t^{p-1}x(t) \tag{1.13}$$

can be written in the form

$$x(t) = c_1 u^+(t) + c_2 v^+(t), \tag{1.14}$$

where c_1, c_2 are arbitrary constants, and the functions u^+, v^+ are defined with the help of the modified Bessel functions of the first and second kind respectively:

$$u^+(t) = \sqrt{t} I_{\frac{1}{p+1}}\big(\frac{\sqrt{2|p|\gamma}}{p+1} t^{\frac{p+1}{2}}\big), \quad v^+(t) = \sqrt{t} K_{\frac{1}{p+1}}\big(\frac{\sqrt{2|p|\gamma}}{p+1} t^{\frac{p+1}{2}}\big)$$

There exist the finite non-zero limits $\lim_{t\to 0} \dfrac{u^+(t)}{t}$ and $\lim_{t\to 0} v^+(t) = v^+(0)$.

The solution to (1.13) satisfies the equation

$$\dot{x}(t) - \frac{\operatorname{sgn}(p)}{2}\gamma t^p x(t) = c_1 u_1^+(t) + c_2 v_1^+(t), \tag{1.15}$$

where

$$u_1^+(t) = -\frac{\operatorname{sgn}(p)}{2}\gamma t^{p+\frac{1}{2}} I_{-\frac{2p+1}{p+1}}\big(\frac{\sqrt{2|p|\gamma}}{p+1} t^{\frac{p+1}{2}}\big),$$

$$v_1^+(t) = -\frac{\operatorname{sgn}(p)}{2}\gamma t^{p+\frac{1}{2}} K_{-\frac{2p+1}{p+1}}\big(\frac{\sqrt{2|p|\gamma}}{p+1} t^{\frac{p+1}{2}}\big)$$

(here we use the recurrent formulae for modified Bessel functions $\Theta'_\nu(z) = \Theta_{\nu-1}(z) - \frac{\nu}{z}\Theta_\nu(z)$) and $\Theta_{\nu-1}(z) = 2\frac{\nu}{z}\Theta_\nu(z) + \Theta_{\nu+1}(z)$). There exists the finite non-zero limits $\lim_{t\to 0} u_1^+(t) = u_1^+(0)$ and $\lim_{t\to 0} v_1^+(t) = v_1^+(0)$ for $-\frac{1}{2} < p$.

If $\frac{1}{p+1}$ is half-integer, then modified Bessel functions and solutions to equation (1.13) can be expressed by elementary functions (see Appendix 1).

Now consider several cases when we can find the explicit form of the solution to (1.7) in elementary functions.

1.1. The constant function $q(t) = q$. The general solution to the problem (1.7), (1.8) is $x(t) = c(t-a)$. Hence,

$$\overline{\gamma} = \frac{2}{(b-a)q}.$$

Note that the sharpness of this constant was established by Beesack [2].

1.2.a. The linear function $q(t) = t - a$. The general solution to the problem (1.7), (1.8) is $x(t) = c\sinh\sqrt{\frac{\gamma}{2}}(t-a)$. Hence,

$$\overline{\gamma} = \frac{2\eta_1^2}{(b-a)^2} \approx \frac{2.87845768}{(b-a)^2},$$

where $\coth\eta_1 = \eta_1$, $\eta_1 \approx 1.19967864$.

1.2.b. The linear function $q(t) = b - t$. The general solution to the problem (1.7), (1.8) is $x(t) = c\sin\sqrt{\frac{\gamma}{2}}(t-a)$. Hence,

$$\overline{\gamma} = \frac{\pi^2}{2(b-a)^2} \approx \frac{4.93480220}{(b-a)^2}.$$

1.2.c. The general linear function $q(t) = k(t-a) + r$. The general solution to the problem (1.7), (1.8) is $x(t) = c \sinh \sqrt{\frac{k\gamma}{2}}(t-a)$ if $k > 0$, and $x(t) = c \sin \sqrt{-\frac{k\gamma}{2}}(t-a)$ if $k < 0$. Thus

$$\overline{\gamma} = \frac{2\upsilon^2}{k(b-a)^2}, \quad \coth(\upsilon) = \upsilon(1 + \frac{r}{k(b-a)})$$

if $k > 0$; and

$$\overline{\gamma} = -\frac{2\upsilon^2}{k(b-a)^2}, \quad \cot(\upsilon) = \upsilon(1 + \frac{r}{k(b-a)})$$

if $k < 0$, where υ is the smallest positive solution to the respective equation.

1.3.a. The power function $q(t) = (t-a)^p$, $p > -1$. Perform the change of variables $t - a = s$, $x(t) = y(s)$. Then equation (1.7) has the form (1.10) for $p < 0$ and the form (1.13) for $p > 0$. The boundary conditions (1.8), (1.9) have the form $y(0) = 0$, $\dot{y}(b-a) - \frac{1}{2}\gamma(b-a)^p y(b-a) = 0$. From (1.11) and (1.14), it follows that $c_2 = 0$. From (1.12) and (1.15), it follows that $u_1^-(b-a) = 0$ and $u_1^+(b-a) = 0$, respectively.

Denote by η_p the smallest positive root of the Bessel function $J_{-\frac{1+2p}{1+p}}$ for $p < 0$ and the smallest positive root of the modified Bessel function $I_{-\frac{1+2p}{1+p}}$ for $p > 0$. The the best constant for every $p > -1$, $p \neq 0$ has the representation

$$\overline{\gamma}_p = \frac{(p+1)^2 \eta_p^2}{2|p|(b-a)^{p+1}}.$$

Whence, in particular, we have $\overline{\gamma}_1 = \frac{2\eta_1^2}{(b-a)^2} \approx \frac{2.87845768}{(b-a)^2}$, where $\coth \eta_1 = \eta_1$; $\overline{\gamma}_{-1/3} = \dfrac{\pi^2}{6(b-a)^{2/3}} \approx \dfrac{1.64493407}{(b-a)^{2/3}}$; $\overline{\gamma}_{-3/5} = \dfrac{2\pi^2}{15(b-a)^{2/5}} \approx \dfrac{1.315947254}{(b-a)^{2/5}}$; $\overline{\gamma}_{-5/7} = \dfrac{2\eta_{-5/7}^2}{35(b-a)^{2/7}} \approx \dfrac{1.15375592}{(b-a)^{2/7}}$, where $\eta_{-5/7}$ is the smallest positive solution to the equation $\tan \eta = \eta$, $\overline{\gamma}_{-7/9} = \dfrac{2\eta_{-7/9}^2}{63(b-a)^{2/9}} \approx \dfrac{1.05452260}{(b-a)^{2/9}}$, where $\eta_{-7/9}$ is the smallest positive solution to the equation $\tan \eta = \dfrac{3\eta}{3-\eta^2}$.

Some approximate values of $\overline{\gamma}_p$ for $b - a = 1$ are given in Appendix 2, and the graph is shown in Figure 1.

If the Bessel functions are not elementary, then we can obtain some lower estimates for the best constant $\overline{\gamma}_p$. From (0.6) for $n = 1, 2, 3, 4$, we get respectively more and more exact estimates:

$\overline{\gamma}_p \geq \gamma_p^{(1)} := \frac{2\sqrt{p+1}}{(b-a)^{p+1}}$ (it was obtained in [10] by W.C. Troy);

$\overline{\gamma}_p \geq \gamma_p^{(2)} := \frac{2}{(b-a)^{p+1}} \left(\frac{(p+1)(p+2)(2p+3)}{p+6}\right)^{1/4}$;

$\overline{\gamma}_p \geq \gamma_p^{(3)} := \frac{2}{(b-a)^{p+1}} \left(\frac{(4+3p)(5+4p)(2p+3)(p+2)^2(p+1)}{6p^3+100p^2+292p+240}\right)^{1/6}$;

$\overline{\gamma}_p \geq \gamma_p^{(4)} := \dfrac{2}{(b-a)^{p+1}} \left(\frac{(7+6p)(6+5p)(p+1)(p+2)^2(2p+3)^2(5+4p)(4+3p)}{180p^5+5431p^4+31882p^3+74652p^2+77832p+30240}\right)^{1/8}$.

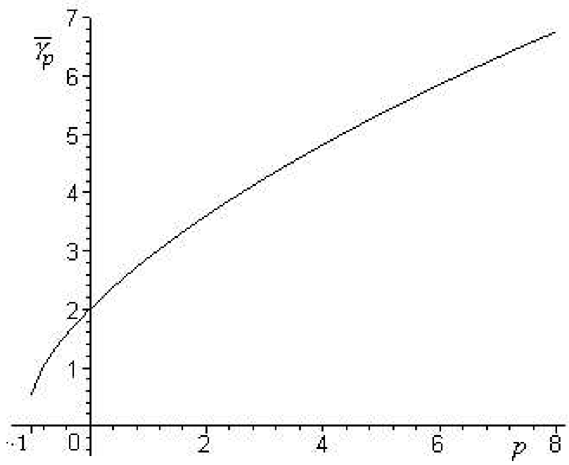

FIGURE 1. The best constant $\overline{\gamma}_p$ in case 1.3.a.

Estimating $\|K\|$ with the help of Lemma 0.1, where $v(t) = \Big(1 - \frac{\mu}{p+2}\Big)\frac{1}{1-\mu} + (\frac{t-a}{b-a})^{p+1}$ and

$$\mu = \frac{4}{(b-a)^{p+1}} \frac{(p+1)(p+2)}{\sqrt{8(p+1)^3 + 9(p+1)^2 - 2p - 1} + 3p + 4},$$

we also get
$\overline{\gamma}_p \geq \gamma_p^{(5)} := 2\mu$ for $p \geq 0$;
estimating $\|K\|$ with the help of Lemma 0.1, where $v(t) = (\frac{t-a}{b-a})^{-\frac{1}{2}+\varepsilon}$ and $\varepsilon > 0$ is small enough, we obtain

$$\overline{\gamma}_p \geq \gamma_p^{(6)} := \frac{1}{2} \frac{1}{|p|(b-a)^{p+1}} \quad \text{for } p \in (-1, -1/2]. \tag{1.16}$$

Note that $\gamma_p^{(1)} < \gamma_p^{(2)} < \gamma_p^{(3)} < \gamma_p^{(4)}$ for all $p > -1$, $p \neq 0$. Thus estimate 1) from [10] is not sharp besides the known case $p = 0$ [2].

In particular, for $p = 1$ and $b-a = 1$, we have $\gamma_1^{(1)} = 2.8284271$, $\gamma_1^{(2)} = 2.8776356$, $\gamma_1^{(3)} = 2.8784392$, $\gamma_1^{(4)} = 2.8784572$ (the sharp constant $\overline{\gamma}_1$ equals 2.8784577 to the last decimal place).

We have $\gamma_p^{(4)} < \gamma_p^{(5)}$ for $p > 50.1$, therefore the estimate $\gamma_p^{(5)}$ yields the best result for large values of p. The estimate $\gamma_p^{(6)}$ is best for small p.

Setting $x(t) = (t-a)^{\frac{1+\sqrt{1+p}}{2}}$ in (0.3), we obtain the upper estimate for $\overline{\gamma}_p$:

$$\overline{\gamma}_p \leq \gamma_p^{(0)} = \frac{(1 + \sqrt{1+p})^2}{2}.$$

The graphics of $\gamma_p^{(k)}$ for $b - a = 1$ are shown in Figure 2.

1.3.b. The singular case $q(t) = \dfrac{1}{t-a}$**.** From (1.16), the passage to the limit $p \to -1$ yields $\overline{\gamma}_{-1} \geq \frac{1}{2}$. On the other hand, setting $x_n(t) = (t-a)^{\frac{1}{n}-\frac{1}{2}}$, $n =$

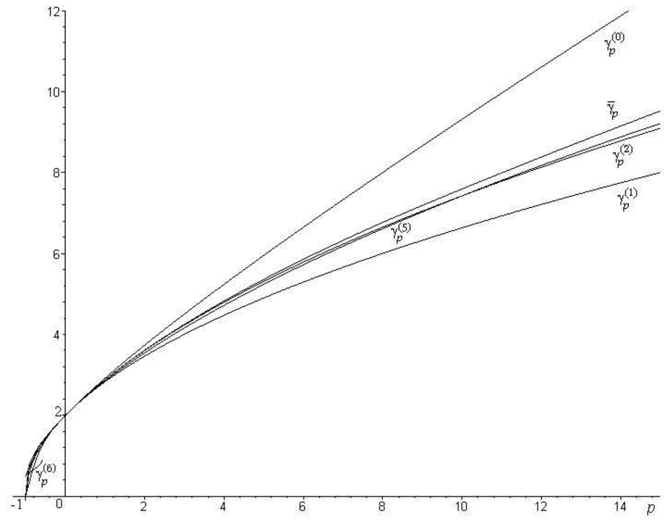

Figure 2. The estimates $\gamma_p^{(k)}$

$2, 3, \dots$, we obtain $\overline{\gamma}_{-1} \le \frac{1}{2} + \frac{1}{n}$. Therefore,

$$\overline{\gamma}_{-1} = \frac{1}{2}.$$

Moreover, it can be shown that

$$\int_a^b \dot{x}^2(t)\,dt > \frac{1}{2}\int_a^b \frac{|\dot{x}(t)x(t)|}{t-a}\,dt$$

for all $x \in \mathbf{W}_a$, $x \not\equiv 0$.

1.3.c. The power function $q(t) = (t-\tau)^p$, $\tau < a$, $p \ne 0$. Perform the change of variables $t - \tau = s$, $x(t) = y(s)$. Then equation (1.7) has the form (1.10) for $p < 0$ and the form (1.13) for $p > 0$. The boundary conditions (1.8), (1.9) have the form $y(a-\tau) = 0$, $\dot{y}(b-\tau) + \frac{1}{2}\gamma(b-\tau)^p y(b-\tau) = 0$.

For $p < 0$ from (1.11) and (1.12) we obtain

$$c_1 u^-(a-\tau) + c_2 v^-(a-\tau) = 0,$$
$$c_2 u_1^-(b-\tau) + c_2 v_1^-(b-\tau) = 0.$$

Therefore, problem (1.7 - 1.9) has a nonzero solution if and only if

$$u^-(a-\tau)v_1^-(b-\tau) - v^-(a-\tau)u_1^-(b-\tau) = 0,$$

that is,

$$J_{\frac{1}{p+1}}\Big(\frac{\sqrt{-2\gamma p}}{p+1}(a-\tau)^{\frac{p+1}{2}}\Big)Y_{-\frac{2p+1}{p+1}}\Big(\frac{\sqrt{-2\gamma p}}{p+1}(b-\tau)^{\frac{p+1}{2}}\Big)$$
$$-J_{-\frac{2p+1}{p+1}}\Big(\frac{\sqrt{-2\gamma p}}{p+1}(b-\tau)^{\frac{p+1}{2}}\Big)Y_{\frac{1}{p+1}}\Big(\frac{\sqrt{-2\gamma p}}{p+1}(a-\tau)^{\frac{p+1}{2}}\Big) = 0.$$

For $p > 0$ from (1.14) and (1.15) we obtain

$$c_1 u^+(a-\tau) + c_2 v^+(a-\tau) = 0,$$
$$c_2 u_1^+(b-\tau) + c_2 v_1^+(b-\tau) = 0.$$

In this case problem (1.7 - 1.9) has a nonzero solution if and only if

$$u^+(a-\tau)v_1^+(b-\tau) - v^+(a-\tau)u_1^+(b-\tau) = 0,$$

that is,

$$\begin{aligned} &I_{\frac{1}{p+1}}\Big(\frac{\sqrt{2\gamma p}}{p+1}(a-\tau)^{\frac{p+1}{2}}\Big)K_{-\frac{2p+1}{p+1}}\Big(\frac{\sqrt{2\gamma p}}{p+1}(b-\tau)^{\frac{p+1}{2}}\Big)\\ &-I_{-\frac{2p+1}{p+1}}\Big(\frac{\sqrt{2\gamma p}}{p+1}(b-\tau)^{\frac{p+1}{2}}\Big)K_{\frac{1}{p+1}}\Big(\frac{\sqrt{2\gamma p}}{p+1}(a-\tau)^{\frac{p+1}{2}}\Big) = 0. \end{aligned}$$

Denote by $\theta_p(k)$ the smallest positive root of the function

$$J_{\frac{1}{p+1}}(\theta k)Y_{-\frac{2p+1}{p+1}}(\theta) - J_{-\frac{2p+1}{p+1}}(\theta)Y_{\frac{1}{p+1}}(\theta k)$$

for $p < 0$ and the smallest positive root of the function

$$I_{\frac{1}{p+1}}(\theta k)K_{-\frac{2p+1}{p+1}}(\theta) - I_{-\frac{2p+1}{p+1}}(\theta)K_{\frac{1}{p+1}}(\theta k).$$

for $p > 0$. Then the best constant is defined by the equality

$$\overline{\gamma} = \frac{(p+1)^2\theta_p^2\big((\frac{a-\tau}{b-\tau})^{\frac{p+1}{2}}\big)}{2|p|(b-\tau)^{p+1}}.$$

Note that $\theta_{-\frac{1}{3}}(k)$ is the smallest positive solution to the equation $\cot(1-k)\theta = k\theta$; $\theta_{-\frac{3}{5}}(k)$ is the smallest positive solution to the equation $\cot(1-k)\theta = \dfrac{k^2\theta^2-3}{3k\theta}$; $\theta_1(k)$ satisfies the equation $\coth(1-k)\theta = \theta$.

The graphics of $\theta_p(k)$ for some k are shown in Figure 3.

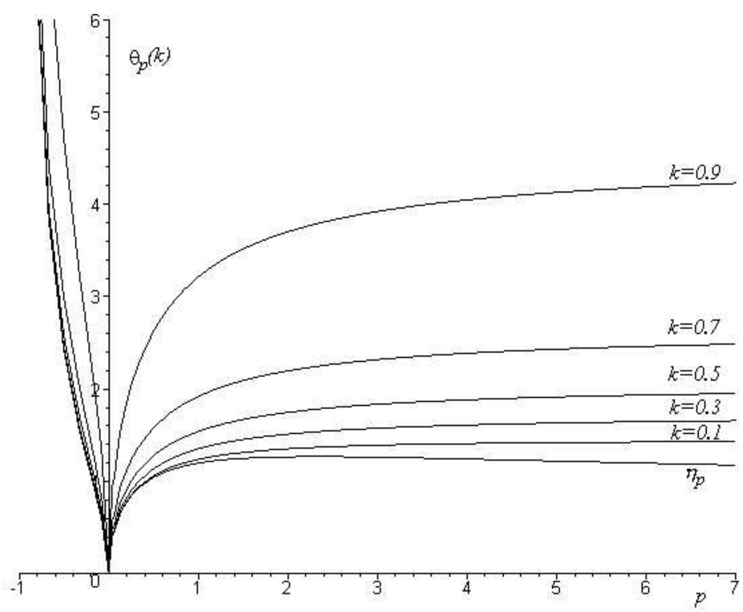

Figure 3. The graphics of $\theta_p(k)$ and η_p

We have $\lim_{k\to 0}\theta_p(k) = \eta_p$. Now we give some estimate for the best constant $\overline{\gamma}$.

By (1.2), it follows that

$$\overline{\gamma} \geq \frac{2\sqrt{2p+1}}{\sqrt{2p+1-2(p+1)\frac{a-\tau}{b-\tau}+(\frac{a-\tau}{b-\tau})^{2p+2}}} \frac{\sqrt{p+1}}{(b-\tau)^{p+1}}.$$

By (1.3), it follows that

$$\overline{\gamma} \geq \frac{2(p+1)}{(b-\tau)^{p+1}-(a-\tau)^{p+1}} \text{ for } p \leq 0,$$

and

$$\overline{\gamma} \geq \frac{2}{(b-\tau)^{p}(b-a)} \text{ for } p \geq 0.$$

1.4.a. The power function $q(t) = (b-t)^p$, $p > 0$. Perform the change of variables $b - t = s$, $x(t) = y(s)$. Then (1.7) has the form (1.10). Boundary condition (1.9) has the form $y(0) = 0$. From (1.12) it follows that $c_1 u_1^-(0) + c_2 v_1^-(0) = 0$. From boundary condition (1.8) and representation (1.11), it follows that $c_1 u^-(b-a) + c_2 v(b-a) = 0$. Therefore $v_1^-(0)u^-(b-a) - u_1^-(0)v^-(b-a) = 0$, that is,

$$\cos(\frac{2p+1}{p+1}\pi)J_{\frac{1}{p+1}}(\frac{\sqrt{2\gamma p}}{p+1}(b-a)^{\frac{p+1}{2}}) - \sin(\frac{2p+1}{p+1}\pi)Y_{\frac{1}{p+1}}(\frac{\sqrt{2\gamma p}}{p+1}(b-a)^{\frac{p+1}{2}}) = 0.$$

Denote by ζ_p the smallest positive root of the function

$$\cos(\frac{2p+1}{p+1}\pi)J_{\frac{1}{p+1}}(\zeta) - \sin(\frac{2p+1}{p+1}\pi)Y_{\frac{1}{p+1}}(\zeta)$$

(the graphic of ζ_p is shown in Figure 4).

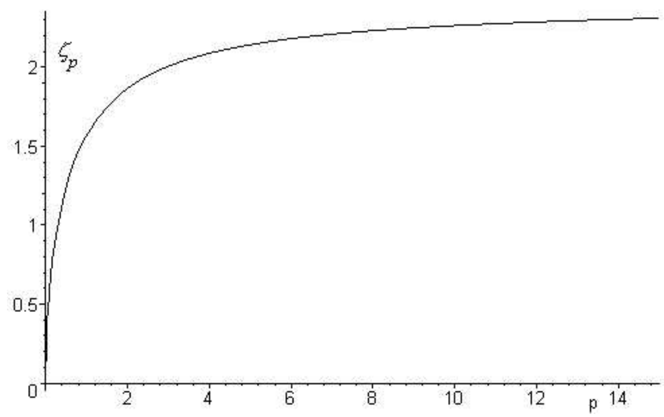

FIGURE 4. The graphic of ζ_p

In this case the best constant is

$$\overline{\gamma}_p = \frac{(p+1)^2\zeta_p^2}{2p(b-a)^{p+1}}.$$

In particular,

$$\overline{\gamma}_1 = \frac{\pi^2}{2(b-a)^2}.$$

Some approximate values of $\overline{\gamma}_p$ for $b-a=1$ are given in Appendix 3, and the graphic is shown in Figure 5.

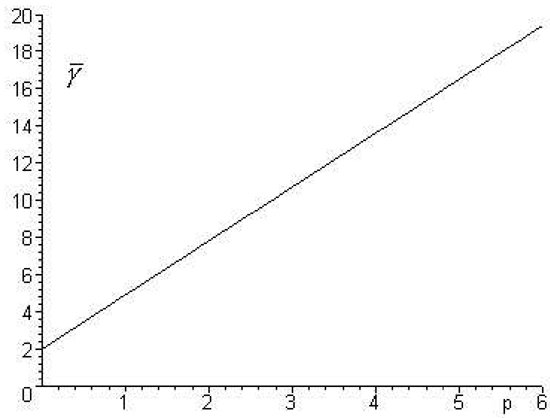

FIGURE 5. The graphic of $\overline{\gamma}_p$ in the case 1.4.a for $b-a=1$

Note that inequality (1.2) implies the lower estimate

$$\overline{\gamma} \geq \frac{2\sqrt{(2p+1)(p+1)}}{(b-a)^{p+1}}.$$

1.4.b. The power function $q(t) = (\tau - t)^p$**,** $b < \tau$**,** $p \neq 0$**.** First let $p < 0$. Perform the change of variables $\tau - t = s$, $x(t) = y(s)$. Then equation (1.7) has the form (1.13). And from condition (1.8), (1.9) and representation (1.14), (1.15), we obtain

$$\begin{gathered} c_1 u^+(\tau - a) + c_2 v^+(\tau - a) = 0, \\ c_2 u_1^+(\tau - b) + c_2 v_1^+(\tau - b) = 0. \end{gathered}$$

Thus problem (1.7) - (1.9) has a nonzero solution if and only if

$$u^+(\tau - a) v_1^+(\tau - b) - v^+(\tau - b) u_1^+(\tau - b) = 0,$$

that is,

$$\begin{aligned} & I_{\frac{1}{p+1}}\Big(\frac{\sqrt{-2\gamma p}}{p+1}(\tau-a)^{\frac{p+1}{2}}\Big) K_{-\frac{2p+1}{p+1}}\Big(\frac{\sqrt{-2\gamma p}}{p+1}(\tau-b)^{\frac{p+1}{2}}\Big) \\ & - I_{-\frac{2p+1}{p+1}}\Big(\frac{\sqrt{-2\gamma p}}{p+1}(\tau-a)^{\frac{p+1}{2}}\Big) K_{\frac{1}{p+1}}\Big(\frac{\sqrt{-2\gamma p}}{p+1}(\tau-a)^{\frac{p+1}{2}}\Big) = 0. \end{aligned}$$

For $p > 0$ the change of variables $t - \tau = s$, $x(t) = y(s)$ reduce equation (1.7) to (1.10). From conditions (1.8), (1.9) and representation (1.11), (1.12) we obtain the conditions

$$\begin{gathered} c_1 u^-(\tau - a) + c_2 v^-(\tau - a) = 0, \\ c_2 u_1^-(\tau - b) + c_2 v_1^-(\tau - b) = 0. \end{gathered}$$

Thus problem (1.7)-(1.9) has a nonzero solution if and only if

$$y^-(\tau - a) v_1^-(\tau - b) - v^-(\tau - a) u_1^-(\tau - b) = 0,$$

that is,

$$J_{\frac{1}{p+1}}\Big(\frac{\sqrt{2\gamma p}}{p+1}(\tau-a)^{\frac{p+1}{2}}\Big)Y_{-\frac{2p+1}{p+1}}\Big(\frac{\sqrt{2\gamma p}}{p+1}(\tau-b)^{\frac{p+1}{2}}\Big) - J_{-\frac{2p+1}{p+1}}\Big(\frac{\sqrt{2\gamma p}}{p+1}(\tau-b)^{\frac{p+1}{2}}\Big)Y_{\frac{1}{p+1}}\Big(\frac{\sqrt{2\gamma p}}{p+1}(\tau-a)^{\frac{p+1}{2}}\Big) = 0.$$

Denote by $\vartheta_p(k)$ the smallest positive root of the function

$$I_{\frac{1}{p+1}}(\vartheta k)K_{-\frac{2p+1}{p+1}}(\vartheta) - I_{-\frac{2p+1}{p+1}}(\vartheta)K_{\frac{1}{p+1}}(\vartheta k)$$

for $p<0$, and the smallest positive root of the function

$$J_{\frac{1}{p+1}}(\vartheta k)Y_{-\frac{2p+1}{p+1}}(\vartheta) - J_{-\frac{2p+1}{p+1}}(\vartheta)Y_{\frac{1}{p+1}}(\vartheta k)$$

for $p>0$. Then the best constant is

$$\overline{\gamma} = \frac{(p+1)^2\vartheta_p^2\big((\frac{\tau-a}{\tau-b})^{\frac{p+1}{2}}\big)}{2|p|(\tau-b)^{p+1}}.$$

In particular, $\vartheta_{-\frac{1}{3}}(k)$ is the smallest positive solution to the equation $\coth(k-1)\vartheta = k\vartheta$, $\vartheta_{-\frac{3}{5}}$ is the smallest positive solution to the equation $\coth(k-1)\vartheta = \frac{k^2\vartheta^2+3}{3k\vartheta}$.

The graphics of $\vartheta_p(k)$ for some k are shown in Figure 6.

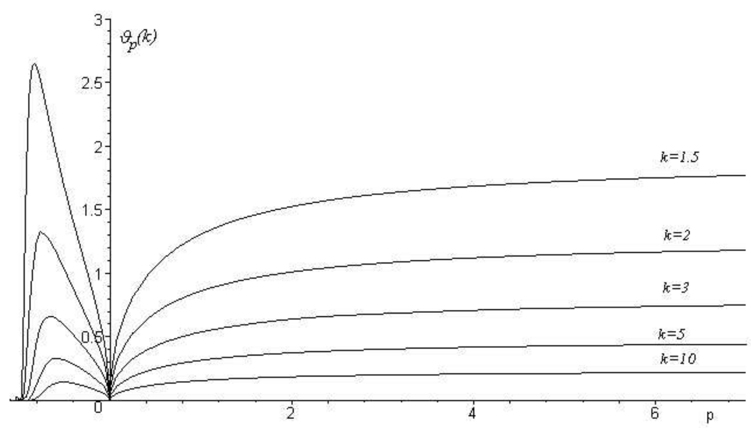

FIGURE 6. The graphic of $\vartheta_p(k)$

Let us get lower estimates for $\overline{\gamma}_p$. By (1.2), it follows that

$$\overline{\gamma} \ge \frac{2\sqrt{p+1}\sqrt{2p+1}}{\sqrt{(2(p+1)a-(2p+1)b-\tau)(b-\tau)^{2p+1}+(\tau-a)^{2p+2}}}.$$

By (1.3), it follows that

$$\overline{\gamma} \ge \frac{2}{(\tau-b)^p(b-a)} \text{ for } p \le 0 \tag{1.17}$$

and

$$\overline{\gamma} \ge \frac{2(p+1)}{(\tau-a)^{p+1}-(\tau-b)^{p+1}} \text{ for } p \ge 0.$$

2. Integro-differential inequalities with the condition $x(b) = 0$

Denote by $\mathbf{W}^b$ the space of absolutely continuous functions $x : [a, b] \to \mathbb{R}$ such that $\dot{x} \in \mathbf{L}_2$ and $x(b) = 0$. Suppose the function $q : [a, b] \to \mathbb{R}$ is measurable, nonnegative on $[a, b]$, and $\int_a^b (b - t)q^2(t)\,dt < \infty$. The proofs of all statements in this section are similar to the proofs of the respective statements in section 1.

Lemma 2.1. *Inequality* (0.3) *holds for all* $x \in \mathbf{W}^b$ *if and only if the zero function is a solution to the variational problem*

$$\begin{gathered}\int_a^b \left(\dot{x}^2(t) + \gamma q(t)\dot{x}(t)x(t)\right)\,dt \to \min,\\ x(b) = 0.\end{gathered} \tag{2.1}$$

The substitution $x(t) = -\int_t^b z(s)\,ds$ reduces problem (2.1) to problem (0.5), where $K = \frac{\gamma}{2}Q^a$, and Q^a is the integral completely continuous operator with the kernel

$$Q^a(t, s) = \begin{cases} q(s) & \text{if } a \le s < t \le b,\\ q(t), & \text{if } a \le t \le s \le b.\end{cases}$$

Theorem 2.1. *Inequality* (0.3) *holds for all* $x \in \mathbf{W}^b$ *if and only if*

$$\gamma \le \overline{\gamma} := \frac{2}{\|Q^a\|}.$$

Corollary 2.1. *If*

$$\gamma \le \frac{\sqrt{2}}{\sqrt{\int_a^b (b - t)q^2(t)\,dt}},$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}^b$.

Corollary 2.2. *If*

$$\gamma \le \frac{2}{\operatorname*{ess\,sup}_{t\in[a,b]}(q(t)(b - t) + \int_a^t q(s)\,ds)}, \tag{2.2}$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}^b$.

Corollary 2.3. *If the function* q *is non-decreasing and*

$$\gamma \le \frac{2}{\int_a^b q(t)\,dt},$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}^b$.

Corollary 2.4. *If*

$$\gamma \le \frac{2}{(b - a)\operatorname{ess\,sup}_{t\in[a,b]} q(t)},$$

then inequality (0.3) *holds for all* $x \in \mathbf{W}^b$.

Theorem 2.2. *Inequality* (0.3) *holds for all* $x \in \mathbf{W}^b$ *if and only if* $\gamma \le \overline{\gamma}$, *where* $\overline{\gamma}$ *equals to the smallest* $\gamma > 0$ *such that the problem*

$$\begin{gathered}\dot{x}(t) = \frac{\gamma}{2}\Big(\int_a^t q(s)\dot{x}(s)\,ds - q(t)x(t)\Big),\\ x(b) = 0\end{gathered} \tag{2.3}$$

has a nonzero solution.

If q is absolutely continuous on $[a,b]$, problem (2.3) can be written in the form

$$\dot{x}(t) = -\frac{\gamma}{2}\Big(\int_a^t \dot{q}(s)x(s)\,ds + q(a)x(a)\Big),$$
$$x(b) = 0.$$

Corollary 2.5. *If the function q is absolutely continuous and*

$$\gamma \le \frac{2}{(b-a)q(a) + \int_a^b (b-s)|\dot{q}(s)|ds},$$

then inequality (0.3) *holds for all $x \in \mathbf{W}^b$.*

Theorem 2.3. *Let the function q be absolutely continuous on $[a,b]$. Then inequality* (0.3) *holds for all $x \in \mathbf{W}^b$ if and only if $\gamma \le \overline{\gamma}$, where $\overline{\gamma}$ is the smallest $\gamma > 0$ such that the boundary value problem for the ordinary differential equation*

$$\ddot{x}(t) + \frac{\gamma}{2}\dot{q}(t)x(t) = 0, \tag{2.4}$$
$$x(b) = 0, \tag{2.5}$$
$$\dot{x}(a) + \frac{\gamma}{2}q(a)x(a) = 0$$

has a nonzero solution.

Now consider several cases when we can integrate equation (2.4) in explicit form.

2.1. The constant function $q(t) = q$. The general solution to the problem (2.4), (2.5) is $x(t) = c(b-t)$. Hence,

$$\overline{\gamma} = \frac{2}{(b-a)q}.$$

2.2.a. The linear function $q(t) = t - a$. The general solution to the problem (2.4), (2.5) is $x(t) = c\sin\sqrt{\frac{\gamma}{2}}(b-t)$. Hence,

$$\overline{\gamma} = \frac{\pi^2}{2(b-a)^2}.$$

2.2.b. The linear function $q(t) = b - t$. The general solution to the problem (2.4), (2.5) is $x(t) = c\sinh\sqrt{\frac{\gamma}{2}}(b-t)$. Hence,

$$\overline{\gamma} = \frac{2\eta_1^2}{(b-a)^2}.$$

2.2.c. The general linear function $q(t) = k(b-t) + r$. The general solution to the problem (2.4), (2.5) is $x(t) = c\sinh\sqrt{\frac{k\gamma}{2}}(b-t)$ if $k > 0$ and $x(t) = c\sin\sqrt{-\frac{k\gamma}{2}}(b-t)$ if $k < 0$. So,

$$\overline{\gamma} = \frac{2\upsilon^2}{k(b-a)^2}, \quad \coth(\upsilon) = \upsilon(1 + \frac{r}{k(b-a)}),$$

if $k > 0$; and

$$\overline{\gamma} = -\frac{2\upsilon^2}{k(b-a)^2}, \quad \cot(\upsilon) = -\upsilon(1 + \frac{r}{k(b-a)}),$$

if $k < 0$, where υ is the smallest positive solution to the respective equation.

2.3.a. The power function $q(t) = (b-t)^p$, $p > -1$, $p \neq 0$. Here we have

$$\overline{\gamma}_p = \frac{(p+1)^2 \eta_p^2}{2|p|(b-a)^{p+1}}. \tag{2.6}$$

2.3.b. The singular case $q(t) = \dfrac{1}{b-t}$. As known, $\lim_{\nu\to\infty} \frac{j_{\nu,1}}{\nu} = 1$, where $j_{\nu,1}$ is the first positive root of the Bessel function J_ν. Whence by the passage to the limit as $p \to -1$ from (2.6) we get $\overline{\gamma}_{-1} = \frac{1}{2}$.

2.3.c. The power case $q(t) = (\tau - t)^p$, $b < \tau$, $p \neq 0$. Here as in 1.3.c. we have

$$\overline{\gamma} = \frac{(p+1)^2 \theta_p^2 ((\frac{\tau-b}{\tau-a})^{\frac{p+1}{2}})}{2|p|(\tau-a)^{p+1}}.$$

2.4.a. The power function $q(t) = (t-a)^p$, $p > 0$. Here we have

$$\overline{\gamma} = \frac{(p+1)^2 \zeta_p^2}{2p(b-a)^{p+1}}.$$

2.4.b. The power function $q(t) = (t-\tau)^p$, $\tau < a$, $p \neq 0$. Here we obtain

$$\overline{\gamma} = \frac{(p+1)^2 \vartheta_p^2 ((\frac{b-\tau}{a-\tau})^{\frac{p+1}{2}})}{2|p|(a-\tau)^{p+1}}.$$

3. Integro-differential inequalities with $x(a) = x(b) = 0$

By $\mathbf{W}_a^b$ denote the space of absolutely continuous function $x : [a,b] \to \mathbb{R}$ such that $\dot{x} \in \mathbf{L}_2$ and $x(a) = x(b) = 0$. Let the function $q : [a,b] \to \mathbb{R}$ be measurable, nonnegative on $[a,b]$ and

$$\int_a^b (t-a)(b-t) q^2(t)\, dt < \infty. \tag{3.1}$$

Consider in the space $\mathbf{W}_b^a$ inequality

$$\int_a^b \dot{x}^2(t)\, dt \geq \gamma \int_a^b q(t)\, |\dot{x}(t) x(t)|\, dt \tag{3.2}$$

for $\gamma > 0$. Following Beesack [2], we partition the interval $[a,b]$ into the intervals $[a,c]$, $[c,b]$ and reduce inequality (3.2) to problems (1.1) on the interval $[a,c]$ and (2.1) on the interval $[c,b]$.

Recall that for any $c \in [a,b]$ by Q_c we denote the integral operator in the space $\mathbf{L}_2[a,c]$ with the kernel

$$Q_c(t,s) = \begin{cases} q(t) & \text{if } a \leq s < t \leq c, \\ q(s) & \text{if } a \leq t \leq s \leq c; \end{cases}$$

by Q^c we denote the integral operator in the space $\mathbf{L}_2[c,b]$ with the kernel

$$Q^c(t,s) = \begin{cases} q(t) & \text{if } c \leq s < t \leq b, \\ q(s) & \text{if } c \leq t \leq s \leq b. \end{cases}$$

By (3.1), the operators Q_c and Q^c are completely continuous. Since $\|Q_c\|$ is non-decreasing and continuous with respect to c, and $\|Q^c\|$ is non-increasing and continuous with respect to c, and moreover, $\|Q_a\| = \|Q^b\| = 0$, it follows that there exists a point $c \in (a,b)$ such that $Q := \|Q_c\| = \|Q^c\|$. Note also that $Q = \max_{c\in[a,b]} \min(\|Q_c\|, \|Q^c\|) = \min_{c\in[a,b]} \max(\|Q_c\|, \|Q^c\|)$.

Theorem 3.1. *Inequality* (3.2) *holds for all* $x \in \mathbf{W}_a^b$ *if and only if*

$$\gamma \le \overline{\gamma} := \frac{2}{Q}.$$

Proof. Let $c \in (a,b)$ be a constant such that $Q = \|Q_c\| = \|Q^c\|$. Suppose inequality (3.2) holds for all functions $x \in \mathbf{W}_a^b$ for some γ. Prove that either for all $x \in \mathbf{W}_a$

$$\int_a^c \dot{x}^2(t)\,dt \ge \gamma \int_a^c q(t)\,|\dot{x}(t)x(t)|\,dt \tag{3.3}$$

or

$$\int_c^b \dot{x}^2(t)\,dt \ge \gamma \int_c^b q(t)\,|\dot{x}(t)x(t)|\,dt \tag{3.4}$$

for all $x \in \mathbf{W}^b$. Suppose that there exist functions $x_a \in \mathbf{W}_a$, $x_b \in \mathbf{W}^b$ such that

$$\begin{gathered}\int_a^c \dot{x}_a^2(t)\,dt < \gamma \int_a^c q(t)\,|\dot{x}_a(t)x_a(t)|\,dt,\\ \int_c^b \dot{x}_b^2(t)\,dt < \gamma \int_c^b q(t)\,|\dot{x}_b(t)x_b(t)|\,dt.\end{gathered}$$

Then if $x_a(c) \ne 0$ and $x_b(c) \ne 0$, for the function

$$x(t) = \begin{cases} \dfrac{x_a(t)}{x_a(c)} & \text{if } t \in [a,c],\\ \dfrac{x_b(t)}{x_b(c)} & \text{if } t \in [c,b], \end{cases}$$

the inequality

$$\int_a^b \dot{x}^2(t)\,dt < \gamma \int_a^b q(t)\,|\dot{x}(t)x(t)|\,dt, \tag{3.5}$$

holds. It's a contradiction. If $x_a(c) = 0$, then inequality (3.5) holds for the function

$$x(t) = \begin{cases} x_a(t) & \text{if } t \in [a,c],\\ 0 & \text{if } t \in [c,b]. \end{cases}$$

And, at last, if $x_b(c) = 0$, then inequality (3.5) holds for the function

$$x(t) = \begin{cases} 0 & \text{if } t \in [a,c],\\ x_b(t) & \text{if } t \in [c,b]. \end{cases}$$

So at least one of the inequalities (3.3), (3.4) holds. Then, by Theorems 1.1 and 2.1, either $\gamma \le \frac{2}{\|Q_c\|}$ or $\gamma \le \frac{2}{\|Q^c\|}$. Therefore, $\gamma \le \frac{2}{\min(\|Q_c\|, \|Q^c\|)} = 2/Q$.

Suppose now that $\gamma \le 2/Q$. Then for any $x \in \mathbf{W}_a^b$ by Theorem 1.1, we have

$$\int_a^c \dot{x}^2(t)\,dt \ge \gamma \int_a^c q(t)\,|\dot{x}(t)x(t)|\,dt,$$

and

$$\int_c^b \dot{x}^2(t)\,dt \ge \gamma \int_c^b q(t)\,|\dot{x}(t)x(t)|\,dt$$

by Theorem 2.1. By summing these inequalities we obtain inequality (3.2). $\square$

It follows from Theorem 3.1 that the best constant for problem (3.2) is defined by equality $\overline{\gamma} = \frac{2}{Q} = \min_{c\in(a,b)} \max(\overline{\gamma}_c, \overline{\gamma}^c) = \max_{c\in(a,b)} \min(\overline{\gamma}_c, \overline{\gamma}^c)$, where $\overline{\gamma}_c = \frac{2}{\|Q_c\|}$ is the best constant of problem (1.1) on the interval $[a, c]$, and $\overline{\gamma}^c = \frac{2}{\|Q^c\|}$ is the best constant of problem (2.1) on the interval $[c, b]$.

The following statements are corollaries of Theorem 3.1. From Corollaries 1.1, 2.1, we get

Corollary 3.1. *If*
$$\gamma \le \sqrt{2} \max_{c\in[a,b]} \frac{1}{\sqrt{\min\Big(\int_a^c (t-a)q^2(t)\,dt, \int_c^b (b-t)q^2(t)\,dt\Big)}},$$
then inequality (3.2) *holds for all* $x \in \mathbf{W}_a^b$.

From Corollaries 1.2, 2.2, we obtain

Corollary 3.2. *Let* $s_1 = \operatorname{ess\,sup}_{t\in[a,c]}(q(t)(t-a) + \int_t^c q(s)\,ds)$ *and* $s_2 = \operatorname{ess\,sup}_{t\in[c,b]}(q(t)(b-t) + \int_c^t q(s)\,ds)$. *If*
$$\gamma \le \max_{c\in[a,b]} \frac{2}{\min\big(s_1, s_2\big)},$$
then inequality (3.2) *holds for all* $x \in \mathbf{W}_a^b$.

From 3.2, it follows

Corollary 3.3. *If an absolutely continuous function q is non-decreasing and*
$$\gamma \le \max_{c\in[a,b]} \frac{2}{\min\big(q(c)(c-a), \int_c^b q(s)\,ds\big)},$$
then inequality (3.2) *holds for all* $x \in \mathbf{W}_a^b$.

Corollary 3.4. *If an absolutely continuous function q is non-increasing and*
$$\gamma \le \max_{c\in[a,b]} \frac{2}{\min\big(\int_a^c q(s)\,ds, q(c)(b-c)\big)},$$
then inequality (3.2) *holds for all* $x \in \mathbf{W}_a^b$.

Assuming $c = \frac{1}{2}(a+b)$ and using Corollaries 1.4, 2.4, we obtain

Corollary 3.5. *If*
$$\gamma \le \frac{4}{(b-a)\operatorname{ess\,sup}_{t\in[a,b]} q(t)},$$
then inequality (3.2) *holds for all* $x \in \mathbf{W}_a^b$.

Consider some particular cases.

3.1. The constant function $q(t) = q$**.** It follows from 1.1 and 2.1 that $\overline{\gamma}_c = \frac{2}{(c-a)q}$, $\overline{\gamma}^c = \frac{2}{(b-c)q}$. Therefore,
$$\overline{\gamma} = \frac{4}{(b-a)q}$$
(this constant was obtained by C. Olech [8] and Z. Opial [9].)

3.2.a. The linear function $q(t) = t - a$. It follows from 1.2.a that $\overline{\gamma}_c = \frac{2\eta_1^2}{(c-a)^2}$, where $\coth \eta_1 = \eta_1$. It follows from 2.2.c that $\overline{\gamma}^c = \frac{2\varsigma^2}{(b-c)^2}$, where $\cot \varsigma = \frac{c-a}{b-c}\varsigma$. From here we have $\cot \varsigma = \eta_1$ and

$$\overline{\gamma} = \frac{2(\eta_1 + \operatorname{arccot} \eta_1)^2}{(b-a)^2} \approx \frac{7.1786291}{(b-a)^2}.$$

3.2.b. The linear function $q(t) = b - t$. It follows from 1.2.c that $\overline{\gamma}_c = \frac{2\varsigma^2}{(c-a)^2}$, where $\cot \varsigma = \frac{b-c}{c-a}\varsigma$. It follows from 1.2.c that $\overline{\gamma}^c = \frac{2\eta_1^2}{(b-c)^2}$, where $\coth \eta_1 = \eta_1$. Therefore, $\cot \varsigma = \eta_1$ and

$$\overline{\gamma} = \frac{2(\eta_1 + \operatorname{arccot} \eta_1)^2}{(b-a)^2} \approx \frac{7.1786291}{(b-a)^2}.$$

3.3.a. The power function $q(t) = (t-a)^p$, $p > -1$, $p \neq 0$. From 1.3.a, it follows that $\overline{\gamma}_c = \frac{(p+1)^2\eta_p^2}{2|p|(c-a)^{p+1}}$. From 2.4.b, it follows that $\overline{\gamma}^c = \frac{(p+1)^2\vartheta_p^2((\frac{b-a}{c-a})^{\frac{p+1}{2}})}{2|p|(c-a)^{p+1}}$. Then the exact constant is equal to

$$\overline{\gamma} = \frac{(p+1)^2(\eta_p \vartheta_p^{-1}(\eta_p))^2}{2|p|(b-a)^{p+1}},$$

where ϑ_p^{-1} is the inverse function to ϑ_p.

In particular, for $p = -\frac{1}{3}$ we have the exact estimate

$$\overline{\gamma} = \frac{2\vartheta^2}{3(b-a)^{\frac{2}{3}}} \approx \frac{2.91592138}{(b-a)^{\frac{2}{3}}},$$

where $\vartheta = \coth(\vartheta - \frac{\pi}{2}) \approx 2.09138281$; for $p = -\frac{3}{5}$ we have the exact estimate

$$\overline{\gamma} = \frac{2\vartheta^2}{15(b-a)^{\frac{2}{5}}} \approx \frac{2.03152023}{(b-a)^{\frac{2}{3}}},$$

where $3\vartheta \coth(\vartheta - \pi) = \vartheta^2 + 3$, $\vartheta \approx 3.90338337$.

Some approximate values of $\overline{\gamma}$ for $b - a = 1$ are given in Appendix. The graphic is shown in , and the graph is shown in Figure 7.

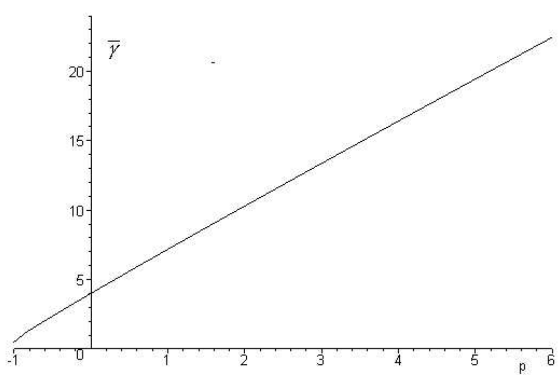

FIGURE 7. The graphic of the best constant in the case 3.3.a for $b - a = 1$

By Corollary 3.1, we have

$$\overline{\gamma} \geq \frac{2^{\frac{3p+2}{2p+1}}(p+1)^{\frac{4p+3}{2(2p+1)}}}{(b-a)^{p+1}} \text{ for } p \neq -\frac{1}{2} \quad (\overline{\gamma} \geq \sqrt{\frac{2e}{b-a}} \text{ for } p = -\frac{1}{2}).$$

By Corollaries 3.3 and 3.4, we obtain

$$\overline{\gamma} \geq \frac{2(p+2)}{(b-a)^{p+1}}, \quad \text{if } p > 0,$$

$$\overline{\gamma} \geq \frac{2(p+2)^{p+1}}{(b-a)^{p+1}(p+1)^p}, \quad \text{if } -1 < p < 0,$$

From estimates (1.16), (1.17), using Theorem 3.1, we obtain the estimate

$$\overline{\gamma} \geq \frac{(1-4p)^{p+1}}{2|p|(b-a)^{p+1}} \text{ for } -1 < p \leq -1/2. \tag{3.6}$$

Putting $x(t) = (t-a)^{\sqrt{p+\frac{5}{4}}}(b-t)$ in (3.2), we obtain the upper estimate

$$\overline{\gamma} \leq \frac{\sqrt{4p+5}\,(\sqrt{4p+5}+p)(\sqrt{4p+5}+p+1)(\sqrt{4p+5}+p+2)}{8(p+1)(p+2\frac{p+2+2\sqrt{4p+5}}{(1+\frac{2}{\sqrt{4p+5}})^{1+p+\sqrt{4p+5}}})}.$$

3.3.b. The singular case $q(t) = \frac{1}{t-a}$. Estimate (3.6) shows that for $\gamma \leq \frac{1}{2(b-a)^{p+1}}$ inequality (3.2) holds for all $x \in \mathbf{W}_a^b$ for $p > -1$. From here we can conclude that

$$\int_a^b \dot{x}^2(t)\,dt \geq \frac{1}{2}\int_a^b \frac{1}{t-a}\,|\dot{x}(t)x(t)|\,dt$$

for all $x \in \mathbf{W}_a^b$. The constant $\frac{1}{2}$ is exact. Note that the best constants in the cases 1.3.b and 3.3.b coincide and do not depend on $b-a$.

3.4.a. The power function $q(t) = (b-t)^p$, $p > -1$. It follows from 1.4.b that $\overline{\gamma}_c = \frac{(p+1)^2\vartheta_p^2((\frac{b-a}{b-c})^{\frac{p+1}{2}})}{2|p|(b-c)^{p+1}}$. Using 2.3.a, we get $\overline{\gamma}^c = \frac{(p+1)^2\eta_p^2}{2|p|(b-c)^{p+1}}$. Then the exact constant equals

$$\overline{\gamma} = \frac{(p+1)^2(\eta_p\vartheta_p^{-1}(\eta_p))^2}{2|p|(b-a)^{p+1}},$$

that is, $\overline{\gamma}$ coincides with the constant from 3.3.a.

3.4.b. The singular case $q(t) = \frac{1}{b-t}$. Similarly 3.3.b we have

$$\int_a^b \dot{x}^2(t)\,dt \geq \frac{1}{2}\int_a^b \frac{1}{b-t}\,|\dot{x}(t)x(t)|\,dt$$

for all $x \in \mathbf{W}_a^b$.

4. Appendix

4.1. Bessel functions for some half-integer indexes.

$J_{-\frac{3}{2}}(z) = -\sqrt{\frac{2}{\pi z}}\frac{1}{z}(z\sin z + \cos z)$, $Y_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{1}{z}(z\cos z - \sin z)$,

$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos z$, $Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z$, $J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z$,

$Y_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}}\cos z$, $J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{1}{z}(-z\cos z + \sin z)$,

$Y_{\frac{3}{2}}(z) = -\sqrt{\frac{2}{\pi z}}\frac{1}{z}(z\sin z + \cos z)$, $J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{1}{z^2}(-z^2\sin z - 3z\cos z + 3\sin z)$,

$Y_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{1}{z^2}(z^2\cos z - 3z\sin z - 3\cos z)$.

$I_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{1}{z}(z\sinh z - \cosh z)$, $K_{-\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2z}}\frac{1}{z}e^{-z}(z+1)$,

$I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cosh z$, $K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$,

$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sinh z$, $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$,

$I_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{1}{z}(z\cosh z - \sinh z)$, $K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2z}}\frac{1}{z}e^{-z}(z+1)$,

$I_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{1}{z^2}(z^2\sinh z - 3z\cosh z + 3\sinh z)$, $K_{\frac{5}{2}}(z) = \sqrt{\frac{\pi}{2z}}\frac{1}{z^2}e^{-z}(z^2 + 3z + 3)$.

4.2. Approximate best constants for some p in the case 1.3.a for $b-a=1$.

$$\overline{\gamma}_p = \frac{(p+1)^2\eta_p^2}{2|p|}, \quad p \neq 0.$$

p	−1.0	−0.99	−0.98	−0.97	−0.95	−0.90	−0.85	−0.8	−0.75	−0.7	−0.6
$\overline{\gamma}_p$	0.500	0.576	0.618	0.653	0.712	0.830	0.929	1.018	1.099	1.175	1.316

p	−0.5	−0.45	−0.4	−0.35	−0.3	−0.25	−0.2	−0.15	−0.1	−0.05	0
$\overline{\gamma}_p$	1.446	1.508	1.567	1.626	1.683	1.738	1.793	1.846	1.898	1.950	2.000

p	0	0.1	0.2	0.3	0.5	0.7	1	1.2	1.5	2	2.5	2.8
$\overline{\gamma}_p$	2	2.098	2.194	2.286	2.465	2.636	2.879	3.033	3.256	3.606	3.935	4.123

p	3	3.5	4	5	6	8	10	20	50	100	1000
$\overline{\gamma}_p$	4.245	4.541	4.824	5.357	5.854	6.766	7.593	10.976	18.000	26.103	86.938

4.3. Approximate best constants for some p in the case 1.4.a for $b-a=1$.

$$\overline{\gamma} = \frac{(p+1)^2\zeta_p^2}{2p}, \quad p \neq 0.$$

p	0	0.1	0.2	0.5	1.0	2	3	4	5	7	10
$\overline{\gamma}$	2.000	2.299	2.595	3.477	4.935	7.837	10.734	13.628	16.522	22.307	30.984

4.4. Approximate best constants for some p in the case 3.3.a for $b-a=1$.

$$\overline{\gamma} = \frac{(p+1)^2(\eta_p\vartheta_p^{-1}(\eta_p))^2}{2|p|}, \quad p \neq 0.$$

p	−1.0	−0.99	−0.97	−0.95	−0.90	−0.7	−0.5	−0.3	−0.1	0
$\overline{\gamma}$	0.500	0.5890	0.6943	0.7833	0.9826	1.69271	2.3657	3.0252	3.6766	4.0000

p	0.1	0.3	0.5	1.0	1.5	2	3	4	5	10
$\overline{\gamma}$	4.3211	4.9630	5.6003	7.7863	8.74181	10.293	13.368	16.415	19.443	34.396

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